SMOOTHNESS OF THE GENERAL ANTICANONICAL DIVISOR 
ON A FANO 3-FOLD

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Abstract. Smoothness of the general anticanonical divisor of a Fano 3-fold is proved. In addition, an analogous result is established for the linear system \( |K| \), where \( rK \sim -K_V \) for some natural number \( r \). The results obtained in the paper can be used to investigate projective imbeddings of Fano 3-folds.

Bibliography: 6 titles.

Following [4], we call a smooth complete irreducible algebraic variety \( V \) of dimension 3 over a field \( k \) which has an ample anticanonical class \( -K_V \) a Fano 3-fold. In [4] projective embeddings of such varieties were considered under the following hypothesis:

HYPOTHESIS (1.14) [4]. There exist an invertible sheaf \( \mathcal{L} \in \text{Pic} \, V \) and a natural number \( r \) such that \( r\mathcal{L} \cong -K_V \) and the linear system \( |\mathcal{L}| \) contains a smooth surface \( H \) (the greatest such \( r \) is called the index of \( V \)).

The purpose of the present work is to show that this hypothesis is satisfied for every Fano 3-fold over an algebraically closed field of characteristic 0. Thus all the results of [4] where Hypothesis (1.14) is assumed remain true also without that assumption.

The question considered in this paper can be given the following more general formulation. Let \( V \) be a complete nonsingular smooth irreducible algebraic variety of dimension \( n \) with an ample anticanonical class \( -K'_V \). Does there exist a smooth divisor in the linear system \( |-K'_V| \)? This problem naturally arises in considering the mapping defined by the linear system \( |-K'_V| \). The answer to this question is affirmative in the case of an algebraically closed field \( k \) of any characteristic if \( n \leq 2 \) and in characteristic 0 for \( n \leq 3 \). In the remaining cases the answer is unknown. In connection with the notion of the index of a variety there arises also an analogous question for \( -K'_V/r \in \text{Pic} \, V \).

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§ 1. The main result

1.1. All the algebraic varieties considered in this paper are defined over an algebraically closed field \( k \) of characteristic zero.

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1.2. Theorem. Let $V$ be a Fano 3-fold, and let $\mathcal{L}$ be an invertible sheaf such that $r\mathcal{L} \cong -K_V$ for some natural $r$. Then in the linear system $|\mathcal{L}|$ there is a smooth surface $D$.

Theorem 1.2 is proved in §3 for the case $r = 1$, and in §4 for $r \geq 2$. §2 is devoted to auxiliary propositions. The general plan of the proof is the following. First we prove that the linear system $|\mathcal{L}|$ is not composite with a pencil. Then using Bertini's theorem we bring the general element of $|\mathcal{L}|$ to the form $D + D_0$ with fixed part $D_0$ and irreducible reduced movable divisor $D$. The dimension of the space $H^0(V, \mathcal{L})$ is known to us from [4]. On the other hand, $h^0(V, \mathcal{O}_V(D)) = h^0(V, \mathcal{L})$. The presence of fixed components or of singularities in the general divisor $D$ reduces the last equality to a contradiction either with the Riemann-Roch theorem on the surface $\overline{D}$, which resolves the singularities of $D$, or with Lemma 2.3. If $r \geq 2$ one shows that the base locus of $|\mathcal{L}|$ consists of no more than a finite number of points. Further, one uses Theorem 4.1 of [3].

§2. Auxiliary lemmas

2.1. Lemma. If $V$ is a Fano 3-fold, then every effective divisor $D$ from the linear system $|-K_V|$ is connected.

Proof. According to (1.4) (i) of [4], $h^0(D, \mathcal{O}_D) = 1$ for $D \in |-K_V|$. Therefore, $D$ is connected. 

2.2. Lemma. Let $D$ be an effective divisor on a K3 surface $X$ such that some multiple of $D$ gives a linear system without fixed components and

$$h^0(X, \mathcal{O}_X(D)) = \frac{D^2}{2} + 2.$$ 

Then the fixed components of $D$ have multiplicity 1.

