THE STUDY OF THE HOMOLOGY OF KUGA VARIETIES

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THE STUDY OF THE HOMOLOGY OF KUGA VARIETIES

UDC 517.4

V. V. SOKUROV

Abstract. The homology of Kuga varieties is studied. A nondegenerate pairing is constructed between certain homology spaces and modular forms.

Bibliography: 10 titles.

This article continues the proof, begun in [7], of a series of results announced in [6] on periodic cusp forms on Kuga varieties. The author thanks Professor Ju. I. Manin, during the course of whose seminar this work was completed.

§0. Main results

Let \( \Gamma \subset \text{SL}(2, \mathbb{Z}) \) be a subgroup of finite index. We denote by \( (\Gamma, w) \) a pair such that either the integer \( w \) is even or the following condition on \( \Gamma \) holds:

\[
-E \not\in \Gamma \quad (\ast)
\]

(see (\ast) of §4 of [5] and §0 of [7]). This article continues [7] and uses its notation. In particular \( \Delta_{\Gamma} \) and \( B_{\Gamma} \) are the modular curve and elliptic modular surface for \( \Gamma \) (see §5 of [7]). The corresponding canonical projection is \( \Phi_{\Gamma} : B_{\Gamma} \to \Delta_{\Gamma} \). In the sequel we will sometimes omit the index \( \Gamma \) for simplicity.

0.1. Let \( S_{w+2}(\Gamma) \) be the space of \( \Gamma \)-cusp forms of weight \( w+2 \) (see §2.1 of [3]). The main goal of this article is to define a canonical pairing

\[
(\, , \, ) : H_1\left(\Delta_{\Gamma}, \Sigma, \left( R_1, \Phi_* \mathbb{Q}\right)^w\right) \times S_{w+2}(\Gamma) \oplus S_{w+2}(\Gamma) \to \mathbb{C},
\]

where \( \Sigma \subset \Delta_{\Gamma} \) is any finite subset.

0.2. Theorem. The canonical pairing \( (\, , \, ) \) is nondegenerate on

\[
H_1\left(\Delta_{\Gamma}, \left( R_1, \Phi_* \mathbb{Q}\right)^w\right) \times S_{w+2}(\Gamma) \oplus S_{w+2}(\Gamma).
\]

The proof of this theorem is given in §6.

0.3. The construction of the pairing \( (\, , \, ) \) is based on the existence of (i) a canonical isomorphism

\[
H^0\left( B_{\Gamma}^w, \Omega^{w+1} \oplus \Omega^{w+1}\right) \simeq S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)},
\]

the proof and construction of which are given in [7]; (ii) a canonical homomorphism

\[
GR_{1,w} : H_1\left(\Delta_{\Gamma}, \Sigma, \left( R_1, \Phi_* \mathbb{Q}\right)^w\right) \to H_{1+w}(B_{\Gamma}^w, \left| B_{\Gamma}^w \right| \Sigma, \mathbb{Q}),
\]

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where \( \Sigma \) is a finite set of points of \( \Delta_\Gamma \) containing the points of singular type (see §1 of [7]), the construction of which is given in §3; and (iii) a canonical pairing

\[
H_{w+1}(\mathcal{E}_\Gamma', \mathcal{E}_\Gamma'|_\Sigma, \mathcal{Q}) \times H^0(\mathcal{E}_\Gamma', \Omega^{w+1} \otimes \overline{\Omega}^{w+1}) \mapsto \mathbb{C},
\]

(homology class \( \sigma, \omega \)) \( \mapsto \int_\sigma \omega \).

**0.4.** In §3 we carry out the construction of the “geometrical realization” homomorphisms \( GR_{i,j} \) \((0 < i < 2, 0 < j < 2w)\). Theorem 1 of [6] corresponds to 3.2, and this result may easily be proved over \( \mathbb{Z} \) by the same methods. We make the change to \( \mathbb{Q} \) for consistency, since in the sequel symmetrization will frequently occur, where division by \( w \) is needed! Theorem 2 of [6] is a simple corollary of Theorem 1 of [6].

Theorem 3 of [6] corresponds to Theorem 4.2 in this article, and Theorem 4 of [6] is a slight variation of Theorem 2.5. Finally, Theorem 6 of [6] corresponds to the special case of Corollary 6.1 with \( K = \mathbb{R} \).

**§1. Neighborhood retracts**

Let \( X \) be an analytic variety, \( D \subset \mathbb{C} \) the disk with center at 0, \( D^* = D - \{0\} \), and \( \Phi : X \to D \) a proper morphism. In addition we assume that the fiber \( \Phi^{-1}(0) \) has normal type. This means that for any point \( x \in \Phi^{-1}(0) \) there exist a neighborhood \( U \subset X \) and coordinates \( X_1, \ldots, X_n \) \((n = \dim X)\) in this neighborhood in which the canonical projection takes monomial form, i.e. \( \Phi|_U = X_1^{m_1} \cdots X_n^{m_n} \) for some positive integers \( m_i \) \((1 < i < n)\). Then by Thom’s isotopy theorem \( X' = \Phi^{-1}(E^*_1) \) is a topological fiber space over \( E^*_1 = E_1 - \{0\} \), where \( E_1 = \{z \in \mathbb{C} | |z| < \varepsilon\} \subset D \) for suitable \( 0 < \varepsilon \).

**1.1. LEMMA.** For any sufficiently small \( \varepsilon \) there exists a deformation retract (see [1], p. 28) of \( \Phi^{-1}(E_1) \) onto \( \Phi^{-1}(0) \).

Corollary 1.2 is obtained from this lemma. Let \( B^w \) be Kuga’s variety corresponding to the elliptic surface \( B \). Consider a pair of topological subvarieties \( \Delta \supset F \supset F' \) with smooth boundary. Then to the mapping of pairs \((\Delta, F') \leftrightarrow (\Delta, F)\) there corresponds the homomorphism in homology

\[
H_i(B^w, B^w|_F, \mathcal{Q}) \to H_i(B^w, B^w|_F, \mathcal{Q}).
\]

In particular, those \( F \) consisting of small closed disks around points of the set \( \Sigma \) give rise to a projective system of vector spaces \( H_i(B^w, B^w|_F, \mathcal{Q}) \) with morphisms (1.1).

**1.2. COROLLARY.** There is a canonical isomorphism

\[
H_i(B^w, B^w|_\Sigma, \mathcal{Q}) \cong \lim \leftarrow H_i(B^w, B^w|_F, \mathcal{Q}).
\]

**PROOF.** If \( N \subset M \) and \( N \) is a deformation retract of \( M \), then \( H_i(M, N, \mathcal{Q}) = 0 \). Therefore, by Lemma 1.1 and 3.4 of [7], \( H_i(B^w|_F, B^w|_\Sigma, \mathcal{Q}) = 0 \) for \( F \) consisting of sufficiently small disks. Then from the exact sequence

\[
\to H_i(B^w|_F, B^w|_\Sigma, \mathcal{Q}) \to H_i(B^w, B^w|_\Sigma, \mathcal{Q}) \to H_i(B^w, B^w|_F, \mathcal{Q}) \to
\]

of the triple \((B^w, B^w|_F, B^w|_\Sigma)\) it follows that (1.1) is an isomorphism for sufficiently small \( F \) and \( F' = \Sigma \).

**PROOF OF LEMMA 1.1.** The only condition on \( \varepsilon + 0 \) is the condition preceding Lemma 1.1, i.e. the local triviality of \( X' = \Phi^{-1}(E^*_1) \) over \( E^*_1 \). Indeed, one can easily show,
because of the normality of the fiber Φ⁻¹(0), that it is a neighborhood deformation retract, i.e. there exists a neighborhood X' ⊃ U ⊃ Φ⁻¹(0) which admits a deformation retract onto Φ⁻¹(0). On the other hand, clearly there exists 0 < ε' < ε such that V = Φ⁻¹(\{z \in \mathbb{C} | |z| < ε'\}) ⊂ U. Also it is easy to construct a deformation retract of X' onto V. Combining the latter deformation with the restriction to V of the first deformation, we obtain the desired one.

§2. Homology with coefficients in the sheaf \( R \Phi^w Q \)

2.1. The sheaf \( R \Phi^w Q \) is obtained by extending from ∆' = Δ - Σ (see §1 of [7]) over Δ the sheaf of local coefficients \( \bigcup_{v \in \Delta'} J_v(B_v^w, Q) \) in the following way: for a small disk E around v \( \in \Sigma \) and \( E' = E - v \)

\[ \Gamma(E, R \Phi^w Q) = \Gamma(E', R \Phi^w Q). \]

For example, \( R \Phi^1 Q = R \Phi Q = G \otimes Q \), where \( G \) is the homological invariant of the elliptic surface \( B \).

2.2. Fix a basis in the lattice \( G|_{u_0} \subset R \Phi^1 Q|_{u_0} \). Then a representation of the group \( SL(2, Q) \) in \( R \Phi^1 Q|_{u_0} \) is determined. For any integer \( w > 0 \) the representation of \( SL(2, Q) \) in the tensor power \( (R \Phi^w Q)^{\otimes w}|_{u_0} \) decomposes into a direct sum of irreducible representations of \( SL(2, Q) \). Each irreducible representation of \( SL(2, Q) \) is a representation in a symmetric power \( (R \Phi^w Q)^m|_{u_0} \); the positive integer \( m \) usually is called the order of the irreducible representation. The identification of the subspace which is the sum of all irreducible representations of order \( m \) in \( (R \Phi^w Q)^{\otimes w}|_{u_0} \) does not depend on the choice of basis in the lattice \( G|_{u_0} \). The dimension \( r_m^w \) of this subspace also is independent of the choice of the point \( u_0 \in \Delta' \). The group \( A_w \) of permutations of \( w \) elements acts naturally on the space \( (R \Phi^w Q)^{\otimes w}|_{u_0} \):

\[ a : x_1 \otimes \ldots \otimes x_w \mapsto x_{a(1)} \otimes \ldots \otimes x_{a(w)}, \]

where \( x_i \in R \Phi^w Q|_{u_0} \) and \( a \in A_w \). The space \( (R \Phi^w Q)^w|_{u_0} \) admits a canonical embedding into \( (R \Phi^w Q)^{\otimes w}|_{u_0} \):

\[ x_1 \ldots x_w \mapsto \frac{1}{w!} \sum_{a \in A_w} a(x_1 \otimes \ldots \otimes x_w). \]

In the sequel \( (R \Phi^w Q)^w|_{u_0} \) will be identified with its canonical image in \( (R \Phi^w Q)^{\otimes w}|_{u_0} \); \( (R \Phi^w Q)^w|_{u_0} \) is an invariant subspace of the representation of \( SL(2, Q) \).

