GEOGRAPHY OF LOG MODELS: THEORY AND APPLICATIONS.

V.V. SHOKUROV AND SUNG RAK CHOI

Abstract. An introduction to geography of log models with applications to positive cones of FT varieties and to geometry of minimal models and Mori fibrations.

Contents

1. Introduction

What is a relation between the LMMP and some concrete geometrical results such as polyhedral properties of effective and mobile cones for FT varieties, finiteness results for modifications of FT varieties and minimal models, factorization of birational isomorphisms of Mori fibrations into Sarkisov links and birational rigidity? One of the possible explanations for those relations can be found by exploring the geography of log models. This paper presents this explanation that was originally introduced in [Sho4] [IskSh] and partially developed in [Choi].

We always assume the LMMP and the semiampleness conjecture. One can find a detailed information about those conjectures and which of their variations are needed in Section 7.

Both authors were partially supported by NSF grant DMS-0701465.

2. Geography of log models

Take a pair $(X/Z, S)$ where X/Z is a proper morphism and $S = \sum S_i$ is a reduced b-divisor of X/Z with the distinct prime components S_i. In this section we introduce 5 equivalence relations on the *unit cube*:

$$\mathfrak{B}_S := \oplus_{i=1}^m [0,1]S_i \cong [0,1]^m.$$

An element $B \in \mathfrak{B}_S$ will be referred to as a *boundary* divisor even when it is actually a b-boundary. For the concepts of the LMMP used in these relations, see Section 7.

Model equivalence. Boundaries $B, B' \in \mathfrak{B}_S$ are *model equivalent* and we write $B \sim_{\text{mod}} B'$ if the pairs $(X/Z, B), (X/Z, B')$ have the same set of resulting models [IskSh, 2.9].

Weakly log canonical equivalence. Boundaries $B, B' \in \mathfrak{B}_S$ are *wlc model equivalent* and we write $B \sim_{\text{wlc}} B'$ if both pairs $(X/Z, B), (X/Z, B')$ have the same wlc models and the models are numerically equivalent, that is: for any model Y/Z of X/Z,

$(Y/Z, B_Y^{\log})$ is a wlc model of $(X/Z, B)$ if and only if $(Y/Z, {B'}_Y^{\log})$ is a wlc model of $(X/Z, B')$; and

the divisors $K_Y + B_Y^{\log}, K_Y + {B'}_Y^{\log}$ have the same signatures with respect to the intersection for all curves on Y/Z.

Log canonical model equivalence. Boundaries $B, B' \in \mathfrak{B}_S$ are *lc model equivalent* and we write $B \sim_{\text{lcm}} B'$ if the pairs $(X/Z, B), (X/Z, B')$ have the same (rational) Iitaka fibrations: for a given (rational) contraction $\varphi\colon X \dashrightarrow Y/Z$, φ is an Iitaka fibration for $(X/Y, B)$ if and only if φ is an Iitaka fibration for $(X/Y, B')$, or equivalently, $(X/Z, B)$ has a (rational) Iitaka fibration $X \dashrightarrow Y$ if and only if $(X/Z, B')$ has a (rational) Iitaka fibration $X \dashrightarrow Y'$ and the fibrations are naturally isomorphic, that is, there is a commutative diagram

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ Y & \xrightarrow{\cong} & Y' \end{array}$$

with the natural isomorphism $Y \cong Y'/Z$.

Mobile equivalence. Boundaries $B, B' \in \mathfrak{B}_S$ are *mobile equivalent* and we write $B \sim_{\text{mob}} B'$ if the positive parts $P(B), P(B')$ of both b-divisors $\mathcal{K} + B^{\log}, \mathcal{K} + {B'}^{\log}$ have the same signatures with respect to the curves on rather high models Y/Z. On a rather high model we can take any model Y/Z over some wlc models of $(X/Z, B), (X/Z, B')$.

Fixed equivalence. Boundaries $B, B' \in \mathfrak{B}_S$ are *fix equivalent* and we write $B \sim_{\text{fix}} B'$ if the fixed parts $F(B), F(B')$ of both b-divisors $\mathcal{K} + B^{\log}, \mathcal{K} + B'^{\log}$ have the same signatures of their multiplicities, that is: for any prime b-divisor D,

$$\operatorname{mult}_D F(B) > 0 \Leftrightarrow \operatorname{mult}_D F(B') > 0.$$

Note that all the relations above are defined on the whole $\mathfrak{B}_S$.

The resulting models obtained by the LMMP are at least projective, and even slt by the slt LMMP. An abstract definition of resulting models gives a larger class of them. However, according to the following it is enough to consider only the good ones: projective $\mathbb{Q}$-factorial wlc models, e.g., slt wlc models.

Proposition 2.1. *The equivalence $\sim_{\text{wlc}}$ for projective $\mathbb{Q}$-factorial wlc models is the same as that for all wlc models.*

Proof. Any non-$\mathbb{Q}$-factorial wlc model can be obtained by a crepant contraction of some $\mathbb{Q}$-factorial wlc model that gives a wlc model by the semiampleness. Any $\mathbb{Q}$-factorial nonprojective wlc model can be obtained from a projective and $\mathbb{Q}$-factorial wlc model by a (generalized) log flop. Both constructions use the slt LMMP and [Sho4]. Thus if the pairs have the same projective and $\mathbb{Q}$-factorial, numerically equivalent models, they have the same numerically equivalent wlc models.

The converse statement means that each projective and $\mathbb{Q}$-factorial wlc model of $(X/Z, B)$ will be also projective and $\mathbb{Q}$-factorial wlc model of $(X/Z, B')$ when $B \sim_{\text{wlc}} B'$ (cf. the proof of Lemma 2.7). ☐

Convention 2.2. The model equivalences $\sim_{\text{mod}}, \sim_{\text{wlc}}$ are defined for projective and $\mathbb{Q}$-factorial resulting models of pairs. In fact, we can even use slt wlc models since Proposition 2.1 holds for slt wlc models too. In most of constructions below, we use slt wlc resulting models. However, slt wlc models are not stable for limits of boundaries (see Lemma 2.9).

The following relations hold for model equivalences.

Proposition 2.3. *We have the following implications:*

$$\sim_{\text{fix}} \Leftarrow \sim_{\text{mod}} \Leftarrow \sim_{\text{wlc}} \Rightarrow \sim_{\text{lcm}} \Leftrightarrow \sim_{\text{mob}} \quad \textit{and}$$

$$\sim_{\text{wlc}} \Leftrightarrow \sim_{\text{fix}} \cap \sim_{\text{lcm}}$$

Proof. Immediate by definition and the semiampleness 7.1. ☐

In general,

$$\sim_{\mathrm{fix}} \not\Rightarrow \sim_{\mathrm{mod}} \not\Rightarrow \sim_{\mathrm{wlc}} \not\Leftarrow \sim_{\mathrm{lcm}} .$$

For example, $\sim_{\mathrm{fix}} \not\Rightarrow \sim_{\mathrm{mod}}$ follows from the fact that the small modifications of flopping facets preserve the $\sim_{\mathrm{fix}}$ equivalence relation but not $\sim_{\mathrm{mod}}$ (see Theorem 5.9, 5.11).

Define the subset of $\mathfrak{B}_S$:

$$\mathfrak{N}_S := \{B \in \mathfrak{B}_S \mid (X/Z, B) \text{ has a wlc model}\}.$$

Equivalently, we can also use the condition: $\nu(X/Z, B) \geq 0$ (see Numerical Kodaira dimension in Section 7). Moreover, by the semiampleness 7.1, this is equivalent to the nonnegativity of (invariant) Kodaira dimension (see Kodaira dimension in Section 7): $\kappa(X/Z, B) \geq 0$.

Proposition 2.3 implies that $\sim_{\mathrm{wlc}}$ is the finest of the model equivalences on $\mathfrak{N}_S$. Thus, for the pairs with wlc models, the finiteness and polyhedral properties of all the equivalence relations follow from those of $\sim_{\mathrm{wlc}}$. All the above equivalence relations, except for $\sim_{\mathrm{mod}}$, are not interesting outside $\mathfrak{N}_S$ because they are trivial on $\mathfrak{B}_S \setminus \mathfrak{N}_S$. It is expected that many similar properties hold for $\sim_{\mathrm{mod}}$ on $\mathfrak{B}_S$ in the case of Mori log fibrations [IskSh] and that plays an important role for the genuine Sarkisov program.

Recall that $\mathfrak{D}_S$ is the $\mathbb{R}$-vector space of b-divisors spanned by S:

$$\mathfrak{D}_S \colon = \oplus_{i=1}^m \mathbb{R} S_i \cong \mathbb{R}^m.$$

The space $\mathfrak{D}_S$ contains the cube $\mathfrak{B}_S$.

A *convex polyhedron* $\mathfrak{P}$ in $\mathfrak{D}_S$ is a set defined by the intersection of finitely many (open or closed) half spaces and hyperplanes in $\mathfrak{D}_S$. A *polyhedron* is a finite union of convex ones. A convex polyhedron $\mathfrak{P}$ is said to be *open* in $\mathfrak{D}_S$ if the half spaces in the intersection are open. The closure $\overline{\mathfrak{P}}$, the interior $\operatorname{Int}\mathfrak{P}$, and the boundary $\partial\mathfrak{P}$ of a convex polyhedron $\mathfrak{P}$ will be taken in its linear span, the minimal intersection of hyperplanes containing $\mathfrak{P}$. A *face* $\mathfrak{F}$ of a convex polyhedron $\mathfrak{P}$ is the intersection $\mathfrak{F} = \overline{\mathfrak{P}} \cap H$ (possibly $\emptyset$ or $\overline{\mathfrak{P}}$ itself) for some hyperplane $H = \{f = 0\}$ such that $\overline{\mathfrak{P}} \subsetneq \{f \geq 0\}$. A *facet* of $\mathfrak{P}$ is a maximal face, that is, a face of dimension $\dim \mathfrak{P} - 1$. A *ridge* is a facet of a facet.

For two distinct vectors $A, B \in \mathfrak{D}_S$, $\overrightarrow{AB}$ denotes the open ray starting from A through B and $\overline{AB}$ denotes the line through A and B. $[A, B]$ ($[A, B), (A, B)$, etc) is defined as the closed (resp. half closed, open, etc) interval between A and B.

Let $\mathfrak{P}$ be a convex polyhedron in $\mathfrak{B}_S$. A facet (face) $\mathfrak{F}$ of $\mathfrak{P}$ is *given by a facet* $\mathfrak{F}_i$ *of* $\mathfrak{B}_S$ if $\mathfrak{F} = \overline{\mathfrak{P}} \cap \mathfrak{F}_i$. The facet (resp. face) $\mathfrak{F}_i$ is given by equation $b_i = 0$ or 1. Thus $\mathfrak{F}$ can be given in $\overline{\mathfrak{P}}$ by one of those equations. Note the following property of the linear function b_i on $\mathfrak{D}_S$: if $b_i = 1$ (respectively 0) in $B \in \operatorname{Int}\mathfrak{F}$ and < 1 (respectively > 0) in some point of $\mathfrak{P}$, then $b_i = 1$ (respectively 0) on $\mathfrak{F}$ and < 1 (respectively > 0) in $\mathfrak{P} \setminus \mathfrak{F}$.

By a *geography* on $\mathfrak{N}_S$, we mean a finite rational polyhedral decomposition of it, and the subsets $\mathfrak{P} \subseteq \mathfrak{N}_S$ of such a decomposition will be referred to as *classes* or just *polyhedrons*. We call a class $\mathfrak{C}$ of maximal dimension, $\dim_{\mathbb{R}} \mathfrak{N}_S$, a *country*, a class $\mathfrak{F}$ of codimension 1 a *facet*, and a class $\mathfrak{R}$ of codimension 2 a *ridge*.

According to the following, each of the above equivalence relations defines a geography on $\mathfrak{N}_S$, but the notations $\mathfrak{N}_S$ will be reserved only for the *wlc* geography given by $\sim_{\mathrm{wlc}}$. Thus we consider $\mathfrak{N}_S$ as a set with the structure.

Let $\mathfrak{F}$ be a minimal face of $\mathfrak{B}_S$ which contains a polyhedron $\mathfrak{P}$. We say that a polyhedron $\mathfrak{P}$ is open in $\mathfrak{B}_S$ if it is open in $\mathfrak{F}$, that is, $\mathfrak{P} = \mathfrak{P}' \cap \mathfrak{F}$, where $\mathfrak{P}'$ is an open polyhedron in $\mathfrak{D}_S$.

Theorem 2.4. *The set $\mathfrak{N}_S$ is closed convex rational polyhedral and it is decomposed into a finite number of $\sim_{\mathrm{wlc}}$ classes $\mathfrak{P}$. Each class $\mathfrak{P}$ of the wlc geography is a convex rational polyhedron which is open in $\mathfrak{B}_S$. For two classes $\mathfrak{P}, \mathfrak{P}'$, $\operatorname{Int}\mathfrak{P}' \cap \overline{\mathfrak{P}} \neq \emptyset$ holds if and only if $\overline{\mathfrak{P}'}$ is a face of $\overline{\mathfrak{P}}$.*

Example 2.5. Let X be a projective surface with a contraction $f : X \to Z$ of a nonsingular rational curve $C = S, C^2 = -n \leq -3$. Consider the pairs $(X/Z, B)$ where $B = aC \in \mathfrak{B}_S = [0,1]C$. Then by the adjunction formula, we have the equality $K \equiv f^*K_Z + \frac{2-n}{n}C$. If $0 \leq a \leq \frac{n-2}{n}$, the pair $(X/Z, aC)$ is a wlc model of itself. If $\frac{n-2}{n} \leq a \leq 1$, $(Z/Z, 0)$ is a wlc model of $(X/Z, aC)$. In total, we have the following geography:

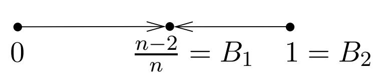


DIAGRAM 1

In this example, we have the following:

(1) $\mathfrak{B}_S = \mathfrak{N}_S$;
(2) there are three $\sim_{\mathrm{wlc}}$ classes in $\mathfrak{N}_S$: $\mathfrak{P}_1 = [0, \frac{n-2}{n})C$, $\mathfrak{P}_2 = \frac{n-2}{n}C$, $\mathfrak{P}_3 = (\frac{n-2}{n}, 1]C$;
(3) there are two $\sim_{\mathrm{lcm}}$ classes: $\mathfrak{P}'_1 = [0, \frac{n-2}{n})C, \mathfrak{P}'_2 = [\frac{n-2}{n}, 1]C$;

(4) $B_1 \sim_{\rm lcm} B_2$, but $\not\sim_{\rm wlc}$: $\sim_{\rm lcm} \not\Rightarrow \sim_{\rm wlc}$.

Example 2.6. Suppose that $X_1 = \mathbb{P}^2$ and $X_2 = \mathbb{P}^2$ are related by a standard quadratic transformation. Choose a general nonsingular prime b-divisor $D_i \in |4H_i|$ where H_i is a straight line on X_i for $i = 1, 2$. Consider the geography in the unit cube $\mathfrak{B}_S$ where $S = D_1 + D_2$.

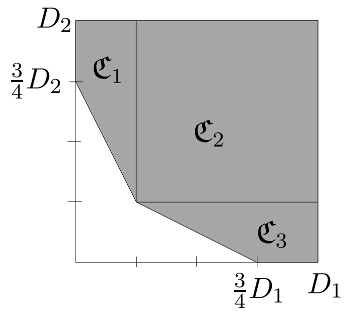


Diagram 2

In particular, $X_1 = \mathbb{P}^2 \approx X_2 = \mathbb{P}^2$ are resulting models for $\mathfrak{C}_1$ and $\mathfrak{C}_3$. Two $\mathbb{P}^2$'s are isomorphic under an unnatural isomorphism (the quadratic transform): $\mathfrak{C}_1$ and $\mathfrak{C}_3$ are different classes for $\sim_{\rm mod}$. However, $\frac{3}{4}D_1 \sim_{\rm lcm} \frac{3}{4}D_2$ and $\sim_{\rm lcm}$ is not convex.

Lemma 2.7. $\sim_{\rm wlc}$ *is convex in* $\mathfrak{N}_S$.

Proof. (Cf. the proof of [Sho6, lemma 2].) If $B \sim_{\rm wlc} B'$ and $(Y/Z, B_Y^{\log})$ is a projective $\mathbb{Q}$-factorial wlc model of $(X/Z, B)$, then $(Y/Z, B'^{\log}_Y)$ is a projective $\mathbb{Q}$-factorial wlc model of $(X/Z, B')$. Moreover, for any $D \in [B, B']$, $(Y/Z, D_Y^{\log})$ is a projective $\mathbb{Q}$-factorial wlc model of $(X/Z, D)$. The models are numerically equivalent to the edge models $(Y/Z, B_Y^{\log})$, $(Y/Z, B'^{\log}_Y)$. Thus all $(Y/Z, D_Y^{\log})$ have the same (generalized) log flops and wlc models. Note that $P(D), F(D)$ are linear with respect to D and by Proposition 2.3, $F(D)$ has the same signatures for all multiplicities with respect to the prime b-divisors. Thus the log flops of $(Y/Z, D_Y^{\log})$ corresponding to wlc models are the same for all $D \in [B, B']$ (cf. Corollary 2.11 (1)). $\square$

We say that a *set* $\mathfrak{P}$ in a finite dimensional $\mathbb{R}$-vector space V is *locally conical* if for any point p in V, there exists an open neighborhood U containing p such that the intersection $U \cap \mathfrak{P}$ is conical. In other words, for any $p' \in U$, either $(p, p') \subseteq \mathfrak{P}$ or $(p, p') \cap \mathfrak{P} = \emptyset$. A *decomposition* of a set into disjoint subsets $\mathfrak{P}$ is said to be *locally conical* if each $\mathfrak{P}$ is locally conical with the same U. As we will see shortly, any wlc geography is a locally conical decomposition of $\mathfrak{N}_S$.

Lemma 2.8. *Let $\mathfrak{D}$ be a closed bounded (convex) set in a finite dimensional $\mathbb{R}$-vector space. Then $\mathfrak{D}$ is locally conical if and only if $\mathfrak{D}$ is polyhedral.*

Proof. We use induction on $\dim_{\mathbb{R}} \mathfrak{D}$. The case $\dim_{\mathbb{R}} \mathfrak{D} \leq 2$ is trivial. Let $\dim_{\mathbb{R}} \mathfrak{D} = n \geq 3$ and assume that the lemma holds in $\dim_{\mathbb{R}} \mathfrak{D} < n$. The property of being locally conical is preserved under taking hyperplane sections. Therefore by the inductive assumption, all the sections of $\mathfrak{D}$ are polyhedral. This implies that the set $\mathfrak{D}$ is polyhedral by the following standard criterion [Klee]:

$\mathfrak{D}$ is polyhedral if and only if all of its j-dimensional plane sections are polyhedral where $2 \leq j \leq n-1$.

The converse is clear. □

Lemma 2.9. *Let $B_i \in \mathfrak{N}_S$ be a sequence of boundaries converging to a limit $B = \lim_{i\to+\infty} B_i$, and Y/Z be a model of X/Z such that $(Y/Z, B_{iY}^{\log})$ is a sequence of wlc models of $(X/Z, B_i)$. Then $(Y/Z, B_Y^{\log})$ is a wlc model of $(X/Z, B)$. In particular, $B \in \mathfrak{N}_S$.*

However, $(Y/Z, B_Y^{\log})$ may not be a slt wlc model of $(X/Z, B)$ even if all models $(Y/Z, B_{iY}^{\log})$ are slt wlc models of $(X/Z, B_i)$. We call a model $(Y/Z, B_Y^{\log})$ in the lemma *limiting* from a class $\mathfrak{P}$ if all $B_i \in \mathfrak{P}$.

Proof. Immediate by definition. □

Proof of 2.4. First, we establish the finiteness, convexity, and polyhedral property for the closures $\overline{\mathfrak{P}}$ of the classes in the geography. Note that the closures do not give a decomposition but, by Lemma 2.9, $\mathfrak{N}_S$ is a finite union of these closures. In particular, $\mathfrak{N}_S$ is closed.

The convexity of wlc classes follows from Lemma 2.7.

The polyhedral property of $\overline{\mathfrak{P}}$ follows from the locally conical property of $\overline{\mathfrak{P}}$ by Lemma 2.8 and its compactness. The conical property for $\overline{\mathfrak{P}}$ follows from that of $\mathfrak{P}$: the closure of a locally conical set is locally conical. But $\mathfrak{P}$ is locally conical by the stability of wlc models. This means the following. Take $B \in \mathfrak{N}_S$, and let $(Y/Z, B_Y^{\log})$ be a *stable* wlc model of $(X/Z, B)$. The stability here means that, for all B' in a neighborhood U of B in $\mathfrak{B}_S$, the pairs $(Y, B_Y'^{\log})$ are slt initial models of $(X/Z, B')$. Such a model $(Y/Z, B_Y^{\log})$ can be constructed from a slt wlc model of $(X/Z, B)$ by additional blowups of lc centers which lie in divisors $(S_i)_Y$ having 0 multiplicities in B. Then, for any boundary $D \neq B \in \mathfrak{N}_S \cap U$, there exists a (generalized) log flop $(Y/Z, B_Y^{\log}) \dashrightarrow (Y'/Z, B_{Y'}^{\log})$ such that, for all $B' \in \overrightarrow{BD} \cap U$, the models $(Y'/Z, B_{Y'}'^{\log})$ are slt wlc models of $(X/Z, B')$, and the boundaries B'

are wlc equivalent, where $\overrightarrow{BD}$ denotes the open ray from B through D. The existence of Y'/Z follows from [Sho6, Corolary 9 and Addendum 5]. The equivalence of B' follows from the construction of the flop and from [Sho6, Corollary 11 and Addendum 6] (cf. the proof of Lemma 2.7 above). Note that $(Y'/Z, B_{Y'}^{\log})$ may not be slt, but by the slt LMMP the dlt condition can be replaced by the lc condition everywhere in [Sho6, remarks after Proposition 1] by [Sho3, Conjecture and Heuristic Arguments].

The finiteness also follows by induction on dimension $\mathfrak{N}_S$ and the stability. See [Choi, Proposition 3.2.5] for details.

These established facts imply all stated properties of $\mathfrak{N}_S$, except for the convexity and rationality of $\mathfrak{N}_S$. Note that convexity follows from the convexity of positive parts: for all $0 \le t, t' \in \mathbb{R}, t + t' = 1$,

$$P(tB + t'B') \ge tP(B) + t'P(B').$$

The slt LMMP assumption is sufficient here. The proof of the rationality (of $\mathfrak{N}_S, \mathfrak{P}$) and the open property (of $\mathfrak{P}$) uses properties of functions $p(C, B), e(D, B)$ introduced below. Actually, for each $\mathfrak{P}$ there exist finitely many curves C_i/Z on a rather high model and finitely many prime b-divisors D_j such that $\mathfrak{P}$ in $\overline{\mathfrak{P}}$ can be given by the system of inequalities:

$$\begin{cases} p(C_i, B) > 0; \\ e(D_j, B) > 0. \end{cases}$$

In particular, those inequalities determine any country $\mathfrak{C}$ in $\mathfrak{N}_S$ while $\mathfrak{N}_S$ itself is given in $\mathfrak{B}_S$ (more accurately, in the minimal face containing $\mathfrak{N}_S$) by nonstrict inequalities $l(C_k, B), l(D_l, B) \ge 0$ for some curves C_k/Z and some b-divisors D_l, where the linear functions $l(C_k, B), l(D_l, B)$ are extensions of linear functions given by $p(C_k, B), e(D_l, B)$, respectively on the linear span of $\mathfrak{N}_S$ as it will be explained in the continuation of the proof by the end of the section, and equations (of the face) $b_i = 0$ or 1. The closure $\overline{\mathfrak{P}}$ of any other lower dimensional class $\mathfrak{P}$ lies in a face of dimension $\dim_{\mathbb{R}} \mathfrak{P}$ of some country $\mathfrak{C}$ of (cf. Corollary 2.12 below) and thus $\mathfrak{P}$ can be given in $\mathfrak{B}_S$ by strict inequalities and equations: $p(C_i, B), e(D_j, B) > 0, l(C_k, B), l(D_l, B) = 0, b_i = 0$ or $b_i = 1$.

(To be continued) □

On a rather high model V/Z of X/Z and for a curve C on V/Z, the function $p(C, -) = p(C; X/Z, -) \colon \mathfrak{N}_S \to \mathbb{R}$ is defined as

$$(C, B) \mapsto p(C; X/Z, B) = (P(B), C) = (P(B)_V, C) = (K_Y + B_Y^{\log}, g_*C),$$

where $(Y/Z, B_Y^{\log})$ is a wlc model of $(X/Z, B)$, and $g\colon V \to Y/Z$ is a birational morphism. A *rather high* model satisfies the following universal property: V dominates some wlc model for any $B \in \mathfrak{N}_S$, and $P(B)$ is $\mathbb{R}$-Cartier over V (cf. definition of $\sim_{\mathrm{mob}}$). Such a universal model V/Z exists by the finiteness in the first half of the proof of the theorem. By [Sho4, 2.4.3], the function is independent of the choice of such a model V/Z.

For a prime b-divisor D of X/Z, the *initial* log discrepancy $\underline{a}(D, X, -)$: $\mathfrak{B}_S \to \mathbb{R}$ is defined as

$$\begin{aligned}\underline{a}(D, X, B) &= 1 - mult_D B^{\log} \\ &= \begin{cases} 1 - mult_D B & \text{if } D \text{ is in } S; \\ 1 & \text{if } D \text{ is nonexceptional on } X, \text{ not in } S; \\ 0 & \text{if } D \text{ is exceptional on } X, \text{ not in } S. \end{cases}\end{aligned}$$

For a prime b-divisor D of X/Z, the *immobility* function $e(D, -) = e(D; X/Z, -)\colon \mathfrak{N}_S \to \mathbb{R}$ is defined as

$$(D, B) \mapsto e(D; X/Z, B) = \mathrm{mult}_D\, F(B) = a(D, Y, B_Y^{\log}) - \underline{a}(D, X, B),$$

where $(Y/Z, B_Y^{\log})$ is a wlc model of $(X/Z, B)$. It is called immobility because $e(D, B)$ is the multiplicity in b-divisors D of the fixed part $F(B)$ of $\mathcal{K} + B^{log}$. By the invariant property of log discrepancies of wlc models, the function $e(D, B)$ is independent of the choice of the wlc model $(Y/Z, B_Y^{\log})$ [Sho4, 2.4.2′].

Proposition 2.10. *The functions $p(C, B), e(D, B)$ on $\mathfrak{N}_S$ are nonnegative continuous rational piecewise linear, and linear on each $\overline{\mathfrak{P}}$, where $\mathfrak{P}$ is a class of $\mathfrak{N}_S$. Moreover, $e(D, B)$ is convex from above, and $p(C, B)$ is convex from below for rather general (mobile) curves, that is, for a curve C/Z through a general point of a rather high model V/Z of X/Z.*

Proof. The function $p(C, B)$ is nonnegative by the nef property of b-divisor $P(B)$. The function $e(D, B)$ is nonnegative by definition or because the fixed part $F(B)$ is effective. The rationality and the linearity on each $\overline{\mathfrak{P}}$ follow from the corresponding properties of the intersection pairing and the discrepancy function by Lemma 2.9. Thus the finiteness established in the first half of the proof of Theorem 2.4 implies the continuity and the rational piecewise linearity of the functions. This includes also the rational property of any maximal linear subset (cf. facets in Corollary 2.13 below).

The convex properties of $p(C,B)$ from above and of $e(D,B)$ from below follow from the inequalities: $P(tB+t'B') \geq tP(B)+t'P(B')$ and $F(tB+t'B') \leq tF(B)+t'F(B')$, where $0 \leq t,t'$ and $t+t'=1$. □

Corollary 2.11. *For any curve C on a rather high model V/Z of X/Z and for any prime b-divisor D of X/Z, the functions $p(C,B), e(D,B)$ satisfy the following.*

(1) *Each function $p(C,B), e(D,B)$ has the constant signature on each class $\mathfrak{P}$: either >0 or $=0$ on $\mathfrak{P}$. The converse holds also, that is, if the values $p(C,B), e(D,B)$ and $p(C,B'), e(D,B')$ have the same signatures, then $B \sim_{\mathrm{wlc}} B'$.*

(2) *$p(C,B)=(P(B),C)=0$ on a class $\mathfrak{P}$ if and only if C is contracted by the lc contraction $I_{\mathfrak{P}}\colon V \to X_{\mathrm{lcm}}/Z$ (cf. Log canonical model in Section 7.) of the pair $(X/Z,B)$ with $B \in \mathfrak{P}$. In particular, the contraction depends only on $\mathfrak{P}$. Equivalently, $p(C,B)>0$ on $\mathfrak{P}$ if and only if $I_{\mathfrak{P}}(C)$ is a curve on X_{lcm}/Z.*

(3) *$e(D,B)=0$ on a class $\mathfrak{P}$ if and only if D can be blown up on a wlc model of this class, i.e., D is nonexceptional on some wlc model Y/Z. Equivalently, $e(D,B)>0$ on a class $\mathfrak{P}$ if and only if D is exceptional on any wlc model of $(X/Z,B)$ for $B \in \mathfrak{P}$.*

Proof. (1) By definition $\sim_{\mathrm{mob}}$ is equivalent to the same signatures for all $p(C,B)$, and so is $\sim_{\mathrm{fix}}$ for $e(D,B)$. Thus by Proposition 2.3 $\sim_{\mathrm{wlc}}$ is equivalent to the same signatures for all $p(C,B), e(D,B)$.