Proof. By Bertini's theorem [1] we may assume that the movable components of $D$ have multiplicity one. We denote by $D_1, \ldots, D_n$ the connected components of the multiplicity one part of the general $D$. Then we have the following representation of $D$ as the sum of effective divisors: $D = \sum_{i=0}^n D_i$, where $D_0$ denotes a multiple of the fixed component of $D$. We need to show that $D_0 = 0$. Let us assume the contrary: $D_0 \neq 0$. By duality and the Riemann-Roch theorem we have

$$h^2(X, \mathcal{O}_X(D)) = h^1(X, \mathcal{O}_X(D)) = 0.$$ 

The latter, using duality and the Ramanujan vanishing theorem for a regular surface (see the remark on page 180 in [2]) implies that

$$h^2(D, \mathcal{O}_D) = h^1(X, \mathcal{O}_X(-D)) + 1 = h^1(X, \mathcal{O}_X(D)) + 1 = 1.$$ 

Therefore $D$ is connected. Consequently, by the nontriviality of $D_0$, $(D_0 \cdot D_0) \geq 2$ for $n \geq i \geq 1$. Hence $(\sum_{i=1}^n D_0, D_0) \geq 2n$.

Using Ramanujan's theorem and duality for the divisor $\sum_{i=1}^n D_i$, we obtain

$$h^1\left(X, \mathcal{O}_X\left(\sum_{i=1}^n D_i\right)\right) = h^0\left(\bigcup_{i=1}^n D_i, \mathcal{O}_{\bigcup_{i=1}^n D_i}\right) - 1 = n - 1.$$
By duality and the nontriviality of $\Sigma_1^n D_i$ (since there exists a movable part), we have $h^2(X, \mathcal{O}_X(\Sigma_1^n D_i)) = 0$. Consequently by the Riemann-Roch theorem

$$h^0(X, \mathcal{O}_X \left( \sum_{i=1}^n D_i \right)) = \frac{\left( \sum_{i=1}^n D_i \right)^2}{2} + n + 1.$$ 

By construction, the movable part of $D$ is contained in the components of multiplicity one. Therefore

$$h^0 \left( X, \mathcal{O}_X \left( \sum_{i=1}^n D_i \right) \right) = h^0(X, \mathcal{O}_X(D)) = \frac{D^2}{2} + 2,$$

whence we obtain the relation

$$\frac{D^2}{2} + 2 = \frac{\left( \sum_{i=1}^n D_i \right)^2}{2} + n + 1,$$

i.e.

$$\frac{D_0 \cdot \sum_{i=1}^n D_i + (D_0, D)}{2} = n - 1.$$

But $(D, D_0) \geq 0$ because of the absence of fixed components in a multiple of the divisor $D$. The latter contradicts the inequality $(\Sigma_1^n D_i, D_0) \geq 2n$. 

**2.3. Lemma.** Let $D$ be an effective divisor on a K3 surface $X$ such that some multiple $mD$, $m$ a natural number, gives a linear system $|mD|$ without base points and such that the image of the corresponding morphism is two-dimensional. Then $D$ can have at most one fixed component, which is a smooth rational curve.

**Proof.** The linear system $|D|$ satisfies the assumptions of Mumford's theorem about degeneration. Hence by duality and the Riemann-Roch theorem we have

$$h^0(X, \mathcal{O}_X(D)) = \frac{D^2}{2} + 2,$$

but then by Lemma 2.2 the fixed part $D_0$ of $D$ has multiplicity one. Every irreducible component of $D_0$ is a smooth rational curve $C$ with $C^2 = -2$. We will show that every connected component $D'_0$ of $D_0$ is a tree such that at every vertex two curves meet and $(D'_0)^2 = -2$. The proof will proceed by induction starting with some curve $C_1$ in $D_0$ and adding curves $C_2, \ldots, C_n$ so that the divisor $\Sigma_1^n C_i$ should be connected and contained in $D'_0$. The first step of the induction is trivial. Therefore we assume that $\Sigma_1^n C_i$ is a connected tree of the kind described above and that $(\Sigma_1^n C_i)^2 = -2$. We also assume that in $D'_0$ there is a curve $C_{n+1}$ which intersects $\Sigma_1^n C_i$; in the contrary case everything is proven. By Ramanujan's...
theorem, since $\sum_{i=1}^{n+1} C_i$ is connected and of multiplicity one, and by the Riemann-Roch theorem,

$$h^0 \left( X, \mathcal{O}_X \left( \sum_{i=1}^{n+1} C_i \right) \right) = \frac{\left( \sum_{i=1}^{n+1} C_i \right)^2}{2} + 2 = \left( \sum_{i=1}^{n} C_i, C_{n+1} \right).$$

Then, because $\sum_{i=1}^{n+1} C_i$ is fixed,

$$\left( \sum_{i=1}^{n+1} C_i \right)^2 = -2, \quad \left( \sum_{i=1}^{n} C_i, C_{n+1} \right) = 1.$$