2.3. Proposition. a. \( r_m^w = 0 \) if \( m \equiv w \) (mod 2).

b. \( r_w^w = 1 \).

c. There is the following direct sum decomposition into subspaces invariant for \( SL(2, Q) \):

\[ (R \Phi^w Q)^{\otimes w}|_{u_0} = (R \Phi^w Q)^w|_{u_0} \oplus \left( \sum_{a \in A_w} a((e_1 \otimes e_2 - e_2 \otimes e_1) \otimes (R \Phi^w Q)^{w-2}|_{u_0}) \right) \]

\( (\Sigma \) is not a direct sum).

d. The space of invariant vectors of \( (R \Phi^w Q)^{\otimes w}|_{u_0} \), i.e. the sum of irreducible subspaces of order 0, is generated by the vectors

\[ a((e_1 \otimes e_2 - e_2 \otimes e_1)^{\otimes w}), \]

where \( a \in A_w \) (by \( a, w \) is even in this case); \( e_1, e_2 \) are a basis of the lattice \( G|_{u_0} \).
2.4. By the Künneth formula, since \(B_{x_0}^w = B_{x_0} \times \cdots \times B_{x_0} \) (w terms), we have

\[
R_j \Phi^{w_0} Q \mid _{x_0} = \bigoplus_{j_1 + \cdots + j_w = j} \bigotimes_{m=1}^w R_{j_m} \Phi^m Q \mid _{x_0},
\]

where \(0 < j_m < 2\). The representation \(S\) (7), 1.4 and the trivial representation \(\pi_1(\Delta')\) in \(R_0 \Phi \cdot Q \mid _{x_0}\) and \(R_2 \Phi \cdot Q \mid _{x_0}\) give a representation of the fundamental group \(\pi_1(\Delta)\) in \(R_j \Phi^{w_0} Q\). This representation, which will also be denoted by \(S\), is uniquely defined by the sheaf \(R_j \Phi^{w_0} Q\). Since \(\dim R_0 \Phi \cdot Q \mid _{x_0} = \dim R_2 \Phi \cdot Q \mid _{x_0} = 1\), there is a noncanonical isomorphism

\[
\bigotimes_{m=1}^w R_{j_m} \Phi^m Q \mid _{x_0} \simeq (R_1 \Phi \cdot Q)^{\otimes w'},
\]

where \(w'\) is the number of \(j_m = 1\), \(w' = j_1 + \cdots + j_w = j\) (mod 2). We have that \(S(\pi_1(\Delta')) \subset SL(2, \mathbb{Q})\), so we may consider the representations of \(\pi_1(\Delta')\) on the subspace \((R_1 \Phi \cdot Q)^{\otimes w'} \mid _{x_0}\) invariant with respect to \(SL(2, \mathbb{Q})\). Below (see Lemma 2.7) we will prove their irreducibility with respect to \(\pi_1(\Delta')\). The decomposition of the space \((R_1 \Phi \cdot Q)^{\otimes w'} \mid _{x_0}\) into irreducible subspaces corresponds to a decomposition of the sheaf \((R_1 \Phi \cdot Q)^{\otimes w'}\) into a direct sum of symmetric sheaves \((R_1 \Phi \cdot Q)^m\), which we will also denote by \(S_m\). We obtain from (2.1), (2.2), and 2.2 a canonical decomposition into a direct sum

\[
R_j \Phi^{w_0} Q = \bigoplus_m S_{m_j}^{\otimes w'}, \tag{1}
\]

where \(r_{m_j}\) is the number of irreducible representations of order \(m\) in \(R_j \Phi^{w_0} Q \mid _{x_0}\), this number not depending on the choice of \(u_0 \in \Delta'\). The decomposition of \(S_{m_j}^{\otimes w'}\) into a sum of sheaves \(S_m\) is not canonical.

2.5. Theorem. a. \(H_1(\Delta, R_j \Phi^{w_0} Q) = \bigoplus_m H_j(\Delta, S_m)^{\gamma_{x_0}}\).

b. \(\dim H_0(\Delta, S_m) = \dim H_2(\Delta, S_m) = \begin{cases} 0 & \text{for } m > 0, \\ 1 & \text{for } m = 0. \end{cases}\)

c. For even \(m > 0\)

\[
\dim H_1(\Delta, S_m) = 2(g - 1)(m + 1) + \sum_{b \geq 1} m(\nu(I_b) + \nu(I_b^*))
\]

\[
+ 2 \left[ \frac{m + 2}{3} \right] (\nu(II) + \nu(II^*) + \nu(IV) + \nu(IV^*))
\]

\[
+ 2 \left[ \frac{m + 2}{4} \right] (\nu(III) + \nu(III^*)).
\]

For odd \(m > 0\)

\[
\dim H_1(\Delta, S_m) = 2(g - 1)(m + 1) + \sum_{b \geq 1} m\nu(I_b)
\]

\[
+ (m + 1) \sum_{b \geq 0} (\nu(I_b^*) + \nu(II^*) + \nu(II) + \nu(III) + \nu(III^*))
\]

\[
+ 2 \left[ \frac{m + 2}{3} \right] (\nu(IV) + \nu(IV^*)).
\]

For \(m = 0\)

\[
\dim H_1(\Delta, S_m) = 2g.
\]

(1) In this article \(V^m\) denotes a direct power, and \((V)^m\) the tensor symmetric power over \(\mathbb{Q}\).
Here \( v(\ast) \) is the number of fibers of type \( \ast \) of the elliptic surface \( B \), and \( \lfloor \cdot \rfloor \) as usual denotes the integer part.

d. \( r_{j,m}^{\ast} = 0 \) for \( j \equiv m \) (mod 2).

2.6. Corollary. \( H_0(\Delta, R_j \Phi_{\ast}^* Q) = H_2(\Delta, R_j \Phi_{\ast}^* Q) = 0 \) for odd \( j \). ■

2.7. Lemma. The representation \( S \) of the fundamental group \( \pi_1(\Delta') \) in \( (R_j \Phi_{\ast} Q)^m |_{u_0} \) is irreducible also with respect to this representation:

a. \( ((R_j \Phi_{\ast} Q)^m |_{u_0})^{\text{inv}} = 0 \).

b. \( ((R_j \Phi_{\ast} Q)^m |_{u_0})^{\text{coinv}} = 0 \) for \( w > 1 \).

c. The following table shows the dimension of the space of sections of the sheaf \( S_m \) over the point \( v \) depending on the type of point.

<table>
<thead>
<tr>
<th>type of point ( v )</th>
<th>I*</th>
<th>I*</th>
<th>I, II*</th>
<th>II, III*</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m &gt; 0 ) even</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( m &gt; 0 ) odd</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

2.8. Following Shioda [5], we construct a complex \( M \) which allows us to compute the dimension of the homology spaces \( H_i(\Delta, S_m) \) (we remark that these spaces are isomorphic to the cohomology spaces \( H^{2-i}(\Delta, S_m) \); see for example §7 of [5]). Fix a point \( u_0 \in \Delta' \). Let \( \Sigma = \{ v_1, \ldots, v_t \} \). As in the proof of Lemma 1.5 of [7], we choose the following system of generators \( \alpha_k, \beta_k \) (1 \( \leq k \leq g \), where \( g \) is the genus of the curve \( \Delta \)) and \( \gamma_l \) (1 \( \leq l \leq t \)) of the fundamental group \( \pi_1(\Delta') = \pi_1(u_0, \Delta) \) with the single relation

\[
\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} \gamma_1 \cdots \gamma_t = 1.
\]  

We consider a small positively oriented disk \( E_i \) around each point \( v_i \in \Sigma \). Set \( \gamma'_l = -\partial E_i \). In each oriented circle \( \gamma'_l \) we fix a point \( u_n \), and then we choose a path \( \delta_i \) from \( u_0 \) to \( u_l \) such that \( \delta_i \gamma'_l \delta_i^{-1} \) is homotopic to \( \gamma_l \). We consider the following complex \( \Delta \): the 0-cells are \( u_l \) (0 \( \leq l \leq t \)), the 1-cells are \( \alpha_k, \beta_k \) (1 \( \leq k \leq g \), \( \delta_i \) and \( \gamma'_l \) (1 \( \leq l \leq t \)), and the 2-cells are \( E_i \) (1 \( \leq l \leq t \)) and \( \Delta_0 = \Delta - \cup E_i \).