(2) Immediate by construction of $V \to X_{\mathrm{lcm}}$ as a composition of $g\colon V \to Y$ followed by the lc contraction of a wlc model $(Y/Z, B_Y^{\log})$, where $(Y/Z, B_Y^{\log})$ is an appropriate wlc model of $(X/Z,B)$. The composition contracts exactly the curves C on V/Z with $(K_Y+B_Y^{\log}, g_*C)=0$ (cf. [IskSh, 1.17 (ii)]).

(3) By definition $e(D,B)=0$ for any wlc model $(Y/Z, B_Y^{\log})$, on which D is nonexceptional.

On the other hand, if $e(D,B)=0$ and D is exceptional on a wlc model $(Y/Z, B_Y^{\log})$, then $a(D,Y,B_Y^{\log}) = \underline{a}(D,X,B) \leq 1$ and one can blow up D to a crepant slt wlc model $(Y'/Z, B_{Y'}^{\log})$ of $(X/Z,B)$ with a (higher) model Y'/Y of Y/Y by the slt LMMP. By (1) this depends only on $\mathfrak{P}$. □

Corollary 2.12. *Let $\mathfrak{P}$ be a class of $\mathfrak{N}_S$ and $\mathfrak{F}$ be its face. Then* $\operatorname{Int}\mathfrak{F} \subseteq \mathfrak{P}'$ *for some class $\mathfrak{P}'$ of $\mathfrak{N}_S$. In particular,* $\operatorname{Int}\overline{\mathfrak{P}} \subseteq \mathfrak{P}' = \mathfrak{P}$.

Thus the geography $\mathfrak{N}_S$ is determined up to finitely many possibilities by its countries $\mathfrak{C}$ and even their closures $\overline{\mathfrak{C}}$.

Note that $\overline{\mathfrak{P}}$ is a convex polyhedron by the first half of the proof of Theorem 2.4.

Proof. Immediate by Proposition 2.10 and Corollary 2.11 (1). An elementary property of linear functions is useful here: a nonnegative linear function on a convex polyhedron $\mathfrak{F} = \overline{\mathfrak{F}}$ has the same signature on $\operatorname{Int}\mathfrak{F}$. □

Corollary 2.13. *Let $\mathfrak{P}$ be a class of $\mathfrak{N}_S$, $\mathfrak{F}$ be its facet, and V/Z be a rather high model of X/Z. Then either*

$\mathfrak{F}$ is given by a facet of $\mathfrak{B}_S$;

there exists a curve C on a rather high model V/Z of X/Z such that $p(C, \mathfrak{P}) > 0$ and $p(C, \mathfrak{F}) = 0$; or

there exists a prime b-divisor D of X/Z such that $e(D, \mathfrak{P}) > 0$ and $e(D, \mathfrak{F}) = 0$.

Moreover, if $\operatorname{Int}\mathfrak{F}\cap\mathfrak{P} = \emptyset$, then $\mathfrak{F}\cap\mathfrak{P} = \emptyset$. However, if $\operatorname{Int}\mathfrak{F}\cap\mathfrak{P} \neq \emptyset$, then $\operatorname{Int}\mathfrak{F} \subseteq \mathfrak{P}$, and this is possible only for facets given by a facet of $\mathfrak{B}_S$. The same holds for the faces of $\mathfrak{P}$.

Proof. Take $B \in \operatorname{Int}\mathfrak{F}$, and an outer direction $\overrightarrow{BC}$ from $\mathfrak{P}$, that is, $\overrightarrow{BC} \cap \mathfrak{P} = \emptyset$ and $\overline{BC} \cap \mathfrak{P} \neq \emptyset$. Let $(Y/Z, B_Y^{\log})$ be a wlc model of $(X/Z, B)$ which is a limit of (slt) wlc model $(Y/Z, B'^{\log}_Y)$ of $(X/Z, B')$ for $B' \in \mathfrak{P}$ (e.g., along $\overline{BC}$; see Lemma 2.9).

By the polyhedral property of lc [Sho2, 1.3.2], the following three cases are only possible:

(i) B' is not a b-boundary for any $B' \in \overrightarrow{BC}$;

(ii) $(Y, B'^{\log}_Y)$ is not lc for any $B' \in \overrightarrow{BC}$;

(iii) for any $B' \in \overrightarrow{BC}$ in a neighborhood of B, B' is a b-boundary and $(Y, B'^{\log}_Y)$ is lc.

In the case (i), there exists prime S_i, such that $b_i' = \operatorname{mult}_{S_i} B'^{\log}_Y < 0$ or > 1. The facet $\mathfrak{F}_i$ of $\mathfrak{B}_S$ with the equation $b_i = 0$ or $b_i = 1$ respectively gives $\mathfrak{F}$ (see the property of function b_i on page 5 above).

In the case (ii), there exists a prime b-divisor D of X/Z such that $a(D, Y, B'^{\log}_Y) < 0$ for any $B' \in \overrightarrow{BC}$. If we do not assume (i), D is exceptional on Y and, by the linear property of discrepancy, $a(D, Y, B_Y^{\log}) = 0$ and $a(D, Y, B'^{\log}_Y) > 0$ for any $B' \in \overline{BC} \cap \mathfrak{P}$. Thus $e(D, B) = 0$ and, if for every $B' \in \overline{BC} \cap \mathfrak{P}$, $e(D, B') = 0$, then for all B' near B on $\overline{BC}$, $\underline{a}(D, X, B') = a(D, Y, B'^{\log}_Y)$ and < 0 for $B' \in \overrightarrow{BC}$, a contradiction again by the negation of (i). Hence $e(D, B') > 0$ for some $B' \in \overline{BC} \cap \mathfrak{P}$. This gives required D by Corollary 2.12.

The case (iii) itself has two subcases: for all $B' \in \overrightarrow{BC}$ in a neighborhood of B,

(iii-1) $(Y/Z, {B'}_Y^{\log})$ is not a wlc model; or

(iii-2) $(Y/Z, {B'}_Y^{\log})$ is a wlc model but not of $(X/Z, B')$.

Indeed, otherwise, for all those B', the pairs $(Y/Z, {B'}_Y^{\log})$ are wlc models of $(X/Z, B')$. By definition all functions $p(C, B'), e(D, B')$ are linear on $\overline{BC}$ in the neighborhood of B. Thus $p(C, B'), e(D, B')$ have the same signatures and all those B' are $\sim_{\text{wlc}}$, a contradiction with our assumptions.

In the case (iii-1), by [Sho6, Corollary 9] there exists a curve C' on Y/Z such that $(K_Y + {B'}_Y^{\log}, C') < 0$ for $B' \in \overrightarrow{BC}$ and $(K_Y + B_Y^{\log}, C') = 0$. Thus $(K_Y + {B'}_Y^{\log}, C') > 0$ for $B' \in \overline{BC} \cap \mathfrak{P}$, and for a curve C on a rather high model V'/Z of X/Z with $g' \colon Z' \to Y$ and $C' = g'_* C$, again by Corollary 2.12, $p(C, \mathfrak{P}) > 0$ and $p(C, \mathfrak{F}) = 0$. The same holds for a birational transform of C on V/Z [Sho4, 2.4.3].

In the case (iii-2), there exists a b-divisor D with $a(D, Y, {B'}_Y^{\log}) < \underline{a}(D, X, B')$. By the linear property of discrepancy and of initial discrepancy, $e(D, B') > 0$ for $B' \in \overline{BC} \cap \mathfrak{P}$ and $e(D, B) = 0$. Thus D is a required divisor.

The results about $\mathfrak{F} \cap \mathfrak{P}$ follow from Corollaries 2.11 (1) and 2.12; in the case of faces by induction. $\square$

Continuation of the proof of Theorem 2.4. Let $\mathfrak{F} \subseteq \mathfrak{B}_S$ be the minimal face of $\mathfrak{B}_S$ containing $\mathfrak{N}_S$. Then the linear span of $\mathfrak{N}_S$ is the linear span of $\mathfrak{F}$ by the monotonicity $\nu(X/Z, B') \geq \nu(X/Z, B)$ for $B' \geq B$ in $\mathfrak{B}_S$ (cf. Proposition 5.1). Moreover, the linear span of $\mathfrak{N}_S$ can be given in $\mathfrak{D}_S$ by equations $b_i = 1$. In particular, the span is rational.

The rationality of $\mathfrak{N}_S$, of the polyhedrons $\overline{\mathfrak{P}}$, and the open property of the classes $\mathfrak{P}$ follow from Corollaries 2.12 and 2.13. This gives also required equations and inequalities. The linear function $l(C, B)$ on the linear span of $\mathfrak{P}$ is determined by the linear function $p(C, B)$ on $\mathfrak{P}$, in particular, those functions with $p(C, \mathfrak{F}) = 0$ for a facet $\mathfrak{F}$ of $\mathfrak{P}$ will suffice. Similarly, one can introduce functions $l(D, B)$. Finally, the interior $\operatorname{Int} \mathfrak{F}$ of a face of $\mathfrak{P}$ given by a facet of $\mathfrak{B}_S$, belongs to $\mathfrak{P}$ or is disjoint from it.

If $\mathfrak{P}, \mathfrak{P}'$ are two classes and $\overline{\mathfrak{P}'}$ is a face of $\overline{\mathfrak{P}}$, then $\operatorname{Int} \mathfrak{P}' \cap \overline{\mathfrak{P}} \neq \emptyset$. Conversely, if $\operatorname{Int} \mathfrak{P}' \cap \overline{\mathfrak{P}} \neq \emptyset$ but $\overline{\mathfrak{P}'}$ is not a face of $\overline{\mathfrak{P}}$, then by Corollary 2.12, $\emptyset \neq \operatorname{Int} \mathfrak{P}' \cap \overline{\mathfrak{P}} \neq \operatorname{Int} \mathfrak{P}'$. Moreover, by the convexity of classes and of $\mathfrak{N}_S$, we can suppose that $\overline{\mathfrak{P}'}$ contains a face $\mathfrak{F}'$ of $\overline{\mathfrak{P}}$ (maybe for different $\mathfrak{P}$) which is a polyhedron of the same dimension as $\mathfrak{P}'$ and with $\mathfrak{F}' = \overline{\mathfrak{P}} \cap \overline{\mathfrak{P}'} \neq \overline{\mathfrak{P}'}$. Hence there exists a facet $\mathfrak{F} \not\supset \mathfrak{F}'$ of

$\mathfrak{P}$ which contains an internal point of $\mathfrak{P}'$. This gives a contradiction with Corollary 2.13. For example, if $\mathfrak{F}$ is given by a facet of $\mathfrak{B}_S$ with an equation $b_i = 0$ or 1, then $b_i = 0$ or 1 respectively on $\overline{\mathfrak{P}'}$, in particular, on $\mathfrak{F}'$, and thus on $\overline{\mathfrak{P}}$, a contradiction. Similarly, for other functions. □

3. Positive cones

For first applications of geography, we recall and define natural cones in $\mathfrak{D}_S$ and $\mathrm{N}^1(X/Z)$. Let W/Z be a model of X/Z, and $S = \sum S_i$ be a reduced b-divisor of X/Z. Our *numerical space* $\mathrm{N}^1(X/Z)$ denotes the space of $\mathbb{R}$-divisors, not necessarily $\mathbb{R}$-Cartier, mudulo the numerical equivalence $\equiv$. Nonetheless, the relative Weil-Picard (class) number $\dim_{\mathbb{R}} \mathrm{N}^1(X/Z) < +\infty$ for proper X/Z.

Semiample cones. In the numerical space $\mathrm{N}^1(X/Z)$ and in the space of b-divisors $\mathfrak{D}_S$, respectively, the cones

$$\begin{aligned} \mathrm{sAmp}(X/Z) &:= \left\{ [D] \in \mathrm{N}^1(X/Z) \;\middle|\; \begin{array}{l} D \equiv D' \text{ for some semiample} \\ \text{divisor } D' \text{ on } X/Z \end{array} \right\}, \\ \mathrm{s}\mathfrak{A}_S(W/Z) &:= \{ D \in \mathfrak{D}_S \mid D_W \text{ is semiample on } W/Z \} \end{aligned}$$

are defined.
Nef cones. In $\mathrm{N}^1(X/Z), \mathfrak{D}_S$, respectively, the cones

$$\begin{aligned} \mathrm{Nef}(X/Z) &:= \{ [D] \in \mathrm{N}^1(X/Z) \mid D \text{ is nef on } X/Z \}, \\ \mathfrak{Nef}_S(W/Z) &:= \{ D \in \mathfrak{D}_S \mid D_W \text{ is nef on } W/Z \} \end{aligned}$$

are defined and closed. In the projective case, they are the closures of the ample (Kähler) cones. The cone $\mathrm{Nef}(X/Z)$ is dual to the Kleiman-Mori cone, the closure of the cone of curves. Thus its polyhedral properties are related to the cone of curves (the Mori theory). However, we consider only cones of divisors (see Corollary 3.5 below).
Mobile cones. In $\mathrm{N}^1(X/Z), \mathfrak{D}_S$, respectively, the cones

$$\begin{aligned} \mathrm{Mob}(X/Z) &:= \left\{ [D] \in \mathrm{N}^1(X/Z) \;\middle|\; \begin{array}{l} D \equiv D' \text{ for some } \mathbb{R}\text{-mobile} \\ \text{divisor } D' \text{ on } X/Z \end{array} \right\}, \\ \mathfrak{M}_S(W/Z) &:= \{ D \in \mathfrak{D}_S \mid D_W \text{ is } \mathbb{R}\text{-mobile on } W/Z \} \end{aligned}$$

are defined.

Effective cones. In $\mathrm{N}^1(X/Z), \mathfrak{D}_S$, respectively, the cones

$$\begin{aligned}\mathrm{Eff}(X/Z) &:= \left\{ D \in \mathrm{N}^1(X/Z) \;\middle|\; \begin{array}{l} D \equiv D' \text{ for some effective divisor} \\ D' \text{ on } X/Z \end{array} \right\}, \\ \mathfrak{E}_S(W/Z) &:= \left\{ D \in \mathfrak{D}_S \;\middle|\; \begin{array}{l} D_W \sim_{\mathbb{R}} D' \text{ for some effective divisor} \\ D' \text{ on } W/Z \end{array} \right\}\end{aligned}$$

are defined.

In general, the numerical equivalence $\equiv$ of divisors does not preserve the semiampleness and mobile properties of divisors while clearly $\sim_{\mathbb{R}}$ does. Thus introducing cones in the numerical space, we had a choice to assume such a property on the whole class $[D]$ or on some representative $D' \in [D]$. We chose the second one to have a larger cone. According to the choice, the linear map $[\ ]$ agrees the $\sim_{\mathbb{R}}$ equivalence property with the numerical one (cf. the proof of Corollary 3.4). It is well-known that the above cones, numerical and linear, are convex and satisfy the inclusions:

$$\begin{array}{ll}\mathrm{sAmp}(X/Z) \subseteq \mathrm{Nef}(X/Z), & \mathrm{sAmp}(X/Z) \subseteq \mathrm{Mob}(X/Z) \subseteq \mathrm{Eff}(X/Z), \\ \mathrm{s}\mathfrak{A}_S(W/Z) \subseteq \mathfrak{Nef}_S(W/Z), & \mathrm{s}\mathfrak{A}_S(W/Z) \subseteq \mathfrak{M}_S(W/Z) \subseteq \mathfrak{E}_S(W/Z).\end{array}$$

We recall also that:

> 0-*log pair* $(X/Z, B)$ is a pair with a proper morphism X/Z and a boundary B on X such that (X, B) is klt and $K + B \sim_{\mathbb{R}} 0$, or equivalently $K + B \equiv 0$ according to [Amb]; and
> (relative) *FT (Fano type)* variety X/Z is a variety such that there exists a boundary B on X such that the pair $(X/Z, B)$ is a klt log Fano variety, in particular, X/Z is projective.

For other equivalent characterizations of relative FT varieties, see [PrSh, Lemma-Definition 2.8]. The most useful one among them to us is the following.

Lemma 3.1. *Let X/Z be be an FT variety. Then for any fixed reduced divisor S of X, there exists an $\mathbb{R}$-boundary B on X such that $(X/Z, B)$ is a 0-log pair and $S \subseteq \operatorname{Supp} B$. Moreover, B is big.*

Conversely, if $(X/Z, B)$ is a 0-log pair with projective X/Z and big B, e.g., prime components of $\operatorname{Supp} B$ *generate* $\mathrm{N}^1(X/Z)$*, then X/Z is FT.*

Proof. See [PrSh, Lemma-Definition 2.8]. The big property for B follows from the big property of $-K$ for FT varieties. □

To obtain polyhedral properties of cones in $\mathfrak{D}_S$ from geography of log models, we translate polyhedrons in the space. Let

$$\begin{array}{rl} -B\colon \mathfrak{D}_S & \to \mathfrak{D}_S \\ D & \mapsto D - B \end{array}$$

be the translation by $-B$. It translates B into the origin $0 \in \mathfrak{D}_S$, the vertex of cones. Geography allows to construct only some polyhedrons in $\mathfrak{B}_S \subset \mathfrak{D}_S$. However each polyhedron near B is conical. Thus to compare the translation of a polyhedron with a cone in a neighborhood U of $0 \in \mathfrak{D}_S$ would be sufficient. In what follows, we say that two sets $\mathfrak{S}_1, \mathfrak{S}_2$ (cones or polyhedrons) coincide in U if

$$\mathfrak{S}_1 \cap U = \mathfrak{S}_2 \cap U.$$

Proposition 3.2. *Let $(X/Z, B)$ be a 0-log pair and put $S = \operatorname{Supp} B$. Then there exists a neighborhood U of 0 in $\mathfrak{D}_S$ in which:*

$$\begin{array}{rl} \mathrm{s}\mathfrak{A}_S(X/Z) & = \mathfrak{Nef}_S(X/Z) = \overline{\mathfrak{P}}_X - B = \cup \mathfrak{P}_{0,X} - B, \\ \mathfrak{M}_S(X/Z) & = \cup \overline{\mathfrak{P}}_X^1 - B = \cup \mathfrak{P}_X^1 - B, \\ \mathfrak{E}_S(X/Z) & = \mathfrak{N}_S - B, \end{array}$$

where $\overline{\mathfrak{P}}_X$ is the closure of a maximal dimensional class $\mathfrak{P}_X$ of wlc models with $Y = X$, $\mathfrak{P}_{0,X}$ are the classes of wlc models with $Y = X$, and $\overline{\mathfrak{P}}_X^1$ are the closures of classes $\mathfrak{P}_X^1$ of wlc models with Y isomorphic to X in codimension 1.

Each divisor $D \in \mathfrak{E}_S(X/Z)$ has a unique mob+exc decomposition: $D \sim_{\mathbb{R}} M + E$ and $\equiv M + E$, where M is $\mathbb{R}$-mobile and E is effective very exceptional with respect to a 1-contraction given by M; the 1-contraction, E are unique and M is unique up to $\sim_{\mathbb{R}}$ and $\equiv$, respectively. In particular, in $\mathfrak{D}_S$ the $\mathbb{R}$-mobile and $\sim_{\mathbb{R}}$ effective properties of divisors are equivalent to those for $\equiv$. Moreover, for $D \in U \cap \mathfrak{E}_S(X/Z)$, $M \sim_{\mathbb{R}} P(B')_X, E = F(B')_X$ and $M \equiv P(B')$, respectively, where $B' = B + D$.

Proof. First we clarify the statement. Let $B' \in \mathfrak{P} \subseteq \mathfrak{N}_S$ be a boundary in a class $\mathfrak{P}$. If the pair $(X/Z, B')$ has a wlc model $(Y/Z, B'_Y)$ for which $X \dashrightarrow Y$ is a small birational isomorphism, that is, an isomorphism in codimension 1, we denote the corresponding class by $\mathfrak{P} = \mathfrak{P}_X^1$. In particular, if $Y = X$, we have isomorphism in codimension $\le \dim_k X$, or in dimension ≥ 0, and we denote it by $\mathfrak{P}_{0,X}$. If X is nonprojective or/and non $\mathbb{Q}$-factorial, we allow such models. But if we have to follow Convention 2.2, we need to replace the models by their small projective $\mathbb{Q}$-factorializations [IskSh, Corollary 6.7] (cf. Proposition 2.1). For $B' \in \mathfrak{P}_{0,X}$, in addition, we suppose that $(X/Z, B') \dashrightarrow (Y/Z, B'_Y)$ is a small log flop. Thus we can replace X/Z by a small projective $\mathbb{Q}$-factorialization.

Now we choose a conical neighborhood U of $0 \in \mathfrak{D}_S$ such that,

for each $D \in U$, the pair $(X/Z, B')$ with $B' = B + D$ is a klt initial model of itself, including the boundary condition;

$(U + B) \cap \mathfrak{P}$ is conical for each class $\mathfrak{P} \subseteq \mathfrak{N}_S$ (this is sufficient for the classes $\mathfrak{P}$ with $B \in \overline{\mathfrak{P}}$).

Such a neighborhood exists by our assumptions and by Theorem 2.4.

For any divisor $D \in U$,

$$D \sim_{\mathbb{R}} K + B + D = K + B'.$$

Thus

$$D \text{ is semiample } \Leftrightarrow K + B' \text{ is semiample;}$$

$$D \text{ is nef } \Leftrightarrow K + B' \text{ is nef;}$$

$$D \text{ is } \mathbb{R}\text{-mobile } \Leftrightarrow K + B' \text{ is } \mathbb{R}\text{-mobile;}$$

$$D \sim_{\mathbb{R}} D' \geq 0 \Leftrightarrow K + B' \sim_{\mathbb{R}} D' \geq 0.$$

This gives the required equalities and mob+exc decompositions. Indeed, since (X, B') is lc, the $\mathbb{R}$-mobile part M of D is $\sim_{\mathbb{R}} P(B')_X$ and respectively, the very exceptional part E is $F(B')_X$. The $\mathbb{R}$-linear decomposition gives the numerical one because the former is essentially numerical by the LMMP (see [Sho5, Definition 3.3 and Proposition 3.4]). The semiampleness gives the equality $\mathrm{s}\mathfrak{A}_S(X/Z) = \mathfrak{Nef}_S(X/Z)$. The closed property and the equalities with closures follow from Lemma 2.9 or Corollary 2.11 (1).

Finally, by Corollary 2.11 (1), by convexity of $\mathfrak{Nef}_S(X/Z)$ and by the open property of classes, $\operatorname{Int} \mathfrak{Nef}_S(X/Z) = \mathfrak{P}_X$. holds for one class $\mathfrak{P}_X$ (cf. [Sho6, Lemma 2]).

To construct all the required models, the klt slt LMMP is sufficient, and for contractions of those models, the semiampleness for klt $\mathbb{Q}$-boundaries is sufficient (see Corollary 7.2). □

Corollary 3.3. *Let $(X/Z, B)$ be a 0-log pair and $S \subseteq \operatorname{Supp} B$ be a reduced divisor on X. Then the cones*

$$\mathrm{s}\mathfrak{A}_S(X/Z) = \mathfrak{Nef}_S(X/Z),\ \mathfrak{M}_S(X/Z), \mathfrak{E}_S(X/Z)$$

in $\mathfrak{D}_S$ are closed convex rational polyhedral.

Each divisor $D \in \mathfrak{E}_S(X/Z)$ has a unique mob+exc decomposition: $D \sim_{\mathbb{R}} M + E$ and $\equiv M + E$, where M is $\mathbb{R}$-mobile and E is effective *very exceptional with respect to a* 1*-contraction given by* M*. Moreover,* $\equiv$ *instead of* $\sim_{\mathbb{R}}$ *gives the same cones in* $\mathfrak{D}_S$*.*

Proof. Immediate by Proposition 3.2 and Theorem 2.4 for $S = \operatorname{Supp} B$. If $S \subseteq \operatorname{Supp} B$, we obtain the required polyhedral properties by intersection with the subspace $\mathfrak{D}_S$. $\square$

Corollary 3.4. *Let* $(X/Z, B)$ *be a* 0*-log pair such that prime components of* $S = \operatorname{Supp} B$ *generate* $\mathrm{N}^1(X/Z)$*. Then the cones*

$$\mathrm{sAmp}(X/Z) = \mathrm{Nef}(X/Z), \mathrm{Mob}(X/Z), \mathrm{Eff}(X/Z)$$

in $\mathrm{N}^1(X/Z)$ *are closed convex rational polyhedral. Thus* $\mathrm{Eff}(X/Z)$ *is the pseudo-effective cone too.*

The relations $\equiv$ *and* $\sim_{\mathbb{R}}$ *coincide on* X/Z*. Each divisor* $D \in \mathrm{Eff}(X/Z)$ *has a unique mob+exc decomposition:* $D \equiv M + E$*, where* M *is* $\mathbb{R}$*-mobile and* E *is effective very exceptional with respect to a* 1*-contraction given by* M*.*

The corollary gives a new result even for $\mathrm{Nef}(X/Z)$ when X/Z is nonprojective.

Proof. The required properties of the linear cones in $\mathfrak{D}_S$ imply the same properties of the numerical cones in $\mathrm{N}^1(X/Z)$. Indeed, the natural linear map

$$\begin{aligned} [\ ]\colon \mathfrak{D}_S &\to \mathrm{N}^1(X/Z) \\ D &\mapsto [D], \text{ the numerical class of } D, \end{aligned}$$

maps the cones

$$\mathrm{s}\mathfrak{A}_S(X/Z), \mathfrak{Nef}_S(X/Z), \mathfrak{M}_S(X/Z), \mathfrak{E}_S(X/Z) \to$$
$$\mathrm{sAmp}(X/Z), \mathrm{Nef}(X/Z), \mathrm{Mob}(X/Z), \mathrm{Eff}(X/Z), \text{ respectively.}$$

Since prime components S_i of S generate $\mathrm{N}^1(X/Z)$, the map [ ] and the induced maps of cones are surjective, that gives the above implication by Corollary 3.3. The surjectivity for cones uses the following: $\sim_{\mathbb{R}}$ can be replaced by $\equiv$ in the definition of cones in $\mathfrak{D}_S$ by the same corollary. Moreover, the coincidence of $\equiv$ and $\sim_{\mathbb{R}}$ holds on X/Z. This is well-known for projective X/Z [Choi, Lemma 4.1.12] (the proof uses the assumption that $\operatorname{char} k = 0$; possibly, Conjecture 7.1 in the positive characteristic allows us to omit the assumption). The general case uses the rationality of singularities of X.

All required models in construction will be FT varieties by Lemma 3.1. Semiampleness holds for FT varieties in characteristic 0: for $\mathbb{Q}$-divisors, it holds by the base freeness of [Sho1], that is sufficient for $\mathbb{R}$-divisors by Corollary 7.2 (cf. [Sho6, Corollary 10]).

By Lemma 3.1, the big klt slt LMMP is sufficient for the FT varieties.

□

Corollary 3.5. *Let X/Z be an FT variety, and S be a reduced divisor of X. Then the cones*

$$\mathrm{sAmp}(X/Z) = \mathrm{Nef}(X/Z), \mathrm{Mob}(X/Z), \mathrm{Eff}(X/Z) \text{ and}$$
$$\mathrm{s}\mathfrak{A}_S(X/Z) = \mathfrak{Nef}_S(X/Z), \mathfrak{M}_S(X/Z), \mathfrak{E}_S(X/Z)$$

in $\mathrm{N}^1(X/Z)$ and in $\mathfrak{D}_S$, respectively are closed convex rational polyhedral.

The relations $\equiv$ and $\sim_{\mathbb{R}}$ coincide on X/Z. Each class $[D] \in \mathrm{Eff}(X/Z)$ (respectively divisor $D \in \mathfrak{E}_S(X/Z)$) has a unique decomposition mob+exc: $[D] = [M] + [E]$ (respectively $D \sim_{\mathbb{R}} M + E$), where M is $\mathbb{R}$-mobile and E is effective very exceptional with respect to a 1-contraction given by M.