This completes the induction. Let us now consider the movable part $D_1$ of $D$. If $D_1$ is not a pencil, then its general element is irreducible and reduced. Hence, again using Ramanujan's theorem and the Riemann-Roch theorem, we obtain

$$h^0 (X, \mathcal{O}_X (D_1)) = \frac{D_1^2}{2} + 2.$$

Let $D'_0$ be a connected component of the fixed part. By the assumption of the lemma on the divisor $D$ we have $(D, D'_0) \geq 0$. On the other hand, $(D, D'_0) = (D_1 + D'_0, D'_0) = (D_1, D'_0) + (D'_0)^2$. Then $(D_1, D'_0) \geq 2$. The divisor $D_1 + D'_0$ is connected and of multiplicity one. Therefore, as above,

$$h^0 (X, \mathcal{O}_X (D_1 + D'_0)) = \frac{(D_1 + D'_0)^2}{2} + 2 = \frac{D_1^2}{2} + 2 + (D_1, D'_0) + \frac{(D'_0)^2}{2},$$

whence $h^0 (X, \mathcal{O}_X (D_1 + D'_0)) > h^0 (X, \mathcal{O}_X (D_1))$. Consequently in this case $D$ has no fixed components. If $D_1$ is a pencil, then $|D_1| = |nE|$, where $|E|$ is an elliptic pencil on the $K3$ surface $X$. In this case because $D$ is connected there must exist at least one fixed component. We will prove that it is unique and that it is a nonsingular rational curve which is a section of $|E|$. Because $D$ is connected there exists a curve $C$ in $D_0$ such that $C$ does not lie in the fibers of $|E|$, i.e. $C \cdot E > 0$. Because $C + E$ is connected and of multiplicity one, we have

$$h^0 (X, \mathcal{O}_X (C + E)) = \frac{(C + E)^2}{2} + 2 = h^0 (X, \mathcal{O}_X (E)) + (C, E) + \frac{C^2}{2};$$

hence $(C, E) = 1$. Consequently $C$ is a section. If in $D_0$ there are two sections $C_1$ and $C_2$, and $n \geq 2$, then

$$h^0 (X, \mathcal{O}_X (C_1 + C_2 + 2E)) = \frac{(C_1 + C_2 + 2E)^2}{2} + 2 = h^0 (X, \mathcal{O}_X (2E)) + \frac{C_1^2}{2} + \frac{C_2^2}{2} + 2 (C_1, E) + 2 (C_2, E) + (C_1, C_2) - 1 \geq h^0 (X, \mathcal{O}_X (2E)) + 1.$$
The latter contradicts the choice of \( C_1 \) and \( C_2 \) from the fixed part of \( D \). Therefore if \( D_0 \) has two sections then \( n \leq 1 \). But \( |D| = |nE + D_0| \) and \( D^2 = \sum_{i=1}^{n} (2nE, D_0^{(i)}) + (D_0^{(i)})^2 > 0 \), where \( D_0^{(i)} \) is a connected component of \( D_0 \). Hence it follows that \( n = 1 \) and that there exists a connected component \( D_0^{(i)} \) with \( (D_0^{(i)}, E) \geq 2 \). From this, as above in the nonpencil case, we derive the inequality

\[
\begin{aligned}
&h^0(X, O_X(E + D_0^{(i)})) > h^0(X, O_X(E)),
\end{aligned}
\]

which leads to a contradiction. Consequently in \( D_0 \) there exists exactly one section \( C \), and the remaining curves \( D_0 \) lie in the fibers. We will assume that the last set of curves is nonempty. Then there exists a curve \( C' \) in \( D_0 \) extreme in some tree, i.e. \( (C', D_0) = -1 \) and \( (C', E) = 0 \). Then \( (D, C') = (nE + D_0, C') = -1 \), which contradicts the choice of \( |D| \). Consequently \( C \) is the only fixed component of \( |D| \) and \( |D| = nE + C \).

**Remark.** Lemma 2.3 in the case of an ample \( D \) was proved in [6].

§3. Proof of the theorem in the case \( r = 1 \)

3.1. We denote by \( W \) the image of the rational map \( V \rightarrow \mathbb{P}^{\dim |-K_V|} \) defined by the linear system \( |K_V| \).

3.2. **Lemma.** \( \dim W \geq 2 \).