The \( i \)-chains \( \sigma_i \) with coefficients in the sheaf \( (R_j \Phi_{\ast} Q)^m = S_m \) have the following form:

\[
\sigma_0 = \sum_{l=0}^{t} m_l u_l,
\]

\[
\sigma_1 = \sum_{k=1}^{g} (a_k \alpha_k + b_k \beta_k) + \sum_{l=1}^{t} (c_l \gamma'_l + d_l \delta_l),
\]

\[
\sigma_2 = e \Delta_0 + \sum_{l=1}^{t} e_l E_l,
\]

where the coefficients \( m_l, a_k, \ldots, e \in (R_j \Phi_{\ast} Q)^m |_{u_0} \) and \( e_l \in ((R_j \Phi_{\ast} Q)^m |_{u_0})^{\text{coinv}} \), i.e. \( e_l = e_i \beta_i \). Let \( \beta_k = S_{\alpha_k}, \beta_k = S_{\beta_k}, \alpha_k = S_{\gamma_k}, \) and \( \beta_k = \partial_k \beta_k \partial_k^{-1}, \beta_{\gamma_k} = \beta_{\gamma_k} \cdots \beta_{\gamma_k} \) and
\( \mathcal{D}^{(0)} = C_1 \cdots C_l \) \((C^{(0)} = \mathcal{C}^{(0)} = 1, C^{(l)} = \mathcal{C})\). The boundary operator is then rewritten in the following form:

\[
\begin{align*}
\partial(a_k \alpha_k) &= a_k(\alpha_k - 1)u_0, \quad \partial(b_k \beta_k) = b_k(\beta_k - 1)u_0, \\
\partial(c_i \gamma_i) &= c_i(\gamma_i - 1)u, \quad \partial(d_i \delta_i) = d_i u - d_i u_0, \\
\partial(e \Delta_0) &= \sum_{k=1}^{g} e \mathcal{E}(k-1)((1 - \mathcal{Q}_k \mathcal{B}_k \mathcal{Q}_k^{-1}) \alpha_k + (\mathcal{Q}_k - \mathcal{Q}_k) \beta_k) \\
&+ \sum_{i=1}^{t} e \mathcal{E}(\mathcal{C}^{(l-1)} - \mathcal{C}^{(l)}) \delta_i + \mathcal{C}^{(l-1)} \gamma_i), \quad \partial(e_i E_i) = -e_i \gamma_i.
\end{align*}
\]

Therefore a complex \( \mathcal{M} \) of vector spaces over \( \mathbb{Q} \)

\[
M_0 \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_2} M_2,
\]

(2.7)
is determined, where

\[
M_0 = S_\infty \bigoplus_{i=1}^{t} S_m |_{v_i}, \quad M_1 = S_{m+1} |_{u_i}, \quad M_2 = S_m |_{u_0},
\]

and \( \partial_1(e, e_1, \ldots , e_t) = (a_k, b_k, c_i) \) for

\[
a_k = e \mathcal{E}(k-1)(1 - \mathcal{Q}_k \mathcal{B}_k \mathcal{Q}_k^{-1}), \\
b_k = e \mathcal{E}(k-1)(\mathcal{B}_k - \mathcal{Q}_k), \\
c_i = e \mathcal{E} \mathcal{C}^{(l-1)} - e_i,
\]

and \( \partial_2 \) is given by

\[
\begin{align*}
\partial_2(a_k, b_k, c_i) &= \sum_{k=1}^{g} (a_k(\alpha_k - 1) + b_k(\beta_k - 1)) + \sum_{i=1}^{t} c_i(\gamma_i - 1).
\end{align*}
\]

From (2.5)–(2.7) it is easy to obtain an isomorphism of the homology spaces \( H_i(\Delta, S_m) \) with the cohomology spaces \( H^{2-i}(M) \) of the complex (2.7).

**Proof of Theorem 2.5.** Part a is an obvious corollary of (2.3).

b. The case \( m = 0 \) is obtained from the fact that \( S_0 = \mathbb{Q} \), the constant sheaf of vector spaces of dimension 1, i.e. there is a canonical isomorphism \( H_i(\Delta, S_m) \cong H_i(\Delta, \mathbb{Q}) \). The case \( m = 0 \) of part c follows obviously from this isomorphism.

By 2.8 there are isomorphisms

\[
\begin{align*}
H_0(\Delta, S_m) \cong H^2(M) &= \text{Coker } \partial_2 = (\langle R_1 \Phi \mathbb{Q} \rangle^m |_{u_0})^{\text{cinv}}, \\
H_2(\Delta, S_m) \cong H^0(M) &= \text{Ker } \partial_1 = (\langle R_1 \Phi \mathbb{Q} \rangle^m |_{u_0})^{\text{inj}}.
\end{align*}
\]

Then by Lemma 2.7 a and b we obtain the proof of part b for \( m > 1 \).

c. Let \( m > 1 \). By the previous part there is an exact sequence

\[
0 \rightarrow M_0 \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_2} M_2 \rightarrow 0.
\]

Then the direct calculation

\[
\dim H_1(\Delta, S_m) = \dim H^1(M) = \dim M_1 - \dim M_0 - \dim M_2
\]

(see 2.8), using the dimension of the space \( S_m |_{v_i} \) given in the table of Lemma 2.7, proves part c.

Part d follows from part a of the theorem, (2.2), and part a of Proposition 2.3.
PROOF OF PROPOSITION 2.3. Part a is proved by induction on $w$ using Theorem 2 in §18.2 of [8]. Similarly we obtain $b$.

c. The action of $SL(2, \mathbb{Q})$ commutes with the action of $A_w$. Moreover,

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} (e_1 \otimes e_2 - e_2 \otimes e_1) = (ad - bc) (e_1 \otimes e_2 - e_2 \otimes e_1) = e_1 \otimes e_2 - e_2 \otimes e_1,
\]

where $\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL(2, \mathbb{Q})$. Therefore the spaces in the decomposition are invariant for the action of $SL(2, \mathbb{Q})$. The exactness of the sequence

\[
0 \to \sum_{a \in A_w} \alpha ((e_1 \otimes e_2 - e_2 \otimes e_1) \otimes (R_1 \Phi_w, \mathbb{Q})^{w-2} \mid u_a) \to (R_1 \Phi_w, \mathbb{Q})^{w} \mid u_a \to 0
\]

is obvious, which proves part c.

Part d is proved by induction for even $w$; the case of odd $w$ is trivial by a. The case $w = 0$ follows because $(R_1 \Phi_w, \mathbb{Q})^{w0} = \mathbb{Q}$ and $a((e_1 \otimes e_2 - e_2 \otimes e_1)^0) = 1$. Further inductive steps are obtained from part c and Lemma 2.7a.

PROOF OF LEMMA 2.7. c. Consider a point $u_0 \in \Delta'$ sufficiently close to $v$, and a small positive circuit $\gamma' \subseteq \Delta'$ around $v$ beginning and ending at $u_0$. In the lattice $G\mid u_0 = H_1(B_u, \mathbb{Z})$ choose a basis $e_1, e_2$ in which the monodromy $s_\beta$ ([7], (1.3)) has the normal form (see §1 of [7]) $\bar{e}_v$.

\[
S_m \mid e = (S_m \mid u_0)^{s_\beta} \simeq (\mathbb{Q} e_1 \oplus \mathbb{Q} e_2)^m \bar{e}_v.
\] (2.8)

From Table 1 of [7] we obtain the following form of the monodromy in the basis $e_\alpha = e_1^\alpha e_2^{m-\alpha}$, $0 < \alpha < m$, of the space $(\mathbb{Q} e_1 \oplus \mathbb{Q} e_2)^m$ for points $v$ of type $I_0$ or $I_0^*$ ($b > 0$):

\[
e_\alpha \mapsto (\pm 1)^m (e_1 + be_2)^\alpha e_2^{m-\alpha} = (\pm 1)^m \left( e_\alpha + \alpha be_\alpha - 1 + \sum_{i \leq \alpha - 2} \ast \cdot e_i \right).
\]

Therefore the monodromy matrix is $(\pm 1)^m$ for $b = 0$ and

\[
(\pm 1)^m \begin{pmatrix} 1 & 0 \\ b & 2b \\ \ast & \pm b \end{pmatrix},
\] (2.9)

for $b > 1$, the action being on the right, with the + sign corresponding to $I_0$ and the – sign corresponding to $I_0^*$. Then by (2.8) we obtain the first four columns of our table.

To compute our table at points with finite monodromy we use the relation

\[
\dim_{\mathbb{Q}}((\mathbb{Q} e_1 \oplus \mathbb{Q} e_2)^m)^{\bar{e}_v} = \dim_{\mathbb{C}}((\mathbb{C} e_1 \oplus \mathbb{C} e_2)^m)^{\bar{e}_v}.
\]

For a given point in $\mathbb{C} e_1 \oplus \mathbb{C} e_2$ there exists a new basis in which $\bar{e}_v$ is diagonal. Depending on the type $II$, $II^*$; $III$, $III^*$; $IV$, $IV^*$ of the point in Table 1 of [7], we obtain a corresponding diagonal matrix:

\[
\begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ \ast & e^{-\frac{2\pi i}{3}} \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}; \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}, \quad \eta = e^{\frac{2\pi i}{3}}.
\]
Therefore in some basis for the space $C_1 \oplus C_2$ the monodromy $\mathcal{G}_v$ has the matrix

$$
\begin{pmatrix}
\varepsilon_\kappa^0 & e_\kappa^{-m} & 0 \\
\vdots & \ddots & \ddots \\
0 & \cdots & e_\kappa^m
\end{pmatrix},
$$

where $\kappa$ corresponds to the type of the point $v$ in Table 2 of [7]. Consequently we obtain by (2.8) that

$$
\dim Q S_m | v = \# \{ 0 \leq \alpha \leq m | e_\kappa^{2\alpha} = e_\kappa^m \},
$$

from which the last three columns of our table follow by an easy computation.

a. The irreducibility of the representation $S$ is obvious for $w = 0$. Suppose $w > 1$. Then to prove part a it suffices to establish the irreducibility of the representation $S$ of the fundamental group $\pi_1(\Delta')$ in $(R, \Phi Q)_{\kappa, \kappa'}^{-w}$. Recall that the matrix of the representation $S$ acts on the right. Since the functional invariant $J \equiv \text{const}$, there exists a point $v \in \Sigma$ of type $I_b$ or $I^*_b$ ($b > 1$) (see the values of $J(v)$ in Table 2 of [7]). Choose a point $u_0 \in \Delta'$ and a basis $e_1, e_2$ of the lattice $G_{1,w_0}$, as was done in part c. Then in the basis $e_0, \ldots, e_w$ (see c) there is a matrix of the representation $S$ of form (2.9). The invariant subspaces for the group generated by the matrix (2.9) have the form $\bigoplus_{\alpha=0}^m Q e_\alpha, 0 < m < w$. Suppose the representation $S$ is reducible. In this case the subspace $e_\alpha Q e_\alpha$ is invariant for $\pi_1(\Delta')$ for some $0 < m < w$. Consider the matrix $S_\gamma = (e_\gamma^b) \in SL(2, \mathbb{Z})$ for any arbitrary $\gamma \in \pi_1(\Delta')$. By the invariance we have

$$
e_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ce_1 + de_2)^w = c^w e_w + \sum_{0 \leq \alpha \leq w} \gamma \cdot e_\alpha \in \bigoplus_{\alpha=0}^m Q e_\alpha,$$

i.e. $c = 0$. It follows that all points of $\Delta$ have either type $I_b$ or type $I^*_b$, and

$$
S_{\gamma I} = \pm \begin{pmatrix} 1 & b_I \\ 0 & 1 \end{pmatrix}^{-1} = \pm \begin{pmatrix} 1 & -b_I \\ 0 & 1 \end{pmatrix}, \quad b_I \geq 0
$$

