Towards a classification of log del Pezzo surfaces of rank one.

$/\mathbb{C}$ work in progress.

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<u>Def)</u> A normal proj. surface S with quot. sing. is called a <u>log del Pezzo surface</u> if $-K_S$ is ample.

- it is <u>of rank one</u> if $\rho(S)=1$
- index $= \min\{r \in \mathbb{Z}_{>0} \mid -rK_S \text{ is Cartier}\}$.

Known results

1) low index r.

① $r=1$ (Gorenstein) Hidaka–Watanabe.

② $r=2$ Alexeev–Nikulin. Nakayama.

③ $r=3$ Fujita–Yasutake.

2) $\rho=1$

- Zhang. Miyanishi–Zhang, Gurjar–Zhang...
- Keel–McKernan. Hacking–Prokhorov.
- <u>Kojima</u> classified the case $\#\mathrm{Sing}=1$.
- <u>Belousov</u> $\#\mathrm{Sing} \le 4$.

<u>Thm1</u>(H) S: a log del Pezzo surface of $\rho=1$.
$f: S' \to S$ min resol
Assume $\rho(S') \ge 3$.
$\Rightarrow \exists$ a log dP surface T of $\rho=1$ of type "(G)" or "(KT)" s.t.
$\exists$ "explicit" birational map $S' \to T'$ where $T' \to T$ min. resol.

i.e.

$$\begin{array}{ccc} S' & \overset{\varphi}{\dashrightarrow} & T' \\ \downarrow & & \downarrow \text{min resol} \\ S & & T \end{array}$$

T ← type (G) or (KT)
(G): Gorenstein 27 cases
(KT): 12 cases.

→ φ can be decomposed by

$$S' = S_0 \overset{\varphi_1}{\dashrightarrow} S_1 \overset{\varphi_2}{\dashrightarrow} S_2 \dashrightarrow \cdots \overset{\varphi_n}{\dashrightarrow} S_n = T'$$

→ s.t. ① φ_i is either a blowdown or a special 'link'
② each S_i is a min. resol of a log del Pezzo surfaces of rank 1.

Rmk ① If $\rho(S') \le 2 \Rightarrow S = \mathbb{P}^2$ or $\overline{\mathbb{F}_n}$ where $\mathbb{F}_n \to \overline{\mathbb{F}_n}$ min. resol.

② φ is mostly blowdowns.

Gives ~~φ is a surjec~~ We can choose φ as a surj. morphism.

Starting from T, by reversing the process in φ, we can enumerate all log del Pezzo surfaces of $\rho=1$.

③ The proof works when S allows rational lc sing [KT]

④ del Pezzo surfaces of $\rho=1$ with nonrat sing are completely classified (Fujisawa, Cheltsov). → 1 simple ell. sing in lc case.

⑤ (H-& Park)

(S, D) : a weak lc del Pezzo pair (no assumption on ρ). & S' has 1 simple ell sing & RDPs..

$$\Rightarrow \begin{array}{ccc} S' & \longrightarrow & T' \\ \downarrow & & \downarrow \text{min resol.} \\ S & & T \end{array}$$

T ← del Pezzo surface

$S' \to T'$: "redundant" blowups.

Notation

S : a log dP of $\rho=1$. $f: S' \to S$ min resol.

$D = \text{reduced}(f^{-1}(\text{Sing}(S)))$. $r = \#\text{Sing}(S)$. ($\to 1 \le r \le 4$)

§ Zhang's theory.

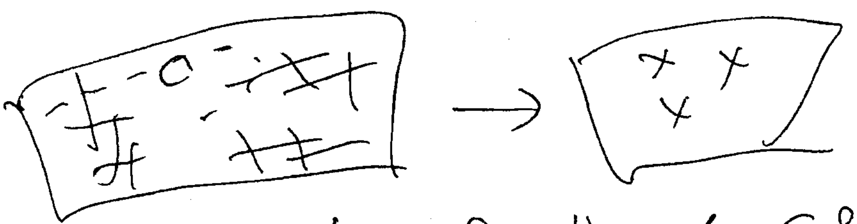
§ Idea of Proof

§ Further Discussion.

§ Zhang's Theory

- $C \subset S'$ ~~a curve~~ an irr. curve.

Def] C is minimal if $C.(f^*(-K_S))$ attains a minimal positive value



→ Classify the configuration of C & D.

Known results

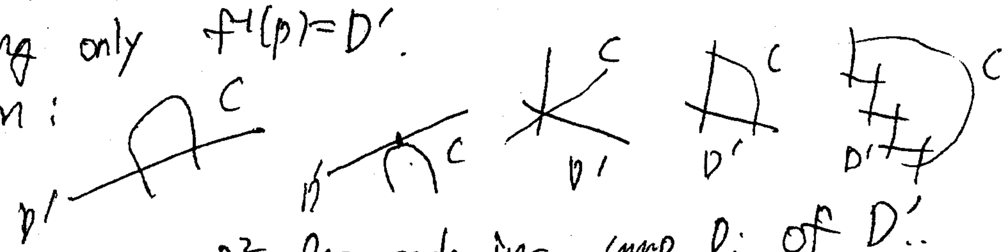
1) the case $|C+D+K_{S'}| \neq \phi$

$\Rightarrow$ · S has $r-1$ RDPs & 1 cyclic sing p

(Z.D) · C : a sm rat curve with $C^2=0$ or -1.

meeting only $f^{-1}(p)=D'$.

· configuration :



→ need to know D_i^2 for each irr comp D_i of D'.