**Proof.** Let the linear system \( |K_V| \) define a mapping onto a curve \( W \) in \( \mathbb{P}^{g+1} \), \( g = (-K_V)^2/2 + 1 \). We denote by \( D_0 \) the fixed component of the system \( |-K_V| \) and by \( D \) the general divisor of the movable part. The curve \( W \) is rational since \( h^1(V, O_V) = 0 \) (see (1.3) in [4]). From linear normality it follows that \( W \) is a smooth rational curve of degree \( g + 1 \) which generates \( \mathbb{P}^{g+1} \). Therefore \( D \sim (g + 1)E \) and the (projectively) one-dimensional system \( |E| \) defines a rational map \( \pi: V \rightarrow W \cong \mathbb{P}^1 \). We have \((D_0 + (g + 1)E)^2, -K_V) = 2g - 2 \) from the definition of \( g \), since \(-K_V \sim D_0 + (g + 1)E \). The following relation is evident:

\[
((D_0 + (g + 1)E)^2, -K_V) = ((g + 1)^2E^2 + (g + 1) (E, D_0) + (D_0, -K_V), -K_V).
\]

The movability of \( E \) and the ampleness of \(-K_V\) implies the inequalities

\[
(E^2, -K_V) \geq 0, \quad (D_0, (-K_V)^2) \geq 0, \quad (E, D_0, -K_V) \geq 0.
\]

If \((E^2, -K_V) > 0 \), then

\[
2g - 2 = ((D_0 + (g + 1)E)^2, -K_V) \geq (g + 1)^2.
\]

The latter leads to a contradiction. Therefore \((E^2, -K_V) = 0 \). Then by the ampleness of \(-K_V\) the general members of \(|E|\) do not intersect and the linear system \(|E|\) defines a morphism \( \pi: V \rightarrow W \cong \mathbb{P}^1 \) whose fibers give \(|E|\). By Lemma 2.1 every divisor in \(|-K_V|\) is connected. Consequently \( D_0 \neq 0 \) and intersects the general member of \(|E|\) along a nontrivial effective one-dimensional algebraic cycle. In addition,

\[
2g - 2 = (g + 1)(E, D_0, -K_V) + (D_0, K_V^2),
\]

where \((E, D_0, -K_V) > 0 \) and \((D_0, K_V^2) > 0 \). That means that \((E, D_0, -K_V) = 1 \) and
(D_0, K_V^2) = g - 3. Then obviously \((E, K_V, K_V) = 1\). This last equality together with the ampleness of \(-K_V\) implies that any fiber (i.e. an element of \(|E|\)) is irreducible and reduced. Therefore \(D_0\) does not have components contained in the fibers of \(\pi\). Since \((E, D_0, -K_V) = 1\) and \(-K_V\) is ample, it follows that \(D_0\) is an irreducible reduced divisor and the fibers of the morphism \(\pi: D_0 \rightarrow W\), which are irreducible and reduced curves, define a linear system \(|(E, D_0)|_{D_0}\) on \(D_0\) whose elements we will call the fibers of \(D_0\). Also, the relation \((E, D_0, -K_V) = 1\) implies the smoothness of the general point of all the fibers of \(D_0\). By the Bertini-Zariski theorem the general fiber \(E\) of \(\pi\) is a smooth irreducible surface. The given surface \(E\) is a del Pezzo surface of degree 1, and \((E, D_0) = (E, -K_V)\) gives an ample anticanonical class of degree 1 on \(E\), \((E, D_0^2) = 1\). Therefore there exists on \(D_0\) a pencil of irreducible reduced curves of arithmetic genus one consisting of the fibers of \(D_0\). Consequently \(h^1(D_0, \mathcal{O}_{D_0}) \leq 1\).

On the other hand, from the long exact cohomology sequence for the triple \(0 \rightarrow \mathcal{O}_V(-D_0) \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_{D_0} \rightarrow 0\) we find that \(h^1(D_0, \mathcal{O}_{D_0}) = h^2(V, \mathcal{O}_V(-D_0))\). By duality

\[h^2(V, \mathcal{O}_V(-D_0)) = h^1(V, \mathcal{O}_V(-(g + 1)E)).\]

From the exact sequence corresponding to

\[0 \rightarrow \mathcal{O}_V(-(g + 1)E) \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_{(g+1)E} \rightarrow 0,
\]

it follows that

\[h^1(V, \mathcal{O}_V(-(g + 1)E)) = h^1((g + 1)E, \mathcal{O}_{(g+1)E}) - 1.\]

Hence, since the general member of the pencil \(|E|\) is irreducible and reduced, we have \(h^1(V, \mathcal{O}_V(-(g + 1)E)) = g\). This means that \(h^1(D_0, \mathcal{O}_{D_0}) = g\). Then because of the above we obtain the inequality \(1 \geq h^1(D_0, \mathcal{O}_{D_0}) = g\). But \((-K_V)^3 = 2g - 2 > 0\) because of the ampleness of \(-K_V\). This contradiction completes the proof of the lemma. ■