(see 2.8). The relation (2.4) then leads to a contradiction, since $\Sigma' b_I > 0$ for $J \equiv \text{const}$.

b. We use the notation and concepts of the preceding part. Since the coinvariant space for the group generated by the matrix (2.9) is

$$
\bigoplus_{\alpha=0}^w Q e_\alpha / \bigoplus_{\alpha=0}^w Q e_\alpha (2.10)
$$

or 0, if $b$ were false then (2.10) would be the coinvariant space of the representation $S$. Suppose that this were so. Then for the matrix

$$
S_\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})
$$

for an arbitrary $\gamma \in \pi_1(\Delta')$ we would have

$$
e_w \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mod \bigoplus_{\alpha=0}^w Q e_\alpha = a^w e_w \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mod \bigoplus_{\alpha=0}^w Q e_\alpha,$$
i.e. \( a = \pm 1 \). Iterating the matrix \( \Theta \) if necessary, we may assume that the representation \( S \) determines some matrix \( \pm \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) \) with \( b > 2 \). Let

\[
S_i = \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right)
\]

be any other matrix of the representation \( S \). Then, since the matrix

\[
\pm \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) S = \pm \left( \begin{array}{cc} a_1 + bc_1 & b_1 \\ c_1 & d_1 \end{array} \right)
\]

is also determined by the representation, we have \( a_1 + bc_1 = \pm 1 \). Consequently \( c_1 = 0 \) and \( d_1 = a_1 = \pm 1 \). As in the proof of part a, this leads to a contradiction. \( \square \)

\section{3. Geometric realizations}

Let \( F \) be a locally constant sheaf of vector spaces over \( \Delta' \). As in 2.1, this sheaf extends to a sheaf \( \mathcal{F} \) over \( \Delta \). In this section \( \Pi \) denotes an arbitrary subset of \( \Sigma \). Let \( F \) and \( F' \) be topological subvarieties of \( \Delta \) with smooth boundary such that \( \Delta \supset F \supset F' \) and \( (\hat{F} \cup \hat{F}') \cap \Sigma = \emptyset \). Then the mapping of pairs \( (\Delta, F') \to (\Delta, F) \) induces a homomorphism

\[
H_i(\Delta, F', \mathcal{F}) \to H_i(\Delta, F, \mathcal{F})
\]

in homology. In particular consider \( F = \bigcup_{\rho \in \Pi} E_\rho \) consisting of small closed disks \( E_\rho \) around the points \( \rho \in \Pi \subset \Delta \). Then a projective system of spaces \( H_i(\Delta, F, \mathcal{F}) \) with morphisms (3.1) is determined. We set

\[
H_i(\Delta, \Pi, \mathcal{F}) = \varprojlim H_i(\Delta, F, \mathcal{F}).
\]

For sufficiently small \( E_\rho \) this projective limit stabilizes and we have the isomorphism

\[
H_i(\Delta, \Pi, \mathcal{F}) \simeq H_i(\Delta, F, \mathcal{F}).
\]

Let \( \Pi \supset \Pi' \). Then the exact sequence of the triple \( \Delta \supset F \supset F' \) induces in the limit the following exact sequence:

\[
0 \to H_i(\Delta, \Pi', \mathcal{F}) \to H_i(\Delta, \Pi, \mathcal{F}) \to H_0(\Pi, \Pi', \mathcal{F}|_\Pi) \to H_0(\Delta, \Pi', \mathcal{F}) \to H_0(\Delta, \Pi, \mathcal{F}) \to 0,
\]

since \( H_i(F, F', \mathcal{F}|_F) = 0 \) (in the future the restriction \( \mathcal{F}|_F \) of the coefficients for the homology of a subvariety will not be indicated). We identify \( H_i(\Delta, \Pi, \mathcal{F}) \) with its image in \( H_i(\Delta, \Sigma, \mathcal{F}) \) under the embedding of 1-dimensional homology from the exact sequence (3.3) for the pair \( \Pi \subset \Sigma \). Then by the functoriality of homology we have the inclusion \( H_i(\Delta, \Pi', \mathcal{F}) \subset H_i(\Delta, \Pi, \mathcal{F}) \) for \( \Pi' \subset \Pi \). In the following considerations the role of the sheaf \( \mathcal{F} \) will be played by a subsheaf of \( R^1\Phi^*\mathbb{Q} \). In contrast to §1 of [7], we will require (unless the contrary is stated) only one \( \Sigma \), namely the finite set consisting of all singular points of \( \Delta \).

We denote by \( \tilde{H}_j(B^w, \mathbb{Q}) \) the image of the homology space \( H_j(B^w, \mathbb{Q}) \) in \( H_j(B^w, B^w|_\Sigma, \mathbb{Q}) \) under the natural homomorphism of the pair \( (B^w, B^w|_\Sigma) \). The aim of
this section is to define natural homomorphisms
\[ GR_{0,j} : H_0(\Sigma, R_j\Omega^w_* Q) \to H_j(B^w|_{\Sigma}, Q), \]
\[ GR_{1,j} : H_1(\Delta, \Sigma, R_j\Omega^w_* Q) \to H_{1+j}(B^w, B^w|_{\Sigma}, Q), \]
\[ GR_{2,j} : H_2(\Delta, R_j\Omega^w_* Q) \to H_{2+j}(B^w, B^w|_{\Sigma}, Q), \]
and to apply them to describe the spaces \( \overline{H}(B^w, Q) \). These homomorphisms will be called geometric realizations. Their definition is given in 3.5, 3.4 and 3.10, and a discussion of the "geometry" in 3.6, 3.7 and 3.10.

From the decomposition into a direct sum of subsheaves \( R_j\Phi^* Q = \Theta \oplus \Theta' \) we have a decomposition of homology spaces \( H_i(\, , R_j\Phi^* Q) = H_i(\, , \Theta) \oplus H_i(\, , \Theta') \). In such a situation we will in what follows identify \( H_i(\, , \Theta) \) with the corresponding subspace of \( H_i(\, , R_j\Phi^* Q) \).

3.2. Theorem. a. \( GR_{0,j}, GR_{1,j} \) and \( GR_{2,j} \) are monomorphisms.
b. The following diagram is commutative:
\[
\begin{array}{ccc}
H_1(\Delta, \Sigma, R_j\Omega^w_* Q) & \to & H_0(\Sigma, R_j\Omega^w_* Q) \\
\downarrow^{GR_{1,j}} & & \downarrow^{GR_{0,j}} \\
H_{1+j}(B^w, B^w|_{\Sigma}, Q) & \to & H_j(B^w|_{\Sigma}, Q)
\end{array}
\]
c. \( GR_{1,j-1}(H_1(\Delta, R_{j-1}\Omega^w_* Q)) \subset \overline{H}_j(B^w, Q), \quad GR_{2,j-2}(H_2(\Delta, R_{j-2}\Omega^w_* Q)) \subset \overline{H}_j(B^w, Q) \) and
\[ \overline{H}_j(B^w, Q) = GR_{1,j-1}(H_1(\Delta, R_{j-1}\Omega^w_* Q)) \oplus GR_{2,j-2}(H_2(\Delta, R_{j-2}\Omega^w_* Q)). \]
d. \( \overline{H}_{w+1}(B^w, Q) = GR_{1,w}(H_1(\Delta, (R_1\Phi^* Q)^w)) \oplus H', \) where each homology class of the subspace \( H' \) decomposes into a sum of classes having some representation as a cyclic proper subvariety of \( B^w \).

Part c of the theorem and Corollary 2.6 imply

3.3. Corollary. For odd \( j \) there is an isomorphism
\[ \overline{H}_j(B^w, Q) \cong H_1(\Delta, R_{j-1}\Omega^w_* Q). \]

3.4. Let \( F = \bigcup E_i \) and \( \Delta_0 = \Delta - \text{Int } F \), where the \( E_i \) are sufficiently small disks around the points \( v_i \in \Sigma \). \( \Delta_0 \) and \( B^w(2) = B^w|_{\Delta_0} \) are compact real varieties with smooth boundary. \( B^w(2) \) is a fiber space over \( \Delta_0 \) with fibers homeomorphic to the product of two circles. Consider a cell decomposition of the pair \( (\Delta_0, \partial \Delta_0) \). To each decomposition corresponds a filtration of cell complexes over the base \( \Delta_0 \):
\[
(\Delta_0, \partial \Delta_0), \quad (\Delta_0(1), \partial \Delta_0), \quad (\Delta_0(0), \partial \Delta_0(0)),
\]
and this means also a filtration of complexes of the bundle \( B^w(2) \):
\[
(B^w(2), \partial B^w(2)), \quad (B^w(1), \partial B^w(1)), \quad (B^w(0), \partial B^w(0)).
\]
Let $E'_{i,j}$ ($r > 0$) be the corresponding spectral sequence (see Chapter 9 of [9]). This sequence reduces to the term $E'_{i,j}$ for $r > 2$, since $E'_{i-r,j+r-1} = 0$ and $E'_{i+r,j-r+1} = 0$ for such $r$. From the assumption $J \equiv \text{const}$ (see the proof of Lemma 2.7a) it follows that $\Sigma \neq \emptyset$, and this means $\partial \Delta_0 \neq \emptyset$. Therefore

$$\text{Im} \left( H_{1+j}(B^w(0), \partial B^w(0), \mathbb{Q}) \to H_{1+j}(B^w(2), \partial B^w(2), \mathbb{Q}) \right) = 0.$$ 