Proof. Immediate by Lemma 3.1 and Corollaries 3.3, 3.4. We use here the finiteness of the Weil-Picard number: $\dim_{\mathbb{R}} \mathrm{N}^1(X/Z) < +\infty$. □

We will give a natural decomposition of the cones $\mathrm{Eff}(X/Z), \mathrm{Mob}(X/Z)$ in $\mathrm{N}^1(X/Z)$ and $\mathfrak{E}_S(X/Z), \mathfrak{M}_S(X/Z)$ in $\mathfrak{D}_S$ in the next section.

4. Finiteness results for log models

The support of a b-divisor $D = \sum d_i D_i$ is defined as the reduced b-divisor $\mathrm{Supp}\, D := \sum_{d_i \neq 0} D_i$. So, a reduced b-divisor $D = \sum D_i$ is identified with its support: $D = \mathrm{Supp}\, D$.

Let $(X/Z, B)$ be a 0-log pair and $S = \mathrm{Supp}\, B$. Then the geography of $\mathfrak{N}_S$ in Proposition 3.2 *induces* a finite decomposition of the cone $\mathfrak{E}_S(X/Z)$ into open convex rational polyhedral subcones $\mathfrak{P}$. The open and polyhedral properties follow from that of classes in $\mathfrak{N}_S$ by Theorem 2.4. Note that B is an internal point of $\mathfrak{B}_S$. This decomposition gives also a decomposition of $\mathfrak{E}_{S'}(X/Z)$ for any reduced $S' \leq S$ by Corollary 3.3; its subcones will be called also classes. To give an internal interpretation of this decomposition, we introduce the following relation.

For divisors $D, D' \in \mathfrak{E}_S(X/Z)$, $D \sim_{\mathrm{m+e}} D'$ if, for their mob+exc decompositions $D \sim_{\mathbb{R}} M + E, D' \sim_{\mathbb{R}} M' + E'$, the rational 1-contractions $X \dashrightarrow Y_{\mathfrak{P}} = Y = Y'/Z$ given by M, M' are the same, and $\mathrm{Supp}\, E = \mathrm{Supp}\, E'$. The relation $\sim_{\mathbb{R}}$ can be replaced by $\equiv$ and used for classes: $[D] \sim_{\mathrm{m+e}} [D']$. The corresponding *induced* classes of $\mathrm{Eff}(X/Z)$ in $\mathrm{N}^1(X/Z)$ will be denoted by P and their contractions by $X \dashrightarrow Y_{\mathrm{P}}/Z$. The rational 1-contraction depends only on the class $\mathfrak{P}$ and P, respectively, and will be referred to as the D-contraction. If X/Z is a birational contraction with respect to D, then Y_P/Z is small and well-known as a D-flip of X/Z [Sho4, p. 2684]. By construction, it is also a log flop of $(X/Z, B)$ in the latter case.

(*Warning*: D-contraction is not a contraction with respect to a divisor D in the D-MMP.)

In particular, this gives polyhedral decompositions of cones $\mathrm{Eff}(X/Z)$, $\mathrm{Mob}(X/Z)$ in $\mathrm{N}^1(X/Z)$ and $\mathfrak{E}_S(X/Z)$, $\mathfrak{M}_S(X/Z)$ in $\mathfrak{D}_S$ for 0-log pairs $(X/Z, B)$, when prime components of $S = \mathrm{Supp}\, B$ generate $\mathrm{N}^1(X/Z)$, and for FT varieties X/Z. The decomposition of mobile cones $\mathrm{Mob}(X/Z)$, $\mathfrak{M}_S(X/Z)$ is interesting for applications (see Proposition 4.14) rather than their general structure: each class of those cones is a class in $\mathrm{Eff}(X/Z)$ or $\mathfrak{E}_S(X/Z)$ respectively and it is determined by its 1-contraction $(E = 0)$. So, we focus on classes of effective cones.

Let $g\colon X \dashrightarrow Y/Z$ be a 1-contraction with projective Y/Z. A polarization on Y/Z is an ample $\mathbb{R}$-divisor H. The polarization is called *supported* in S if $g^*H + E \sim_{\mathbb{R}} D \in \mathfrak{D}_S$ for some effective divisor E on X, very exceptional on Y, where g^*H is the pull back (transform) of H [Sho5, p. 84]; in general, g^*H is an $\mathbb{R}$-Weil divisor.

Corollary 4.1. *Let $(X/Z, B)$ be a 0-log pair and $S \subseteq \mathrm{Supp}\, B$ be a reduced divisor on X. Then the decomposition of the cone $\mathfrak{E}_S(X/Z)$ induced by $\mathfrak{N}_S$ can be given by the mob+exc relation $\sim_{\mathrm{m+e}}$.*

There are finitely many rational 1-contractions $X \dashrightarrow Y_{\mathfrak{P}}/Z$, and $Y_{\mathfrak{P}}/Z$ is projective with a polarization supported in S. Conversely, any such contraction corresponds to a class $\mathfrak{P}$.

Proof. Immediate by Proposition 3.2, Corollary 3.3 and Corollary 2.11 (1). Note that, for $D, D' \in U$ such that $D + B, D' + B \in \mathfrak{N}_S$, in the proposition, the contraction condition of $\sim_{\mathrm{m+e}}$ means that $B + D \sim_{\mathrm{mob}} B + D'$, and the support condition that $\mathrm{Supp}\, F(B + D)_X = \mathrm{Supp}\, F(B + D')_X$. But for any prime divisor E exceptional on X, $e(E, B + D), e(E, B + D') > 0$ by the klt property in U. Thus the support condition means that $B + D \sim_{\mathrm{fix}} B + D'$, or equivalently, $F(B + D), F(B + D')$ have the same signatures.

The finiteness of rational 1-contractions $X \dashrightarrow Y_{\mathfrak{P}}/Z$ follows from the finiteness of geography by Theorem 2.4. By construction, each $Y_{\mathfrak{P}}/Z$ is projective with a polarization supported in S. The converse is by definition. □

Corollary 4.2. *Let $(X/Z, B)$ be a 0-log pair such that components of $S = \mathrm{Supp}\, B$ generate $\mathrm{N}^1(X/Z)$. Then the decomposition of the cone $\mathrm{Eff}(X/Z)$ induced by $\mathfrak{N}_S$ can be given by the mob+exc relation $\sim_{\mathrm{m+e}}$.*

There are finitely many rational 1*-contractions* $X \dashrightarrow Y_{\mathrm{P}}/Z$*, and* Y_{P}/Z *is projective. Conversely, any such contraction corresponds to a class* P*.*

Proof. Immediate by the above corollary and Corollary 3.4. ☐

Corollary 4.3. *Let* X/Z *be an FT variety and* S *be a reduced divisor on* X*. Then the decomposition of cones* $\mathrm{Eff}(X/Z), \mathfrak{E}_S(X/Z)$ *induced by* $\mathfrak{N}_S$ *can be given by the mob+exc relation* $\sim_{\mathrm{m+e}}$*.*

There are finitely many rational 1*-contractions* $X \dashrightarrow Y_{\mathrm{P}}, Y_{\mathfrak{P}}/Z$*, and* $Y_{\mathrm{P}}, Y_{\mathfrak{P}}/Z$ *are projective with a polarization supported in* S *for* $Y_{\mathfrak{P}}$*. Conversely, any such contraction corresponds to a class* $\mathrm{P}, \mathfrak{P}$*.*

Proof. Immediate by Lemma 3.1 and Corollary 4.2. ☐

Corollary 4.4. *Let* $(X/Z, B)$ *be a* 0*-log pair such that components of* $S = \mathrm{Supp}\, B$ *generate* $\mathrm{N}^1(X/Z)$*. Then there are only finitely many rational* 1*-contractions* $X \dashrightarrow Y/Z$ *with projective* Y/Z*.*

Proof. Immediate by Corollary 4.2. ☐

Corollary 4.5. *Let* X/Z *be an FT variety. Then there are only finitely many rational* 1*-contractions* $X \dashrightarrow Y/Z$ *with projective* Y/Z*.*

Proof. Immediate by Lemma 3.1 and Corollary 4.4. ☐

Theorem 4.6. *Let* $(X/Z, B)$ *be a klt log pair such that* $K + B$ *is big. Then there are only finitely many projective wlc models* $(Y_i/Z, B_{Y_i}^{\log})$ *of* $(X/Z, B)$*.*

Proof. Let $(Y/Z, B_Y^{\log})$ be a slt wlc model of $(X/Z, B)$. Then by the monotonicity [IskSh, Lemma 2.4], $(Y/Z, B_Y^{\log})$ is also klt and thus $B_Y^{\log} = B_Y$. Let $(X_{\mathrm{lcm}}/Z, B_{\mathrm{lcm}})$ be its lc model. By construction, $(X_{\mathrm{lcm}}/Z, B_{\mathrm{lcm}})$ is also klt and projective. It is known also that any wlc model $(Y/Z, B_Y)$ of $(X/Z, B)$ is a generalized log flop of $(X_{\mathrm{lcm}}/Z, B_{\mathrm{lcm}})$. Generalized means that it can blow up divisors D of X_{lcm}, but only with log discrepancies $a(D, X_{\mathrm{lcm}}, B_{\mathrm{lcm}}) \le 1$. Thus any wlc model $(Y/Z, B_Y)$ of $(X/Z, B)$ is a 1-contraction $Y' \dashrightarrow Y/X_{\mathrm{lcm}}$ of a projective *terminalization* $(Y'/Z, B')$ where $(Y'/X_{\mathrm{lcm}}, B')$ is a crepant blow up of all prime D with $a(D, X_{\mathrm{lcm}}, B_{\mathrm{lcm}}) \le 1$. Since the set of those D is finite [Sho4, Corollary 1.7], such a blow up exists. The terminalization $(Y'/X_{\mathrm{lcm}}, B')$ is FT by Lemma 3.1 (for birational contraction Y'/X_{lcm} any divisor is big) and the number of 1-contractions into projective models is finite by Corollary 4.5. ☐

Corollary 4.7. *Let* X/Z *be a relative variety of general type. Then there are only finitely many projective minimal models of* X/Z*.*

Proof. Immediate by Theorem 4.6 applied to a resolution of singularities of X/Z. □

D-contractions can be constructed by the D-MMP with the initial model (X, D) [IskSh, 1.1].

Corollary 4.8. *Let $(X/Z, B)$ be a 0-log pair with projective X/Z and D be an $\mathbb{R}$-Cartier divisor on X supported in $\operatorname{Supp} B$. Then the D-MMP holds for (X, D) and, if D is pseudo-effective, then the D-MMP followed by the D-contraction of a resulting model gives a rational 1-contraction $X \dashrightarrow Y_{\mathfrak{P}}/Z$ for the class $\mathfrak{P}$ of D.*

Proof. It is well-known that, in this situation, the D-MMP is equal to the LMMP with an initial model $(X/Z, B + \varepsilon D)$ for $0 < \varepsilon$ and $\varepsilon D \in U$ of Proposition 3.2. Each birational transformation is a rational 1-contraction and they are not isomorphic as log pairs (monotonicity) of log discrepancies). Existence of transformations (divisorial contractions and flips) and their termination hold by the klt slt LMMP. In the pseudo-effective case, we get a wlc model and the required rational 1-contraction by the semiampleness. Otherwise, D will be negative on some fibration and thus not pseudo-effective. □

Corollary 4.9. *Let $(X/Z, B)$ be a 0-log pair with projective X/Z and such that the components of $S = \operatorname{Supp} B$ generate $\mathrm{N}^1(X/Z)$. Then for any $\mathbb{R}$-Cartier divisor D on X, the D-MMP holds for (X, D) and, if D is pseudo-effective, then the D-MMP followed by the D-contraction of a resulting model gives a rational 1-contraction $X \dashrightarrow Y_{\mathfrak{P}}/Z$ for the class $\mathfrak{P}$ given by D. The class $\mathfrak{P}$ can be replaced by numerical P.*

Termination in this case is universally bounded (the bound depends only on the variety X/Z).

Proof. Immediate by Corollary 4.8. The boundedness for termination follows from Corollary 4.5, because X/Z is FT (cf. the following lemma). □

Corollary 4.10. *Let X/Z be an FT variety. Then the D-MMP holds for (X, D) and, if D is pseudo-effective, then the D-MMP followed by the D-contraction of a resulting model gives a rational 1-contraction $X \dashrightarrow Y_{\mathrm{P}}/Z$ for the class P of $\operatorname{Eff}(X/Z)$ containing D.*

Termination in this case is universally bounded (the bound depends only on variety X/Z).

Proof. Immediate by Lemma 3.1 and Corollary 4.9. □

Theorem 4.6 gives the finiteness of projective wlc models for a klt pair $(X/Z, B)$ of general type. It is well-known that all wlc models are crepant birationally isomorphic. Moreover, if $(Y/Z, B_Y)$ and $(Y'/Z, B_{Y'})$ are two wlc models of $(X/Z, B)$, then their natural (identical on rational functions) birational isomorphism $Y \dashrightarrow Y'$ can be factored into projective$(/Y, /Y')$ crepant minimizations $(V, B_V) \to (Y, B_Y)$, $(V', B_{V'}) \to (Y', B_{Y'})$ and a small log flop $(V/Z, B_V) \dashrightarrow (V'/Z, B_{V'})$:

$$\begin{array}{ccc} V & \dashrightarrow & V' \\ \downarrow & & \downarrow \\ Y & \dashrightarrow & Y'/Z. \end{array}$$

A minimal model (V, B_V) is maximal for the contraction order $Y \to Y'$ for wlc models of $(X/Z, B)$: V is $\mathbb{Q}$-factorial, projective$/Y$ and blows up all prime D with $e(D, B) = 0$ (cf. the proof of Theorem 4.6). Moreover, $(V/Z, B_V)$ is a slt wlc model of $(X/Z, B)$ exactly when V/Z is projective. In its turn, each crepant birational contraction can be factored into a sequence of extremal divisorial and small contractions and their number is bounded by the relative Picard number $\rho(V_{\mathrm{lcm}}/X_{\mathrm{lcm}})$ for a projective$/Z$ minimization $(V_{\mathrm{lcm}}, B_{V_{\mathrm{lcm}}})$ of the log canonical model. Thus the remaining factorization concerns the log flop. Note also that V_{lcm}/Z is projective but V/Z may be not. If $\dim_k X = 2$, any klt model, in particular, above minimization is projective$/Z$. However, if $\dim_k X \geq 3$, V/Z can be nonprojective and factorization of birational isomorphism $V \dashrightarrow V_{\mathrm{lcm}}$ or V' into elementary (extremal) flops is out of our grasp. Thus we consider only projective wlc models and focus on minimal (slt) models $(Y/Z, B_Y), (Y'/Z, B_{Y'})$. Then their birational isomorphism $Y \dashrightarrow Y'$ gives a small log flop and can be factored into elementary flops. (Note that, for an elementary flop, the flopping locus can be very nonelementary, e.g., not an irreducible curve in dimension 3.) The general case, not necessarily of general type models, will be considered in Corollary 5.10 below. Here we have a little bit stronger result.

It is natural to consider the factorization problem for flops of projective 0-log pairs $(X/Z, B)$. However to control polarization in $\operatorname{Supp} B$ we need to consider big B and thus FT varieties by Lemma 3.1. Note that a $\mathbb{Q}$-factorialization of an FT variety is again an FT variety, and, for any rational 1-contraction $X \dashrightarrow Y/Z$ with projective Y/Z, Y/Z is FT [PrSh, Lemma 2.8].

Corollary 4.11. *Let $Y, Y'/Z$ be two $\mathbb{Q}$-factorial FT varieties. Then any small birational isomorphism $Y \dashrightarrow Y'$ can be factored into elementary small birational (flips, antiflips, or flops) transformations/Z. The number of such transformations is bounded (the bound depends only on Y/Z).*

Proof. The factorization is a general fact. For instance, we can use the D-MMP, where D is the birational transform of any polarization from Y'. Each transformation will be an elementary D-flip. The number of flips is bounded by the number of rational 1-contractions with projective Y_{P}/Z according to Corollary 4.5. □

Corollary 4.12. *Let $(Y/Z, B_Y), (Y'/Z, B_{Y'})$ be two klt slt wlc models of general type. Then any (small) log flop $(Y/Z, B_Y) \dashrightarrow (Y'/Z, B_{Y'})$ can be factored into elementary log flops and the number of such flops is bounded (the bound depends only on $(Y/Z, B_Y)$).*

Proof. Immediate by Corollary 4.11 for FT variety Y/Y_{lcm}. □

We conclude the section with results that shed a light on terminology: *a mobile cone* $\mathrm{Mob}(X/Z)$ *as a polarizations cone.*

Proposition 4.13. *Let X/Z be a variety with a reduced divisor S and $g\colon X \dashrightarrow Y/Z$ be a small isomorphism. Then g induces an isomorphism $\mathfrak{D}_S(X/Z) = \mathfrak{D}_S(Y/Z)$ that preserves cones $\mathfrak{E}_S(X/Z) = \mathfrak{E}_S(Y/Z), \mathfrak{M}_S(X/Z) = \mathfrak{M}_S(Y/Z)$. If the isomorphism is a rational 1-contraction of a class $\mathfrak{P}$ in $\mathfrak{E}_S(X/Z)$, then under the above identification: $\mathfrak{P} = \mathfrak{A}_S(Y/Z), \overline{\mathfrak{P}} = \mathfrak{Nef}_S(Y/Z)$. For FT X/Y, the last subcone $= \mathrm{s}\mathfrak{A}_S(Y/Z)$.*

Proof. Uses the birational invariance of mob+exc decomposition. Of course, all definitions work and the result holds for general X/Z but cones are not necessarily closed nor polyhedral. □

Proposition 4.14. *Let X/Z be an FT variety, and* P *be the classes in* $\mathrm{Mob}(X/Z)$ *with rational* 1*-contractions* $X \dashrightarrow Y_{\mathrm{P}}/Z$*. Then* $\mathrm{Mob}(X/Z) = \coprod_{\mathrm{P}} \mathrm{Amp}(Y_{\mathrm{P}}/Z)$ *in* $\mathrm{N}^1(X/Z)$*, where* $\mathrm{Amp}(Y_{\mathrm{P}}/Z)$ *is the ample cone. Furthermore,* $\mathrm{P} = \mathrm{Amp}(Y_{\mathrm{P}}/Z)$, $\overline{\mathrm{P}} = \mathrm{sAmp}(Y_{\mathrm{P}}/Z) = \mathrm{Nef}(Y_{\mathrm{P}}/Z)$ *in* $\mathrm{N}^1(X/Z)$*.*

Proof. Transform the decomposition into the classes P of $\mathrm{Mob}(X/Y)$ into their polarization cones: $\mathrm{P} = \mathrm{Amp}(Y_{\mathrm{P}}/Z)$. The equation is given by the pull back $g^*[H]$ of the numerical classes of ample divisors H on Y_{P}/Z by $g\colon X \dashrightarrow Y_{\mathrm{P}}$. □

5. Geography generalities

The points of polyhedron $\mathfrak{N}_S$ fills up the the cube $\mathfrak{B}_S$.

Proposition 5.1. *$\mathfrak{N}_S$ satisfies the following:*

(1) *(monotonicity) if $B \in \mathfrak{N}_S, B' \in \mathfrak{B}_S$ and $B' \geq B$, then $B' \in \mathfrak{N}_S$;*
(2) *(the face of geography) the linear span of $\mathfrak{N}_S$ is the linear span of a minimal face $\mathfrak{F}_S$ of $\mathfrak{B}_S$ containing $\mathfrak{N}_S$; moreover, equations of the span have the form $b_i = 1$, and are the linear equations for $B \in \mathfrak{N}_S$;*
(3) $\dim_{\mathbb{R}} \mathfrak{N}_S = \dim_{\mathbb{R}} \mathfrak{F}_S$.

Proof. The monotonicity is immediate by the inequality for Kodaira dimensions: $\nu(B') = \kappa(B') \geq \kappa(B) = \nu(B)$. However, the slt LMMP is sufficient. For this, one can use the stability of wlc models [Sho6]. Indeed, by the compact and convex property of $\mathfrak{N}_S$, there exists a maximal $D \in \mathfrak{N}_S$ in the direction $\overrightarrow{BB'}$, that is, $\mathfrak{N}_S \cap \overrightarrow{BB'} = (B, D]$. If $\mathfrak{B}_S \cap \overrightarrow{BB'} = (B, D]$, the monotonicity holds in the direction. Otherwise, let $(Y/Z, D_Y^{\log})$ be a slt wlc model of $(X/Z, D)$. Then by [Sho6, Corollary 9 and Addendum 5] after finitely many log flops of this model, it will become a wlc model such that, for each

$$D' \in \mathfrak{B}_S \cap \overrightarrow{BB'} \setminus (B, D]$$

near D, $(Y/Z, {D'}_Y^{\log})$ has a Mori log fibration (cf. the proof of Theorem 2.4). Then, since $D' > D$, for a generic curve C/Z of the fibration,

$$0 > (C, K_Y + {D'}_Y^{\log}) \geq (C, K_Y + D_Y^{\log}) \geq 0,$$

a contradiction.

The other two properties follow from the monotonicity and definitions. $\square$

Separatrix. So, to find the largest lower bound of $\mathfrak{N}_S$, or the minimum divisors (by the closedness) of $\mathfrak{N}_S$, is sufficient to determine $\mathfrak{N}_S$. Such a bound $\mathfrak{S}_S$ will be called a *separatrix*. It consists of boundaries $B \in \mathfrak{N}_S$ which *separate* $\mathfrak{N}_S$: for any neighborhood U of B in the face $\mathfrak{F}_S$, there exists a boundary $B' \in U \setminus \mathfrak{N}_S$, that is, $B' \in U$ and $\nu(B') = -\infty$. Equivalently, $B \in \mathfrak{S}_S$ if and only if $B \in \mathfrak{N}_S$ and is a limit of boundaries $B' \in \mathfrak{F}_S$ with $\nu(B') = -\infty$.

Proposition 5.2. *$\mathfrak{S}_S$ is a closed rational polyhedron (usually nonconvex), and*

$$\mathfrak{S}_S = \overline{\partial \mathfrak{N}_S \cap \operatorname{Int} \mathfrak{F}_S} = \mathfrak{N}_S \cap \overline{\mathfrak{F}_S \setminus \mathfrak{N}_S}.$$

More precisely, a bordering facet of $\mathfrak{N}_S$ lies either

in a facet of $\mathfrak{B}_S$ (or of $\mathfrak{F}_S$), or
in $\mathfrak{S}_S$, and $\mathfrak{S}_S$ is the closure of union of those facets.

A facet in the proposition is a facet of geography $\mathfrak{N}_S$. (However, the statement holds also for facets of the polyhedron $\mathfrak{N}_S$.) A facet $\mathfrak{F}$ of *geography* $\mathfrak{N}_S$ is *bordering* if $\mathfrak{F} \subset \partial\mathfrak{N}_S$.

Proof. Immediate by Theorem 2.4 and Proposition 5.1. □

The separatrix plays an important role in birational geometry of unir-uled varieties, because it divides Mori log fibrations from wlc models (see Corollary 5.6 below). To factorize the birational transformations of those fibrations into more elementary standard transformations (links), we will use a path on $\mathfrak{S}_S$ connecting corresponding points. The best one would be a segment on $\mathfrak{S}_S$ connecting the points but the separatrix is usually not convex. Thus we can use *geodesics* on $\mathfrak{S}_S$ according to the Euclidean metric of the ambient space $\mathfrak{D}_S$ or induced by a projection. We prefer a natural projection from the *origin* 0_S of the face of geography $\mathfrak{F}_S$: $0_S = \sum b_i S_i$ where $b_i = 1$ for the equations of $\mathfrak{N}_S$ and $= 0$ otherwise.

Lemma 5.3. *$\mathfrak{S}_S \neq \emptyset$ if and only if $0_S \notin \mathfrak{N}_S$ and $\mathfrak{N}_S \neq \emptyset$, or equivalently, $\nu(S) \geq 0$ and $\nu(0_S) = -\infty$.*

Proof. Immediate by Proposition 5.2 and definition. □

Let

$$\mathrm{pr}\colon \mathfrak{F}_S \setminus \{0_S\} \to H$$

be a projection from 0_S onto a hyperplane H in $\mathfrak{F}_S$ given by the equation $\sum b_i = 1$ (here we discard all b_i corresponding to the equations of $\mathfrak{N}_S$). (See Diagram 3 below.)

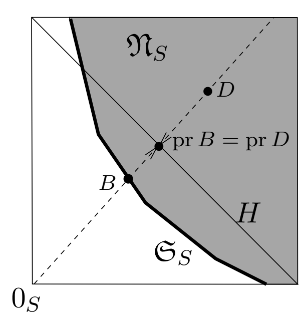


Diagram 3

By Lemma 5.3, pr is well-defined on $\mathfrak{S}_S$.

Proposition 5.4. *If $\mathfrak{S}_S \neq \emptyset$, the image*

$$\operatorname{pr} \mathfrak{N}_S = \operatorname{pr} \mathfrak{S}_S$$

is a closed convex rational polyhedron. Moreover, pr *is* 1*-to-*1 *on* $\mathfrak{S}_S$*.*

Proof. Immediate by Theorem 2.4 and Proposition 5.1. The 1-to-1 property follows from definition. □

Geography of general type. It is a geography $\mathfrak{N}_S$ containing a boundary B such that the pair $(X/Z, B)$ is of general type. This assumes that the geography $\mathfrak{N}_S$ is equipped with certain functions, e.g., $p(C, B), e(D, B)$ of Section 2, $\nu(B), \kappa(B)$, etc. One of those functions is the numerical Kodaira dimension $\nu(B)$. (For other functions, see Section 2.) It behaves almost constant.

Proposition 5.5. *For each $B \in \mathfrak{N}_S \setminus \operatorname{pr}^{-1} \partial \operatorname{pr}(\mathfrak{N}_S)$ (in particular, $B \in \operatorname{Int} \mathfrak{N}_S$),*

$$\nu(B) = \nu = \max\{\nu(B) | B \in \mathfrak{B}_S\} = \nu(S).$$

The same holds for κ.

Proof. Immediate by Theorem 2.4 and the following fact on a rather high projective model V/Z. If $D' \geq D$ are nef divisors on V/Z, then for any subvariety $W \subseteq V$ and any real numbers $a \geq 0, a' > 0$ such that $a + a' = 1$, D' is big on W if and only if $aD + a'D'$ is big on W, and if D is big on W, then D' is big on W. Thus $\nu(aD + a'D') = \nu(D') \geq \nu(D)$. For κ, one can use semiampleness. □

Corollary 5.6. *Let $\mathfrak{N}_S$ be a geography of general type. Then $\mathfrak{F}_S = \mathfrak{B}_S, \dim_{\mathbb{R}} \mathfrak{N}_S = \dim_{\mathbb{R}} \mathfrak{B}_S$ (the number of prime components of S), in particular, $\mathfrak{N}_S \neq \emptyset$ and, for any $B \in \mathfrak{S}_S$, $0 \leq \nu(B) < \nu = \dim_k X/Z$.*

Proof. Immediate by Proposition 5.1 and the open property of general type: if $(X/Z, B), B \in \mathfrak{B}_S$ is of general type, then $(X/Z, B')$ is also of general type for any $B' \in \mathfrak{B}_S$ near B. For the Kodaira dimension, this follows from definition, for numerical one from the slt LMMP. □

If $S' \geq S$ are two reduced b-divisors on X/Z, then there exists a natural inclusion of geographies:

$$\mathfrak{N}_S \to \mathfrak{N}_{S'}, B \mapsto B' = B + S'',$$

and $\mathfrak{N}_S + S'' = \mathfrak{N}_{S'} \cap (\mathfrak{B}_S + S''), \mathfrak{S}_S + S'' = \mathfrak{S}_{S'} \cap (\mathfrak{B}_S + S'')$ where $S'' = S' - S - (S' - S)_X$, the exceptional part of $S' - S$. Indeed, by definition $(Y/Z, B_Y^{\log})$ is a wlc model of $(X/Z, B)$ if and only if this holds for $(X/Z, B')$. This correspondence preserves $\sim_{\text{wlc}}$. The transition from S to S' will be called an *extension*. It allows us to perturb any wlc model to a much better model as follows. For instance, if $(Y/Z, B_Y^{\log})$ is a wlc model of $(X/Z, B)$ not slt or not of general type, e.g., for $B \in \mathfrak{S}_S$, we can convert it to a polarized wlc model. Indeed, if S' is sufficiently larger than S, the geography $\mathfrak{N}_{S'}$ is of general type, and B' belongs to a face of a country $\mathfrak{C} \subseteq \mathfrak{N}_{S'}$. Thus $(X/Z, B'')$ where $B'' \in \mathfrak{C}$, has a lc model $(X_{\mathrm{lcm}}/Z, B''_{\mathrm{lcm}})$, and by Lemma 2.9 a limit of such models gives a wlc model $(Y/Z, B_Y^{\log})$ of $(X/Z, B)$ (might be different from Y above or/and not slt; cf. with Convention 2.2) with a polarization of $Y/Z = X_{\mathrm{lcm}}/Z$. Moreover, as we will see below, for a good choice of S', Y/Z will be $\mathbb{Q}$-factorial. Of course, the variety X_{lcm}/Z obtained by a perturbation is not unique. Nonetheless it is useful (see Corollary 5.10 and Theorem 6.2 below).

