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ON THE CLOSED CONE OF CURVES OF ALGEBRAIC 3-FOLDS

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ABSTRACT. In this paper the author establishes, under natural conditions, the local polyhedrality of the closed cone of curves of a three-dimensional algebraic variety in the part that is negative with respect to the canonical class. In particular, it is shown that there always exists an extremal ray giving a contraction. The results can be used in three-dimensional birational geometry.

Bibliography: 10 titles.

$X$ denotes throughout a normal projective 3-fold defined over an algebraically closed field $k$ of characteristic 0. We recall the terminology of Mori [4] and Kawamata [3]. There are two real vector spaces associated with the variety $X$.

$N(X) = (\{1\text{-cycles on } X \} / \equiv ) \otimes \mathbb{R}$

and

$N(X)^0 = (\{\text{Cartier divisors on } X \} / \equiv ) \otimes \mathbb{R},$

where $\equiv$ denotes numerical equivalences; the intersection pairing

$(\cdot) : N(X)^0 \times N(X) \to \mathbb{R}$

is nondegenerate by definition of $\equiv$. On $N(X)$ and $N(X)^0$ we fix a Euclidean norm $|| \cdot ||$. This defines the closed cone of curves $\overline{NE}(X) \subset N(X)$, which is the closure with respect to $|| \cdot ||$ of the cone $NE(X)$ of effective 1-cycles on $X$. This cone is obviously independent of the choice of $|| \cdot ||$.

$K_X$ denotes the canonical Weil divisor of $X$ [5]. By definition $\partial_{\text{Reg}(X)}(K_X) = \Omega_{\text{Reg}(X)}^1$, where $\text{Reg}(X) = X - \text{Sing}(X)$ is the set of nonsingular points of $X$.

By a $\mathbb{Q}$-Cartier divisor we mean a linear combination of Cartier divisors with rational coefficients. We suppose furthermore that $X$ is $\mathbb{Q}$-factorial. This means that every Weil divisor $D$ on $X$ is a rational multiple of a Cartier divisor; that is, there exists an integer $n$ such that $nD$ is a Cartier divisor on $X$. On such a variety each Weil divisor $D$ corresponds to a $\mathbb{Q}$-Cartier divisor, and has a numerical class $(D) \in N(X)^0$. In particular, we can take the intersection of Weil divisors with 1-cycles. The Weil divisor $K_X$ defines a $\mathbb{Q}$-Cartier
divisor which we continue to denote by $K_X$. We recall that $X$ is said to have canonical singularities (respectively terminal singularities) if for some resolution $h: X' \to X$,

$$K_{X'} = h^*K_X + \sum a_i E_i,$$

where the $E_i$ are the exceptional divisors of $h$, and all the $a_i$ are $\geq 0$ (respectively $> 0$). It is easy to check that this definition is independent of the resolution $h$. We assume from now on that $X$ is a variety with canonical singularities.

We let $\overline{NE}(X)^-$ denote the cone \{ $\mathbf{Z} \in \overline{NE}(X) | \mathbf{Z} \cdot K_X < 0$ \}

By an extremal ray of $\overline{NE}(X)^-$ we mean a ray $R \subset \overline{NE}(X)^-$ such that

1. If $Z_1, Z_2 \in \overline{NE}(X)$ and $Z_1 + Z_2 \in R$, then $Z_1, Z_2 \in R$.

A ray $R$ is said to be locally polyhedral if there exists a divisor $D \in N(X)^0$ and a finite collection of curves $C_1, \ldots, C_r$ on $X$ such that $\overline{NE}(X) = \overline{NE}(X, D)^+ + \sum R_+(C_i)$ and $D \cdot Z < 0$ for all $Z \in R - \{0\}$; here $\overline{NE}(X, D)^+ = \{ Z \in \overline{NE}(X) | D \cdot Z \geq 0 \}$. In this case the ray $R$ satisfies Mori's conditions, namely

1. $R$ is rational; that is, $R = R_+(C_R)$ for some curve $C_R \subset X$.
2. $R^\perp = \{ D \in N(X)^0 | D \cdot R = 0 \}$ contains an open subset of numerically effective divisors $D \in N(X)^0$ for which $D^\perp \cap \overline{NE}(X) = R$.

To a locally polyhedral extremal ray $R \subset \overline{NE}(X)^-$ we can apply Kawamata’s technique [3], and so $R$ determines a morphism $\text{cont}_R: X \to Y$ contracting the extremal ray $R$. (Kawamata’s preprint in fact assumes that $X$ has terminal singularities, but this condition is not used in an essential way in his proof; see [7].)

We say that $R$ is a ray of type (a) (respectively of type (b)) if $R$ is a locally polyhedral extremal ray of $\overline{NE}(X)^-$ such that the morphism $\text{cont}_R: X \to Y$ is birational and contracts a surface $S$ of $X$ (respectively contracts only a finite set of curves of $X$).