Then we obtain the isomorphisms

$$H_1(\Delta_0, \partial \Delta_0, R_j\Phi^w_\ast \mathbb{Q}) \simeq E_{1,j} \simeq E_{1,j}^\infty,$$

$$\text{Im} \left( H_{1+j}(B^w(1), \partial B^w(2), \mathbb{Q}) \to H_{1+j}(B^w(2), \partial B^w(2), \mathbb{Q}) \right).$$

Consequently, there is a natural homomorphic embedding

$$H_1(\Delta_0, \partial \Delta_0, R_j\Phi^w_\ast \mathbb{Q}) \subset H_{1+j}(B^w|_{\Delta_0}, B^w|_{\partial \Delta_0}, \mathbb{Q}). \quad (3.4)$$

Moreover, there are isomorphisms

$$H_1(\Delta_0, \partial \Delta_0, R_j\Phi^w_\ast \mathbb{Q}) \simeq H_1(\Delta, F, R_j\Phi^w_\ast \mathbb{Q}),$$

$$H_{1k+j}(B^w|_{\Delta_0}, B^w|_{\partial \Delta_0}, \mathbb{Q}) \simeq H_{1+j}(B^w, B^w|_F, \mathbb{Q})$$

by the excision theorem. Then the monomorphism (3.4) determines the canonical monomorphism

$$H_1(\Delta, F, R_j\Phi^w_\ast \mathbb{Q}) \subset H_{1+j}(B^w, B^w|_F, \mathbb{Q}). \quad (3.5)$$

Passing to the projective limit on both sides of (3.5), we obtain by Lemma 1.2 a canonical mapping $GR_{1,j}$. Obviously $GR_{1,j}$ is injective.

3.5. In analogy with 3.4, the spectral sequence of the filtration of the bundle $B^w|_{\partial \Delta_0}$ induced by the filtration of the skeletons of the base $\partial \Delta_0$, reduces to the term $E_{0,j}^r$ for $r > 0$. Therefore there is a canonical monomorphism

$$H_0(\partial \Delta_0 = \partial F, R_j\Phi^w_\ast \mathbb{Q}) \subset H_1(\partial \Delta_0, \partial F, \mathbb{Q}). \quad (3.6)$$

It is to establish the isomorphism $H_0(\partial F, R_j\Phi^w_\ast \mathbb{Q}) \simeq H_0(F, R_j\Phi^w_\ast \mathbb{Q})$ for the natural mapping of the pair $(F, \partial F)$. For the proof it suffices to consider a simple cell decomposition of the pair $(F = \bigcup_i E_i, \partial F = \bigcup_i \partial E_i)$; for example, 0-cells $u_i$ ($1 < l < t$), 1-cells $y_i$ ($1 < l < t$) and 2-cells $E_i$ ($1 < l < t$) (see (2.8)). The composition of this isomorphism, the mapping (3.6), and the natural homomorphism $H_j(\partial \Delta_0, \mathbb{Q}) \to H_j(\partial \Delta_0, \mathbb{Q})$ of the pair $(\partial \Delta_0, \partial F)$ determines the canonical homomorphism

$$H_j(\partial \Delta_0, \mathbb{Q}) \to H_j(\partial \Delta_0, \mathbb{Q}). \quad (3.7)$$

Passing to the projective limit, we obtain the homomorphism $GR_{0,j}$, since

$$\lim H_j(\partial \Delta_0, \mathbb{Q}) = H_j(\partial \Delta_0, \mathbb{Q}).$$

Indeed, $H_j(\partial \Delta_0, \mathbb{Q}) = 0$ for sufficiently small $F$ (see the proof of Corollary 1.2). Then from the exact sequence of the pair $(\partial \Delta_0, \mathbb{Q})$ we get the isomorphism

$$H_j(\partial \Delta_0, \mathbb{Q}) \simeq H_j(\partial \Delta_0, \mathbb{Q}),$$

i.e.

$$\lim H_j(\partial \Delta_0, \mathbb{Q}) = H_j(\partial \Delta_0, \mathbb{Q}). \quad (3.8)$$
3.6. We will give an explicit geometric description of the mapping $GR_{0,j}$. First we describe (3.7). Fix a cell decomposition of $\partial F$. The 2-cells $E_i$ augment this complex to a decomposition of $F$. Let $u_i$ be the 0-cells of the given complex. Then a 0-cycle with coefficients in $R^*_\Phi Q$ has the following form:

$$\sigma_0 = \sum m_i u_i,$$

where $m_i \in R^*_\Phi Q |_{u_i} = H_j(B^w_{u_i}, Q)$. Consider an arbitrary representative $[m_i]$ of the homology class $m_i$ in the fiber $B^w_{u_i}$. Then the homology class $\Sigma [m_i]$ in $B^w | F$ is the image of the homology class of the 0-cycle $\sigma_0$ under the mapping (3.7). Further, for sufficiently small $F$ the retraction of the cycle $\Sigma [m_i]$ in the fiber $B^w | F$ and the isomorphism (3.8) describe the mapping $GR_{0,j}$.

3.7. Consider a cell decomposition of the pair $(\Delta, F)$ for sufficiently small $F$. We require that the intersection of this complex with $F$ provide $F$ with a cell decomposition of the type described in 3.6. Let $\Delta_0(1)$ be the 1-skeleton of the cell complex of $(\Delta, F)$. We denote one-dimensional cells by $\gamma$. We construct a cell decomposition of the bundle $B^w |_{\Delta_0(1)}$ over the cell complex $\Delta_0(1)$. To do this, fix in each one-dimensional cell an arbitrary point $u_0$ and a basis $e_1, e_2$ in the lattice $G |_{u_0} = H_1(B^w_{u_0}, \mathbb{Z})$, as in §1 of [7]. Then canonical periods $z$ and $1, z \in H$, are determined, and $B^w_{u_0} \simeq \mathbb{C}/z \mathbb{Z} + \mathbb{Z}$. The lattice $z \mathbb{Z} + \mathbb{Z}$ determines a cell decomposition of the elliptic curve $B^w_{u_0}$: the 0-cell $e$ is the image of 0, the 1-cells $e_1$ and $e_2$ are the images of $z \times [0, 1]$ and $[0, 1]$ respectively, and the 2-cell $\epsilon$ is the image of $z \times [0, 1] \oplus [0, 1]$. We will call the dimension of the cells $e, e_1$ and $\epsilon$ their degree. Then the concept of degree is defined in the free tensor algebra over $Q$ for these cells. The cell complex $e, e_1, \epsilon$ induces a cell decomposition of $B^w_{u_0}$ since $u_0 \in \Delta'$, and consequently

$$B^w_{u_0} = B^w_{u_0} \times \cdots \times B^w_{u_0}.$$ 

We will call this cell decomposition of $B^w_{u_0}$ the cell decomposition corresponding to the choice of basis in the lattice $G |_{u_0}$ (note that the basis must be chosen with negative orientation). The cells of this decomposition will be written as $w$-fold free tensor products of the cells $e, e_1$ and $\epsilon$. The dimension of the cell coincides with the degree of the corresponding tensor product. To each homology class $m \in H_j(B^w_{u_0}, Q)$ there corresponds a unique representation $[mu]$, a cycle in the cell decomposition corresponding to the choice of basis in $G |_{u_0}$. In the future by the representative $[m_i]$ in 3.6 we will mean the cycle described in this form. Continuation of the cell decomposition of $B^w_{u_0}$ along $\gamma$ in both directions by the linear connection gives a cell decomposition of $B^w | F$, over $\gamma$, and continuation of the representative $[mu]$ gives the representative $[c\gamma]$ of the chain $c\gamma$, $c \in R^*_\Phi Q | \text{Int}, \gamma \simeq H_j(B^w_{u_0}, Q)$. “Sections” of the cell decomposition over each point $u_0 \in \gamma$ are also cell decompositions corresponding to a choice of basis in $G |_{u_0}$. Each cell lies either over $\gamma$ or over one of the ends $\partial \gamma$. A complete cell decomposition of $B^w |_{\Delta_0(1)}$ is obtained by taking the union of the cell complexes formed over $\gamma$ by 1-cells and the intersection of terminal cell decompositions over each 0-cell of $\Delta_0(1)$. For an arbitrary 1-chain $\sigma_1 = \Sigma c\gamma$ with coefficients in $R^*_\Phi Q$ we set $[\sigma_1] = \Sigma [c\gamma]$. The geometric realization $[c\gamma]$ is a relative cycle of the pair $(B^w | \gamma, B^w | \partial \gamma)$. Therefore $[\sigma_1]$ is a relative cycle of the pair $(B^w |_{\Delta_0(1)}, B^w |_{\Delta_0(0)})$. If $\sigma_1$ is a cycle of the pair $(\Delta, F)$, then the boundary of the chain $[\sigma_1]$ is homologous to 0 over the interior 0-cells of $\Delta_0 = \Delta - F$. Therefore in
this case the chain \([\sigma_1]\) may be completed to a relative cycle \((\sigma_1)\) of the pair \((B^w|_{\Delta_0}, B^w|_{\Delta_0})\) over the interior 0-cells of \(\Delta_0\). The mapping \((\sigma_1)\) induces the mapping (3.5). From this description of the mapping (3.5) and the description 3.6 of the mapping (3.7) we obtain the commutativity of the diagram

\[
\begin{align*}
H_1(\Delta, F, R\Phi_0^w\mathbb{Q}) &\xrightarrow{\partial} H_0(F, R\Phi_0^w\mathbb{Q}) \\
H_1(B^w, B^w|_F, \mathbb{Q}) &\xrightarrow{\partial} H_1(B^w|_F, \mathbb{Q}).
\end{align*}
\]

The boundaries of \((\sigma_1)\) are situated over \(F\). Retracting the boundaries of \((\sigma_1)\) to \(B^w\), we obtain a description of the mapping \(GR^X\), thanks to the isomorphism (3.2) for \(\Pi = \Sigma\).