Generality conditions. In what follows we assume that S is a reduced divisor, $(X/Z, S)$ is a slt initial pair of general type, and components of S generate $\mathrm{N}^1(X/Z)$. It is easy to satisfy these conditions on a log resolution of $(X/Z, S)$ after an extension of S.

Each country in such a geography has a unique model.

Theorem 5.7. *Under the generality conditions on $(X/Z, S)$, let $B \in \operatorname{Int}\mathfrak{C}$ be an internal point of a country $\mathfrak{C}$ in the geography $\mathfrak{N}_S$. Then $(X/Z, B)$ has a unique wlc model $(Y/Z, B_Y^{\log})$. Moreover, the model is a klt slt wlc model of $(X/Z, B)$, in particular, $B_Y^{\log} = B_Y$, Y/Z is projective $\mathbb{Q}$-factorial, depending only on $\mathfrak{C}$, and the model is lc; components of S_Y generate $\mathrm{N}^1(Y/Z)$.*

The key instrument to investigate generic geographies is the following.

Lemma 5.8. *Under the generality conditions on $(X/Z, B)$, let $(Y/Z, B_Y)$ be a klt slt wlc model of $(X/Z, B)$ for a boundary B in a subclass $\operatorname{Int}\mathfrak{B}_S \cap \mathfrak{P}$ and $Y \to T = X_{\mathrm{lcm}}$ be its lc contraction. Then Y/T is FT, components of S_Y generate $\mathrm{N}^1(Y/Z)$, and*

$$\rho(Y/T) \le \dim_{\mathbb{R}} \mathfrak{B}_S - \dim_{\mathbb{R}} \mathfrak{P}.$$

Proof. Since the initial model $(X/Z, B)$ is klt slt and $\operatorname{Supp} B = S$, by the klt slt LMMP there exists a klt slt wlc model $(Y/Z, B_Y)$ of $(X/Z, B)$ and components of $\operatorname{Supp} B_Y = S_Y$ generate $\mathrm{N}^1(Y/Z)$. Otherwise, we get a Mori log fibration and nothing to prove. By construction of the lc model, $(Y/T, B_Y)$ is a 0-log pair and by Lemma 3.1, Y/T is FT. The contraction exists by the base freeness [Sho1], because B_Y is big. The klt slt properties imply also that the image $\mathfrak{P}_Y$ of $\mathfrak{P}$ under the natural linear map

$$\mathfrak{D}_S \to \mathfrak{D}_{S_Y}, D \mapsto D_Y,$$

has the same codimension: $\dim_{\mathbb{R}} \mathfrak{B}_S - \dim_{\mathbb{R}} \mathfrak{P} = \dim_{\mathbb{R}} \mathfrak{B}_{S_Y} - \dim_{\mathbb{R}} \mathfrak{P}_Y$ [Choi, Lemma 3.3.2]. By the open property in Theorem 2.4, $\operatorname{Int} \mathfrak{P} = \operatorname{Int} \mathfrak{B}_S \cap \mathfrak{P}$ when $\operatorname{Int} \mathfrak{B}_S \cap \mathfrak{P} \neq \emptyset$. On the other hand, the generation property of S_Y gives the surjectivity of the natural linear map

$$\mathfrak{D}_{S_Y} \twoheadrightarrow \mathrm{N}^1(Y/T), D \mapsto [K_Y + D].$$

By construction and the definition of $\sim_{\mathrm{wlc}}$, $\mathfrak{P}_Y$ lies in the kernel of the last map. Thus

$$\rho(Y/T) = \dim_{\mathbb{R}} \mathrm{N}^1(Y/T) \leq \dim_{\mathbb{R}} \mathfrak{D}_{S_Y} - \dim_{\mathbb{R}} \mathfrak{P}_Y = \dim_{\mathbb{R}} \mathfrak{B}_S - \dim_{\mathbb{R}} \mathfrak{P}.$$

Finally, note that the constructed model Y/T can be different from the one in the statement of lemma. However, the difference is in a log flop. By Corollary 4.11, it is a composition of elementary small birational transformations which preserve the required properties. □

Proof of Theorem 5.7. Let $(Y/Z, B_Y^{\log})$ be a slt wlc model of $(X/Z, B)$. Since the latter pair is klt initial, the former pair is klt and $B_Y^{\log} = B_Y$ [IskSh, Lemma 2.4]. By our assumptions and Proposition 5.5, the pair $(Y/Z, B_Y)$ has general type and a lc model $(T = X_{\mathrm{lcm}}, B_{\mathrm{lcm}})$. Thus by Lemma 5.8 and Corollary 5.6, $\rho(Y/T) \leq 0$ and $Y = T$, the uniqueness.

The last uniqueness was established only for slt models. Any other wlc model is a rational 1-contraction of Y/T and thus coincide with Y (see the projective case in Corollary 4.3; in general, cf. the proof of Theorem 4.6). □

Models of facets and ridges for a generic geography also behave quite controllable. In dimension $d = \dim_k X = 1$, all facets are bordering and ridges only cube bordering. In dimension $d = 2$, flopping facets are impossible and the birational modifications of ridges are only divisorial (see the classification of ridges below).

We divide facets into two types according to their location in $\mathfrak{N}_S$: *bordering* and *internal*. Furthermore, they are divided into two subtypes, respectively: *cube bordering, (Mori) fibering* and *flopping, divisorial*. Let $\mathfrak{F}$ be a facet of $\mathfrak{N}_S$ under the generality conditions, $B \in \operatorname{Int} \mathfrak{F}$, and $(Y/Z, B_Y^{\log})$ be a projective wlc model of $(X/Z, B)$. First, we describe neighboring classes and properties of their models (only projective, possibly not $\mathbb{Q}$-factorial) and then prove them (see the proof of Theorem 5.9 below).

Cube bordering. The facet $\mathfrak{F}$ is bordering for $\mathfrak{N}_S$ and its closure $\overline{\mathfrak{F}}$ lies in a facet of $\mathfrak{B}_S$, or satisfies an equation $b_i = 0$ or 1 in $\mathfrak{N}_S$. In this case, $(Y/Z, B_Y^{\log})$ can be non klt. If it is klt, the wlc models can be classified as types below. We are not interested in this because after a perturbation we have internal boundaries (see the proof of Corollary 5.10 below).
(Mori) Fibering. The facet $\mathfrak{F} \subseteq \mathfrak{S}_S$ is also bordering for $\mathfrak{N}_S$ but $B \in \operatorname{Int}\mathfrak{B}_S$. The pair $(Y/Z, B_Y^{\log})$, $B_Y^{\log} = B_Y$, is a klt slt wlc model of $(X/Z, B)$ and Y has a Mori log fibration $Y \to T = X_{\text{lcm}}/Z$ given by the lc contraction. The base T is $\mathbb{Q}$-factorial. Such a Mori fibration will be called *polarized* by the boundary B_Y. The model $(Y/Z, B_Y)$ and the fibration are unique for B and both morphisms $Y/Z, Y/T$ depend only on the facet $\mathfrak{F}$. Moreover, there exists a unique (neighboring) country $\mathfrak{C}$ of $\mathfrak{N}_S$ such that $\overline{\mathfrak{F}}$ is its facet. For any $B' \in \mathfrak{C}$, $(Y/Z, B'_Y)$ is a lc model of $(X/Z, B')$, and $(Y/Z, B_Y)$ is limiting from $\mathfrak{C}$. On the other side, if $B' \in \mathfrak{B}_S \setminus \mathfrak{N}_S$ and is sufficiently close to B, the Mori log fibration $Y \to T/Z$ is a slt Mori log fibration of $(X/Z, B')$. The facet is given by the equation $p(C, B) = 0$ in $\overline{\mathfrak{C}}$ where C is a curve on a rather high model V/Z of X/Z, contracted on T but not on Y. The function is > 0 on $\mathfrak{C}$ and < 0 in the points of $\mathfrak{B}_S \setminus \mathfrak{N}_S$ near the points of $\mathfrak{F}$ for the linear extension of $p(C, B')$ from $\mathfrak{C}$. The functions $e(D, B')$ have the same signatures on $\mathfrak{C}, \mathfrak{F}$.

The next two types are possible respectively only in dimension $d \geq 3$ and $d \geq 2$.
Flopping. The facet is internal for $\mathfrak{N}_S$: $\operatorname{Int}\mathfrak{F} \subset \operatorname{Int}\mathfrak{N}_S$, and $\overline{\mathfrak{F}}$ is a facet of two (neighboring) countries $\mathfrak{C}_1, \mathfrak{C}_2$ in $\mathfrak{N}_S$. The pair $(X/Z, B)$ has exactly three wlc models which are klt of general type and related by an elementary small flop, extraction or contraction. Countries $\mathfrak{C}_1, \mathfrak{C}_2$ give exactly two slt wlc models $(Y_1/Z, B_1)$, $B_1 = B_{Y_1} = B_{Y_1}^{\log}$, $(Y_2/Z, B_2)$, $B_2 = B_{Y_2} = B_{Y_2}^{\log}$ of $(X/Z, B)$, where $(Y_1/Z, B'_{Y_1})$, $(Y_2/Z, B'_{Y_2})$ are lc models of $(X/Z, B')$ for $B' \in \mathfrak{C}_1, \mathfrak{C}_2$, respectively. Those models of $(X/Z, B)$ are limiting from $\mathfrak{C}_1, \mathfrak{C}_2$, respectively. The third model $(T/Z, B_T), B_T = B_T^{\log}$, of $(X/Z, B)$ is a lc one. Both contractions $Y_1, Y_2 \to T$ are elementary small. In particular, T is not $\mathbb{Q}$-factorial and the third model is not slt. The facet is given by equations $p(C_1, B) = p(C_2, B) = 0$ in $\overline{\mathfrak{C}_1} \cup \overline{\mathfrak{C}_2}$ where C_1, C_2 are curves on a rather high model V/Z of X/Z contracted on T but not on Y_1, Y_2, respectively; $p(C_1, B') > 0$ on $\mathfrak{C}_1$ and ≥ 0 on $\mathfrak{C}_2$; $p(C_2, B') \geq 0$ on $\mathfrak{C}_1$ and > 0 on $\mathfrak{C}_2$. The functions $e(D, B')$ have the same signatures on $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{F}$.

Divisorial. The facet is also internal for $\mathfrak{N}_S$: $\operatorname{Int}\mathfrak{F} \subset \operatorname{Int}\mathfrak{N}_S$, and $\overline{\mathfrak{F}}$ is a facet of two (neighboring) countries $\mathfrak{C}_1, \mathfrak{C}_2$ in $\mathfrak{N}_S$. The pair $(X/Z, B)$ has exactly two wlc models which are klt of general type and related by an elementary divisorial extraction or contraction of a prime divisor D. The country $\mathfrak{C}_1$ gives exactly one slt wlc model $(Y_1/Z, B_1), B_1 = B_{Y_1} = B_{Y_1}^{\log}$, of $(X/Z, B)$, where $(Y_1/Z, B'_{Y_1})$ is a lc model of $(X/Z, B')$ for $B' \in \mathfrak{C}_1$. The model $(Y_1/Z, B_1)$ is limiting from $\mathfrak{C}_1$. The second model $(T/Z, B_T), B_T = B_T^{\log}$, of $(X/Z, B)$ is the lc one, the contraction $Y_1 \to T$ of D is elementary divisorial. In particular, T is $\mathbb{Q}$-factorial, $(T/Z, B_T)$ is a slt wlc model, but not a slt wlc model of $(X/Z, B)$. The model $(T/Z, B_T)$ is limiting from $\mathfrak{C}_2$. The facet $\mathfrak{F}$ is given by the equation $p(C, B) = 0$ in $\overline{\mathfrak{C}_1}$ where C is a curve on a rather high model V/Z of X/Z contracted on T but not on Y_1; $p(C, B') = 0$ on $\overline{\mathfrak{C}}_2$ and > 0 on $\mathfrak{C}_1$. Similarly, $\mathfrak{F}$ can also be given by the equation $e(D, B) = 0$ in $\overline{\mathfrak{C}_2}$: $e(D, B') = 0$ on $\overline{\mathfrak{C}}_1$ and > 0 on $\mathfrak{C}_2$.

Theorem 5.9. *Under the generality conditions on $(X/Z, S)$, let $\mathfrak{F}$ be a facet of the geography $\mathfrak{N}_S$. Then it has one of four above types: cube bordering, fibering, flopping or divisorial, and behaves accordingly. (See Diagram 4 below.)*

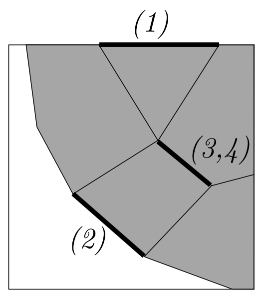


Diagram 4

(1) Cube bordering type : $\mathfrak{F} \subset \partial \mathfrak{B}_S$;
(2) (Mori) Fibering type : $\mathfrak{F} \subseteq \mathfrak{S}_S$;
(3,4) Flopping or divisorial type : $\operatorname{Int}\mathfrak{F} \subseteq \operatorname{Int}\mathfrak{N}_S$.

Proof. A bordering facet $\mathfrak{F}$ of $\mathfrak{N}_S$ has the cube bordering or fibering type by definition, Proposition 5.2 and Corollary 5.6.

Cube bordering type: $\mathfrak{F}$ satisfies an equation $b_i = 0$ or 1 and lies in the corresponding facet of $\mathfrak{F}_S = \mathfrak{B}_S$ by Corollary 5.6.

For the boundaries $B \in \operatorname{Int}\mathfrak{F}$ of the other type, B is *internal*: $B \in \operatorname{Int}\mathfrak{B}_S$. By the generality assumption, $(X/Z, B)$ is a klt slt initial pair of itself and components of $\operatorname{Supp} B = S$ generate $\mathrm{N}^1(X/Z)$. In particular B is big. Thus any wlc model $(Y/Z, B_Y^{\log})$ of $(X/Z, B)$

is klt [IskSh, Lemma 2.4] with big $B_Y^{\log} = B_Y$ and has a lc contraction $Y \to T = Y_{\mathrm{lcm}}$ [Sho1]. If Y/T is projective, it is FT by Lemma 3.1. Such a model $(Y/Z, B_Y)$ exists because $B \in \mathfrak{N}_S$ and it satisfies Lemma 5.8 when it is a slt wlc model of $(X/Z, B)$. The slt model can be constructed by the slt LMMP from the above initial model.

Fibering type. Suppose that $\mathfrak{F}$ is fibering and $(Y/Z, B_Y)$ is a slt wlc model of $(X/Z, B)$. By construction and Lemma 5.8, Y/T is a Mori log fibration. Indeed, it is not isomorphism and an extremal fibration by Corollary 5.6. The model (in the projective class) and contraction are unique by Corollary 4.3. Since Y is $\mathbb{Q}$-factorial and Y/T is FT, T is $\mathbb{Q}$-factorial too (cf. [Sho1, 2.13.5]). The boundary B_Y is a polarization of Y/T. Thus if we perturb B to B' in a positive direction, $B' \in \operatorname{Int} \mathfrak{N}_S$, then $B' \in \mathfrak{C}$ for a unique neighboring country $\mathfrak{C}$ of $\mathfrak{N}_S$ and $(Y/Z, B_Y')$ is a wlc (actually lc) model of $(X/Z, B')$ with a polarization $K_Y + B_Y'$ over T. Hence the functions $e(D, B')$ have the same signatures on $\mathfrak{C}, \mathfrak{F}$. By Lemma 2.9, $(Y/Z, B_Y)$ is limiting from $\mathfrak{C}$, and by Theorem 2.4, $\overline{\mathfrak{F}}$ is a facet of $\mathfrak{C}$. A perturbation $(Y/Z, B_Y')$ in a negative direction, $B' \in \mathfrak{B}_S \setminus \mathfrak{N}_S$, gives a slt Mori log fibration $Y \to T/Z$ of $(X/Z, B')$. By construction and Theorem 5.7, a required equation $p(C, B) = 0$ can be given by a curve C on V/Z with the curve image in a fiber of Y/T.

The next two types are possible only in dimension $d \geq 2$ and internal: $\operatorname{Int} \mathfrak{F} \subset \operatorname{Int} \mathfrak{N}_S$, by Theorem 2.4 $\overline{\mathfrak{F}}$ is a facet of two (neighboring) countries $\mathfrak{C}_1, \mathfrak{C}_2$ in $\mathfrak{N}_S$, and $B \in \operatorname{Int} \mathfrak{N}_S$. (According to the open property in Theorem 2.4, $\mathfrak{C}_1 \cap \mathfrak{F} = \mathfrak{C}_1 \cap \mathfrak{F} = \emptyset$.) By our assumptions and Proposition 5.5, any wlc model $(Y/Z, B_Y)$ of $(X/Z, B)$ has general type and a lc model $(T/Z, B_T) = (X_{\mathrm{lcm}}/Z, B_{\mathrm{lcm}})$ with the lc contraction $Y \to T$. Let $(Y/Z, B_Y)$ be a slt wlc model of $(X/Z, B)$. Note that it is not lc: $Y \neq T$. Otherwise, any small perturbation B' of B in $\mathfrak{B}_S$ is $\sim_{\mathrm{wlc}} B$: $\mathfrak{F} \subset \mathfrak{C}_1, \mathfrak{C}_2$, a contradiction. Hence $\rho(Y/T) = 1$, or equivalently, Y/T is elementary by Lemma 5.8.

Flopping type. Now we claim that $\mathfrak{F}$ is of flopping type if Y/Z is small. It is possible only in dimension $d \geq 3$. By Corollary 4.3 the pair $(X/Z, B)$ has exactly three projective wlc models $Y_1/Z, Y_2/Z, T/Z$ which are related by an elementary small flop, extraction or contraction:

$$\begin{array}{ccc} Y_1 & \dashrightarrow & Y_2 \\ & \searrow \quad \swarrow & \\ & T/Z. & \end{array}$$

(The cone $\mathrm{Eff}(Y/T)$ is a line $\mathbb{R}$ with three subcones (classes): $(-\infty, 0), 0, (0, +\infty)$.) The boundary B_Y is a polarization of Y/T. If we perturb B to B' in a positive direction, $B' \in \mathrm{Int}\,\mathfrak{N}_S$, that is, $K_Y + B'_Y$ becomes a polarization, then $Y/Z = Y_1/Z$ is limiting from $\mathfrak{C}_1$ if $B' \in \mathfrak{C}_1$, otherwise $Y/Z = Y_2/Z$ is limiting from $\mathfrak{C}_2$. So, the boundary $B' \in \mathfrak{C}$ for one of the two neighboring countries $\mathfrak{C}$ of $\mathfrak{N}_S$ and $(Y/Z, B'_Y)$ is a wlc (actually lc) model of $(X/Z, B')$ with a polarization $K_Y + B'_Y$. The same holds for another neighboring country after a log flop of Y/T. By Theorem 2.4, $\overline{\mathfrak{F}}$ is a facet of $\mathfrak{C}_1$ and of $\mathfrak{C}_2$. The third wlc model of $(X/Z, B)$ is $(T/Z, B_T)$. In particular, T is not $\mathbb{Q}$-factorial and the model is not slt. By construction, the functions $e(D, B)$ have the same signatures on $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{F}$. By construction and Theorem 5.7, required equations $p(C_1, B) = p(C_2, B) = 0$ can be given by curves C_1, C_2 on V/Z with the curve image in a fiber of $Y_1, Y_2/T$, respectively.

Divisorial type. In the remaining last type case, Y/T is a divisorial contraction of a unique prime divisor D by the extremal and $\mathbb{Q}$-factorial properties. Again by Corollary 4.3 the pair $(X/Z, B)$ has exactly two projective wlc models which are related by an elementary divisorial extraction or contraction. (The cone $\mathrm{Eff}(Y/T)$ is a line $\mathbb{R}$ with two subcones (classes): $(-\infty, 0], (0, +\infty)$.) The boundary B_Y is a polarization of Y/T. Thus if we perturb B to B' in a positive direction, $B' \in \mathrm{Int}\,\mathfrak{N}_S$, that is, $K_Y + B'_Y$ becomes a polarization, say for $B' \in \mathfrak{C}_1$, then $Y/Z = Y_1/Z$ is limiting from $\mathfrak{C}_1$. The second wlc model of $(X/Z, B)$ is $(T/Z, B_T)$. In particular, T is $\mathbb{Q}$-factorial, $(T/Z, B_T)$ is a slt wlc model, but not a slt wlc model of $(X/Z, B)$: $e(D, B) = 0$. The model $(T/Z, B_T)$ is limiting from $\mathfrak{C}_2$. By construction and Theorem 5.7, required equations $e(D, B) = 0, p(C, B) = 0$ can be given respectively by the contracted divisor D and by a curve C on V/Z with the curve image in a fiber of Y/T. □

Corollary 5.10. *Let $(Y/Z, B_Y), (Y'/Z, B_{Y'})$ be two klt slt wlc models. Then any small log flop $(Y/Z, B_Y) \dashrightarrow (Y'/Z, B_{Y'})$ can be factored into elementary log flops.*

It is sufficient to assume that the transformation is small and B'_Y is the birational image of B_Y.

Proof. We can prove the corollary using elementary log flops of the D-MMP with $D = H'_Y$ where H' is an effective polarization on Y' [IskSh]. According to the stability of [Sho6, Addendum 5], the D-MMP of Y/Z is equivalent to the klt slt LMMP of the initial model $(Y/Z, B_Y + \varepsilon D)$ for some $0 < \varepsilon \ll 1$.

We can use a geography instead of the D-LMMP which is more symmetric. Pick up two effective (generic) polarizations H, H' on

Y, Y' respectively. (In dimension ≥ 2, one can suppose that H, H' are prime.) Take a log resolution $(X/Z, S)$ of $(Y/Z, B_Y + H + H'_Y)$ where $S = \mathrm{Supp}(B_Y) + H + H' + \sum E_i$ and E_i are the exceptional divisors of $X \to Y$. Then for an appropriate klt boundary $B \in \mathfrak{B}_S$, both models $(Y/Z, B_Y)$ and $(Y'/Z, B_{Y'})$ are klt slt wlc models of $(X/Z, B)$. After a perturbation by polarizations H, H', they will become klt log canonical models $(Y/Z, B_Y + \varepsilon H), (Y'/Z, B_{Y'} + \varepsilon' H')$ respectively of $(X/Z, D), (X/Z, D')$ where D, D' are perturbations of B in $\mathfrak{B}_S$. In other words, there are two classes $\mathfrak{C}, \mathfrak{C}'$ in the geography $\mathfrak{N}_S$ such that $D \in \mathfrak{C}, D' \in \mathfrak{C}'$ and $(Y/Z, B_Y), (Y'/Z, B_{Y'})$ are limiting from $\mathfrak{C}, \mathfrak{C}'$ respectively; $B \in \overline{\mathfrak{C}}, \overline{\mathfrak{C}}'$. As above by the stability [Sho6, Addendum 5], the geography is conical near B. Hence for a perturbation, $\mathfrak{C}, \mathfrak{C}'$ are actually countries and the segment $[D, D']$ intersects only facets $\mathfrak{F}$, which are also adjacent to B: $B \in \overline{\mathfrak{F}}$ is limiting for the facets. By Theorem 5.9, this gives a required factorization. But for this we need to expand S to fulfil the generality conditions. We also need to verify that all intersections $[D, D'] \cap \mathfrak{F}$ have flopping type. In other words, if a prime divisor F is not exceptional on Y, then F should not be exceptional on the wlc models of all $B' \in [D, D']$ and vice versa. By construction, H, H' are mobile and big on the divisors of all those wlc models. Thus if all divisors in the extension of S are also mobile and big (e.g., ample on X; cf. the perturbation in the proof of Theorem 6.2) on those divisors, we secure the required nonexceptionality: each wlc model for $B' \in [D, D']$ corresponds to an effective perturbation of B_Y. On the other hand, for B with all $\mathrm{mult}_{E_i} B = 1$ or close to 1 and any exceptional on Y b-divisor E, by the klt assumption for (Y, B_Y) the inequality $e(E, B') > 0$ holds near B. Hence E will be exceptional on the wlc models of all $B' \in [D, D']$.

Finally, note that this proof allows us to relax the klt slt LMMP to the klt slt big one.

A birational transformation $Y \dashrightarrow Y'/Z$ gives a (small) log flop if and only if the transformation is small and $B_{Y'}$ is the birational image of B_Y [Sho4, Proposition 2.4]. □

We divide ridges also into two types according to their location in $\mathfrak{N}_S$: *bordering* and *internal*. Furthermore, the bordering ones are divided into two subtypes: *cube bordering* and *(Mori) fibering*. The internal ridges correspond to general type and will be also referred to as *birational.* In its turn, fibering and birational types have three subtypes each. Essentially they can be divided into types as *only flopping*, *divisorial and flopping*, etc. But we prefer just a conventional ordering 2A, 2B, etc: e.g., 2B means fibering type B.

So, let $\mathfrak{R}$ be a ridge of $\mathfrak{N}_S$ under the generality conditions, $B \in \operatorname{Int}\mathfrak{R}$, and $(Y/Z, B_Y^{\log})$ be a projective wlc model of $(X/Z, B)$. First, we describe the neighboring classes and properties of their models (only projective, possibly not $\mathbb{Q}$-factorial) and then prove the statements (see the proof of Theorem 5.11 below).

Cube bordering. The ridge $\mathfrak{R}$ is *bordering*, that is, $\mathfrak{R} \subset \partial\mathfrak{N}_S$, and the closure $\overline{\mathfrak{R}}$ either lies in a ridge of $\mathfrak{B}_S$, or in a facet $\mathfrak{F}$ of $\mathfrak{B}_S$. In the former case, $\mathfrak{R}$ satisfies two equations $b_i = 0$ or 1, $b_j = 0$ or 1 in $\mathfrak{N}_S$. In the latter case, $\mathfrak{F}$ satisfies an equation $b_i = 0$ or 1 in $\mathfrak{B}_S$ and $\mathfrak{R}$ is a ridge of $\mathfrak{N}_S$ in $\mathfrak{F}$. In both cases, $(Y/Z, B_Y^{\log})$ can be non klt. If it is klt, the wlc models can be classified as types below.

For the internal types, the wlc models $(Y/Z, B_Y^{\log})$ of $(X/Z, B), B \in \mathfrak{R}$, are klt and $B_Y^{\log} = B_Y$. Each of them has the lc contraction $Y \to T = X_{\mathrm{lcm}}/Z$ and Y/T is FT. The base T depends only on the ridge $\mathfrak{R}$. The pair $(Y/Z, B_Y)$ with projective Y/Z is a slt wlc model of $(X/Z, B)$ if and only if $\rho(Y/T) = 2$ and there exist exactly two extremal contractions of Y over T.

Fibering or 2. The ridge $\mathfrak{R} \subset \mathfrak{S}_S$ is also bordering for $\mathfrak{N}_S$ but $B \in \operatorname{Int}\mathfrak{B}_S$. Equivalently, $\operatorname{Int}\mathfrak{R} \subset \mathfrak{S}_S \setminus \partial\mathfrak{B}_S$. The closure $\overline{\mathfrak{R}}$ is a facet of $m+1 \ge 2$ (neighboring) facets $\mathfrak{F}_1, \dots, \mathfrak{F}_{m+1}$ in $\mathfrak{N}_S$, and a ridge of m countries $\mathfrak{C}_1, \dots, \mathfrak{C}_m$; $\mathfrak{C}_i$ has (closed) facets $\overline{\mathfrak{F}}_i, \overline{\mathfrak{F}}_{i+1}$ (see Diagram 5 below).