**Proof of Theorem 1.2 (case \(r = 1\)).** By Lemma 3.2, \(\dim W \geq 2\). Then by Bertini’s theorem [1] the general element of the linear system \(|-K_V|\) is of the form \(D + D_0\), where \(D_0\) is the fixed component of \(|-K_V|\) and \(D\) is the movable irreducible reduced divisor normally intersecting \(D_0\) (\(\dim D \cap D_0 \leq 1\)) and having singular points only at the base points of the linear system \(|D|\). We will resolve the points of indeterminacy of \(|D|\) (in the Hironaka-Zariski sense) by monoidal transformations with smooth centers in the base locus. We denote a general resolution by \(\sigma: \tilde{V} \rightarrow V\). By Bertini’s theorem the strict transform \(\widetilde{D}\) of a generic \(D\) is nonsingular and \(\sigma^*(D) = \widetilde{D} + \sum E_i\), where \(E_i\) is the surface corresponding to the \(i\)th monoidal transformation (the strict transform on \(\tilde{V}\) of the \(i\)th center of blowing up). Also \(\widetilde{D}\) is the maximal movable part of the linear system \(|\sigma^*(D)|\). Consequently

\[h^0(\widetilde{V}, \mathcal{O}_{\widetilde{V}}(\widetilde{D})) = h^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(\sigma^*(D))) = h^0(V, \mathcal{O}_V(D)) = \frac{-K_V^2}{2} + 3\]

(the last part because of (1.3) (ii) of [4]). The canonical class of \(\tilde{V}\) is computed from the
where $\alpha_i \geq 1$. Under blowing up a curve the canonical class changes according to the formula $K_{\tilde{V}} \sim \sigma^*(K_V) + E$. In our case the blowing up is carried out only at the base curves and points. Hence by induction we obtain $n_i \geq \alpha_i$, if $\alpha(E_i)$ is a curve on $V$. Then

$$-K_{\tilde{V}} \sim \tilde{D} + \sigma^*(D_0) + \sum_{i=1}^m (n_i - \alpha_i) E_i.$$ 

By the adjunction formula

$$K_{\tilde{D}} \sim -\left( \tilde{D}, \sum_{i=1}^m (n_i - \alpha_i) E_i + \sigma^*(D_0) \right).$$

Let us consider on $\tilde{D}$ the divisors $F = (\tilde{D}, \sigma^*(D + D_0))$ and $L = (\tilde{D}, \tilde{D} + \sum_{i=1}^m \alpha_i E_i)$. Then $K_{\tilde{D}} + F \sim L$. A multiple of $F$ comes from a hyperplane section because of the ampleness of $-K_{\tilde{V}}$. The sheaf $\mathcal{O}_{\tilde{D}}(-F)$ satisfies the conditions of Mumford's vanishing theorem [5], $H^1(\tilde{D}, \mathcal{O}_{\tilde{D}}(-F)) = 0$, since $\sigma_* \mathcal{O}_{\tilde{D}}(F)$ is ample on $D$. Consequently

$$h^1(\tilde{D}, \mathcal{O}_{\tilde{D}}(L)) = h^1(\tilde{D}, \mathcal{O}_{\tilde{D}}(K_{\tilde{D}} - L)) = h^1(\tilde{D}, \mathcal{O}_{\tilde{D}}(-F)) = 0.$$

Also it is obvious that $h^2(\tilde{D}, \mathcal{O}_{\tilde{D}}(L)) = 0$ since $h^0(\tilde{D}, \mathcal{O}_{\tilde{D}}(-F)) = 0$, whence by the Riemann-Roch theorem we obtain

$$h^0(\tilde{D}, \mathcal{O}_{\tilde{D}}(L)) = \frac{L(L - K_{\tilde{D}})}{2} + 1 - q + p_g.$$

Using the zero part of the cohomology sequence corresponding to the short exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{V}} \rightarrow \mathcal{O}_{\tilde{V}}(\tilde{D}) \rightarrow \mathcal{O}_{\tilde{D}}((\tilde{D}, \tilde{D})) \rightarrow 0,$$

we obtain the inequality

$$h^0(\tilde{D}, \mathcal{O}_{\tilde{D}}(L)) \geq h^0(\tilde{D}, \mathcal{O}_{\tilde{D}}((\tilde{D}, \tilde{D}))) \geq \frac{(K_V)^3}{2} + 2.$$

The latter together with the previous computations gives

$$\frac{(K_V)^3}{2} + 2 \leq \frac{L(L - K_{\tilde{D}})}{2} + 1 - q + p_g. \quad (3.3)$$