The isogeny of multiplication of the fiber \(B^w_0, u_0 \in \Delta',\) by any integer \(n\) induces an analytic mapping of the pair \(B^w|_{\Delta_0}, B^w|_{\Delta_0}\). The corresponding mapping in homology we denote by \(n_*\). We easily get the following result from the explicit description of the mapping (3.5), which of course applies also to (3.4).

**3.8. Lemma.** \(n_*|_{\text{im}(3.4)} = n!\).

For the proof it suffices to take a cell decomposition of the pair \((\Delta, F)\) such that all the 0-cells lie in \(F\).

Fix a point \(u_0 \in \Delta'\) and a basis \(e_1, e_2\) in the lattice \(G|_{u_0}\). Then \(B^2_{u_0}\) has a cell decomposition corresponding to the choice of a negative basis. We denote by \(D\) the homology class of the diagonal of \(B^2_{u_0}\) with the natural analytic orientation.

**3.9. Lemma.** \(D = e \otimes e + e \otimes e - (e_1 \otimes e_2 - e_2 \otimes e_1)\).

**3.10.** Consider an \(F\) such that \(u_0 \in \Delta_0\). Then by Theorem 2.5b

\[
H_2(\Delta, R\Phi_0^w\mathbb{Q}) = H_2(\Delta, S_{01}^{(f, a)}) \simeq (R\Phi_0^w\mathbb{Q})_{u_0}^{\text{inv}},
\]

where the invariant subspace is taken relative to the representation of \(SL(2, \mathbb{Q})\) analogous to the representation (2.4) of the fundamental group \(\pi_1(\Delta')\) (the action of the matrix is on the left in this case). This representation is defined by a componentwise K"unneth decomposition (2.1) in the following way: it is induced by the choice of basis \(G|_{u_0}\) for \(R_1\Phi_0^w\mathbb{Q}|_{u_0}\) (see (2.2)) and it is trivial for \(R_0\Phi_0^w\mathbb{Q}|_{u_0}\) and \(R_2\Phi_0^w\mathbb{Q}|_{u_0}\). Fix generators \(e\) and \(e\) in the spaces \(R_0\Phi_0^w\mathbb{Q}|_{u_0}\) and \(R_2\Phi_0^w\mathbb{Q}|_{u_0}\) respectively. Let \(a \in A_w\) be a permutation. It determines the analytic mapping \(a : B^w|_{\Delta_0} \to B^w|_{\Delta_0}\) which permutes the components of the fiber, the \(i\)th component mapping to the \(a(i)\)th. We denote by \(a_*\) the corresponding mapping in homology. We denote the corresponding action on the sheaf \(R\Phi_0^w\mathbb{Q}\) the same way. This mapping is connected as follows with the mapping \(a\) defined in 2.2 of the space of sections of \((R_1\Phi_0^w\mathbb{Q})\otimes^w|_{u_0}: a = \text{sign}(a)a_*\). Then from the decomposition (2.1), the isomorphism (2.2), and Lemma 2.3d we find that the space \((R\Phi_0^w\mathbb{Q}|_{u_0})^{\text{inv}}\) has the following generators: the vectors

\[
a_*((e \otimes k \otimes e \otimes l \otimes (e_1 \otimes e_2 - e_2 \otimes e_1)^{\otimes m})
\]

of degree \(j\), where \(a \in A_w, k, l\) and \(m\) are positive integers, and \(k + l + 2m = w, \ l + m = j/2\). We put in correspondence with the vector

\[
e^{\otimes k} \otimes e^{\otimes l} \otimes (e_1 \otimes e_2 - e_2 \otimes e_1)^{\otimes m}
\]

of degree \(j\) the relative algebraic cycle \(D_{k,l,m}\) of dimension \(j + 2\) for the pair \((B^w, B^w|_{\Delta})\).
This cycle is uniquely determined by the following property:

\[
D_{k,l,m} \big|_{u=\Delta} = e \times \cdots \times e \times B_{u_k} \times \cdots \times B_{u_l} \times \left( B_{u_k} \times e + e \times B_{u_k} - D \right) \times \cdots \times \left( B_{u_l} \times e + e \times B_{u_l} - D \right).
\]

It is easy to verify, using the symmetric compactification of \( B^w \), that the mapping \( a \) extends to a regular morphism \( a : B^w \rightarrow B^w \) (for the sequel its birationality and regularity over \( \Delta' \) suffice, and they are obvious). We obtain the mapping \( GR_{2,j} \) by putting the relative cycle \( a_*(D_{k,l,m}) \) in correspondence with the vector

\[
a_*(e^{a_k} \otimes e^{a_l} \otimes (e_1 \otimes e_2 - e_2 \otimes e_1))^{sm}.
\]

We show that it is well defined. For odd \( j \), the mapping \( GR_{2,j} \) is trivial by Corollary 2.6. Therefore we assume that \( j \) is even, unless the contrary is stated. Consider the spectral sequence of (3.4). This sequence reduces to the term \( E_{2,j} \) for \( r > 2 \). For \( r > 3 \) this is obvious. For \( r = 2 \) we have

\[
E_{2,j} = \ker d_{2,j+1} \cap \text{im} d_{2,j+1} = \ker d_{2,j} = E_{2,j}.
\]

Since \( H_0(\Delta, R_j, \Omega_\Sigma) = 0 \) by 2.6, we have

\[
E_{2,j+1} = H_0(\Delta, \partial \Delta, R_j, \Omega_\Sigma) \simeq H_0(\Delta, F, R_j, \Omega_\Sigma) \simeq H_0(\Delta, R_j, \Omega_\Sigma) = 0.
\]

Since the spectral sequence reduces to the term \( E_{2,j} \), by the excision theorem we obtain the isomorphisms

\[
H_2(\Delta, F, R_j, \Omega_\Sigma) \simeq H_2(\Delta, \partial \Delta, R_j, \Omega_\Sigma) \simeq E_{2,j} \simeq E_{2,j} \simeq E_{2,j}.
\]

Consequently there is a natural isomorphism

\[
H_2(\Delta, F, R_j, \Omega_\Sigma) \simeq H_2(\Delta, F, R_j, \Omega_\Sigma) \simeq H_2(\Delta, F, R_j, \Omega_\Sigma) \simeq H_2(\Delta, F, R_j, \Omega_\Sigma) = 0.
\]

Passing to the projective limit on both sides of (3.10), we obtain the natural isomorphism

\[
H_2(\Delta, F, R_j, \Omega_\Sigma) \simeq H_2(\Delta, F, R_j, \Omega_\Sigma) \simeq H_2(\Delta, F, R_j, \Omega_\Sigma) \simeq H_2(\Delta, F, R_j, \Omega_\Sigma) = 0.
\]

By Lemma 3.9 and the geometric description of the mapping \( GR_{2,j} \) given above we obtain the congruence (3.11) \( \equiv GR_{2,j} \) (mod \( \text{im} GR_{1,j+1} \)). Therefore to prove that \( GR_{2,j} \) is well defined it suffices to establish the triviality of the intersection \( H'' \cap \text{im} GR_{1,j+1} = 0 \), where \( H'' \) is the subspace of \( H_{2+j}(B^w, B^w|_{\Sigma}, \Omega) \) generated by the algebraic cycles \( a_*(D_{k,l,m}) \) of dimension \( j + 2 \). Using the stability of the projective limits and the excision theorem, this problem may be reduced to proving the triviality of the intersection \( H''|_{\Delta_0} \cap (3.4) = 0 \) for sufficiently small \( F \), where the subspace \( H''|_{\Delta_0} \subset H_{2+j}(B^w|_{\Delta_0}, B^w|_{\Delta_0}, \Omega) \) generated by the restrictions of the algebraic cycles \( a_*(D_{k,l,m}) \) of dimension \( j + 2 \). The last is obvious from the relation \( n_\Sigma H''_{|\Delta_0} = n'' \), and, by Lemma 3.8, \( n_\Sigma \text{im}(3.4) = n'' + 1 \). The operator \( n_\Sigma \) on the homology space \( H_{2+j}(B^w|_{\Delta_0}, B^w|_{\Delta_0}, \Omega) \) is induced by the fiberwise isogeny of multiplication by \( n \).

Now we assume \( j \) arbitrary, not just even.

3.11. Lemma. a. \( GR_{1,j} \) and \( GR_{2,j} \) are monomorphisms.

b. \( H_{2+j}(B^w, B^w|_{\Sigma}, \Omega) = \text{im} GR_{2,j} \oplus \text{im} GR_{1,j+1} \).
THE HOMOLOGY OF KUGA VARIETIES 413

PROOF. The injectivity of \( GR_{1,j} \) comes from the process of defining the homomorphism in 3.4. For even \( j \) the injectivity of \( GR_{2,j} \) and the decomposition \( b \) are immediate corollaries of (3.11), since the intersection 

\[ \text{Im } GR_{2,j} \cap \text{Im } GR_{1,j+1} = H'' \cap \text{Im } GR_{1,j+1} = 0 \]

is trivial, and the mapping (3.11) is induced by \( GR_{2,j} \). Suppose \( j \) is odd. In this case the injectivity is obvious because of the triviality of \( GR_{2,j} \) (see 2.6). Since \( H_2(\Delta, R_j \Phi^w Q) = 0 \) and \( H_1(F, R_j \Phi^w Q) = 0 \) for sufficiently small \( F \), we get the triviality of \( H_2(\Delta, F, R_j \Phi^w Q) = 0 \) from the exact sequence of the pair \( (\Delta, F) \). Then \( H_2(\Delta_0, \partial \Delta_0, R_j \Phi^w Q) = 0 \) by the excision theorem. Consequently the spectral sequence of (3.4) reduces to the term \( E_{2,j} \) for \( r > 2 \), and

\[ 0 = H_2(\Delta_0, \partial \Delta_0, R_j \Phi^w Q) = E_{2,i} \simeq E_{2,i} \]

\[ \simeq H_{2+i}(B^w(2), \partial B^w(2) Q) / \text{Im } (H_{2+i}(B^w(1), \partial B^w(2), Q) \]

\[ \longrightarrow H_{2+i}(B^w(2), \partial B^w(2), Q)). \]