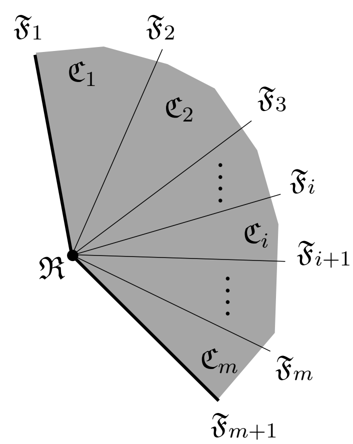


Diagram 5

Facets $\mathfrak{F}_1, \mathfrak{F}_{m+1}$ are fibering, and $\mathfrak{F}_2, \dots, \mathfrak{F}_m, m \ge 2$, are *birational*: flopping or divisorial. Let $Y_1/T_1, Y_m/T_{m+1}$ be Mori log fibrations of facets $\mathfrak{F}_1, \mathfrak{F}_{m+1}$, $T_i, 2 \le i \le m$, be lc models of the birational facets, and $Y_i, 1 \le i \le m$, be lc models of countries $\mathfrak{C}_i$. If $\mathfrak{F}_m$ is divisorial, we suppose that $\mathfrak{F}_2$ is divisorial (after interchanging $\mathfrak{C}_1$ with $\mathfrak{C}_m$).

Each $\mathfrak{F}_i, 3 \le i \le m-1$, is flopping, and corresponding T_i are not $\mathbb{Q}$-factorial. The varieties T_1, T_{m+1} are $\mathbb{Q}$-factorial. The varieties T_2, T_m are $\mathbb{Q}$-factorial if and only if $\mathfrak{F}_2, \mathfrak{F}_m$ respectively are divisorial. More precisely, if $\mathfrak{F}_2$ is divisorial, then $Y_2 \to Y_1 = T_2$ is an elementary divisorial contraction of a prime divisor D_2. The similar holds for divisorial $\mathfrak{F}_m$ with contraction $Y_{m-1} \to Y_m = T_m$ of a prime divisor D_m (possibly the same as b-divisor D_2). For $2 \le i \le m-1$, the variety Y_i/T has (only) two elementary contractions $Y_i/T_i, T_{i+1}$. The same holds for Y_1, Y_m if $\mathfrak{F}_2, \mathfrak{F}_m$ respectively are flopping. All contractions $Y_i/T_i, T_{i+1}, 3 \le i \le m-2, m \ge 5$ are small. The same holds for Y_2, Y_{m-1} if $\mathfrak{F}_2, \mathfrak{F}_m$ respectively are flopping.

Each $T_i, 1 \le i \le m+1$, has a natural contraction to T, T_i/T is extremal, FT and fibered for $2 \le i \le m$. The base T is not $\mathbb{Q}$-factorial exactly when T_1/T is small. The base T is $\mathbb{Q}$-factorial for $\dim T \le 2$, in particular, in dimension $d \le 3$, and for types 2A with $m = 1$, 2B-C. In all other cases, T can be non $\mathbb{Q}$-factorial.

The pair $(X/Z, B)$ has exactly two wlc models $(Y_1/Z, B_1)$, $B_1 = B_{Y_1}$, $(Y_m/Z, B_m)$, $B_m = B_{Y_m}$, $m \ge 2$, equipped with a Mori log fibration: Y_1/T_1, Y_m/T_{m+1} over T, respectively. The fibrations $Y_1/T_1, Y_m/T_{m+1}$ are *square* birational, that is, both contractions $T_1, T_{m+1}/T$ are birational if and only if the facets $\mathfrak{F}_1, \mathfrak{F}_{m+1}$ are coplanar. For $m = 1$, 2A is only possible, the pair $(X/Z, B)$ has exactly one wlc model $(Y_1/Z, B_1)$, the latter is slt and has two Mori log fibrations: $Y_1/T_1, T_2$ over T, the base T is $\mathbb{Q}$-factorial and $T_1, T_2/T$ are Mori log fibrations too, $\mathfrak{F}_1, \mathfrak{F}_2$ are never coplanar.

The flopping (small) contractions exist only for $d \ge 3$. Thus, $m \le 3$ in dimension $d = 2$ (there are no fibering ridges in dimension $d = 1$). In dimension $d \ge 3$, the only nonflopping cases are: 2A with $m = 1$, 2B with $m = 2$ and 2C with $m = 3$.

Each pair $(Y_i/Z, B_i)$, $B_i = B_{Y_i}$, $2 \le i \le m-1$, is a klt slt wlc model of $(X/Z, B)$, limiting from $\mathfrak{C}_i$. The pairs $(Y_1/Z, B_1), (Y_m/Z, B_m)$ satisfy exactly the same when respectively $\mathfrak{F}_2, \mathfrak{F}_m$ are flopping, and they are limiting from $\mathfrak{C}_1, \mathfrak{C}_m$, respectively. Each pair $(T_i/Z, B_{T_i}), 3 \le i \le m-1$, is a non $\mathbb{Q}$-factorial nonslt projective wlc model of $(X/Z, B)$, limiting from $\mathfrak{F}_i$. The pairs $(T_2/Z, B_{T_2}), (T_m/Z, B_{T_m})$ satisfy exactly the same when respectively $\mathfrak{F}_2, \mathfrak{F}_m$ are flopping. If $\mathfrak{F}_2$ is divisorial, $(Y_1/Z, B_1) = (T_2/Z, B_{T_2})$ is an $\mathbb{Q}$-factorial projective wlc model of $(X/Z, B)$ but not a slt wlc model of $(X/Z, B)$. The similar holds for $\mathfrak{F}_m$. There are no other projective wlc models of $(X/Z, B)$ than above.

The closures $\overline{\mathfrak{F}}_1, \overline{\mathfrak{F}}_{m+1}$ are given by equations $p(C_1, B') = 0, p(C_{m+1}, B') = 0$ in $\overline{\mathfrak{C}}_1, \overline{\mathfrak{C}}_m$, respectively where C_1, C_{m+1} are curves on a rather high model V/Z of X/Z, contracted on T_1, T_{m+1} but not on Y_1, Y_m, respectively. Moreover, if C_1 is rather general, then $p(C_1, B') > 0$ on $\mathfrak{C}_i, 1 \le i \le m, \mathfrak{F}_j, 2 \le j \le m$, and ≥ 0 on $\overline{\mathfrak{F}}_{m+1}$ whereas $= 0$ if and only if the fibration Y_1/T_1 factors through Y_m/T_{m+1}: the diagram

$$\begin{array}{ccc} Y_1 & \dashrightarrow & Y_m \\ \downarrow & & \downarrow \\ T_1 & \dashrightarrow & T_{m+1}, \end{array}$$

with the natural birational isomorphism $Y_1 \dashrightarrow Y_m$ and a (natural) rational contraction $T_1 \dashrightarrow T_{m+1}$, is commutative.

The similar holds for general C_{m+1}. The models $Y_1/T_1, Y_m/T_{m+1}$ are square birational if and only if $p(C_1, \mathfrak{F}_{m+1}) = p(C_{m+1}, \mathfrak{F}_1) = 0$. Moreover, $Y_1/T_1, Y_m/T_{m+1}$ are *generically isomorphic*, that is, they are natural isomorphic over the general points of T_1, T_{m+1}, if and only if the facets $\mathfrak{F}_1, \mathfrak{F}_{m+1}$ are coplanar.

If $\mathfrak{F}_2$ is divisorial, its closure $\overline{\mathfrak{F}}_2$ is given by the equation $e(D_2, B') = 0$ in $\overline{\mathfrak{C}}_1$: $e(D_2, B') > 0$ on $\mathfrak{F}_1, \mathfrak{C}_1$ and $= 0$ on $\mathfrak{C}_i, 2 \le i \le m-1, \mathfrak{F}_j, 2 \le j \le m, \mathfrak{R}$; $e(D_2, B') = 0$ on $\mathfrak{C}_m, \mathfrak{F}_{m+1}$ if $D_2 \ne D_m$ as b-divisor or $\mathfrak{F}_m$ is flopping. If $D_2 = D_m$, then $e(D_2, B') > 0$ on $\mathfrak{C}_m, \mathfrak{F}_{m+1}$. The similar holds for D_m if $\mathfrak{F}_m$ is divisorial.

Other equations for the divisorial and flopping facets were given in Theorem 5.9 above.

Fibering A or 2A. (The Sarkisov link of type IV.) Both $T_1, T_{m+1} \not\cong T, m \ge 1$. The pair $(Y_1/Z, B_1)$ is a slt wlc model of $(X/Z, B)$. The base T is possibly not $\mathbb{Q}$-factorial for $m \ge 2$. All intermediate facets $\mathfrak{F}_i, 2 \le i \le m$, are flopping. The only possible other slt wlc models $(Y_i/Z, B_i)$ of $(X/Z, B)$ correspond to the countries $\mathfrak{C}_i, 2 \le i \le m$, and they are log flops of $(Y_1/Z, B_1)$, directed by a polarization on Y_i/T. More precisely, those flops are composition of elementary ones:

$$Y_1 \dashrightarrow Y_2 \dashrightarrow \cdots \dashrightarrow Y_i/T.$$

The total birational transformation of fibrations:

$$\begin{array}{ccccccccc} Y_1 & \dashrightarrow Y_2 \dashrightarrow \cdots \dashrightarrow & Y_i & \dashrightarrow \cdots \dashrightarrow & Y_{m-1} & \dashrightarrow & Y_m \\ \downarrow & & & & & & \downarrow \\ T_1 & & & & & & T_{m+1} \\ & \searrow^{\not\cong} & & & & {}^{\not\cong}\swarrow & \\ & & & T/Z. & & & \end{array}$$

Fibering B or 2B. (The Sarkisov link of type I and its inverse type III.) $T_1 \cong T$ but $T_{m+1} \not\cong T, m \geq 2$. The base T is $\mathbb{Q}$-factorial. The facet $\mathfrak{F}_2$ is divisorial with the elementary divisorial contraction $Y_2 \to T_2 \cong Y_1$ and the only possible extremal contraction of Y_1/T is its Mori fibration. All other intermediate facets $\mathfrak{F}_i, 3 \leq i \leq m$, are flopping. The only slt wlc models of $(X/Z, B)$ are $(Y_i/Z, B_i), 2 \leq i \leq m$, limiting from the countries $\mathfrak{C}_i$, and they are log flops of $(Y_2/Z, B_2)$, directed by a polarization on Y_i/T. More precisely, those flops are composition of elementary ones:

$$Y_2 \dashrightarrow \cdots \dashrightarrow Y_i/T.$$

The total birational transformation of fibrations:

$$\begin{array}{ccccccccc}
Y_1 & \longleftarrow & Y_2 \dashrightarrow \cdots \dashrightarrow & Y_i & \dashrightarrow \cdots \dashrightarrow & Y_{m-1} & \dashrightarrow & Y_m \\
\downarrow & & & & & & & \downarrow \\
 & & & & & & & T_{m+1} \\
 & & & & & \swarrow_{\not\cong} & & \\
T_1 & \xrightarrow{\cong} & & T/Z. & & & &
\end{array}$$

Fibering C or 2C. (The Sarkisov link of type II.) $T_1 \cong T_{m+1} \cong T, m \geq 3$. The base T is $\mathbb{Q}$-factorial. Facets $\mathfrak{F}_2, \mathfrak{F}_m$ are divisorial with elementary divisorial contractions $Y_2 \to T_2 \cong Y_1, Y_{m-1} \to T_m \cong Y_m$, respectively. The only possible extremal contractions of $Y_1, Y_m/T$ are their Mori fibrations. All other intermediate facets $\mathfrak{F}_i, 3 \leq i \leq m-1$, are flopping. The only slt wlc models of $(X/Z, B)$ are $(Y_i/Z, B_i), 2 \leq i \leq m-1$, which are limiting from the countries $\mathfrak{C}_i$, and they are log flops of $(Y_2/Z, B_2)$, directed by a polarization on Y_i/T. More precisely, those flops are composition of elementary ones:

$$Y_2 \dashrightarrow \cdots \dashrightarrow Y_i/T.$$

The total birational transformation of fibrations:

$$\begin{array}{ccccccccc}
Y_1 & \longleftarrow & Y_2 \dashrightarrow \cdots \dashrightarrow & Y_i & \dashrightarrow \cdots \dashrightarrow & Y_{m-1} & \longrightarrow & Y_m \\
\downarrow & & & & & & & \downarrow \\
T_1 & & & & & & & T_{m+1} \\
 & \searrow^{\cong} & & & & \swarrow_{\cong} & & \\
 & & & T/Z. & & & &
\end{array}$$

Internal, or birational, or 3. The ridge $\mathfrak{R}$ is *internal* if $\operatorname{Int}\mathfrak{R} \subset \operatorname{Int}\mathfrak{N}_S$. In this case, as for internal facets, all wlc models $(Y/Z, B_Y^{\log})$ are klt with $B_Y^{\log} = B_Y$ and of general type with the lc model $(T/Z, B_T) = (Y_{\text{lcm}}/Z, B_{\text{lcm}})$. The model T/Z of X/Z depends only on $\mathfrak{R}$. The closure $\overline{\mathfrak{R}}$ is a facet of $m \ge 3$ (neighboring) facets $\mathfrak{F}_1, \ldots, \mathfrak{F}_m$ in $\mathfrak{N}_S$, and a ridge of m countries $\mathfrak{C}_1, \ldots, \mathfrak{C}_m$; $\mathfrak{C}_i$ has (closed) facets $\overline{\mathfrak{F}}_i, \overline{\mathfrak{F}}_{i+1}, m+1 = 1$ (see Diagrams 6-8 below). Any two subsequent facets $\mathfrak{F}_i, \mathfrak{F}_{i+1}$ are strictly in a half-plane. All facets are birational: flopping or divisorial. Let $T_i, Y_i, 1 \le i \le m$, be lc models of the birational facets $\mathfrak{F}_i$ and of countries $\mathfrak{C}_i$, respectively. If there is a divisorial facet, we suppose that $\mathfrak{F}_2$ is divisorial with contraction $Y_2 \to T_2 \cong Y_1$ after shifting indices i.

Any variety Y_i is $\mathbb{Q}$-factorial, and T_i is $\mathbb{Q}$-factorial if and only if $\mathfrak{F}_i$ is divisorial. The base T is $\mathbb{Q}$-factorial if and only if there are two subsequent divisorial facets $\mathfrak{F}_1, \mathfrak{F}_2$ with contractions $Y_2 \to T_2 \cong Y_1 \to T \cong T_1 \cong Y_m$.

All varieties $Y_i/T_i, T_{i+1}/T$ are FT extremal or identical, and Y_i/T are FT with $\rho(Y_i/T) \le 2$. The pairs $(T_i/Z, B_{T_i}), (T/Z, B_T)$ are the nonslt wlc models of $(X/Z, B)$.

Birational A or 3A. The Weil-Picard number $\dim_{\mathbb{R}} \mathrm{N}^1(T/T) = 2$ all facets $\mathfrak{F}_i$ are flopping, and all contractions $Y_i/T_i, T_{i+1}, T_i/T$ respectively elementary and extremal small. The pairs $(Y_i/Z, B_i)$, $B_i = B_{Y_i}$, are the only slt wlc models of $(X/Z, B)$. The models T_i and base T are not $\mathbb{Q}$-factorial. There exists a sequence of elementary log flops in the facets $\mathfrak{F}_i$:

$$Y_1 \dashrightarrow Y_2 \dashrightarrow \cdots \dashrightarrow Y_m \dashrightarrow Y_1/T.$$

A subsequence $Y_i \dashrightarrow \cdots \dashrightarrow Y_j$ of flops is directed with respect to a polarization on Y_j/T if the flopping facets $\mathfrak{F}_l$ are strictly in a half-plane. In general, the configuration of facets and countries is not symmetric, in particular, facets are not coplanar. However two facets $\mathfrak{F}_i, \mathfrak{F}_j$ are coplanar if and only if $T_i \dashrightarrow T_j$ is a generalized extremal and directed log flop$/T$ with respect to their polarizations. Otherwise the span of $\mathfrak{F}_i$ intersects $\mathfrak{C}_j$ and $T_i \dashrightarrow Y_j$ is a generalized non-extremal and directed log flop$/T$ with respect to a polarization. An equation $p(C_i, B) = 0$ for the closure $\overline{\mathfrak{F}}_i$ in $\overline{\mathfrak{C}}_i, \overline{\mathfrak{C}}_{i-1}, 1 - 1 = m$, can be given by a curve C_i on a rather high model V/Z of X/Z. This allows us to find coplanar facets in terms of linear extensions for functions $p(C_i, B')$ from certain countries. The commuting, centrally symmetric, case is possible only for $m = 4$ with coplanar pairs $\mathfrak{F}_1, \mathfrak{F}_3$ and $\mathfrak{F}_2, \mathfrak{F}_4$; both flopping loci are disjoint. See Diagram 6 below:

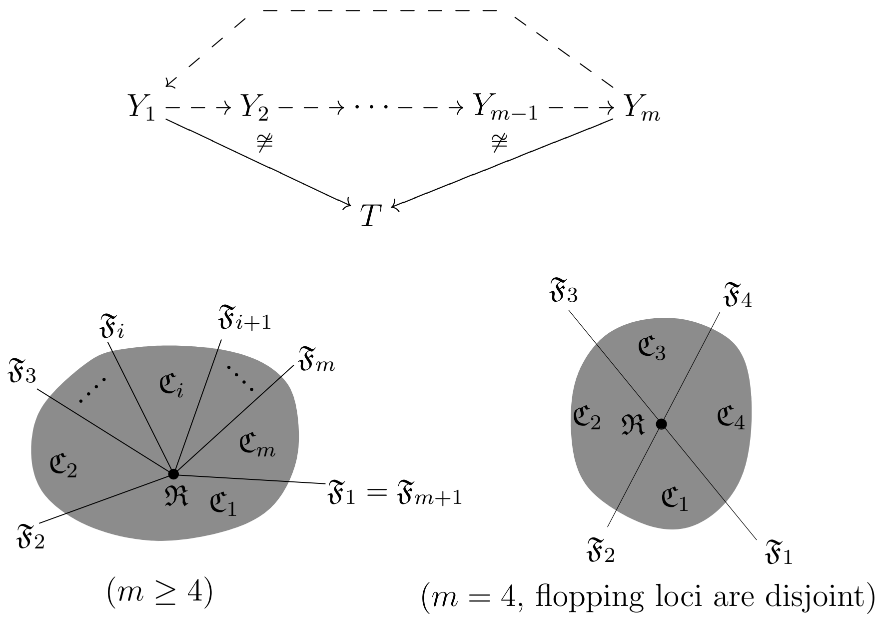
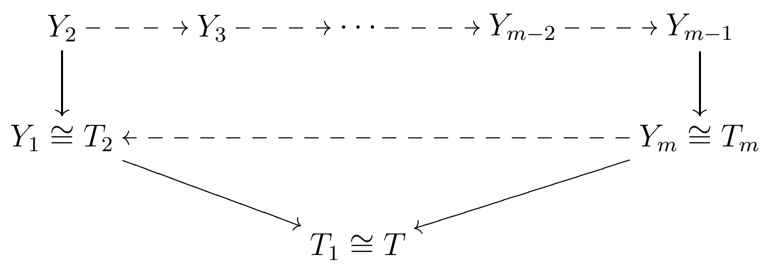


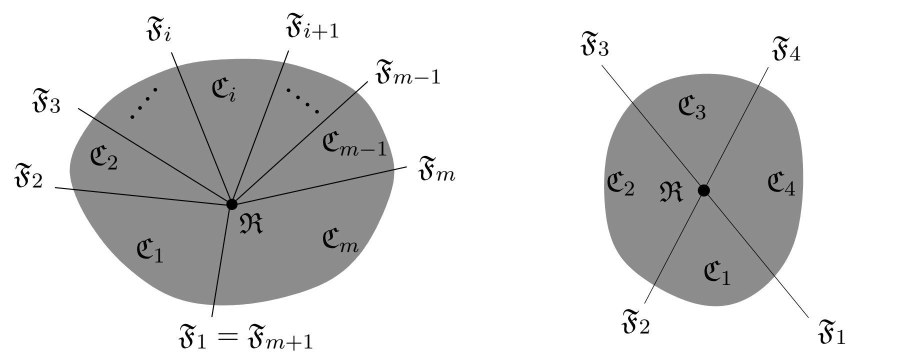
Diagram 6

Birational B or 3B. The Weil-Picard number $\dim_{\mathbb{R}} \mathrm{N}^1(T/T) = 1$, and the only divisorial facets are $\mathfrak{F}_2, \mathfrak{F}_m$ with elementary divisorial contractions $Y_2 \to T_2 \cong Y_1, Y_{m-1} \to T_m \cong Y_m$ of a unique prime b-divisor D_2. All other facets $\mathfrak{F}_i$ are flopping with elementary small contractions $Y_{i-1}, Y_i/T_i, i = 1$ with $1 - 1 = m$ or $3 \leq i \leq m-1$, and with extremal divisorial contractions T_i/T, $3 \leq i \leq m-1$. The pairs $(Y_i/Z, B_i)$, $B_i = B_{Y_i}$, $2 \leq i \leq m-1$, are the only slt wlc models of $(X/Z, B)$. The models T_2, T_m are $\mathbb{Q}$-factorial. All other models T_i and the base T are not $\mathbb{Q}$-factorial. There exists a sequence of an elementary divisorial extraction, elementary log flops in the facets $\mathfrak{F}_1, \mathfrak{F}_i, 3 \leq i \leq m-1$, and of an elementary divisorial contraction:

$$Y_1 \leftarrow Y_2 \dashrightarrow \cdots \dashrightarrow Y_{m-1} \to Y_m \dashrightarrow Y_1/T.$$

Any subsequence $Y_i \dashrightarrow \cdots \dashrightarrow Y_j$ of flops is directed with respect to a polarization on Y_j/T, and the flopping facets $\mathfrak{F}_l$ are strictly in a half-plane. Moreover, the facets $\mathfrak{F}_i, 2 \leq i \leq m$, are in a half-plane and strictly exactly when $\mathfrak{F}_2, \mathfrak{F}_m$ are not coplanar. In general, the configuration of facets and countries is not symmetric, in particular, pairs of facets $\mathfrak{F}_1, \mathfrak{F}_j$ and $\mathfrak{F}_2, \mathfrak{F}_m$ are not coplanar. However, two facets $\mathfrak{F}_1, \mathfrak{F}_j$ are coplanar if and only if $T_j \to T_1 \cong T$ is a generalized divisorial contraction, that is, for $B' \in \mathfrak{F}_1$, $K_{T_j} + B'_{T_j}$ is negative on T_j/T, and





$(m \geq 4)$ $(m = 4, \text{flopping locus} \cap D_2 = \emptyset)$

Diagram 7

$K_T + B'_T$ is $\mathbb{R}$-Cartier. If $\mathfrak{F}_1, F_j, 3 \leq j \leq m-1$, are not coplanar and the span of $\mathfrak{F}_j$ intersects $\mathfrak{C}_1$ (respectively $\mathfrak{C}_m$), then $T_j \dashrightarrow T_2$ (respectively $T_j \dashrightarrow T_m$) is a generalized extremal and directed log flop$/T$ with respect to a polarization. The facets $\mathfrak{F}_2, \mathfrak{F}_m$ are coplanar if and only if the flopping locus of $Y_m \dashrightarrow Y_1$ does not contain $\operatorname{center}_{Y_1} D_2$. This is possible only for $m \geq 4$. Moreover, the commuting, centrally symmetric, case is possible only for $m = 4$ with coplanar pairs $\mathfrak{F}_1, \mathfrak{F}_3$ and $\mathfrak{F}_2, \mathfrak{F}_4$; the flopping locus is disjoint from D_2. An equation $p(C_i, B) = 0$ for the closure $\overline{\mathfrak{F}}_i$ in $\overline{\mathfrak{C}}_i, \overline{\mathfrak{C}}_{i-1}$, can be given by a curve C_i on a rather high model V/Z of X/Z. For closures $\overline{\mathfrak{F}}_2, \overline{\mathfrak{F}}_m$ respectively in $\overline{\mathfrak{C}}_1, \overline{\mathfrak{C}}_m$, the equations $e(D_2, B) = 0$ are given by the contractible divisor D_2. This allows us to find coplanar facets in terms of linear extensions for functions $p(C_i, B'), e(D_2, B')$ from certain countries.

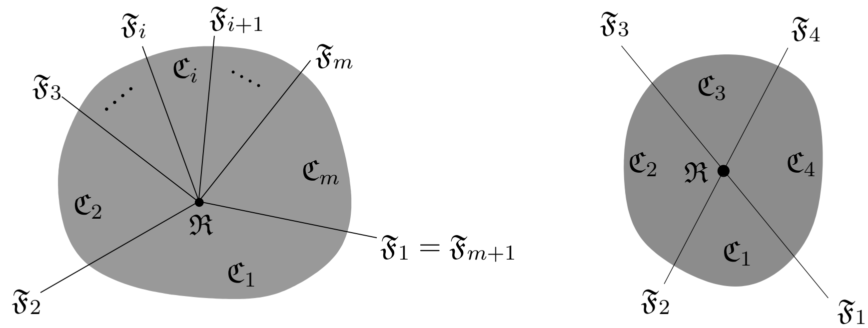
Birational C or 3C. The Weil-Picard number $\dim_{\mathbb{R}} \mathrm{N}^1(T/T) = 0$, $m \geq 4$, the only divisorial facets are $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_m$ with elementary divisorial contractions $Y_3 \to T_3 \cong Y_2 \to T_2 \cong Y_1 \cong T_1 \cong T$, $Y_{m-1} \to T_m \cong Y_m \to Y_1$ of two prime divisors D_3, D_2, respectively. The other contractions $T_i/T, 4 \leq i \leq m-1$, are extremal divisorial with both exceptional divisors D_2, D_3. The contractions $Y_{i-1}, Y_i/T_i$ corresponding to $\mathfrak{F}_i, 4 \leq$

$i \leq m-1$, are elementary small. The pairs $(Y_i/Z, B_i)$, $B_i = B_{Y_i}$, $3 \leq i \leq m-1$, are the only slt wlc models of $(X/Z, B)$. The models $T_3, T_m, T \cong T_1 \cong T_2$ are $\mathbb{Q}$-factorial. All other models $T_i, 4 \leq i \leq m-1$, are not $\mathbb{Q}$-factorial. There exists a sequence of elementary divisorial extractions, elementary log flops in the facets $\mathfrak{F}_i, 4 \leq i \leq m-1$, and of elementary divisorial contractions:

$$Y_1 \leftarrow Y_2 \leftarrow Y_3 \dashrightarrow \cdots \dashrightarrow Y_{m-1} \rightarrow Y_m \rightarrow Y_1/T.$$

Any subsequence $Y_i \dashrightarrow \cdots \dashrightarrow Y_j$ of flops is directed with respect to a polarization on Y_j/T, and the flopping facets $\mathfrak{F}_l$ are strictly in a half-plane. Moreover, the facets $\mathfrak{F}_i, 3 \leq i \leq m+1 = 1$, and $2 \leq i \leq m$, respectively, are in a half-plane and strictly exactly when $\mathfrak{F}_1, \mathfrak{F}_3$ and $\mathfrak{F}_2, \mathfrak{F}_m$, respectively, are not coplanar. In general, the configuration of facets and countries is not symmetric, in particular, pairs of facets $\mathfrak{F}_1, \mathfrak{F}_3$ and $\mathfrak{F}_2, \mathfrak{F}_m$ are not coplanar. However, pairs $\mathfrak{F}_1, \mathfrak{F}_3$ and $\mathfrak{F}_2, \mathfrak{F}_m$ are coplanar if and only if respectively $\operatorname{center}_T D_3 \not\subset \operatorname{center}_T D_2$ and $\operatorname{center}_T D_2 \not\subset \operatorname{center}_T D_3$. Each case or both cases simultaneously can occur for $m \geq 5$. However, for $m = 4$, both pairs are coplanar or none of the pairs are coplanar. The former is possible only for the commuting centrally symmetric case with disjoint contractible divisors D_2, D_3. An equation $p(C_i, B) = 0$ for the closure $\overline{\mathfrak{F}}_i$ in $\overline{\mathfrak{C}}_i, \overline{\mathfrak{C}}_{i-1}$, can be given by a curve C_i on a rather high model V/Z of X/Z. For closures $\overline{\mathfrak{F}}_2, \overline{\mathfrak{F}}_m$ respectively in $\overline{\mathfrak{C}}_1, \overline{\mathfrak{C}}_m$, the equations $e(D_2, B) = 0$ are given by the contractible divisor D_2. For closures $\overline{\mathfrak{F}}_3, \overline{\mathfrak{F}}_1$ respectively in $\overline{\mathfrak{C}}_1, \overline{\mathfrak{C}}_2$ the equations $e(D_3, B) = 0$ are given by the contractible divisor D_3. This allows to find coplanar facets in terms of linear extensions for functions $e(D_2, B'), e(D_3, B')$ from certain countries.

$$\begin{array}{ccccccc}
Y_3 & \dashrightarrow & Y_4 & \dashrightarrow \cdots \dashrightarrow & Y_{m-2} & \dashrightarrow & Y_{m-1} \\
\downarrow & & & & & & \downarrow \\
Y_2 \cong T_3 & & & & & & Y_m \cong T_m \\
& \searrow & & & & \swarrow & \\
& & & Y_1 \cong T_1 \cong T_2 \cong T & & &
\end{array}$$



($m = 4$, D_2, D_3 are disjoint.)