**Main Theorem.** Let $X$ be a projective normal $\mathbb{Q}$-factorial 3-fold with canonical singularities, and suppose that any compact subset of the cone $\overline{NE}(X)^-$ has at most a finite number of extremal rays of type (b). Then $\overline{NE}(X)$ is locally polyhedral in $\overline{NE}(X)^-$; that is, for any ample divisor $A$ and any $\varepsilon > 0$ there exists a finite set of curves $C_1, \ldots, C_r$ such that

$$\overline{NE}(X) = \overline{NE}_\varepsilon(X, A) + \sum R_+(C_i),$$

where $\overline{NE}_\varepsilon(X, A) = \{ Z \in \overline{NE}(X) | (K_X + \varepsilon A) \cdot Z > 0 \}$.

**Corollary.** If $K_X$ is not numerically effective, then $\overline{NE}(X)^-$ always contains a locally polyhedral extremal ray $R$.

This result was proved independently (but later) by Reid [7], using a closely related method.*

**§2. The main lemma**

2.1. **Lemma.** Let $X$ be a projective normal $\mathbb{Q}$-factorial 3-fold with canonical singularities, let $A$ be an ample Cartier divisor, and suppose that for some $a \in \mathbb{R}$, $D \in (A + aK)$ is a numerically effective divisor such that

1. the face of $\overline{NE}(X)$ given by $R = D^\perp \cap \overline{NE}(X)$ satisfies $R \subset \overline{NE}(X)^-$, and
2. either $D^3 > 0$ or $-D^2 K_X > 0$.

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*Translator's note. Much progress has been made on this problem in recent months; see [8], [9] and [10]. Both the Contraction Theorem and the Theorem on the Cone are now known in all dimensions.
Then either $\overline{\text{NE}}(X)$ is locally polyhedral in a neighborhood of $R$ (that is, there exist a finite set of curves $C_1, \ldots, C_r$ and an $\varepsilon > 0$ such that

$$
\overline{\text{NE}}(X) = \overline{\text{NE}}_r(X, D) + \sum_{i=1}^r \mathbf{R}_+(C_i),
$$

(2.2)

where $\overline{\text{NE}}_r(X, D) = \{ Z \in \overline{\text{NE}}(X) | (D + \varepsilon K_X \cdot Z) \geq 0 \}$, or there exists a morphism $\varphi: X \rightarrow Y$ making $X$ into a conic fibration, such that $(C) \in R$ for a general fiber $C = \varphi^{-1}(y)$.

**Proof.** Let $\alpha = m/n - \delta$, where $m$ and $n$ are positive integers and $0 < \delta \leq 1/n$. Then

$$D = A + (m/n)K_X - \delta K_X.
$$

From (i), the divisor $D_{m/n} = A + (m/n)K_X$ is numerically negative on $R$. By virtue of the proof of Theorem 1 in [3], in order to establish the decomposition (2.2) it is enough to check that, for some integer $N > 0$,

$$|ND_{m/n}| \neq \emptyset.
$$

(2.4)

We will prove this using Riemann-Roch and vanishing; consider a resolution $h: X' \rightarrow X$ which is the standard resolution along the curves of canonical singularities, and is otherwise arbitrary. Then the exceptional divisors $E_i$ which map to curves of $X$ have discrepancy $a_i = 0$. We also assume that all exceptional divisors of $h$ are nonsingular and interest transversally. Set

$$\overline{h}(mK_X) = -[mK_X + \sum (m-1)a_i E_i] = mK_X - \sum [(m-1)a_i] E_i,$

where $[\ ]$ denotes the integral part of a number or a divisor. For $n \gg 0$ the divisor $D = (1/n - \delta)K_X = A + ((m-1)/n)K_X$ will satisfy the hypothesis of the Kawamata-Viehweg vanishing theorem [1], except in the case $D^3 = 0$ and $\delta = 1/n$. However, in this case, by (ii), $D$ is a Q-Cartier divisor with $D^3 = 0$ and $-D^2 K_X > 0$; then $D$ defines a conic fibration $\varphi_{|ND|}: X \rightarrow Y$. This is proved in [2] and [3] assuming terminal singularities, and in general using Kawamata’s technique in [6] and [7]. The general fiber $C = \varphi^{-1}(y)$ obviously has class $(C) \in R$ (by the definition of $R$; see (i)). In this case we have one of the conclusions of the lemma, so that from now on we can assume that it does not occur. Then, by Kawamata-Viehweg vanishing,

$$h^i(X', \mathcal{O}_{X'}(nh*A + \overline{h}(mK_X))) = h^i(X', \mathcal{O}_{X'}(-[nh*A - (m-1)h*K_X] + K_{X'})) = 0$$

for all $i > 0$. Hence

$$h^0(X', \mathcal{O}_{X'}(nh*A + \overline{h}(mK_X))) = \chi(\mathcal{O}_{X'}(nh*A = \overline{h}mK_X)) = \text{R-R expression}.
$$