2) the case $|C+D+K_{S'}| = \phi$

$\Rightarrow$ · C : a sm rat curve with $C^2=-1$

· C meets each conn. comp of D at most once (RR)

· $C.D \le 3$.

Let D_i $(i=1,2,3)$ be the irr comp of D meeting C.

① $C.D=1 \Rightarrow D_1^2=-2$

② $C.D=2 \Rightarrow (D_1^2, D_2^2) = \begin{cases} (-2,-2) \\ (-2,-n) \quad (n\ge 3) \end{cases}$

③ $C.D=3 \Rightarrow (D_1^2, D_2^2, D_3^2) = \begin{cases} (-2,-2,-n) \\ (-2,-3,-n) \quad (n=3,4,5) \end{cases}$

Completly classified in [KT] ← type (KT)

$\varphi = \mathrm{contr}_C : S' \to S'$

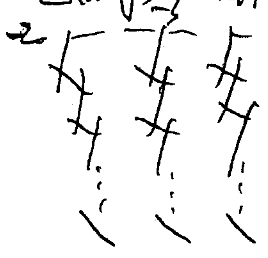
$\downarrow \quad \downarrow$

$S \quad S_1$ ← ⇒ a log dP of $\rho=1$

hardest

(Gurjar–Zhang)

$\Rightarrow$



& possibly 1 more sing pt.

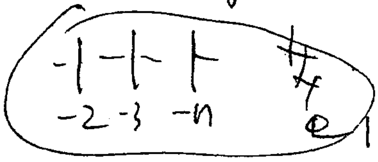
New results (H−)

Lemma] $|C+D+K_{S'}| = \emptyset$

1) If $CD=3$,

$\Rightarrow$ ① one of 2 cases in [KT]

or ② #Sing $=4$ & the configuration is as follows:



1 RDP

← all classified!

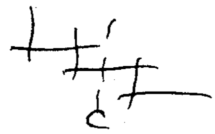
($\Rightarrow$ all maps to type (G) ~~[illegible]~~)

2) If $CD=2$,

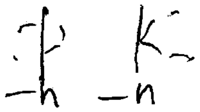
$\Rightarrow$ ① one of 10 cases in [KT]

or ② $(D_1^2, D_2^2) = (-2, -n)$ $(n \geq 3)$

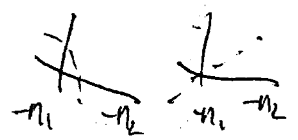
3) If $CD=1$, $\Rightarrow$ $D_1^2 = -2$ & $D_1(D-D_1) \geq 2$.

eg) 

Lemma] $|C+D+K_{S'}| \neq \emptyset$

1) 

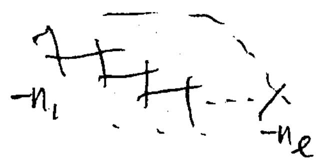
$\Rightarrow$ $C^2=-1$ & $n=2,3$. ← all classified.

2) 

$\Rightarrow$ · $n_1=2$. & $n_2=2,3,4$.

& · 2 cases in [KT]

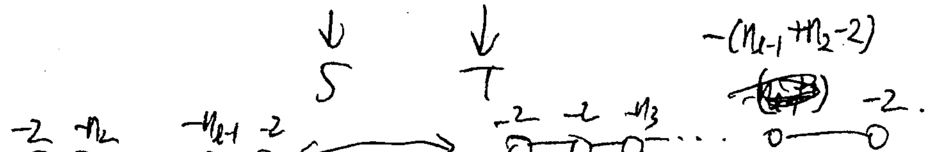
or · ≤ 20 cases ← can be classified.

3) 

$\Rightarrow$ ~~either~~ $(n_1, n_e) = \begin{cases} (2, n) & (n>2) \text{ [← use induction} \\ (2,2) \end{cases}$

$\boxed{\text{"trick"}}$

$$\begin{array}{ccc} S' & \dashrightarrow & T' \\ \downarrow & & \downarrow \\ S & & T \end{array}$$

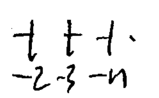


§ Idea of Proof

<u>Induction</u> on $\#\mathrm{comp}(D)$

- OK if it is of type (G) or (KT).
- If not,

  1) choose a min curve C on S' with $|C+D+K| = \emptyset$

  $\Rightarrow$ • $CD = 3$ &  $\Rightarrow$ maps to Type G_1

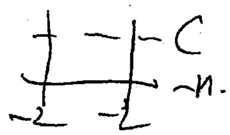
  • $CD \leq 2 \Rightarrow$

$$\begin{array}{ccc} S' & \xrightarrow{\varphi} & T' \\ \downarrow & & \downarrow \text{min resol} \\ S & & T \end{array}$$

  $T \leftarrow$ a log dP of rk 1 with $\rho(T') = \rho(S') - 1$.

  2) $C.D = 2$.

  $\Rightarrow$ similarly we can reduce ρ by $\underline{\underline{3}}$.. using similar method as "trick".

<u>Lemma</u>) $\nexists$ no minimal curve with 

Since $\rho(S')$ is finite, it ends with f.m. steps. $\boxed{A}$

§ Further Discussion

Further steps to complete classification.

① Find all $\left\{ \begin{array}{l} (-1)\text{-curves on the "base" surfaces.} \\ (n)\text{-curves } C \ (n \geq 0) \text{ with } C+D' \text{ forms a circle} \\ \text{for a conn. comp } D' \text{ of } D. \end{array} \right.$

② Consider all possible reverse operations.

③ Enumerate all surfaces.

possible by using formular

using <u>"formula"</u>.

eg) A_8