DIAGRAM 8

Theorem 5.11. *Under the generality conditions on $(X/Z, S)$, let $\mathfrak{R}$ be a ridge of the geography $\mathfrak{N}_S$. Then it is one of the above types: cube bordering, fibering, or birational, and behaves accordingly. (See Diagram 9 below.)*

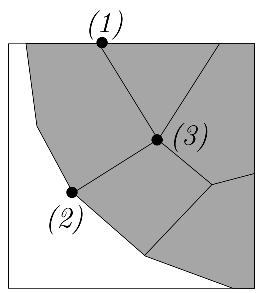


DIAGRAM 9

(1) Cube bordering type : $\mathfrak{R} \subset \partial\mathfrak{B}_S$;
(2) (Mori) Fibering type : $\operatorname{Int}\mathfrak{R} \subset \mathfrak{S}_S \setminus \partial\mathfrak{B}_S$;
(3) Birational type : $\operatorname{Int}\mathfrak{R} \subset \operatorname{Int}\mathfrak{N}_S$.

Proof. A bordering ridge $\mathfrak{R}$ of $\mathfrak{N}_S$ has the cube bordering or fibering type by definition, Proposition 5.2 and Corollary 5.6. Indeed, if $\mathfrak{R} \subset \partial\mathfrak{B}_S$, it is cube bordering. Otherwise, $\operatorname{Int}\mathfrak{R} \subset \mathfrak{S}_S \setminus \partial\mathfrak{B}_S$ and $\mathfrak{R}$ is fibering.

Cube bordering type: $\mathfrak{R}$ satisfies at least one equation $b_i = 0$ or 1. If $\mathfrak{R}$ satisfies another equation $b_j = 0$ or 1, it lies in the ridge of $\mathfrak{B}_S$ given by these equations. Otherwise, $\mathfrak{R} \subset \partial\mathfrak{N}_S \cap \mathfrak{F}$ for a facet $\mathfrak{F}$ of $\mathfrak{B}_S$ which satisfies the equation $b_i = 0$ or 1.

For the boundaries $B \in \operatorname{Int}\mathfrak{R}$ of the other type ridges, B is internal: $B \in \operatorname{Int}\mathfrak{B}_S$. As in the proof of Theorem 5.9, any wlc model $(Y/Z, B_Y^{\log})$ of $(X/Z, B)$ is klt with big $B_Y^{\log} = B_Y$ and has a lc contraction $Y \to$

$T = X_{\rm lcm}$. If Y/T is projective, it is FT by Lemma 3.1 with $\rho(Y/T) \le 2$ and with at most two extremal contractions/T by Lemma 5.8. As we see from the following, $\rho(Y/T) = 2$ holds and there exist two extremal contractions of Y over T exactly when $(Y/Z, B_Y)$ is a slt wlc model of $(X/Z, B)$. In particular, $d \ge 2$. Such a model $(Y/Z, B_Y)$ exists because $B \in \mathfrak{N}_S$ and it satisfies the assumptions of Lemma 5.8. The slt model can be constructed by the slt LMMP from the above initial model.

Fibering type. Suppose that $\mathfrak{R}$ is fibering. Then by the open property of classes and the last statement in Theorem 2.4, there exist required countries $\mathfrak{C}_i$ and facets $\mathfrak{F}_j$. Most of stated properties follow from Theorem 5.9 and standard facts.

Type 2A. The pair $(Y_1/Z, B_1)$ is a wlc model of $(X/Z, B)$ by Lemma 2.9, limiting from $\mathfrak{C}_1$. By definition it has a Mori log fibration $Y_1 \to T_1/T$ and $T_1 \ne T$. Thus $\rho(Y/T) \ge 2$. On the other hand, since there exists a slt wlc model of $(X/Z, B)$ over Y_1, this model is $(Y_1/Z, B_1)$ and $\rho(Y_1/T) = 2$ by Lemma 5.8. Similarly, $(Y_m/Z, B_m)$ is a slt wlc model of $(X/Z, B)$ too. Moreover, all intermediate facets $\mathfrak{F}_j, 2 \le j \le m$, are flopping and the models $(Y_i/Z, B_i), 1 \le i \le m$, are the only slt wlc models of $(X/Z, B)$. If $m = 1$, Y_1/T has two Mori log fibrations $Y_1 \to T_1, T_2/T$, which are the only extremal contractions/T, and by Corollary 5.10, $(Y_1/Z, B_1)$ is the only wlc model of $(X/Z, B)$. In the last case, the facets $\mathfrak{F}_1, \mathfrak{F}_2$ are noncoplanar by Corollary 2.12. For $m \ge 2$, another extremal contraction $Y_1 \to T_2/T$ is birational and flopping by Corollary 5.10. Indeed, if it is divisorial, $Y_1 \to T_2 = Y_2$ and Y_2/T has a Mori log fibration $Y_2/T_3 = T$, a contradiction. Similarly, all other intermediate facets are flopping. Now the description of all wlc models can be obtained from Corollary 4.3 and Theorem 5.9.

Type 2B. Again the pair $(Y_1/Z, B_1)$ is a wlc model of $(X/Z, B)$, limiting from $\mathfrak{C}_1$. By definition, it has a Mori log fibration $Y_1 \to T_1 \cong T$. Thus $\rho(Y_1/T) = 1$ and $Y_1 \to T_1 \cong T$ is the only extremal contraction for Y_1/T. By Theorem 5.9, $\mathfrak{F}_2$ is divisorial. Otherwise, Y_1 has another extremal contraction to T_2/T. In particular, $m \ge 2$. So, the pair $(Y_2/Z, B_2)$ is a wlc model of $(X/Z, B)$, limiting from $\mathfrak{C}_2$, with an elementary divisorial contraction $Y_2 \to T_2 \cong Y_1$. Again $\rho(Y_2/T) = 2$ and $(Y_2/Z, B_2)$ is a slt model of $(X/Z, B)$. Moreover, all intermediate facets $\mathfrak{F}_j, 3 \le j \le m$, are flopping and the models $(Y_i/Z, B_i), 2 \le i \le m$ are the only slt wlc models of $(X/Z, B)$. If $m = 2$, Y_2/T has exactly one more contraction/T, the Mori log fibration $Y_2/T_3 \ncong T$, and by Corollary 5.10, $(Y_2/Z, B_2)$ is the only slt wlc model of $(X/Z, B)$. For $m \ge 3$, another extremal contraction $Y_2 \to T_3/T$ is birational and flopping by Corollary 5.10. Indeed, if it is divisorial, $Y_2 \to T_3 \cong Y_3$ and Y_3/T has a Mori log fibration $Y_3/T_4 \cong T$, a contradiction. Similarly, all other intermediate facets are flopping. As above the description of all wlc models can be obtained from Corollary 4.3 and Theorem 5.9.

Type 2C. As in type 2B, $\mathfrak{F}_2$ is divisorial and $(Y_2/Z, B_2)$ is a slt model of $(X/Z, B)$ with an elementary divisorial contraction $Y_2 \to T_2 \cong Y_1$. Similarly, $\mathfrak{F}_m$ is divisorial and $(Y_{m-1}/Z, B_{m-1})$ is a slt model of $(X/Z, B)$ with an elementary divisorial contraction $Y_{m-1} \to T_m \cong Y_m$ and the subsequent Mori log fibration $Y_m/T_{m+1} \cong T$. In particular, $m \geq 3$. As above all intermediate facets $\mathfrak{F}_j, 3 \leq j \leq m-1$, are flopping and the models $(Y_i/Z, B_i), 2 \leq i \leq m-1$ are the only slt wlc models of $(X/Z, B)$. The description of all wlc models can be obtained from Corollary 4.3 and Theorem 5.9.

Note that flopping facets only possible for $d \geq 3$.

The properties of equations given by $p(C_1, B'), p(C_{m+1}, B')$ and $e(D_2, B')$, $e(D_m, B')$ follow from Theorem 5.9. Moreover, if C_1 is a sufficiently general curve of a Mori log fibration Y_1/T_1 (from a rather high model V/Z of X/Z), it is not contracted on all models $Y_i, T_i, 2 \leq i \leq m$. Hence $p(C_1, B') > 0$ on $\mathfrak{C}_i, 1 \leq i \leq m, \mathfrak{F}_j, 2 \leq j \leq m$, and ≥ 0 on $\overline{\mathfrak{F}}_{m+1}$. More precisely, $p(C_1, B') = 0$ on $\overline{\mathfrak{F}}_{m+1}$ if and only if the birational image of C_1 belongs to a fiber of Y_m/T_{m+1}, that is, Y_1/T_1 factors through the last fibration. The similar holds for a general curve C_{m+1} on Y_m/T_{m+1}. Thus the fibrations $Y_1/T_1, Y_m/T_{m+1}$ are square birational to each other if and only if $T_1, T_{m+1}/T$ are birational (the same general curves in fibers), or $p(C_1, \mathfrak{F}_{m+1}) = p(C_{m+1}, \mathfrak{F}_1) = 0$ for general C_1, C_{m+1}. Moreover, $Y_1/T_1, Y_m/T_{m+1}$ are generically isomorphic if and only if the facets $\mathfrak{F}_1, \mathfrak{F}_{m+1}$ are coplanar. Indeed, if the isomorphism holds then $C_1 = C_{m+1}$ and the natural modification $Y_1 \dashrightarrow Y_m$ does not touch the general fiber. Thus $p(C_1, B')$ is linear on $\mathfrak{N}_S$ near $\mathfrak{R}$ and $\mathfrak{F}_1, \mathfrak{F}_{m+1}$ are coplanar. Conversely, if $\mathfrak{F}_1, \mathfrak{F}_{m+1}$ are coplanar, the convexity of Proposition 2.10 implies that $p(C_1, \mathfrak{F}_{m+1}) = p(C_{m+1}, \mathfrak{F}_1) = 0$ and $p(C_1, B')$ is linear on $\mathfrak{N}_S$ near $\mathfrak{R}$. Thus the modifications $Y_1 \dashrightarrow Y_2 \dashrightarrow \cdots \dashrightarrow Y_m$ do not touch C_1 and $Y_1/T_1, Y_m/T_{m+1}$ are generically isomorphic.

Birational type. Suppose that $\mathfrak{R}$ is birational. Then by the open property of classes and the last statement in Theorem 2.4, there exist required countries $\mathfrak{C}_i$ and facets $\mathfrak{F}_i$. Most of stated properties follow from Theorem 5.9 and standard facts. We discuss the others below.

In this case as for internal facets, all wlc models $(Y/Z, B_Y^{\log})$ are klt with $B_Y^{\log} = B_Y$, but now of general type with the lc model $(T/Z, B_T) = (X_{\mathrm{lcm}}/Z, B_{\mathrm{lcm}})$ by our assumptions and Proposition 5.5. The facets $\mathfrak{F}_i, \mathfrak{F}_{i+1}$ are in a half-plane by convexity of $\mathfrak{C}_i$ and strictly by Corollary 2.12. In particular, $m \geq 3$.

Type 3A. Since $\dim_{\mathbb{R}} \mathrm{N}^1(T/T) = 2$ and by Lemma 5.8, for a slt wlc model $(Y/Z, B_Y)$ of $(X/Z, B)$, the contraction Y/T is small with $\rho(Y/T) = 2$ and two extremal contractions/T are also elementary small. As above, for some i, $Y = Y_i$, limiting from a country $\mathfrak{C}_i$ with two flopping facets $\mathfrak{F}_i, \mathfrak{F}_{i+1}$, corresponding to two extremal contractions/T by Theorem 5.9. The rest also follows from the classification of facets in the theorem and Corollaries 4.3, 5.10. Note also, that if two facets $\mathfrak{F}_i, \mathfrak{F}_j$ are coplanar, then $T_i \dashrightarrow T_j$ is a generalized extremal and directed log flop/T with respect to a polarization. Indeed, if the facets are coplanar, the one dimensional subspace of $\mathbb{R}$-Cartier divisors in $\mathrm{N}^1(T_i/T)$ is generated by $B' \in \mathfrak{F}_i$ or $B' \in \mathfrak{F}_j$, and respectively $K_{T_i} + B'_{T_i}$ is ample or antiample/T. The converse follows from definition and the description of all projective rational 1-contractions of Y/T. Similarly, one can treat the case with the span of $\mathfrak{F}_i$ intersecting $\mathfrak{C}_j$. An equation $p(C_i, B) = 0$ for a facet $\mathfrak{F}_i$ in $\overline{\mathfrak{C}}_i, \overline{\mathfrak{C}}_{i-1}$ can be given by a curve C_i on a rather high model V/Z of X/Z which is contracted on T_i but not on Y_i, Y_{i-1}, respectively. The commuting case can be treated with the convexity of Proposition 2.10.

Type 3B. Since $\dim_{\mathbb{R}} \mathrm{N}^1(T/T) = 1$, in particular, T is not $\mathbb{Q}$-factorial, and by Lemma 5.8, for a slt wlc model $(Y/Z, B_Y)$ of $(X/Z, B)$, the contraction Y/T is not small with $\rho(Y/T) = 2$ but two extremal contractions/T can be small. However, applying the D_2-LMMP for a prime divisor D_2 of Y exceptional on T, we obtain a new slt wlc model of $(X/Y, Z)$, say, $(Y_2/Z, B_2)$, limiting from $\mathfrak{C}_2$ and with an elementary divisorial contraction $Y_2 \to T_2 \cong Y_1$ of D_2. Equivalently, $\mathfrak{F}_2$ is divisorial. Thus Y_1 is $\mathbb{Q}$-factorial and $(Y_1/Z, B_1)$ is a nonslt model of $(X/Z, B)$, limiting from $\mathfrak{C}_1$. The second facet $\mathfrak{F}_1$ of $\mathfrak{C}_1$ is flopping by Theorem 5.9, because it corresponds to a small extremal contraction Y_1/T. In particular, D_2 is the only prime divisor of Y exceptional on T. The log flop in $\mathfrak{F}_1$ gives another $\mathbb{Q}$-factorial nonslt model of $(X/Z, B)$: $(Y_m/Z, B_m)$, limiting from $\mathfrak{C}_m$. Again by Theorem 5.9, the facet $\mathfrak{F}_m$ is divisorial with an elementary divisorial extraction $T_m \cong Y_m \leftarrow Y_{m-1}$ of D_2. Moreover, $\mathfrak{F}_2, \mathfrak{F}_m$ are the only divisorial facets, e.g., by Corollary 5.10. The rest follows from the classification of facets and Corollaries 4.3, 5.10. By construction and definition, $e(D_2, B') > 0$ on $\mathfrak{C}_1, \mathfrak{F}_1, \mathfrak{C}_m$ and $= 0$ on the other countries and facets. Thus by the convexity of $e(D_2, B')$, the facets $\mathfrak{F}_i, 2 \le i \le m$, are in a half-plane and strictly exactly when $\mathfrak{F}_2, \mathfrak{F}_m$ are not coplanar. The facets $\mathfrak{F}_2, \mathfrak{F}_m$ are coplanar if and only if the function $e(D_2, B')$ is linear on the half-plane of $\mathfrak{F}_1$, or equivalently, the flop in $\mathfrak{F}_1$ does not touch $\operatorname{center}_{Y_1} D_2$ generically. This is possible only for $m \ge 4$ by

Corollary 2.12. By construction facets $\mathfrak{F}_1, \mathfrak{F}_j$ are coplanar if and only if $T_j \to T_1 \cong T$ is a generalized divisorial contraction of D_2. Otherwise, $T_j \dashrightarrow T_2$ or T_m is a generalized extremal and directed log flop/T with respect to a polarization [Sho5, Proof-Explanation of Corollary 1.9]. Note that the contraction T_j/T is not small, but extremal, and $K_{T_j} + B'_{T_j}$ is negative on T_j/T for $B' \in \mathfrak{C}_1 \cup \mathfrak{F}_1 \cup \mathfrak{C}_m$ when $K_{T_j} + B_{T_j}$ is $\mathbb{R}$-Cartier, or equivalently B' is coplanar with $\mathfrak{F}_j$. In this situation, for $B' \in \mathfrak{C}_1$ (respectively $B' \in \mathfrak{C}_m$), a generalized log flip with respect to B' or a generalized directed log flop of $(T_j/T, B)$ with the polarization $K_{T_2} + B'_{T_2}$ (respectively $K_{T_m} + B'_{T_m}$) is $T_j \dashrightarrow T_2$ (respectively $T_j \dashrightarrow T_m$); it is neither a divisorial contraction or a small flip/flop. For $B' \in \mathfrak{F}_1$, a log flip or a log flop of T_j/T is a generalized divisorial contraction $T_j \to T_1$. The commuting case can be treated as above.

Type 3C. Since $\dim_{\mathbb{R}} \mathrm{N}^1(T/T) = 0$, T is $\mathbb{Q}$-factorial. As in type 2C, we can construct a slt wlc model of $(X/Z, B)$, say, $(Y_3/Z, B_3)$ with $\rho(Y_3/T) = 2$, limiting from $\mathfrak{C}_3$ and with an elementary divisorial contraction $Y_3 \to T_3 = Y_2$ of a prime divisor D_3. Thus $\mathfrak{F}_3$ is divisorial, Y_2 is $\mathbb{Q}$-factorial and $(Y_2/Z, B_2)$ is a limiting from $\mathfrak{C}_2$ nonslt model of $(X/Z, B)$. However, now by the $\mathbb{Q}$-factorial property of T, the second facet $\mathfrak{F}_2$ of $\mathfrak{C}_2$ is also divisorial with an elementary divisorial contraction $Y_2 \to T = T_2 = Y_1$ of a prime divisor D_2. Again by Theorem 5.9 the facets $\mathfrak{F}_1, \mathfrak{F}_m$ are divisorial with elementary divisorial contractions $Y_{m-1} \to T_m = Y_m \to T = T_1 = Y_1$ of divisors D_2, D_3, respectively. In particular, $m \geq 4$. All other facets are flopping. The rest follows from the classification of facets and Corollaries 4.3, 5.10. By construction and definition, $e(D_2, B') > 0$ on $\mathfrak{C}_1, \mathfrak{F}_1, \mathfrak{C}_m$ and 0 on the other countries and facets. Thus by the convexity of $e(D_2, B')$, the facets $\mathfrak{F}_i, 2 \leq i \leq m$, are in a half-plane and strictly exactly when $\mathfrak{F}_2, \mathfrak{F}_m$ are not coplanar. The facets $\mathfrak{F}_2, \mathfrak{F}_m$ are coplanar if and only if the function $e(D_2, B')$ is linear on the half-plane of $\mathfrak{F}_1$, or equivalently, the extraction D_3 in $\mathfrak{F}_1$ does not touch $\mathrm{center}_{Y_1} D_2$ generically, that is, $\mathrm{center}_T D_2 \not\subset \mathrm{center}_T D_3$. Similarly, the facets $\mathfrak{F}_i, 3 \leq i \leq m+1 = 1$, are in a half-plane and strictly exactly when $\mathfrak{F}_1, \mathfrak{F}_3$ are not coplanar. The facets $\mathfrak{F}_1, \mathfrak{F}_3$ are coplanar exactly when $\mathrm{center}_T D_3 \not\subset \mathrm{center}_T D_2$. If D_2, D_3 are disjoint, then $m = 4$ and the commuting case holds with coplanar pairs $\mathfrak{F}_1, \mathfrak{F}_3$ and $\mathfrak{F}_2, \mathfrak{F}_4$. Conversely, if $m = 4$, then Y_3 has two extremal divisorial contractions/T of D_2, D_3. If $D_2 \cap D_3 = \emptyset$, the commuting case holds with the coplanar pairs. Otherwise, the extremal property implies that $\mathrm{center}_T D_2 = \mathrm{center}_T D_3$ and both pairs $\mathfrak{F}_1, \mathfrak{F}_3$ and $\mathfrak{F}_2, \mathfrak{F}_4$ are not coplanar. □

Central model. Let $\mathfrak{R}$ be a fibering or birational ridge of a geography under the generality conditions, $B \in \operatorname{Int}\mathfrak{R}$ be a boundary with a decomposition $B = F + M$, where $F \geq 0$ and $M \geq 0$ is big and $\mathbb{R}$-mobile over the lc model T of $(X/Z, B)$. Such a decomposition always exists and, in applications, the decomposition mob+exc is typical. A klt slt wlc model $(Y/Z, B_Y)$ of $(X/Z, B)$ is *central* with respect to the decomposition $B = F + M$ if $(Y/T, F_Y)$ is a weak log Fano [PrSh, Definition 2.5]. Since $M_Y \equiv -(K_Y + F_Y)/T$, Y depends only on $\mathfrak{R}$ when $M_Y = \operatorname{Mob}(B_Y) = \operatorname{Mob}(-K_Y)$ (and $F_Y = 0$) for the mob+exc decomposition of B_Y. Theorem 5.11 implies the following properties of a central model $(Y/Z, B_Y)$:

(i) Y/T is a $\mathbb{Q}$-factorial FT contraction with $\rho(Y/T) = 2$.

(ii) $(Y/Z, B_Y) = (Y_i/Z, B_i)$, and all other slt wlc models $(Y_j/Z, B_j)$ can be reconstructed from $(Y/Z, B_Y)$ by elementary log flops in both directions:

$$Y_1, Y_2 \text{ or } Y_3 \dashleftarrow \cdots \dashleftarrow Y_i \dashrightarrow \cdots \dashrightarrow Y_{m-1} \text{ or } Y_m;$$

$Y_i \dashrightarrow \cdots \dashrightarrow Y_3, Y_2, Y_1$ and $Y_i \dashrightarrow \cdots \dashrightarrow Y_{m-1}, Y_m$ are sequences of a log flop and log flips with respect to F_{Y_i} (the modifications are optional), with at most one log flop for both chains. In particular, this restores all other divisorial and fibering contractions over T: $Y_3 \to Y_2, Y_2 \to Y_1, Y_{m-1} \to Y_m, Y_m \to T_{m+1}$, $Y_1 \to T_1$ and all other wlc models of $(X/Z, B)$. Thus a modification (link) of Y_1 (respectively Y_2, Y_3) into $Y = Y_{m-1}$ (respectively Y_m) is factored into log antiflips, a log flop (optional) and log flips with respect to F_Y. The pair $(Y/T, F_Y)$ is a log Fano exactly when there are no flops; flops are possible only in dimension $d \geq 3$.

(iii) A prime divisor D of Y is contractible/T by $-(K_Y + F_Y)$ if and only if there exists a $\mathbb{Q}$-factorial wlc model $(Y'/Z, B_{Y'})$ of $(X/Z, B)$ where D is exceptional on Y' and with $a(D, Y', F_{Y'}) = 1 - \operatorname{mult}_D F \geq 0$. In particular, there are no such divisors if, for any $\mathbb{Q}$-factorial wlc model $(Y'/Z, B_{Y'})$ of $(X/Z, B)$, $(Y', F_{Y'})$ is terminal (the last condition can be extended for non $\mathbb{Q}$-Cartier $K_Y' + F_{Y'}$, assuming that $(Y', F_{Y'})$ is terminal when it is terminal for a $\mathbb{Q}$-factorialization).

(iv) If $(Y/Z, B_Y)$ is ε-lc (in codimension ≥ 2), then $(Y/Z, F_Y)$ is also ε-lc (respectively, in codimension ≥ 2). For example, this holds if (X, B) is ε-lc (respectively, in codimension ≥ 2, $\varepsilon \leq 1$ and the components of B are nonexceptional on Y).

Corollary 5.12. *Let $\mathfrak{R}$ be a fibering or birational ridge in $\mathfrak{N}_S$ under the generality assumptions and $B \in \operatorname{Int}\mathfrak{R}$. Then for any decomposition $B = F + M$ as above, $(X/Z, B)$ has a central model $(Y/Z, B_Y)$. For such a decomposition, the model $(Y/Z, B_Y)$ is unique up to a log flop of $(Y/T, F_Y)$ over T. Conversely, any klt weak log Fano $(Y/T, F_Y)$ with the property (i) in Central model above corresponds to such a ridge.*

Proof. To construct a central model one can use the M_Y-MMP$/T$ with any slt wlc model as an initial one. Since M_Y is $\mathbb{R}$-mobile, the modifications are small. Note also that $M_Y \equiv -(K_Y + F_Y)$ by definition of T: $K_Y + B_Y \equiv 0/T$. Thus a required model is defined up to a log flop of $(Y/T, F_Y)$.

Conversely, take $Z = T$, $B = F_Y + M$ where $M \geq 0$ and $\sim_{\mathbb{R}} -(K_Y + F_Y)$ ($\mathbb{R}$-complement). The generality conditions can be satisfied near B according to the big and the semiample property of M. Note that, for $S = \operatorname{Supp} B$, (Y, S) is possibly non slt. Thus to apply Theorem 5.11, we can consider its local version near B or we can replace (Y, S) by an appropriate log resolution (X, S). The latter uses the klt property of (Y, B): for some $D \in \mathfrak{B}_S$, $\operatorname{Supp} D_Y = S_Y$ and $(Y/T, B = D_Y)$ is a slt wlc model of $(X/T, D)$ corresponding to a ridge. □

The following finiteness is a key to birational factorizations.

Corollary 5.13. *Let ε be a positive real number. Suppose that* [PrSh, Conjecture 1.1] *holds for ε-lc varieties of dimension d. Then the family of central models Y/T and thus their links in dimension d for ε-lc slt models $(Y/Z, B_Y)$ are bounded over $k(T)$.*

Proof. Immediate by Corollary 5.12 and by Conjecture 1.1 in [PrSh]. For links, see Section 6 below. □

Example 5.14. All possible central models with links in dimension 2 with $Z = \mathrm{pt}, F = 0, k = \overline{k}$, e.g., $k = \mathbb{C}$, and terminal singularities i.e., nonsingular, (see links in Mori category in Section 6 below) are as follows:

2A: $\mathbb{P}^1 \leftarrow \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1/T = \mathrm{pt}$, a nonsingular quadric;

2B: $\mathbb{P}^2 \leftarrow \mathbb{F}_1 \to \mathbb{P}^1/T = \mathrm{pt}$, a blowup of the plane $\mathbb{P}^2$;

2C: $\mathbb{F} \leftarrow \operatorname{Bl}\mathbb{F} \to \mathbb{F}'/T = C$, the standard fiber modification, where $\mathbb{F}, \mathbb{F}'$ are $\mathbb{P}^1$-fibrations over a nonsingular curve C and $\operatorname{Bl}\mathbb{F}$ is a blowup of $\mathbb{F}$ with blowdown of another curve to $\mathbb{F}'$. Note that the last model is $\cong \mathbb{P}^1/k(C)$.

There are much more such links (≥ 70) and central models in dimension 3 in the nonsingular case, and thousands are expected for terminal singularities.

Links. Let $Y_1/T_1, Y_2/T_2$ be two extremal contractions (possibly birational) of $\mathbb{Q}$-factorial projective/T_1, T_2 birationally equivalent varieties Y_1, Y_2. Their *link* $Y_1 \dashrightarrow Y_2$ is a composition (chain) of extremal divisorial extractions, contractions and small modifications (e.g., flips, flops or antiflips) over another variety T with contractions $T_1, T_2 \to T$:

$$
\begin{array}{ccccc}
Y_1 & \dashrightarrow & \cdots & \dashrightarrow & Y_2 \\
\downarrow & & & & \downarrow \\
T_1 & & & & T_2 \\
& \searrow & & \swarrow & \\
& & T. & &
\end{array}
$$

Such a link from Y_1/T_1 into Y_2/T_2 is *elementary* if Y_1, Y_2 and all intermediate models Y/T in the link are $\mathbb{Q}$-factorial FT with $\rho(Y/T) \le 2$, and modifications are elementary in the following order: (at most 2) divisorial extractions, small modifications, (at most 2) divisorial contractions; all modifications are optional. The sequences of modifications of fibering 2A-C and birational 3A-C ridges are elementary.