We now prove that $p_g - q - 1 \geq 0$. We have $L - K_{\tilde{D}} \sim F$, and by (3.3)

$$\frac{\sigma^*(-K_V)^3}{2} - \frac{\left( \sigma^*(-K_V), \tilde{D}, \tilde{D} + \sum_{i=1}^m \alpha_i E_i \right)}{2} \leq p_g - q - 1. \quad (3.4)$$

The left-hand side of (3.4) can be written in the form

$$\left( \frac{\sigma^*(-K_V)}{2}, \left( \tilde{D}, \sum_{i=1}^m (n_i - \alpha_i) E_i + \sigma^*(D_0), \sum_{i=1}^m n_i E_i + \sigma^*(D_0), \sigma^*(-K_V) \right) \right).$$
Here $\sigma^*(-K_V)$ is the lifting of the ample divisor, and $\tilde{D}$ is movable, irreducible and reduced. Hence the left-hand side of (3.4) is obviously positive in all its terms except perhaps $(\sigma^*(-K_V)/2, \tilde{D}, (n_i - \alpha_i)E_i)$ in the case when $a(E_i)$ is a point of $V$, since in the opposite case $n_i \geq \alpha_i$. But then in this case the corresponding term is equal to 0 by the projection formula. From (3.4) we obtain the desired inequality. Let us now consider a divisor $D$ such that for some smooth model of it the inequality $p_g - q - 1 \geq 0$ is satisfied, and in addition let $D$ be chosen so general that its singularities lie only in the base locus of its complete linear system. We resolve the singularities of $D$ by monoidal transformations centered in singular sets of $D$.

We will denote the new resolution by $\sigma': V' \to V$. Accordingly $\sigma'(\mathcal{O}_D) = \mathcal{O}_{\sigma'(V')} + \sum n_i E_i$ and $K_{V'} \sim \sigma^*(K_V) + \sum \alpha_i' E_i$, where $\alpha_i' \geq 1$, since this time we perform monoidal transformations only in singular sets $n_i \geq \alpha_i'$. Therefore by the adjunction formula $K_{D'} \leq 0$. Consequently $p_g \leq 1$, whence because $p_g - q - 1 \geq 0$ we have $p_g = 1$, $q = 0$ and $K_{D'} = 0$. This means that $D'$ is a $K3$ surface. From the latter one easily concludes by Lemma 2.1 that $D_0 = 0$ and $K_{V'} \sim -D'$. Consequently,

$$\sigma'^*(-K_V) \sim D' + \sum_{i=1}^m \alpha_i' E_i.$$ 

We have

$$h^0(V', \mathcal{O}_{V'}(D' + \sum_{i=1}^m \beta_i E_i')) = h^0(V, \mathcal{O}_V(-K_V)) = \frac{(-K_V)^3}{2} + 3$$

for the maximal movable part $|D' + \sum_{i=1}^m \beta_i E_i'|$ in $|D' + \sum_{i=1}^m \alpha_i' E_i'|$, where $\beta_i \leq \alpha_i'$. For $L' = (D', D' + \sum_{i=1}^m \alpha_i' E_i')$ we have

$$h^0(D', \mathcal{O}_{D'}(L')) \geq h^0\left(D', \mathcal{O}_{D'}\left(D' + \sum_{i=1}^m \beta_i E_i'\right)\right) \geq \frac{-(K_V)^3}{2} + 2.$$ 

Hence, as above, using Mumford's theorem about degeneration and the Riemann-Roch theorem we obtain the inequality

$$\frac{-(K_V)^3}{2} + 2 \leq h^0(D', \mathcal{O}_{D'}(L')) \leq \frac{(L')^2}{2} + 2,$$ 

(3.5)

since in the last case $K_{D'} = 0$, $q = 0$ and $p_g = 1$. Considering the difference in the left-hand side of the corresponding inequality analogous to (3.4), we find that it is positive, which means that our inequality (3.5) becomes an equality. Hence the linear system $|L'|$ on $D'$ has a fixed component $\sum_{i=1}^m (\alpha_i' - \beta_i')(E_i' \cdot D')$. Obviously the first resolution in $\sigma'$ as well as all the others resolve an isolated quadratic singularity, i.e. $\alpha_i' = 2$ according to the formula $-D' \sim K_{V'}$ for the canonical class of $V'$. Hence, by Lemma 2.3, $\beta_1 \geq 1$. This means that the first resolved singularity is movable. By the requirement that singularities should be at the base points we obtain that $\beta_1 = 1$, $\alpha_1' - \beta_1 = 1$, and $|D|$ and $|D' + \sum_{i=1}^m \alpha_i' E_i'|$ have a fixed curve outside of $E_1'$. Hence $|L'|$ has at least two distinct fixed curves: $(E_1', D')$ and one lying
outside $E_i$. The latter is impossible by Lemma 2.3. Consequently the general element $D = D'$, and it is nonsingular. This completes the proof of the theorem for the case $r = 1$. 