This proves the surjectivity of (3.4) for \( j + 1 \), and similarly the surjectivity of (3.5). Therefore \( GR_{1,j+1} \) is an isomorphism for odd \( j \), which together with the triviality of \( GR_{2,j} \) proves \( b \). □

Consider an arbitrary point \( v \in \Delta \). Let \( u_0 \in \Delta' \) be a point sufficiently close to \( v \), i.e. \( u_0 \in E_1 \), a small closed disk around \( v \) satisfying Lemma 1.1. Then the composition of the embedding \( B^w_{u_0} \rightarrow B^w|_{E_1} = B^w_0 \) and the retraction \( B^w_0 \rightarrow B(1) = B^w|_v \) determines the following homomorphism:

\[ H_j(B^w|_{u_0}, Q) \rightarrow H_j(B^w|_v, Q). \quad (3.12) \]

We denote by \( \beta \) a single positive circuit around the point \( v \), \( \beta \subset \Delta' \), with origin at the point \( u_0 \). To this circuit there corresponds an endomorphism \( s_\beta \) of the space \( H_j(B^w_{u_0}, Q) \) defined as in (1.3) of [7] by the natural connection on \( B^w|_\Delta' \). Then (3.12) determines the specialization homomorphism

\[ \text{Sp} : (H_j(B^w_{u_0}, Q))^\text{coinv} \rightarrow H_j(B^w|_v, Q), \]

where the space of coinvariant vectors is taken with respect to the endomorphism \( s_\beta \).

3.12. PROPOSITION. \( \text{Sp} \) is a monomorphism.

PROOF OF THEOREM 3.2. a. The injectivity of \( GR_{1,j} \) and \( GR_{2,j} \) was proved in Lemma 3.11a. We prove injectivity for \( GR_{0,j} \). Because of the stability of the projective limit it suffices to prove this for (3.7) for sufficiently small \( F \). In this case \( F = \bigcup E_i \) decomposes into the connected components \( E_i \). Consequently, (3.7) also decomposes into a direct sum of natural homomorphisms

\[ H_0( E_i, R_j \Phi^w Q) \rightarrow H_j( B^w|_{E_i}, Q) \quad (3.13) \]

and it suffices to establish their injectivity for small \( E_i \). Consider one of the disks, say \( E_1 \), and assume it is so small that Lemma 1.1 holds. Let \( v \) be the center of the disk \( E_1 \). Then we get a natural isomorphism \( H_j(B^w_0, Q) \simeq H_j(B^w|_v, Q) \) as in the proof of the isomorphism (3.8). Proving the injectivity of \( GR_{0,j} \) reduces to checking the injectivity of the composition

\[ H_0( E_1, R_j \Phi^w Q) \rightarrow H_j( B^w|_v, Q) \quad (3.14) \]
of this isomorphism and the mapping (3.13) for \( l = 1 \). Consider the point \( u_0 \in \partial E_1 \) in the boundary of \( E_1 \). This last choice determines a cell decomposition of \( E_1 \): 0-cell \( u_0 \), 1-cell \( \partial E_1 \) and 2-cell \( E_1 \). It follows immediately from 3.6 that for this cell decomposition the mapping (3.14) assumes the form \( \text{Sp} \). Therefore the injectivity of (3.14) follows from Proposition 3.12.

Part b follows from the commutative diagram (3.9) by passing to the projective limit.

c. From the construction 3.10 of the mapping \( GR_{2,j-2} \) we have the inclusion \( \text{Im} \, GR_{2,j-2} \subseteq \overline{H}_j(B^w, \mathbb{Q}) \). Therefore this part of the theorem is an immediate corollary of 3.4 and 3.2a,b, since

\[
\text{Ker} \left( H_j(B^w, B^w|_\Sigma, \mathbb{Q}) \to H_{j-1}(B^w|_\Sigma, \mathbb{Q}) \right) = \overline{H}_j(B^w, \mathbb{Q})
\]

from the exact sequence of the pair \( (B^w, B^w|_\Sigma) \).

d. By the construction of the mapping \( GR_{2,w-1} \) we have \( \text{Im} \, GR_{2,w-1} \subseteq H' \). Therefore by 3.2c it suffices to establish the analogous decomposition for \( GR_{1,w}(H_1(\Delta, R_\ast, \Phi^w, \mathbb{Q})) \). The Künneth formula (2.1) for \( j = w \) reduces this problem to the decomposition of \( GR_{1,w}(H_1(\Delta, (R_1\Phi, \mathbb{Q})^{\otimes w})) \). If even one \( j_m \neq 1 \), then by the description 3.7 of the mapping \( GR_{1,w} \) we have

\[
GR_{1,w} \left( H_1(\Delta, \bigotimes_{m=1}^w R_{1m}\Phi, \mathbb{Q}) \right) \subseteq H'.
\]

The decomposition

\[
GR_{1,w}(H_1(\Delta, (R_1\Phi, \mathbb{Q})^{\otimes w})) = GR_{1,w}(H_1(\Delta, (R_1\Phi, \mathbb{Q})^w)) \oplus H_1,
\]

where \( H_1 \subseteq H' \), is an immediate corollary of 2.3c, 3.9 and 3.7. ■

**Proof of Lemma 3.9.** We have

\[
D = (e_2 \otimes e + e \otimes e_2) \otimes (e_1 \otimes e + e \otimes e_1) = e \otimes e - e_1 \otimes e_2 + e_2 \otimes e_1 + e \otimes e. \]

**Proof of Proposition 3.12.** a. We denote by

\[
\overline{\text{Sp}} : H_j(B_{m_{00}}^w, \mathbb{Q})^{\text{coinv}} \to H_j(B_{m_0}^w|_\nu, \mathbb{Q})
\]

the composition of \( \text{Sp} \) with the natural homomorphism in homology induced by the projection \( B^w|_\nu \to B^w|_\nu \). We note that \( B_{m_0}^w = \overline{B}_{m_0}^w \) since \( u_0 \in \Delta' \). We will show below that \( \overline{\text{Sp}} \) is a monomorphism, from which the injectivity of \( \overline{\text{Sp}} \) follows immediately. The projection \( \Psi^w \) (see §3 of [7]) of the deformation of Lemma 1.1 determines a deformation retract of \( B_1^w = B^w|_{E_1} \) onto \( \overline{B^w} \). Therefore we have the canonical isomorphism

\[
H_j(B_1^w, \mathbb{Q}) \simeq H_j(\overline{B}_1^w, \mathbb{Q}).
\]

This isomorphism shows the equivalence of the injectivity of \( \overline{\text{Sp}} \) and

\[
H_j(B_{m_0}^w, \mathbb{Q})^{\text{coinv}} \subseteq H_j(\overline{B}_{m_0}^w, \mathbb{Q}),
\]

where the last homomorphism is induced by the natural mapping of the pair \( (\overline{B}_{m_0}^w, \overline{B}_{m_0}^w) \).

b. **Reduction to the case** \( I_\nu \) (\( b \geq 1 \)). We know that \( \overline{B}_{m_0}^w \simeq C \setminus F^w \), where \( C \) is a finite cyclic group of order \( \kappa \), with action compatible with the projection of \( \sigma \) on the base \( D \) (see [7], 2.2). In the case under consideration \( D = \{ |\sigma^k| < \epsilon \} \) is a closed disk. In the base
D the generator $e_\epsilon = e^{2\pi i/\epsilon}$ of the group C acts by multiplication. Therefore there is an isomorphism

$$(H_1(F^w_\infty, Q)^{\text{coinv} (D)})^{\text{coinv} (C)} = H_1(B_{u_0}, Q)^{\text{coinv} (E_1)},$$

where $\sqrt{\epsilon}$ is the arithmetic root, and coinv(D) and coinv(E_1) denote the coinvariants of the circuits around the boundaries $\partial D$ and $\partial E_1$; coinv(C) the coinvariants of the group $C$; and $\tau(u_0) = \sigma^*(u_0) = \epsilon$. On the other hand,

$$H_1(B_1^w, Q) \cong H_1(F^w, Q)^{\text{inv}} \cong H_1(F^w, Q)^{\text{coinv}}.$$ 

The last isomorphism follows from the semisimplicity of the representation of the finite cyclic group $C$ of automorphisms in the homology space $H_j(F_w^w, Q)$. Consequently, to prove the injectivity of (3.15) it suffices to show the injectivity of

$$H_1(F^w_d, Q) \subseteq H_1(F^w, Q)$$

for the natural mapping of the pair $(F^w, F^w_d)$, where $d \in \partial D$. Then from 2.2 of [7] and Chapter 8 of [10] it follows that $F^w_0$ is the only singular fiber of $F^w$ of type $I_b$ ($b > 0$).