Now note that

$$\overline{h}(mK_X) = mK_{X'} - \sum (m-1)a_j E_j - \sum \{(m-1)a_j\} E_j = mh*K_X + \sum (a_j - \{(m-1)a_j\}) E_j.
$$

(2.5)

Hence

$$nh*A + \overline{h}(mK_X) = h^*(nA + mK_X) + \sum b_j E_j.$$
where \( b_j = O(1) \) as \( n \gg 0 \). By (2.3), \( nA = mK_X = nD + \delta nK_X \). Writing down only the cubic and quadratic terms in the Riemann-Roch formula, and using the fact that \( |\delta n| \leq 1 \), we get
\[
h^0(X', \mathcal{O}_{X'}(nA^*A = \overline{h}(mK_X))) = \frac{1}{2} (nD + \delta nK_X)^3 - \frac{1}{4} (\overline{h}^*(nD + \delta nK_X))^2 K_X + \cdots.
\] (2.6)
where the dots denote terms bounded by a linear function of \( n \). We now prove that the right-hand side of (2.6) is strictly positive if \( n \gg 0 \). If \( \exists \eta > 0 \) this is obvious. Suppose then that \( D^3 = 0 \) and \(-D^2K_X > 0\). If \( \alpha \) is rational, we have seen above that \( X \) is not a conic fibration, \( \alpha \) is irrational. Then letting \( m/n \) be a continued fraction approximation of \( \alpha \), we can assume that \( \delta n < \sqrt{n} \), and then for \( n \gg 0 \) we get
\[
h^0(X', \mathcal{O}_{X'}(nh^*A + \overline{h}(mK_X))) = -\frac{1}{4} n^2 D^2 K_X + \cdots > 0,
\]
with the dots as before. Thus \( |nh^*A + \overline{h}(mK_X)| \neq \emptyset \) for suitable \( n \gg 0 \), and using (2.5) we get the required nonemptiness assertion (2.4).

\section{Proof of the main theorem}

3.1. \textit{Choice of the curves \( C \).} The cone \( \overline{NE}(X)^- \) can have at most a finite set of extremal rays of type (a) which “contract to a point”, since the exceptional surfaces \( E \) corresponding to these rays are disjoint in pairs, so that their classes in \( N(X)^0 \) are linearly independent. We also have outside \( \overline{NE},(X, A) \) a finite set of extremal rays of type (a) which “contract onto a curve”, since there is a curve \( C \) in such rays with \( CK_X = -1 \). So first of all we assume that \( \{C_i\} \) includes a finite set of curves \( C \), giving the extremal rays \( R^+(C) \) of type (a) outside \( \overline{NE},(X, A) \).

We can also see that the cone \( \{Z \in \overline{NE}(X)(K_X + \varepsilon A \cdot Z) \leq 0\} \) can have at most a finite set of rays of the form \( R^+(C) \) where \( C = \varphi^{-1}(y) \) is the general fiber of a conic fibration \( \varphi: X \to Y \). Indeed, then \( CK_X = -2 \), so that, assuming \( (K_X + \varepsilon A \cdot C) < 0 \), the degree \( (A \cdot C) < 2/\varepsilon \) is bounded, so that such curves belong to a bounded family. We include in \( \{C_i\} \) a finite set of curves which exhausts this set of rays.

By hypothesis, the half-cone \( \{Z \in \overline{NE}(X)(K_X + \varepsilon A \cdot Z) < 0\} \) has only a finite number of rays of type (b), and we add to \( \{C_i\} \) the curves corresponding to these.

Now consider the cone
\[
V = \overline{NE},(X, A) + \sum_{i=1}^r R^+(C_i) \subset \overline{NE}(X).
\]
If \( V = \overline{NE}(X) \) then the theorem is proved. Otherwise \( \overline{NE}(X) \) contains a rational ray \( Z = R^+(C) \subset V \), and obviously \( (C \cdot K_X) < 0 \).