§4. Proof of the theorem for $r \geq 2$

4.1. We denote by $W$ the image of the rational map $V \dashrightarrow \mathbb{P}^{\dim |H|}$ defined by the linear system $|H|$, where $H$ is an effective divisor in $|\mathcal{L}|$.

4.2. Lemma. $\dim W \geq 2$.

Proof. Let us assume the contrary; then, as in the proof of Lemma 3.2, we obtain the decomposition $|H| = |D_0 + nE|$, $n = h^0(V, \mathcal{O}_V(H)) - 1$, and the one-dimensional linear system $|E|$ without fixed components gives a rational mapping $\pi: V \dashrightarrow W \cong \mathbb{P}^1$. According to (1.9) (ii) of [4],

$$n = \frac{(r+1)(r+2)}{2} H^3 + \frac{2}{r} \geq 2;$$

hence

$$H^3 = \frac{12n}{(r+1)(r+2)} - \frac{24}{r(r+1)(r+2)} < n$$

for $r \geq 2$. Using the relation $H^3 = (H, n^2 E^2 + nED_0 + D_0 H)$, the ampleness of $H$ and the absence of fixed components in $|E|$, we show as in the case $r = 1$ that $(H, E^2) = 0$. Because of the connectedness of the divisors in $|H|$ (a simple consequence of 2.1) we have $(H, E, D_0) \geq 1$ and $(H^2, D_0) \geq 1$. Therefore $n > H^3 \geq n + 1$, a contradiction. 

4.3. Lemma. For $r \geq 2$ the linear system $|H|$ can only have base points in the absence of a fixed component.

Proof. By Theorems 1.2 ($r = 1$) and 1.5 of [4] the general surface $D$ of the linear system $|-K_V|$ is a smooth $K3$ surface. Let us assume that the linear system $|H|$ has a fixed curve. Then by the ampleness of $D$ we obtain fixed points of the restricted system $|(H, D)|_D$. After restricting to $D$ one obtains a complete linear system. The latter follows from the exact cohomology sequence of the short exact sequence

$$0 \rightarrow \mathcal{O}_V((1-r)H) \rightarrow \mathcal{O}_V(H) \rightarrow \mathcal{O}_D((D, H)) \rightarrow 0,$$

since $h^1(V, \mathcal{O}_V((1-r)H)) = 0$ by (1.9) (i) of [4]. The restricted linear system is ample. In [6] it is shown that for every ample sheaf $\mathcal{L}$ on a $K3$ surface $D$ the linear system $|\mathcal{L}|$ has no base points if it has no fixed components. Therefore the linear system $|(H, D)|_D$ has a fixed component. Consequently by Lemma 2.3 the linear system $|(H, D)|_D = |nE + Z|$, with $Z$ a fixed curve. Then either $|H|$ has a fixed component or $|-K_V|$ has the fixed curve $Z$. We will show that the last case is impossible. Indeed, assuming the contrary we obtain for the restricted linear system $|(\mathcal{-K}_V, D)|_D$ on $D$ a representation of the form $|Z + n'E'|$, where $E'$ is a fiber of the elliptic pencil $|E'|$ on $D$. Consequently $rZ + nE \sim Z + n'E'$. $Z$ is a section of both pencils. Intersecting both sides of the last equivalence with $E'$, we obtain a contradiction for $r \geq 2$. 

4.4. Lemma. Let the linear system $|H|$ (4.1) have only fixed points and $H^3 < 8$. Then the general element of $|H|$ is smooth.

Proof. By Bertini's theorem [1] singular points of the general surface $H$ can only be among the fixed base points. If there exists a singular base point, then $H^3 \geq 8$ since at that singular point the general surfaces from $|H|$ have intersection index $\geq 8$.