This concludes the reduction to the case $I_b$.

c. If $\nu$ has type $I_0$, then $\sigma_b^B = \text{id}$ and $\tau^B$ is an isomorphism, since the bundle $B_1^w$ is topologically trivial for sufficiently small $E_1$.

d. Suppose the point $\nu$ has type $I_b$ ($b > 1$). It is easy to check that

$$H_0 (B_{u_0}, Q) \cong H_0 (B_v, Q), \quad H_1 (B_{u_0}, Q)^{\text{coinv}} \cong H_1 (B_v, Q),$$

$$H_2 (B_{u_0}, Q) \subseteq H_2 (B_v, Q),$$

where in the first and last cases $\sigma_b^B = \text{id}$. Hence by 2.2(ii) of [7] and by the Künneth decomposition (2.1) at the points $u_0$ and $v$, we obtain the injectivity of $\tau^B$, since by (2.10)

$$(H_1 (B_{u_0}, Q)^{\otimes m})^{\text{coinv}} \simeq (H_1 (B_{u_0}, Q)^{\text{coinv}})^{\otimes m}. \quad \square$$

§4. Nondegeneracy conditions of the canonical pairing

4.1. If $\omega \in H^0 (B^w, \Omega^{w+1})$ is a first order differential form on Kuga’s variety $B^w$, then its integrals are trivial along every chain of fibers over points of the base. Therefore a pairing

$$(\cdot, \cdot): H_{w+1} (B^w, B^w|_\Sigma, Q) \times H^0 (B^w, \Omega^{w+1} \oplus \bar{\Omega}^{w+1}) \to C,$$

(homology class $\sigma, \omega = \int_\sigma \omega$, is defined, where $\sigma$ represents a homology class of the space $H_{w+1} (B^w, B^w|_\Sigma, Q)$ for some cell decomposition, and $\omega \in H^0 (B^w, \Omega^{w+1} \oplus \bar{\Omega}^{w+1})$). The pairing $(\cdot, \cdot)$ and the monomorphism $GR_{1,w}$ determine the pairing

$$\langle \cdot, \cdot \rangle: H_1 (\Delta, \Sigma, R_{u} (\Phi^w_\bullet Q)|_\Sigma) \times H^0 (B^w, \Omega^{w+1} \oplus \bar{\Omega}^{w+1}),$$

$$\langle \cdot, \cdot \rangle = (GR_{1,w}).$$

The pairing $\langle \cdot, \cdot \rangle$ is always nondegenerate on the right.

4.2. Theorem. $H_1 (\Delta, (R_1 \Phi_\bullet Q)^w)^{\perp} = 0$, i.e. $\langle H_1 (\Delta, (R_1 \Phi_\bullet Q)^w), \omega \rangle = 0$ implies $\omega = 0$ for any $\omega \in H^0 (B^w, \Omega^{w+1} \oplus \bar{\Omega}^{w+1})$.

The proof follows immediately from de Rham’s theorem, Theorem 3.2d, and the triviality of the pairing of $H^0 (B^w, \Omega^{w+1} \oplus \bar{\Omega}^{w+1})$ with $H'$. $\quad \square$
4.3. Corollary. If
\[
\dim H^0(B^w, \Omega^{w+1}) = (g - 1) (w + 1) + \sum_{b \geq 1} \frac{w}{2} (\nu (I_b) + \nu (I_b^*)) + \left[ \frac{w + 2}{3} \right] (v (II) + v (III)) + \left[ \frac{w + 2}{4} \right] (v (IV) + v (IV^*))
\]
for even \(w > 0\),
\[
\dim H^0(B^w, \Omega^{w+1}) = (g - 1) (w + 1) + \sum_{b \geq 1} \frac{w}{2} (\nu (I_b) + \left( \frac{w + 1}{2} \right) \left( \sum_{b \geq 0} \nu (I_b) \right) + v (II^*) + v (II) + v (III) + v (III^*) + \left[ \frac{w + 2}{3} \right] (v (IV) + v (IV^*))
\]
for odd \(w > 0\), or
\[
\dim H^0(B^w, \Omega^{w+1}) = g
\]
for \(w = 0\), then the pairing
\[
\langle , \rangle : H_1(\Delta, (R_1, \Phi, Q)^\omega) \times H^0(B^w, \Omega^{w+1} + \overline{\Omega}^{w+1})
\]
is nondegenerate.

The proof follows directly from Theorems 3.2a, 2.5c, and 4.2.

§5. Application. The Shimura torus

Consider the Hodge decomposition of the \((w + 1)\)-cohomology of \(B^w\):
\[
H^{w+1}(B^w, Q) \otimes C = H^{w+1,0}(B^w) \oplus \ldots \oplus H^{0,q}(B^w) \oplus \ldots \oplus H^{0, w+1}(B^w).
\]
We project \(H^{w+1}(B^w, Q)\) onto \(H^{0, w+1}(B^w)\) and denote the resulting \(Q\)-subspace by \(Q\).

On the other hand, we have
\[
H_{w+1}(B^w, Q) \xrightarrow{D} H^{0, w+1}(B^w, Q) \xrightarrow{pr_{0, w+1}} H^{0, w+1}(B^w),
\]
where the mapping \(D\) comes from Poincaré duality. The mapping (5.1) may be realized as follows. An element \(c \in H_{w+1}(B^w, Q)\) determines on \(H^{w+1,0}(B^w)\) the \(C\)-functional \(f_c \omega, \omega \in H^{w+1,0}(B^w), (H^{w+1,0}(B^w))^* = H^{0, w+1}(B^w)\). Hence an element corresponding to \(f_c\) is determined in \(H^{0, w+1}(B^w)\). This element is the image of \(c\) under the homomorphism (5.1). Since \(f_c \omega = 0\) for any \(\omega\) of type \((w + 1, 0)\) and
\[
c \in \text{Im}(H_{w+1}(B^w, Q) \rightarrow H_{w+1}(B^w, Q)),
\]
(5.1) determines the mapping
\[
H_{w+1}(B^w, Q) \rightarrow H^{0, w+1}(B^w).
\]
By 3.2c, we may compose the mapping (5.2) and \(GR_{1,w}\) to obtain a mapping
\[
H_1(\Delta, (R_1, \Phi, Q)^\omega) \rightarrow H^{0, w+1}(B^w).
\]

5.1. Proposition. \(\text{Im} (5.3) = Q\).

The proof follows directly from 3.2d and the fact that \(D\) in (5.1) is an isomorphism.

5.2. Definition. If \(\dim Q = \dim \mathbb{R} H^{0, w+1}(B^w)\), then the torus
\[
T(B^w) = H^{0, w+1}(B^w) / \mathbb{Z}
\]
is determined up to isogeny, where \( Z \subset \mathbb{Q} \subset H^{0,w+1}(B^w) \) is some lattice. \( T(B^w) \) is called the Shimura torus. A Kuga variety for which this condition in the definition of the Shimura torus is satisfied will be called a special Kuga variety.

**5.3. Theorem.** If \( w \) is even and \( B^w \) is a special Kuga variety, then \( T(B^w) \) is an abelian variety.

This theorem is a generalization of Theorem 2 of [4] (see Theorem 7 of [6]).

**Proof.** We interpret cohomology in terms of harmonic forms. Then \( (\alpha, \beta)_{B^w} = \int_{B^w} \alpha \wedge \beta \). Consider the hermitian form

\[
H(\omega_1, \omega_2) = 2i \int_{B^w} \omega_1 \wedge \overline{\omega_2}
\]
on \( H^{0,w+1}(B^w) \). The form \( H \) is real, hermitian, positive definite for \( w + 2 \equiv 2 \pmod{4} \) and negative definite for \( w + 2 \equiv 0 \pmod{4} \). Therefore by the Riemann-Frobenius condition (see §6 of [2]) it remains to verify the rationality of \( \text{Im} \, H \) in \( \mathbb{Q} \). If \( \alpha, \beta \in \mathbb{Q} \), then

\[
A = \alpha + \overline{\alpha}, \quad B = \beta + \overline{\beta}
\]
are rational cohomology classes. Hence

\[
A \wedge B = 2 \text{Re} (\alpha \wedge \overline{\beta})
\]
and therefore

\[
\text{Im} \, H(\alpha, \beta) = \text{Im} \, 2i \int_{B^w} \alpha \wedge \overline{\beta} = (A, B)_{B^w} \in \mathbb{Q}. \]

**5.4. Remarks.** 1. The Jacobi variety \( \Theta_{w/2}(B^w) \) ([2], §6) admits a canonical projection onto \( T(B^w) \) in the category of complex tori up to isogeny.

2. Corollary 4.3 gives a sufficient condition for \( B^w \) to be a special Kuga variety.

**§6. Application. The modular case**

**Proof of Theorem 0.2.** This assertion is a direct corollary of the definition from (0.1) of the pairing (\( , \)):

\[
(\sigma, (\varphi, \psi)) = \langle \sigma, \omega_\varphi + \omega_\psi \rangle,
\]
where \( \sigma \in H_1(\Delta, \Sigma, (R_1 \Phi_* Q)^w) \), \( \varphi, \psi \in S_{w+2}(\Gamma) \), and \( \omega_\varphi, \omega_\psi \in H^0(B^w, \Phi^w) \) are the corresponding regular differentials of Theorem 0.3 of [7], and also Corollary 4.3 of this paper and Corollary 5.3 and Theorem 0.3 of [7].

Let \( C \supset K \supset \mathbb{Q} \) be an arbitrary field. Then we may define a canonical pairing

\[
( \cdot, \cdot ) : H_1(\Delta, \Sigma, (R_1 \Phi_* K)^w) \times S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)} \rightarrow C,
\]
since

\[
H_1(\Delta, \Sigma, (R_1 \Phi_* K)^w) \simeq H_1(\Delta, \Sigma, (R_1 \Phi_* Q)^w) \otimes_q K.
\]

**6.1. Corollary.** The pairing \(( \cdot, \cdot )\) on

\[
H_1(\Delta, (R_1 \Phi_* K)^w) \times S_{w+2}(\Gamma) \oplus \overline{S_{w+2}(\Gamma)}
\]
is nondegenerate.
Moreover, we have

**6.2. Corollary.** i) \(B^w_T\) is a special Kuga variety.

ii) \(T(B^w_T)\) is an abelian variety for even \(w\).

**Proof.** ii) follows from i) and 5.3. The proof of i) follows from the fact that

\[
\dim_{\mathbb{Q}} Q = \dim_{\mathbb{R}} H^{0, w+1}(B^w_T),
\]

since (5.3) is an embedding: by the nondegeneracy of \((\ , \ )\) and the relation \((\ , \overline{\omega}) = (\ , \omega)\), and also from the equations

\[
\dim_{\mathbb{Q}} H_1(\Delta, (R_1 \Phi_\ast Q)^w) = 2 \dim_{\mathbb{C}} H^{0, w+1}(B^w_T)
\]

\[
= 2 \dim_{\mathbb{C}} H^0(B^w_T, \Omega^{w+1}) = 2 \dim_{\mathbb{C}} S_{w+2}(\Gamma)
\]

(see Theorem 0.3 and Corollary 5.3 of [7], and Theorem 2.5c of this paper).

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**Bibliography**


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