\[
V_Z = \overline{NE},(X, A) + \sum_{i=1}^r R^+(C_i) + Z \subset \overline{NE}(X),
\]
so that \( Z \) is an edge of \( V_Z \), and take a Cartier divisor \( D \) such that the hyperplane \( D^\perp \) passes through this edge, with \( D^\perp \cap V_Z = Z \). Corresponding to \( D \) we have an affine line \( \{D, K_X\} \subset N(X)^0 \) and this line contains a divisor \( L_1 \) such that \( L_1^+ \) is a supporting hyperplane of \( \overline{NE}(X) \), with \( L_1 \) numerically effective and positive on \( V \); this \( L_1 \) can be written as a combination \( L_1 = D + \alpha K_X \), with \( \alpha > 0 \). By construction the cone \( R = L_1^+ \cap \overline{NE}(X) \) is nonempty and is contained strictly inside the half-cone \( \overline{NE}(X)^- \). Moreover, a suitable small neighborhood of \( R \) does not contain any of the rays \( R^+(C_i) \), and the divisor
mL_1 - K_X is ample for \( m \gg 0 \). It follows that \( L_1^3 \geq 0 \). If \( L_1^3 > 0 \) then it follows from the main lemma that \( \overline{NE}(X) \) is locally polyhedral in a neighborhood of \( R \). Then \( L_1 \) can be taken to be \( \mathbb{Q} \)-rational; but then \( R \) contains an extremal ray \( R' \) of type (a) or (b), which is impossible by construction. Hence \( L_1^3 = 0 \). Then \(-L_1^3 K_X \geq 0\) if \(-L_1^3 K_X > 0\) then again using the main lemma we see that either \( R \) contains a ray of the form \( R_\gamma(C) \) where \( C = \varphi^{-1}(y) \) is the general fiber of a conic fibration, which is impossible by construction, or \( \overline{NE}(X) \) is locally polyhedral in a neighborhood of \( R \). In this final case we again get either a ray of type (a) or (b), or a ray corresponding to a conic fibration, any of which are impossible by construction. Hence \(-L_1^3 K_X = 0\).

3.2. We have thus arrived at the situation that \( L_1^3 = L_2^3 K_X = 0 \). Using Mori's argument from [4], §6, we see that \( L_2^3 = 0 \). If \( \rho(X) \geq 3 \) then there exists another \( L_2 \) so that \( L_2^3 \) is a supporting hyperplane of \( \overline{NE}(X) \) similar to \( L_1 \), but \( L_2 \) are not proportional. Again \( L_2^3 = 0 \). By the numerical effectivity of \( L_1 \) and \( L_2 \) we have \( L_1 L_2 \in \overline{NE}(X) \). On the other hand, \( L_2^2 L_2 = L_1^2 L_2 = 0 \), and hence \( L_1 L_2 \in R_1 \cap R_2 \). If \( R_1 \cap R_2 = 0 \), then \( L_1 L_2 \equiv 0 \), and that by Mori's arguments it follows that \( L_1 \) and \( L_2 \) are proportional, which is impossible by assumption. If \( R_1 \cap R_2 \neq 0 \) then \( L_1 + L_2 \) again satisfies the same conditions as \( L_1 \), and then \((L_1 + L_2)^2 = 0\). Hence \( L_1 L_2 \equiv 0 \), which again leads to a contradiction.

3.3. Finally it remains to consider the case \( L_1^3 = L_2^3 K_X = 0 \) and \( \rho(X) = 2 \), the case \( \rho(X) = 1 \) being trivial. Here the extremality condition is trivial, and according to the results of [3] we need only the rationality of \( L_1 \). Indeed, if \( L_1 \) is rational, then by Kawamata's results \( L_1^4 \) is a supporting hyperplane for a ray \( R \) specifying a fibration of del Pezzo surfaces. But as with the rays giving conic fibrations, there are only a finite number of such rays outside \( \overline{NE}_e(X,A) \). Thus we could have added to \( \{C_i\} \) the classes of curves \( C_i \) of general del Pezzo's surfaces in such fibrations having \(-C_i K_X \leq 9\).

Thus \( L_1 \) is an irrational divisor, so that we can assume that \( L_1 = D + \alpha K_X \) with \( \alpha \) irrational, and \( D \) an ample Cartier divisor. The equations \( L_1^3 = L_1^2 K_X = 0 \) give polynomial equations of degree \( \leq 3 \) and \( 2 \) in \( \alpha \). Hence \( \alpha \) is a quadratic irrationality. Let \( \alpha' \) be the conjugate irrationality, and \( L_2 = D + \alpha' K_X \). Now \( L_2 \) must satisfy both the equations, since they have rational coefficients. It is easy to check that the cycle \( L_1 L_2 \) is rational. But \( L_1^2 L_2 = 0 \). Hence \( L_1 L_2 \equiv 0 \), since otherwise by irrationality of \( \alpha \) we would have \( L_1 L_2 K_X = 0 \); but if \( Z K_X = Z L_1 = 0 \) then \( Z \equiv 0 \). The relation \( L_1 L_2 \equiv 0 \) again leads to a contradiction, since \( L_1 \) and \( L_2 \) are not proportional, so that \( D = BL_1 + \gamma L_2 \), and hence \( D^3 = (BL_1 + \gamma L_2)^3 = 0 \). This contradiction completes the proof of the main theorem.

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** The Russian original cites a preprint of this article.


Translated by M. Reid

***Added by translator.