Proof of Theorem 1.2 (case $r \geq 2$). By Lemma 4.2 and Bertini's theorem [1] the general element of the linear system $|H|$ has the form $D + D_0$, where $D_0$ is the fixed component of $|H|$ and $D$ is a movable irreducible and reduced divisor normally intersecting $D_0$ and having singular points only at the base points of the linear system $|D|$, $K_V \sim -rD - rD_0$. We resolve by monoidal transformations the points of indeterminacy of $|D|$. We denote the general resolution by $\sigma: \tilde{V} \to V$. The strict transform for the general divisor $D$, by Bertini's theorem, will be a smooth divisor $\tilde{D} \subset \tilde{V}$, and $\sigma^*(D) = \tilde{D} + \sum_{i=1}^{m} n_iE_i$, where $E_i$ is the surface corresponding to the $i$th transform and $n_i \geq 1$. We may assume that $\tilde{D}$ is the maximal movable part in $\sigma^*(D)$. Hence by (1.9) (ii) of [4] we have

$$h^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(\tilde{D})) = h^0(V, \mathcal{O}_V(H)) = \frac{(r+1)(r+2)}{12} H^3 + \frac{2}{r} + 1.$$ 

The canonical classes that we need have the form

$$-K_{\tilde{V}} \sim r\tilde{D} + \sum_{i=1}^{m} (rn_i - \alpha_i) E_i + r\sigma^*(D_0),$$

and

$$K_{\tilde{D}} \sim -\left(\tilde{D}, (r - 1)\tilde{D} + \sum_{i=1}^{m} (rn_i - \alpha_i) E_i + r\sigma^*(D_0)\right),$$

where $n_i \geq \alpha_i$ if $\sigma(E_i)$ is not a point of $V$ and all $\alpha_i \geq 1$. We consider on the surface $\tilde{D}$ the following divisors:

$$F = (\tilde{D}, \sigma^*(D + D_0)) \quad \text{and} \quad L = \left(\tilde{D}, \tilde{D} + \sum_{i=1}^{m} \alpha_iE_i\right).$$

Then $K_{\tilde{D}} + rf \sim L$.

Further using the degeneration theorem as in §3 for the sheaf $\mathcal{O}_{\tilde{D}}(F)$, we obtain the inequalities

$$\frac{(r+1)(r+2)}{12} H^3 + \frac{2}{r} \leq h^0(\tilde{D}, \mathcal{O}_{\tilde{D}}((\tilde{D}, \tilde{D}))) = \frac{L \cdot (I - K_{\tilde{V}})}{2} + 1 - q. \quad (4.5)$$

In contrast to §3, in (4.5) we have $p_g = 0$, as it is easy to check that $K_{\tilde{D}} < 0$. The extreme terms of (4.5) give the inequality

$$\left(\sigma^*(H), \frac{(r+1)(r+2)}{12} \sigma^*(H)^2 - \frac{r}{2} \left(\tilde{D}, \tilde{D} + \sum_{i=1}^{m} \alpha_iE_i\right)\right) \leq 1 - q - \frac{r}{2}. \quad (4.6)$$
Substituting in (4.6) the expression for $\sigma^*(H) = \sigma^*(D + D_0)$ and collecting like terms, we obtain

$$
\frac{(r-1)(r-2)}{12} \left( \sigma^*(H), \tilde{D}, \tilde{D} + \sum_{i=1}^{m} a_i E_i \right) + \frac{(r+1)(r+2)}{12} \left( \sigma^*(H)^2, \sum_{i=1}^{m} n_i E_i + \sigma^*(D_0) \right) \leq 1 - q \cdot \frac{2}{r}.
$$

As in §3, one proves the positivity of the left-hand side of (4.7). Therefore $q = 0$ and $D_0 = 0$. The latter follows from the fact that $(\sigma^*(H), \tilde{D}, \sigma^*(D_0)) = (H, D, D_0) \geq 1$ by the connectedness of $H$. We now note that if $D_0 = 0$ then by Lemma 4.3 $|H|$ has only base points. Then by Lemma 4.4 we may assume that $H^3 \geq 8$. Let $d = H^3 > 0$ and $\Delta = 3 + d - h^0(V, \mathcal{O}_V(H))$, and let $g$ be defined by the relation $2g - 2 = (K_V + 2H)H^2 = (2 - r)H^3$, i.e. $g = ((2 - r)d + 2)/2$. Knowing $h^0(V, \mathcal{O}_V(H))$ from (1.9) in [4], we can easily check that $\Delta \leq g$ for $d = H^3 \geq 2$. Therefore, by Theorem 4.1 of [3], there are no base points in $|H|$ if $d \geq 2\Delta$. This inequality fails to be satisfied only for $r = 2, d = 1$ and $r = 3, d = 1$. In our case $d = H^3 \geq 8$. Consequently there are no base points in this case. Therefore by Bertini’s theorem the general member of $|H|$ is smooth. □

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