If, in addition, $Y_1/T_1, Y_2/T_2$ are Mori fibrations and all the models in the link are terminal, it is one of Sarkisov links of types I,II,III, or IV [Cor1]. If $(Y_1/T_1, B_1), (Y_m/T_{m+1}, B_m)$ are two wlc models of $(X/Z, B)$ for B in a fibering ridge $\mathfrak{R}$ connected by generalized log flops as in types 2A-C, then these log flops give an elementary link from Y_1/T_1 into Y_m/T_{m+1} by Theorem 5.11. We can take also their inverses but the only new will be for nonsymmetric type 2B. They are Sarkisov links when all varieties Y_i are terminal. Those links will be referred to as *cte* (*centralized terminal elementary*) if they have a terminal *central* model Y/T with $F_T = 0$, in particular, $-K_Y \equiv B_Y/T$ is big and $\mathbb{R}$-mobile itself/T on any wlc model $(Y/Z, B_Y)$ of $(X/Z, B)$. In addition, Y should be terminal for all $\mathbb{Q}$-factorial wlc models $(Y/Z, B_Y)$ of $(X/Z, B)$. E.g., this is true if a slt model $(Y_i/Z, B_i)$ in the link is terminal and $-K_{Y_i}$ or B_i is big/T on any divisor of Y_i (cf. the mobility below). Conversely, the Sarkisov links are cte. The terminal and elementary properties follow from definition. The central property also follows from definition. More precisely, if the first small modification is a flip or a flop, Y_1/T is a central model. Otherwise, a central model Y_i/T is a result of maximal chain of antiflips; if the next modification is a flop, Y_i/T is a weak Fano fibration, otherwise, Y_i/T is a Fano fibration. In total, the link is factored into a divisorial extraction, antiflips, a flop, flips and a divisorial contraction (all factors are optional; cf. (ii) in Central model). Flops are possible only in dimension $d \geq 3$.

The contractions Y/T in a cte or Sarkisov link satisfy the *mobility*$/T$: $-K_Y$ is $\mathbb{R}$-mobile and big$/T$ in codimension 1. The latter means that $-K_Y$ is big$/T$ and big$/T$ on any divisor of Y. Thus

$$K_Y^2 \geq 0/T \text{ as a cycle modulo} \equiv \text{ and even } > 0/T \text{ for } \dim_k X/T \geq 2.$$

For $\dim_k X/T = 1$, $-K_Y$ is nonzero effective$/T$ modulo $\sim_{\mathbb{R}}$.

The boundedness of Corollary 5.13 with $\varepsilon = 1$ holds also for the Sarkisov links (cf. [BCZ, Proposition 6]).

Corollary 6.1. *(Global) Cte links with* $T = \mathrm{pt}$ *are bounded in dimension* $d = 3$*.*

Proof. Immediate by Corollary 5.13 with $\varepsilon = 1$ and [KMMT]. □

Mori category. Let X/Z be a variety. Its Mori category is the category of minimal resulting models of X/Z with birational transformations as morphisms. So, an object is a projective terminal $\mathbb{Q}$-factorial model Y/Z of X/Z with nef K_Y, a minimal model, or a projective terminal $\mathbb{Q}$-factorial model Y/Z of X/Z equipped with a Mori fibration $Y \to T/Z$. A factorization of morphisms in the case of minimal models is known (see Corollary 5.10 above). A factorization between Mori fibrations was suggested by Sarkisov.

Theorem 6.2. *A morphism between Mori fibrations can be factorized into cte links, or equivalently, into Sarkisov links.*

However, this is not a factorization of the genuine Sarkisov program: the Sarkisov's invariants may not decrease. So, this is not a solution to the Sarkisov program.

Proof. Follows from a linkage [IskSh, Example 2.11] after a perturbation. For simplicity, we assume the absolute case: $Z = \mathrm{pt} = \operatorname{Spec} k$, and omit the notation $/Z$ below. We suppose also that the dimension $d = \dim Y \geq 2$. So, let

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ T & & T' \end{array}$$

be a birational transformation for Mori fibrations $Y/T, Y'/T'$.

We can convert each Mori fibration Y/T into a polarized one $(Y/T, B_Y)$ with an ample boundary B_Y. Take a sufficiently (very) ample divisor H on T and put $B_Y = D/N$ where D is a generic reduced irreducible divisor in a linear system $|N(-K_Y + g^*H)|$, $N \gg 0$, and $g\colon Y \to T$. By construction (Y, B_Y) is a terminal (in codimension ≥ 2) slt model with the lc contraction g, the Mori fibration. Similarly, Y'/T' has a polarization $B'_{Y'}$ with $B'_{Y'} = D'/N'$. We can assume also that $(Y, B_Y), (Y', B'_{Y'})$ have a common log resolution X on which (the birational transformations of) D, D' are big, mobile (even free) and disjoint.

According to [IskSh, Example 2.11] and Corollary 5.10 (see also factorization for a birational transformation of wlc models in Section 4), there exists a factorization of $Y/T \dashrightarrow Y'/T'$ into links. The intermediate models (Y_i, B_i) in the factorization are slt wlc models of (X, B_i) for $B_i \in \mathfrak{P}_i$, a class (an open interval) in the separatrix $\mathfrak{S}_S, S = D + D'$. (Since each B_i is $\mathbb{R}$-mobile, we identify B_i with its birational transform $(B_i)_{Y_i}$ on Y_i by writing the same B_i.) The union $\cup\overline{\mathfrak{P}}_i$ gives a path from B to B' in $\mathfrak{S}_S$. By construction each Y_i has only terminal singularities and has a Mori fibration Y_i/T_i (possibly after a log flop of (Y_i, B_i)) by Corollary 5.6. We can suppose that $Y_1/T_1 = Y/T$ for the first interval $\mathfrak{P}_1$ and $Y'/T' = Y_l/T_l$ for the last interval $\mathfrak{P}_l$. See Diagram 10 below. A link from Y_i/T_i into Y_{i+1}/T_{i+1} corresponds to the point V_i of intersection $\overline{\mathfrak{P}}_i \cap \overline{\mathfrak{P}}_{i+1}$. Note that $(Y_i, V_i), (Y_{i+1}, V_i)$ are wlc models of (X, V_i) by Lemma 2.9. The link may not be elementary, in particular, T_i is not necessarily the lc model of (Y_i, B_i), and the Picard ranks for intermediate models Y_i, Y_{i+1} over the lc model of (X, V_i) can be rather high.

To attain cte links we use a perturbation and Theorem 5.11. To fulfil the generality conditions for (X, S), we add to $D + D'$ generic (very ample) divisors S_i so that the conditions hold for a suitable extension $S := D + D' + \sum S_i$. Note that after a perturbation our new subclasses $\mathfrak{P}_i$ and V_i are embedded into $\mathfrak{S}_S \subset \mathfrak{N}_S$. (If $S \neq D + D'$, the subclasses are cube bordering: $\mathfrak{P}_i \subset \partial\mathfrak{B}_S$.) Now we perturb the whole path $\cup\overline{\mathfrak{P}}_i$ into a path in $\mathfrak{S}_S$ away from $\partial\mathfrak{B}_S$. We perturb the path in such a way that it goes only through internal points of fibering facets and ridges. (This path can be straight under the projection of Proposition 5.4.) Thus now we suppose that new intervals $\mathfrak{P}_i$ are in fibering facets $\mathfrak{F}_i$ and new vertexes V_i in fibering ridges $\mathfrak{R}_i$ (we use the same notations for the perturbed path even though there is no 1-to-1 correspondence for intervals and vertexes under the perturbation). By construction $Y = Y_1/T_1, Y' = Y_l/T_l$ are the Mori log fibrations of $\mathfrak{F}_1, \mathfrak{F}_l$, Y_1, Y_l are terminal and Y_i is terminal for other Mori log fibrations of facets $\mathfrak{F}_i$.

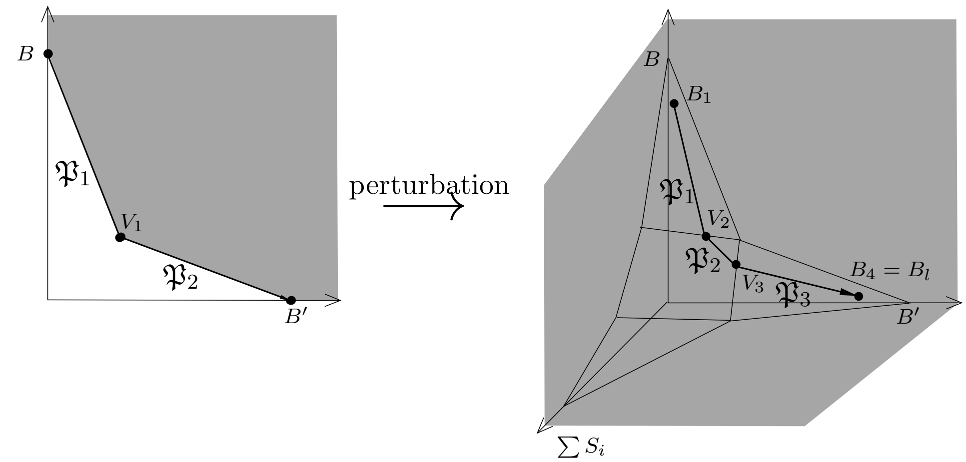


Diagram 10

Indeed, for the perturbation $B_1 \in \mathfrak{S}_S \setminus \partial\mathfrak{B}_S$ of B, the pair (Y_1, B_1) has the same contraction Y/T because the contraction is extremal FT. The same works for Y': for the perturbation B_l in the facet $\mathfrak{F}_l$ close to B', a slt model (Y_l, B_l) is terminal, and so is $Y_l = Y'$. The terminal property is also stable under perturbation. Each slt wlc model (W, D''_W) of (X, D'') for a (klt) boundary $D'' = bD + b'D', b, b' \in (0, 1)$, is terminal because (X, D'') is terminal and D'' is non contractible. (This model may have cn singularities on the lc model.) Similarly, the terminal property holds for intermediate Y_i. Moreover, the same holds for any intermediate $\mathbb{Q}$-factorial model of a link from Y_i/T_i into Y_{i+1}/T_{i+1} by Theorem 5.9. Indeed, if we have the lc model (W, D''_W) for D'' in a divisorial facet of $\mathfrak{N}_S$, with a contracted divisor E of the facet, then for pair (W, D''_W), the only cn singularity is $\operatorname{center}_W E$. But by our assumptions $\operatorname{Supp} D''_W$ contains the center, and W is terminal. Since $V_i \in \operatorname{Int} \mathfrak{R}_i$ is big and $\mathbb{R}$-mobile, the link is cte by Corollary 5.12 and a required factorization is $Y = Y_1 \dashrightarrow \cdots \dashrightarrow Y_l = Y'$. $\square$

Polarized linkage. So, we have constructed a chain of links with *polarizations* as follows. There exist a model (X, S) of Y, where $S = D + D' + \sum S_j$ is a reduced divisor with only big and mobile prime components, with components of S_{Y_i} generating each $N^1(Y_i)$, and intervals $\mathfrak{P}_i = (V_i, V_{i+1}) \subset \mathfrak{S}_S \setminus \partial\mathfrak{B}_S$ such that:

all boundaries in the interval $\mathfrak{P}_i$ are $\sim_{\mathrm{wlc}}$ equivalent with terminal klt wlc models $(Y_i, B_i), B_i = B_{Y_i}$, of (X, B_i); actually B_i on Y_i is the birational transform of B_i from X;

$Y_i \to T_i$ is the lc contraction of (Y_i, B_i), that is, $K_{Y_i} + B_i$ is a pull back of an ample divisor from T_i; and

each vertex V_i gives a cte or Sarkisov link between limiting models $(Y_i, V_i), (Y_{i+1}, V_i)$; they can be cn and with lc models Z_i over which the link goes: for V_i,

2A: $T_i \not\cong Z_i \not\cong T_{i+1}$ and each Y_i, Y_{i+1} have at most one another elementary small and numerically trivial contractions/Z_i with respect to $K_{Y_i} + V_i, K_{Y_{i+1}} + V_i$;

2B: $T_i \cong Z_i \not\cong T_{i+1}$, an elementary divisorial extraction $Y_i' \to Y_i$ corresponds to the unique *maximal* center D_i (exceptional divisor) with cn singularity: $a(D_i, Y_i, V_i) = 1$ with center $\operatorname{center}_{Y_i} D_i$ (a small linear prolongation in direction $\overrightarrow{V_{i-1}V_i}$ gives a unique noncn center), and Y_{i+1} has at most one another elementary small and numerically trivial contraction/Z_i with respect to $K_{Y_{i+1}} + V_i$; the same can hold after interchanging i and $i + 1$;

2C: $T_i = Z_i = T_{i+1}$ with elementary divisorial extractions $Y_i' \to Y_i, Y_{i+1}' \to Y_{i+1}$ as in 2B;

each of those links can be centralized (see Links above).

A limit to boundaries in $\mathfrak{S}_{D+D'}$ supported in $D + D'$ (the original path) gives similar conditions with cn singularities of pairs instead of terminal and semiampleness instead of ampleness on T_i and Z_i.

Rigidity. A Mori fibration Y/T is *rigid* if, for any other Mori fibration Y'/T', any birational isomorphism $Y \dashrightarrow Y'$ gives an isomorphism of the fibrations $Y/T, Y'/T'$. The definition and the definitions below work over any Z. However, if $Z = \mathrm{pt}$ the rigidity is only possible when $\dim_k T \leq 1$. Thus, for $\dim_k T \geq 2$, it is reasonable to use a weaker version: Y/T is birationally rigid (see below) and any Mori fibration Y'/T satisfies the above property. There are other weaker versions of the rigidity. If it holds over the general points of T, T' we say that Y/T is *generically* rigid. For example, so do the conic bundles satisfying Sarkisov's criterion below. A Mori fibration Y/T (or over the general point of T) is *birationally* rigid if, for any other Mori fibration Y'/T', any birational isomorphism $Y \dashrightarrow Y'$ gives a birational isomorphism of the fibrations Y/T and Y'/T', that is, they are square birational under the isomorphism. The fibration is *weakly* rigid if, in addition, any Mori fibration Y'/T' square birational to Y/T is isomorphic to Y/T over the general point of T (possibly under a different birational isomorphism $Y \dashrightarrow Y'$). Such an isomorphism allows us to treat the birational isomorphism $Y \dashrightarrow Y'$ as a birational automorphism of Y/T and vice versa. Del Pezzo fibrations of degree 2 and 3 give nontrivial examples of the last rigidity (see Corollary 6.4 below). In general, it is expected that a Mori fibration Y/T is weakly rigid if its general fiber is weakly rigid and Y/T is sufficiently twisted, e.g., the discriminant locus is large or the map of T into moduli of fibers has a large degree. Moreover, in higher dimensions, a rather general Mori fibration Y/T is usually rigid$/T = Z$ if its general fiber is rigid. We will treat this with more details and applications elsewhere. Here we only illustrate this in some special cases usually assuming $Z = \text{pt}$.

Corollary 6.3. *Suppose* [PrSh, Conjecture 1.1] *for terminal weak Fano varieties of dimension* $d \geq 4$. *In dimension* d*, there exists a rational number* $4 > \mu_d \geq 1$ *(*the birational rigidity threshold for conic bundles of dimension d*) satisfying the following. A Mori conic bundle* X/T *with* $\dim X = d$ *is generically rigid if, for some real number* $\mu > \mu_d$*,* $(T, C/\mu)$ *is cn and* $\mu K_T + C$ *is pseudo-effective where* C *is the discriminant locus of the conic bundle* X/T*.*

Moreover, for $\mu \geq 4$*, we can omit* [PrSh, Conjecture 1.1].

For $\mu = 4$, the corollary becomes the Sarkisov theorem [Sar1] [Sar2]. For any $\mu > 0$ with cn $(T, C/\mu)$, the condition $\mu K_T + C$ is pseudo-effective is equivalent to the condition $\mu K_T + C \in \overline{\text{Eff}}(T)$ by Conjecture 7.1. According to Zagorskii [Zag] in dimension 3 and to Sarkisov [Sar2, Theorem 1.13] in higher dimensions, any conic bundle X/T has a Mori conic bundle model Y/T (square birational to X/T) with cn $(T, C/\mu)$ for any $\mu \geq 2$. Actually, in the corollary one can suppose that X/T is a morphism which is a Mori or standard conic bundle over codimension 1. Such a weak standard model with cn $(T, C/\mu)$ can be easily constructed for any conic bundle and any $\mu \geq 2$.

Proof. Suppose that X'/T' is another Mori fibration, X and X' are birational, but not square birational. Note that square birational links preserve all conditions except for the cn property $(T, C/\mu)$. However, if $K_T + C/\mu$ is pseudo-effective on such a cn model, it is true on any model with the birational transform of C plus any exceptional divisors (cf. [Sar2, Lemma 2.2] [Isk3, Lemma 4]). Now consider a cte link from X/T into X'/T' which is not square birational, that is, over some W with $\dim_k X/W \geq 2$, there exists a fibering contraction $T \to W$. According to the big and $\mathbb{R}$-mobile property of $-K/W$, $K^2 > 0/W$. Hence, for $\mu \geq 4$, by the key formula $\frac{4}{\mu}(\mu K_T + C) \leq 4K_T + C \equiv -g_* K^2 < 0/W$ [Isk3, Lemma 2], where $g\colon X \to T$, a contradiction.

Now by Corollary 5.13, the FT varieties X/W and thus T/W are bounded generically$/W$. Note that $\rho(T/W) = 1$ and they are Mori log fibrations/W with $B_T = 0$. This implies our improvement for

$\mu_d = \max\{\tau_{T/W} < 4\}$ where $\tau_{T/W} = \max\{r < 4 \mid\ rK_{T/W} + C \equiv 0/W\}$,

for all Weil reduced divisors C on T and assuming $\dim_k T/W \le d-1$. Actually, $\mu_d \ge \mu_2 = 7/2 = \tau_{\mathbb{P}^1/\mathrm{pt}}$. Indeed, the cn property of $(T, C/\mu)$ and the pseudo-effective property of $\mu K_T + C$ for any $\mu > \mu_d$ implies that of $(T, C/\mu')$ and of $\mu' K_T + C$ for $\mu' = \max\{\mu, 4\} \ge 4$. □

Corollary 6.4. *Let X/C be a 3-fold Mori fibration of degree $d = 1, 2, 3$ over a curve C: $\dim X = 3$, C is a nonsingular curve (not necessarily complete), X/C is a Mori fibration whose general fibers are Del Pezzo surfaces of degree d. Then X/C is weakly rigid if one of the following holds:*

(i) *X/C has sufficiently many singular fibers over a nonsingular completion $\overline{C}$ of C, that is, there exists a natural number n_d such that the number of nonsmoothable fibers on the completion of C is $\ge n_d$;*

(ii) *the map $C_{\overline{k}} \dashrightarrow \mathcal{M}_d$, $c \mapsto$ the class of the smooth geometric fiber X_c of $X_{\overline{k}}/C_{\overline{k}}$, has a sufficiently large degree, where $\mathcal{M}_d$ is a coarse moduli space of smooth Del Pezzo surfaces of degree d;*

(iii) *for each Mori model $Y/\overline{C}$ of X/C over the completion $\overline{C}$,*

$$-K_Y \notin \operatorname{Int}\operatorname{Mob}(Y)$$

holds; or

(iv) *for each Mori model $Y/\overline{C}$ of X/C,*

$$K_Y^2 \notin \operatorname{Int}\operatorname{NE}(Y)$$

holds, where $\operatorname{NE}(Y)$ denotes the cone of curves on Y.

Moreover, for degree 1, X/C is generically rigid and, for degree 2, 3, any birational automorphism of X/C is a composition of (Bertini and Geiser) involutions and of isomorphisms over the general point of C.

The number in (i) is the sum of degrees of points in $\overline{C}$, the fibers over which are not smoothable. The condition (ii) is geometrical and, according to the proof below, (i) can be treated geometrically too. The condition in (iii) for $Y/\overline{C}$ was introduced by Grinenko [Grin] and is known as the K-*condition* (cf. [Isk2, Conjecture 1.2 (ii)]). For a Mori fibration Y/C over a nonsingular projective curve C, it is equivalent to the inclusion:

$$\operatorname{Mob}(Y) \subseteq \mathbb{R}_+[-K_Y] \oplus g^* \operatorname{Eff}(C)$$

where $g\colon Y \to C$ and $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \ge 0\}$. The condition in (iv) for $Y/\overline{C}$ was introduced by Pukhlikov [Pukh] and is known as the K^2-*condition*. The K^2-condition implies the K-condition.

Proof. After a completion we can suppose that X and $C = \overline{C}$ are projective. By the weak rigidity of minimal Del Pezzo surfaces of degree $d \leq 3$ [Grin2, Proposition 3.3], each link over $T = C$ is square birational and preserves all conditions (i-iv). For $d = 1$, the links do not change X/C generically over C. For $d = 2, 3$, the only nontrivial fiberwise links are (Bertini and Geiser) involutions and they give the same fibration generically. But the (global) links/$T =$ pt are bounded by Corollary 6.1 that contradicts (i) and (ii) with sufficiently large constants. The property (ii) is geometrical because the degree is geometrical and, actually, we need to consider moduli over an algebraically closed field. The property (i) is geometrical too because the smoothness is geometrical due to Park: the smooth compactification is unique for $d \leq 3$ (even ≤ 4) [Park].

By the mobility of cte links, there are no links of Y/C under the conditions (iii-iv). More precisely, the K^2-condition implies the K-condition and the latter contradicts the mobility. Indeed, the link has type 2B or 2C. Let Y' be its central model and Y'' be an anticanonical model of Y' with a small contraction $Y' \to Y''$. Thus there exists a (natural) small modification $\varphi\colon Y \dashrightarrow Y''$ into a Fano 3-fold Y''. This gives a decomposition $-K_Y \sim_{\mathbb{Q}} M + fF$, that contradicts the K-condition, where M is a (big) $\mathbb{Q}$-mobile divisor, F is a fiber of Y/C, and f is a positive rational number. Note that $-K_{Y''} = -\varphi(K_Y)$ is an ample divisor on Y'' and $\varphi(F)$ is a Weil divisor on Y''. □

Corollary 6.5. *Let X/C be a 3-fold Mori fibration of degree 1 over a nonsingular projective curve C. Then X/C is generically rigid if and only if, for each Mori model Y/C of X/C, the K-condition holds.*

This is a weak version of the K-conjecture [Grin2, Conjecture 4.4] for Del Pezzo fibrations of degree 1.

Proof. Immediate by Corollary 6.4 (iii) and [Grin2, Theorem 4.5]. □

It is not difficult to construct many examples of weakly rigid 3-fold Mori fibrations satisfying Corollary 6.4 (i-ii).

Those examples and Corollary 6.5 allow us to construct many rigid Mori fibrations of degree 1 satisfying Corollary 6.4 (iii). Examples of such fibrations satisfying Corollary 6.4 (iv) for degree 1 or (iii-iv) for degrees $2, 3$ are unknown at present, but they are expected [Cor2, Problem 5.9 (3)].

Questions. For what degrees d, does the K-, or K^2-condition hold, or the mobility not hold for almost all Del Pezzo fibrations with terminal singularities? Equivalently, for what degrees d, does the condition

$-K \in \operatorname{Int}\operatorname{Mob}(X), K^2 \in \operatorname{Int}\operatorname{NE}(X)$, or the mobility is *bounded*, that is, the Del Pezzo fibrations X/C of degree d with given property and only with terminal singularities form a bounded family? The same questions are especially important for Mori fibrations.

For example, the condition $K^2 \in \operatorname{Int}\operatorname{NE}(X)$ is unbounded for cubic Mori fibrations [BCZ, Lemma 3.6]. This and Corollary 6.4 give an unbounded family of terminal Gorestein weakly rigid cubic Mori fibrations/$\mathbb{P}^1$ without the K^2-condition. On the other hand, all nonsingular Mori fibrations with K-condition of degrees $1, 2$ and rather general of degree 3 are weakly rigid [Grin2, Theorems 5.6, 6.4 and 7.1]. Some of them does not satisfy the K^2-condition. However, it is unknown whether the condition (iii) of Corollary 6.4 holds for some of them. The proofs of these rigidity results use the maximal singularity method and some elements of the Sarkisov program [Pukh] [Grin] [Cor2]. Probably, the latter can be replaced by Polarized linkage. Maximal singularities play a crucial role in birational geometry. Anyway, our approach improves the maximal singularity method.

Theorem 6.6. *Let X/T be a Mori fibration. The following properties are equivalent:*

(i) *X/T is nonrigid/$T = Z$;*
(ii) *there exists a mobile/T linear system $L \subseteq |-dK|$ of degree (possibly a rational number) $d > 0$ which has a single exceptional maximal b-divisor;*
(iii) *there exists a mobile/T linear system $L \subseteq |-dK|$ of degree $d > 0$ which has an exceptional maximal b-divisor;*
(iv) *there exists a big mobile/T linear system $L \subseteq |-dK|$ of degree $d > 0$ which has a single exceptional cn b-divisor and is big/T on all its cn b-divisors;*
(v) *there exists a big mobile/T linear system $L \subseteq |-dK|$ of degree $d > 0$ which has an exceptional cn b-divisor and is big/T on all its cn b-divisors;*
(vi) *X/T has a model Y/T such that Y/T is a weak projective terminal $\mathbb{Q}$-factorial Fano variety with $\rho(Y/T) = 2$ and with big/T $-K_Y$ in codimension 1;*
(vii) *X/T has a model Y/T such that Y/T is a terminal Fano variety with $\dim_{\mathbb{R}} \mathrm{N}^1(Y/T) = 2$;*
(viii) *X/T has a model Y/T such that Y/T is a weak projective terminal $\mathbb{Q}$-factorial Fano variety with $\rho(Y/T) \geq 2$ and with big/T $-K_Y$ in codimension 1; and*

(ix) *X/T has a model Y/T such that Y/T is a terminal Fano variety with* $\dim_{\mathbb{R}} \mathrm{N}^1(Y/T) \geq 2$.

In the statement and below $\subseteq$ means that, for any Weil divisor $D \in L$, $|D| = |-dK|$, that is, $D \sim -dK$. In other words, $-dK$ denotes an integral Weil divisor D such that $D/d \sim_{\mathbb{Q}} -K$. (Of course, for the integral numbers d, dK has a canonical choice up to $\sim$.) Note that D is not unique even if d is integral. The linear system L *has a cn b-divisor* E if, for general $D \in L$, the cn threshold t of $(X, D/d)$ is ≤ 1 and $a(E, X, tD/d) = 1$. The cn *b*-divisor is *maximal* if $t < 1$. Note that t is independent of general D and each nonexceptional divisor is cn if $t \leq 1$.

Proof. Suppose that X/T is nonrigid$/T = Z$ and $X/T \dashrightarrow X'/T'$ with T'/T is the first cte link. The link goes over T and let Y/T be its central model. The model Y/T satisfies (vi) and (viii), and (iv-v) with L, the birational transformation of $|-NK_Y|$ for some $N \gg 0$. Taking L with the fixed exceptional divisor for $Y \dashrightarrow X$, we get (iii). The contraction of flopping curves of Y/T in (vi) (the anticanonical model) gives a model in (vii) and (ix). Conversely, the $\mathbb{Q}$-factorialization of models in (vii) and (ix) gives models in (vi) and (viii) respectively. Thus (vi), (viii) and (vii), (ix) are equivalent respectively. Note that, (i), (iv) and (vi) are equivalent (see the proof of Corollary 5.12), and follow from (ii). Similarly, (v) and (viii) are equivalent and follow from (iii). A perturbation of the linear system $|L|$ in (iii) allows us to construct a linear system in (ii). Hence (ii) and (iii) are equivalent. Similarly, (iv) and (v) are equivalent. Finally, (ii) implies (iv) by definition. ☐

Corollary 6.7. *A Mori fibration X/T is rigid$/T$ if and only if any mobile$/T$ linear system $L \subseteq |-dK|$ of rational degree $d > 0$ has no exceptional maximal b-divisors.*

Proof. Immediate by Theorem 6.6. ☐

This implies that rigidity and nonrigidity are constructive conditions (cf. [ChGr]). Recall that a *Mori-Fano variety* is a Mori fibration over a point $T = \mathrm{pt}$.

Corollary 6.8. *Suppose that the base field k is algebraically closed and assume* [PrSh, Conjecture 1.1] *for projective weak Fano varieties of dimension $d \geq 4$. Then the rigid (respectively nonrigid) Mori-Fano varieties of dimension d form a constructive set in the coarse moduli of Mori-Fano varieties of dimension d.*

Proof. It is enough to consider nonrigidity. By Theorem 6.6, the nonrigid Mori-Fano varieties are the images of the terminal Fano varieties

X with $\dim_{\mathbb{R}} \mathrm{N}^1(X) = 2$. More precisely, the latter should have a fixed $\mathbb{Q}$-factorialization and a fixed extremal ray on it. Then the image is given by a modifications starting from the ray. The image gives Mori-Fano varieties for a constructive subset and is constructive itself. $\square$

7. LMMP and semiampleness

This section gives an overview of the LMMP and semiampleness and detailed assumptions for the statements of the paper. We use the basic notions and notations from [KMM] [KoMo] [IskSh]. Below we recall some of them.

