Russian Math. Surveys **60**:5 875–965

Uspekhi Mat. Nauk **60**:5 71–160 DOI 10.1070/RM2005v060n05ABEH003736

Birationally rigid Fano varieties

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Abstract. The birational superrigidity and, in particular, the non-rationality of a smooth three-dimensional quartic was proved by V. Iskovskikh and Yu. Manin in 1971, and this led immediately to a counterexample to the three-dimensional Lüroth problem. Since then, birational rigidity and superrigidity have been proved for a broad class of higher-dimensional varieties, among which the Fano varieties occupy the central place. The present paper is a survey of the theory of birationally rigid Fano varieties.

Contents

AMS 2000 Mathematics Subject Classification. Primary 14J45, 14E05, 14E30; Secondary 14G22, 14J30, 14E07, 14M20, 14M10.

Introduction

§ 0.1. Non-rationality

The rationality problem for algebraic varieties[1] is one of the most deep and interesting problems in algebraic geometry. The global holomorphic differential forms give a natural birational invariant for smooth surfaces and completely solve the rationality problem for algebraic curves and surfaces (see [182], [160], [91]). However, even in the three-dimensional case there are non-rational varieties that are close to rational varieties in many respects, and the known discrete invariants are insufficient to establish whether or not these varieties are rational. In particular, the following well-known result, which was announced already in [61], was proved in [94].

Theorem 0.1.1. *Let V be a smooth hypersurface in $\mathbb{P}^4$ of degree 4. Then the group* $\mathrm{Bir}(V)$ *of birational automorphisms coincides with the group* $\mathrm{Aut}(V)$ *of biregular automorphisms.*

One can readily see that Theorem 0.1.1 implies the non-rationality of every smooth quartic threefold in $\mathbb{P}^4$. Indeed, in the notation of Theorem 0.1.1, the linear system $|\mathcal{O}_{\mathbb{P}^4}(1)|_V|$ is invariant under the action of the group $\mathrm{Aut}(V)$, because the divisor $-K_V$ is linearly equivalent to a hyperplane section of the quartic V. Therefore, the group of biregular automorphisms of the quartic hypersurface V consists of projective automorphisms, and hence is finite (see [127]). Thus, the group of birational automorphisms of the smooth quartic threefold V is finite, which implies that V is non-rational, because the group $\mathrm{Bir}(\mathbb{P}^3)$ is infinite. Later, the technique of [94] was usually called the *method of maximal singularities* (see 0.2).

The non-rationality of any smooth quartic threefold immediately implied the negative solution of the Lüroth problem in dimension 3. We recall that the Lüroth problem in dimension n is as follows: Is it true that all subfields of the field $\mathbb{C}(x_1,\dots,x_n)$ that contain the field $\mathbb{C}$ are of the form $\mathbb{C}(f_1,\dots,f_k)$, where $f_i = f_i(x_1,\dots,x_n)$ is a rational function? Thus, there are non-rational threefolds that are unirational.[2] For example, the quartic

$$x_0^4 + x_0x_4^3 + x_1^4 - 6x_1^2x_2^2 + x_2^4 + x_3^4 + x_3^3x_4 = 0 \subset \mathrm{Proj}(\mathbb{C}[x_0,x_1,x_2,x_3,x_4]) \cong \mathbb{P}^4$$

is unirational (see [87] and [123]), smooth, and hence non-rational by Theorem 0.1.1. We note that, by the rationality criterion in [182], every unirational curve or surface is rational. The existence of counterexamples to the Lüroth problem was conjectured long ago (see [61]). For example, in the book [16] it is claimed that the three-dimensional Lüroth problem has a negative solution and, as an argument,

[1] All varieties under consideration are assumed to be projective, normal, and defined over the field of complex numbers. A variety V is said to be *rational* if the field of rational functions on V is isomorphic to the field $\mathbb{C}(x_1,\dots,x_n)$, or, equivalently, if there is a birational map $\rho\colon \mathbb{P}^n \dashrightarrow V$. By a *divisor* we always mean a $\mathbb{Q}$-divisor, that is, a formal finite $\mathbb{Q}$-linear combination of subvarieties of codimension one.

[2] A variety V is said to be *unirational* if there is a dominant rational map $\rho\colon \mathbb{P}^n \dashrightarrow V$ or, equivalently, if the field of rational functions of V is a subfield of $\mathbb{C}(x_1,\dots,x_n)$. Some non-trivial constructions of higher-dimensional unirational varieties can be found in [87], [113], [123], [44].