Model. Let X/Z be a morphism of varieties. A variety means a geometrically reduced and irreducible normal algebraic variety over some field k, usually, of characteristic 0 (not necessarily algebraically closed). The role of the last assumption will be explained below. According to the following we can suppose that X/Z is proper. A model of X/Z is another *proper* morphism Y/Z of varieties such that X and Y are birationally isomorphic$/Z$. Moreover, the birational isomorphisms $\chi\colon X \dashrightarrow Y/Z$ is fixed and called *natural* for the model Y/Z. In other words, a *model* is such an isomorphism $X \dashrightarrow Y/Z$. A *natural birational isomorphism* of models $Y_1 \dashrightarrow Y_2$ for X/Z is a unique birational isomorphism in the commutative diagram:

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ Y_1 & \dashrightarrow & Y_2 \end{array}$$

with natural birational isomorphisms $X \dashrightarrow Y_1, Y_2$. Similarly, if $Y_1 \to T_1$, $Y_2 \to T_2$ are two proper morphisms of models$/Z$, their *(rational) morphism* is a commutative square:

$$\begin{array}{ccc} Y_1 & \dashrightarrow & Y_2 \\ \downarrow & & \downarrow \\ T_1 & \dashrightarrow & T_2 \end{array}$$

with the natural birational isomorphism $Y_1 \dashrightarrow Y_2$; the *natural* rational morphism $T_1 \dashrightarrow T_2$ is unique. Two models $Y_1, Y_2/Z$ are considered as the *same (equal)* if the natural birational isomorphism is biregular, that is, they are naturally *isomorphic*. Respectively, $Y_1/T_1, Y_2/T_2$ are *naturally isomorphic* if $Y_1 \dashrightarrow Y_2, T_1 \dashrightarrow T_2$ are biregular isomorphisms. Usually, everybody omit the adjective "natural". The symbol $\cong$ or $=$ denotes a natural isomorphism and $\approx$ any other. For example, a quadratic transformation $X = \mathbb{P}^2 \dashrightarrow Y = \mathbb{P}^2$ gives a model $Y = \mathbb{P}^2$

of $X = \mathbb{P}^2$ which is unnaturally isomorphic to $X = \mathbb{P}^2$ (see Example 2.6 above).

There are two subtleties in the model theory. First, we can consider local and even formal theory, that is, we can replace Z by a local or formal germ of Z at some scheme point $z_0 \in Z$. This gives *local* and *formal* models over Z whereas models over a variety Z, especially for projective Z, are known as *global*. For simplicity, a reader can suppose that all models are global. However, everything works ditto for local and formal models. Second, that is more important for us, we consider models of pairs. This leads to log models.

Pair. A pair $(X/Z, D)$ consists of a proper morphism X/Z and a b-divisor D/Z. Usually, we suppose that $D = B$ is a b-boundary. A b-boundary means a finite b-$\mathbb{R}$-divisor $B = \sum b_i D_i$ where the sum runs over a finite set of distinct prime b-divisors D_i with multiplicities b_i in the unit segment $[0, 1]$ of real numbers. The divisors D_i are supposed to be fixed. For example, all $D_i = S_i$ and

$$B \in \mathfrak{B}_S = \oplus_{i=1}^m [0, 1] S_i \cong [0, 1]^m,$$

where $S = \sum_{i=1}^m S_i$ is a fixed finite reduced b-divisor. An element $B \in \mathfrak{B}_S$ will be referred to as a *boundary* divisor even when it is actually a b-boundary.

The relations $\equiv, \sim_{\mathbb{R}}$ and the properties such as ampleness, nefness, $\equiv 0$, bigness for divisors on X/Z will always be treated relatively over Z.

Log model. Let $(X/Z, B)$ be a pair with a boundary $B \in \mathfrak{B}_S$. A log pair $(Y/Z, B_Y^{\log})$ is a *log model* of $(X/Z, B)$ if Y/Z is a model of X/Z, with a natural birational isomorphism $\chi\colon X \dashrightarrow Y/Z$, and $B_Y^{\log}$ is the *log transformation* of B on Y, that is, $B_Y^{\log} = B_Y + \sum D_i$ where B_Y is the trace of B on Y and the summation runs over the distinct χ^{-1}-exceptional prime divisors D_i on Y, not components of S. Since $(Y/Z, B_Y^{\log})$ is a log pair, for any canonical divisor K_Y of Y, the divisor $K_Y + B_Y^{\log}$ is $\mathbb{R}$-Cartier. Note that the boundary $B_Y^{\log}$ is also the trace on Y of an *infinite* b-boundary $B^{\log} := B + \sum D_i$ where D_i are the distinct prime divisors exceptional on X and not prime components of S. Both boundaries $B^{\log}$ and $B_Y^{\log}$ depend on S. However, if S is a usual divisor (i.e., a b-divisor without exceptional components) on X, the divisors $B^{\log}$ and $B_Y^{\log}$ are independent of S.

Initial model. It is a lc pair with a boundary. Such a pair $(Y/Z, B_Y^{\log})$ is called an *initial model* of $(X/Z, B)$ if $(Y/Z, B_Y^{\log})$ is a log model of $(X/Z, B)$ and the following inequalities hold: for each prime χ-exceptional divisor E of X, $a(E, X, B) \le a(E, Y, B_Y^{\log})$ and, for each prime component S_i of S exceptional on Y, $1 - b_i = 1 - \operatorname{mult}_{S_i} B \le a(E, Y, B_Y^{\log})$. Using the initial log discrepancy $\underline{a}(D, X, B)$ defined in Section 2, we can combine the the two inequalities into one: $\underline{a}(E, X, B) \le a(E, Y, B_Y^{\log})$.

An initial *strictly lt (slt)* model is a slt pair with a boundary. Recall that the slt property means that the variety is $\mathbb{Q}$-factorial and projective over the base, the pair is lt. An initial model $(Y/Z, B_Y^{\log})$ of $(X/Z, B)$ is called an initial *slt* model of $(X/Z, B)$ if $(Y/Z, B_Y^{\log})$ is a slt pair and the above inequalities are strict for all divisors E of X and all prime components of S exceptional on Y.

Existence of an initial model for some pair $(X/Z, B)$ is not obvious and related to log resolutions. If a pair (X, S) is log nonsingular and the divisor $S_X = S$ is nonexceptional, then $(X/Z, B)$ is initial for any boundary $B \in \mathfrak{B}_S$. For any pair $(X/Z, S)$ and for any of its projective/Z log resolution $(Y, S_Y^{\log})$, the pair $(Y/Z, B_Y^{\log})$ is a slt initial model of $(X/Z, B)$ for all $B \in \mathfrak{B}_S$.

Weakly log canonical model. It is an initial model $(X/Z, B)$ with a nef divisor $K + B$. The pair is a *log canonical (lc) model* if $K + B$ is ample. An initial model $(Y/Z, B_Y^{\log})$ of $(X/Z, B)$ is called a *weakly log canonical (wlc) model* of $(X/Z, B)$ if $(Y/Z, B_Y^{\log})$ is a wlc model. It will be a *log canonical (lc) model* of $(X/Z, B)$ if $(Y/Z, B_Y^{\log})$ is a lc model. A lc model of $(X/Z, B)$ is unique (up to a natural isomorphism) if it exists and will be denoted by $(X_{\mathrm{lcm}}/Z, B_{\mathrm{lcm}})$ where $B_{\mathrm{lcm}} := B_{X_{\mathrm{lcm}}}^{\log}$.

A wlc model is called a *strictly log terminal (slt)* wlc model if it is also slt. Respectively, a *strictly log terminal (slt)* wlc model $(Y/Z, B_Y^{\log})$ of $(X/Z, B)$ is a slt wlc model which is a slt initial model of $(X/Z, B)$.

Mori log fibration. It is an elementary Fano log fibration [IskSh, Definition 1.6 (v)]. More precisely, it is an initial model $(X/Z, B)$ with a contraction $X \to T/Z$ such that: (a) $\dim_k T < \dim_k X$; (b) $(X/Z, B)$ has only $\mathbb{Q}$-factorial and lt singularities; (c) the relative Picard number $\rho(X/T) := \rho(X/Z) - \rho(T/Z) = 1$; and (d) $-(K + B)$ is relatively ample/T. Such a model without (b-c) (respectively only without (b)) is known as an (elementary) Fano log fibration. Note that the contraction X/T is a fixed structure of the Mori log fibration. It is an *slt* Mori log fibration if, additionally, T/Z is projective. A Mori log fibration is *generic* if it is considered over the general point of T.

An initial model $(Y/Z, B_Y^{\log})$ of $(X/Z, B)$ with a contraction $Y \to T/Z$ is called a *Mori log fibration of* $(X/Z, B)$ if $(Y/Z, B_Y^{\log})$ is a Mori log fibration with the contraction Y/T. Similarly, one can define a *Fano log fibration of* $(X/Z, B)$. The model $(Y/Z, B_Y^{\log})$ is an *slt* Mori log fibration of $(X/Z, B)$ if $(Y/Z, B_Y^{\log})$ is a slt initial model of $(X/Z, B)$ and $(Y/Z, B_Y^{\log})$ is also a slt Mori log fibration. Thus, for each slt Mori log fibration of $(X/Z, B)$, the pair $(Y/Z, B_Y^{\log})$ is a slt Mori log fibration, but not conversely in general even when $(Y/Z, B_Y^{\log})$ is a Mori log fibration of $(X/Z, B)$. Similarly, for a wlc model $(Y/Z, B_Y^{\log})$ of $(X/Z, B)$, a slt wlc model $(Y/Z, B_Y^{\log})$ is not the same as a slt model of $(X/Z, B)$, in general.

If $B = 0$ and X is trm then a slt Mori log fibration $(X/Z, B)$ is simply a *Mori fibration*. Respectively, if $B = B_Y^{\log} = 0$ and Y is trm then a slt Mori log fibration $(Y/Z, 0)$ is a *Mori fibration* Y/Z or a *Mori model of* X/Z.

Resulting model. It is a wlc model or a Mori log fibration (or even a Fano log fibration). A *resulting model* of a pair $(X/Z, B)$ is either a wlc model or a Mori log fibration (or even a Fano log fibration) of $(X/Z, B)$. According to [Sho4, 2.4.1], any pair $(X/Z, B)$ cannot have both resulting models simultaneously. However, a pair can have many wlc models or Mori (Fano) log fibrations. It is expected that every pair $(X/Z, B)$ with a boundary divisor B has at least one of them, that is, a resulting model.

If S is a usual divisor, that is, nonexceptional on X then B is a boundary and the above concepts are the same as in [Sho4] [IskSh].

LMMP. The Log Minimal Model Program (LMMP) allows us to construct a resulting model. The *LMMP* is a special case of the D-Minimal Model Program (D-MMP) [IskSh, 1.1]. An initial model for the LMMP is an initial model $(X/Z, B)$ and $D = K + B$ (usually, it works with $\equiv$ instead of $=$). We suppose also that X/Z is projective for reasons explained (see Characteristic 0 below). Then a D-minimal model is a projective wlc model and a nonbirational D-contraction $X \to T$ is a projective elementary Fano log fibration. The program itself is a chain of birational transformations of the initial model which are supposed to exist and terminate. Each transformation is a (generalized) log flip of a projective negative extremal contraction [Sho4], but not necessarily elementary (see below). Intermediate models obtained after each transformation are again projective initial models. The last model under the LMMP is either a projective wlc model or a projective elementary Fano log fibration of the initial model. The last model under the LMMP starting from some initial model of $(X/Z, B)$ is a resulting model of $(X/Z, B)$.

However, the slt LMMP is easier and more common. An initial model of the *slt LMMP* is a slt initial model $(X/Z, B)$, in particular, X/Z is projective. Each transformation is either a log flip or a divisorial contraction which are supposed to be elementary. Intermediate models obtained after each transformations are again slt initial. The last model under the LMMP is either a slt wlc model or a slt Mori log fibration of the initial model. The last model under the LMMP starting from some slt initial model of $(X/Z, B)$ is a slt resulting model of $(X/Z, B)$: either a slt wlc model or a slt Mori log fibration of $(X/Z, B)$.

An *elementary* contraction (extraction) or *elementary* small transformation (modification) is respectively a projective contraction (extraction) $Y \to Y' = T$ or a small nonregular projective modification $Y \dashrightarrow Y'/T$ such that Y, Y' are $\mathbb{Q}$-factorial and $\rho(Y/T) = 1, \rho(Y'/T) \leq 1$. Actually, for an elementary contraction (extraction), $\rho(Y/T) = 1, \rho(Y'/T) = 0$ and, for an elementary small transformation, $\rho(Y/T) = \rho(Y'/T) = 1$.

An *extremal* contraction or *extremal* transformation (modification) is respectively a contraction (extraction) $Y \to Y' = T$ or a transformation $Y \dashrightarrow Y'/T$ such that $\rho(Y/T) = 1, \rho(Y'/T) \leq 1$. Actually, for an extremal contraction (extraction), $\rho(Y/T) = 1, \rho(Y'/T) = 0$ and, for an extremal transformation, $\rho(Y/T) = \rho(Y'/T) = 1$

Thus each pair $(X/Z, B)$ has a resulting model by the LMMP and a slt resulting model of $(X/Z, B)$ by the slt LMMP, of course, if such a program exists for $(X/Z, B)$, including existence of an initial model (see Characteristic 0 below). For such a construction, a *weak* termination is enough, that is, some sequence of birational transformations terminates, gives a resulting model. For example, the (slt) LMMP is established over any perfect field in dimension 2: for any algebraic surface over such a field, the LMMP exists.

Conjecture 7.1 (semiampleness)**.** [Sho4, 2.6] [IskSh, 1.16] *Let $(X/Z, B)$ be a wlc model. Then $K + B$ is semiample.*

One of the major applications of geography:

Corollary 7.2. *In any fixed dimension d, the slt LMMP in dimension d and the semiampleness with $\mathbb{Q}$-boundaries imply the semiampleness with $\mathbb{R}$-boundaries. Moreover, for the klt semiampleness with $\mathbb{R}$-boundaries, the klt slt LMMP and the klt semiampleness with $\mathbb{Q}$-boundaries are sufficient.*

Proof. Immediate by Theorem 2.4. By definition, semiampleness for vertexes of geography implies that of for all boundaries. □

Log canonical model [IskSh, 1.17 (ii)]. Let $(X/Z, B)$ be a wlc model. Then by Conjecture 7.1 the $\mathbb{R}$-divisor $K + B$ defines a *lc (Iitaka) contraction* $I\colon X \to X_{\mathrm{lcm}}/Z$ onto a normal projective variety X_{lcm}/Z, called a *lc (Iitaka) model* of $(X/Z, B)$. In particular, if I is birational, the pair $(X_{\mathrm{lcm}}/Z, B_{\mathrm{lcm}})$ with $B_{\mathrm{lcm}} = B^{\log}_{X_{\mathrm{lcm}}}$ is the lc model of $(X/Z, B)$ (see Weakly log canonical model above).

If both the LMMP and Conjecture 7.1 hold for a pair $(X/Z, B)$ and the pair has a wlc model $(Y/Z, B^{\log}_Y)$, then one can define a rational Iitaka contraction $I : X \dashrightarrow X_{\mathrm{lcm}}/Z$ as a composition of the birational transformation $X \dashrightarrow Y$ and of the Iitaka contraction of $Y \to Y_{\mathrm{lcm}} = X_{\mathrm{lcm}}$. If (X, B) is lc, the rational Iitaka contraction is a 1-contraction.

By [Sho4, 2.4.3-4], the image X_{lcm} and the rational map $X \dashrightarrow X_{\mathrm{lcm}}$ depend only on the pair $(X/Z, B)$. Thus we can write X_{lcm} instead of Y_{lcm}. Furthermore, it is known that the rational Iitaka contraction I and its image X_{lcm} depend only on the $\sim_{\mathrm{lcm}}$-class by definition(see Corollary 2.11). In particular, the contraction I is independent of B modulo $\sim_{\mathrm{wlc}}$.

Mobile decomposition. The *positive part* of $\mathcal{K} + B^{\log}$ is the b-Cartier divisor $P(B) = \overline{K_Y + B^{\log}_Y}$ where $(Y/Z, B^{\log}_Y)$ is a wlc model of $(X/Z, B)$. By definition $P(B)$ is nef and $\mathbb{R}$-mobile by the semiampleness 7.1. If $(X/Z, B)$ does not have a wlc model, that is, $\nu(B) = -\infty$, put $P(B) = -\infty$.

We can define the *mobile decomposition*: $\mathcal{K} + B^{\log} = P(B) + F(B)$. Here, $F(B)$ is called the *fixed part* of $\mathcal{K} + B^{\log}$. If $\nu(B) = -\infty$, put $F(B) = +\infty$.

Numerical Kodaira dimension. [Sho4, 2.4.4] Given a pair $(X/Z, B)$ with $B \in \mathfrak{B}_S$, we define the *relative numerical Kodaira dimension* $\nu(X/Z, B)(= \nu(B))$ as $-\infty$ if the resulting model of $(X/Z, B)$ is a Mori log fibration and as the integer

$$\nu(B) = \max\{m \in \mathbb{Z} \mid (K_Y + B^{\log}_Y)^m \not\equiv 0 \text{ generically over } Z\}$$

if $(X/Z, B)$ has a wlc model $(Y/Z, B^{\log}_Y)$.

By definition, the following are equivalent: (i) $\nu(X/Z, B) \geq 0$, (ii) there exists a wlc model of $(X/Z, B)$, (iii) $B \in \mathfrak{N}_S$ (see Section 2 above), (iv) $\kappa(X/Z, B) \geq 0$ (if the semiampleness holds 7.1). If this is the case, the integer $\nu(B)$ is independent of the choice of the wlc model Y [Sho4, 2.4.4].

Kodaira dimension. [Choi, Definition 2.2.3] Let B be an $\mathbb{R}$-boundary on X/Z and Y/Z be a model of X/Z such that the log pair $(Y/Z, B_Y^{\log})$ is lc. Then the *relative invariant log Kodaira dimension* $\kappa(X/Z, B)$ of the pair $(X/Z, B)$ is defined as the integer:

$$\kappa(X/Z, B) := \iota(Y/Z, K_Y + B_Y^{\log}),$$

where $\iota(X/Z, D)$ denotes the relative Iitaka dimension of D. For a $\mathbb{Q}$-divisor D, this is the same as the usual definition, but for an $\mathbb{R}$-divisor D, see [Choi, Definition 2.2.3].
Characteristic 0. Most of the results in this paper, if not all, can be established under the following assumptions: existence of log resolutions, the LMMP and the semiampleness. Of course, by the subsection LMMP above we can include log resolutions into the LMMP. Indeed, to construct an (slt) initial model of $(X/Z, B)$, one can use a log resolution (see Initial model above). However, usually it is not useful especially in characteristic 0, because in that case a log resolution always exists. Moreover, in characteristic 0 some parts of the LMMP are established or announced: the cone theorem, the contraction of an extremal ray and existence of some special flips (e.g., slt). According to the proofs, these results require projectivity of Y/Z. The main remaining problem is termination. A *weak* termination, that is, some sequence of flips terminates, is sufficient for many results in birational geometry, in particular, for this paper (cf. Assumptions for our results below). Thus we suppose the weak termination in the LMMP. Note also, that in characteristic 0, the semiampleness holds for any nef divisor of relative FT varieties.

The LMMP with usual termination is established only in dimension ≤ 3 [Sho4] and the slt LMMP with weak termination in dimension 4 [Sho6]. For the klt big LMMP with weak termination in all dimensions see in [BCHM]. The *"big"* means that B is big for an initial model, or equivalently, the same holds for each initial model.

Assumptions for our results. Since the LMMP is still not established even in characteristic 0 and some of our results can be use in its proof, we prefer to put explicitly what is needed in each statement even some of assumptions is already proved. In the low dimension applications, we do not do this. So, in addition to the assumptions in Characteristic 0, we need the following. Note that the big Conjecture 7.1 means that of for big $K + B$. Needed assumptions are mentioned after colons and each assumption is sufficient in the dimension of varieties of the statement and the dimension ≥ 4.

Proposition 2.1: the slt LMMP and big Conjecture 7.1.
Proposition 2.3: Conjecture 7.1.
Theorem 2.4: the slt LMMP; the klt (big) LMMP for any convex closed polyhedral subset of klt (respectively big) boundaries in $\mathfrak{B}_S$ on some slt initial model.
Proposition 2.10: the slt LMMP; etc as Theorem 2.4.
Corollary 2.11: the slt LMMP; etc as Theorem 2.4; Conjecture 7.1 in (2) (not needed for the klt big case).
Corollary 2.12: the slt LMMP; etc as Theorem 2.4.
Corollary 2.13: the slt LMMP; etc as Theorem 2.4.

Proposition 3.2: the klt slt LMMP and Conjecture 7.1 for klt pairs with $\mathbb{Q}$-boundaries; the klt slt big LMMP for any convex closed polyhedral subset of klt big boundaries.
Proposition 3.3: the klt slt LMMP and Conjecture 7.1 for klt pairs with $\mathbb{Q}$-boundaries; etc as Proposition 3.2.
Corollary 3.4: the klt slt big LMMP. Comment: It looks that char $k = 0$ is only needed for the coincidence $\equiv$ and $\sim_{\mathbb{R}}$. For arbitrary characteristic, one can use Conjecture 7.1.
Corollary 3.5: the klt slt big LMMP. Comment as in Corollary 3.4.

Corollary 4.1: the klt slt LMMP and Conjecture 7.1 for klt pairs with $\mathbb{Q}$-boundaries; etc as Proposition 3.2.
Corollary 4.2: the klt slt big LMMP.
Corollary 4.3: the klt slt big LMMP.
Corollary 4.4: the klt slt big LMMP.
Corollary 4.5: the klt slt big LMMP.
Theorem 4.6: the klt slt big LMMP.
Corollary 4.7: the klt slt big LMMP.
Corollary 4.8: the klt slt LMMP and Conjecture 7.1 for klt pairs with $\mathbb{Q}$-boundaries; etc as Proposition 3.2.
Corollary 4.9: the klt slt big LMMP.
Corollary 4.10: the klt slt big LMMP.
Corollary 4.11: the klt slt big LMMP.
Corollary 4.12: the klt slt big LMMP.
Proposition 4.13: for FT varieties, the klt slt big LMMP.
Proposition 4.14: the klt slt big LMMP.

Proposition 5.1*: the slt LMMP.
Proposition 5.2*: the slt LMMP.
Lemma 5.3*: the slt LMMP.
Proposition 5.4*: the slt LMMP.
Proposition 5.5*: the slt LMMP.

Corollary 5.6*: the slt LMMP.

(* can be treated as Theorem 2.4 for any convex closed polyhedral subset in $\mathfrak{B}_S$, respectively any closed polyhedral subset in $\mathfrak{S}_S$.)

Theorem 5.7: the slt LMMP; for any closed convex polyhedral subset in $\mathfrak{B}_S$ with klt boundaries, the klt slt big LMMP.

Lemma 5.8: the klt slt big LMMP.

Theorem 5.9: the slt LMMP; etc as Theorem 5.7.

Corollary 5.10: the klt slt big LMMP.

Theorem 5.11: the slt LMMP; etc as Theorem 5.7.

Corollary 5.12: the klt slt big LMMP.

Corollary 5.13: the klt slt big LMMP.

Theorem 6.2: the trm slt big LMMP. Comment: trm slt implies klt in dimensions ≥ 2. We prefer slt because it includes the $\mathbb{Q}$-factorial and projective properties.

Corollary 6.3: the trm slt big LMMP.

Theorem 6.6: the trm slt big LMMP.

Corollary 6.7: the trm slt big LMMP.

Corollary 6.8: the trm slt big LMMP.

References

[Amb] F. Ambro, *The moduli b-divisor of an lc-trivial fibration.* Compos. Math. 141 (2005), no. 2, 385-403.

[BCHM] C. Birkar, P. Cascini, C. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. 23(2010), no. 2, 405-468.

[BCZ] G. Brown, A. Corti, F. Zucconi, *Birational geometry of 3-fold Mori fibre spaces.* The Fano Conference, 235-275, Univ. Torino, Turin, 2004.

[ChGr] I. Cheltsov, M. M. Grinenko, *Birational rigidity is not an open property*, arXiv:math/0612159v1, Preprint. 2006.

[Choi] S. Choi, *Geography of log models and its applications*, Thesis. Johns Hopkins University, 2008.

[Cor1] A. Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. 4 (1995), no. 2, 223-254.

[Cor2] A. Corti, *Singularities of linear systems and* 3*-fold birational geometry.* In Explicit birational geometry of 3-folds. 259-312. Cambridge University Press, 2000.

[Grin] M.M. Grinenko, *Birational properties of pencils of del Pezzo surfaces of degrees* 1 *and* 2. Sb. Math. 191 (2000), no. 5-6, 633-653.

[Grin2] M.M. Grinenko, *Fibrations into del Pezzo surfaces*, Russian Mathematical Surveys, 61 (2006), no. 2, 255-300.

[Isk1] V.A. Iskovskih, *Rational surfaces with a sheaf of rational curves and with a positive square of canonical class.* (Russian) Mat. Sb. (N.S.) 83 (125) 1970, 90-119.

[Isk2] V.A. Iskovskih, *On the rationality problem for three-dimensional algebraic varieties fibered over del Pezzo surfaces.* Trudy Mat. Inst. Steklov. 208 (1995), Teor. Chisel, Algebra i Algebr. Geom., 128-138.

[Isk3] V.A. Iskovskih, *A rationality criterion for conic bundles*, Sb. Mat. 187 (1996), 1021-1038.

[IskSh] V.A. Iskovskikh and V.V. Shokurov, *Birational models and flips*, Russian Mathematical Surveys, 60 (2005), 27-94.

[KMMT] J. Kollár, Y. Miyaoka, S. Mori and H.Takagi, *Boundedness of canonical $\mathbb{Q}$-Fano 3-folds.* Proc. Japan Acad. Ser. A Math. Sci. 76 (2000), no. 5, 73-77.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model program*, Algebraic Geometry, Sendai (T. Oda, ed.), Kinokuniya-North- Holland, 1987, Adv. Stud. Pure Math, vol 10, pp. 283-360.

[Klee] V. Klee, *Some characterizations of convex polyhedra.* Acta mathematica, 102, 79-107. 1959.

[KoMo] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge University Press 134, 1998.

[Park] J. Park, *Birational maps of del Pezzo fibrations*, J. Reine Angew. Math. 538 (2001), 213-221.

[PrSh] Yu. G. Prokhorov and V. V. Shokurov, *Towards the second main theorem on complements*, J. Algebraic Geom. 18 (2009), no. 1, 151-199.

[Pukh] A.V. Pukhlikov, *Birational automorphisms of three-dimensional algebraic varieties with a pencil of del Pezzo surfaces*, Izv. Math. 62 (1998), no. 1, 115-155.

[Sar1] V.G. Sarkisov, *Birational automorphisms of conic bundles*, Math. USSR-Izv. 17 (1981), 177-202.

[Sar2] V.G. Sarkisov, *On the structure of conic bundles*, Math. USSR-Izv. 20 (1982), 355-390.

[Sho1] V.V. Shokurov, *The nonvanishing theorem*, Math. USSR Izv. 26 (1986), 591-604.

[Sho2] V.V. Shokurov, *Three-dimensional log perestroikas*, Izv. Ross. Akad. Nauk Ser. Mat. 56 (1992), no. 1, 105-203.

[Sho3] V.V. Shokurov, *Anticanonical boundedness for curves.* Appendix to Nikulin V.V. Hyperbolic reflection group methods and algebraic varieties in Higher dimensional complex varieties (Trento, June 1994) ed. Andreatta M., and Peternell T. Berlin: New York: de Gruyter, 1996, 321-328.

[Sho4] V.V. Shokurov, *3-fold log models.* Algebraic Geometry, 4, J.Math. Sci. 81, Consultants Bureau, New York, 1996, 2667-2699.

[Sho5] V.V. Shokurov, *Prelimiting flips.* Tr. Mat. Inst. Steklova 240 (2003), Proc. Steklov Inst. Math. 2003, no. 1 (240), 75-213

[Sho6] V.V. Shokurov, *Letters of a bi-rationalist VII: Ordered termination.* Proc. Steklov Inst. 2009, Volume 264, 184-208.

[Zag] A.A Zagorskii, *On 3 dimensional conical bundles.* Mat Zametki. 21 (1977):6, 745-757.

Department of Mathematics
Johns Hopkins University
Baltimore, MD 21218, USA

Steklov Mathematical Institute
Russian Academy of Sciences
Gubkina str. 8, 119991, Moscow, Russia
E-mail address: shokurov@math.jhu.edu

Department of Mathematics
University of California, Riverside
900 University Ave
Riverside, CA 92521, USA
E-mail address: schoi@math.ucr.edu