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On a smooth quintic 4-fold

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Abstract. The birational geometry of an arbitrary smooth quintic 4-fold is studied using the properties of log pairs. As a result, a new proof of its birational rigidity is given and all birational maps of a smooth quintic 4-fold into fibrations with general fibre of Kodaira dimension zero are described.

In the Addendum similar results are obtained for all smooth hypersurfaces of degree n in $\mathbb{P}^n$ in the case of n equal to 6, 7, or 8.

Bibliography: 11 titles.

All the varieties considered in this paper are assumed to be projective and defined over $\mathbb{C}$. The main definitions, notation, and concepts are contained in [1] and [2].

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Main results and their historical background

In 1971 Iskovskikh and Manin proved in [3] the following result.

Quartic theorem. *Let X be a smooth quartic 3-fold in $\mathbb{P}^4$. Then*

$$\operatorname{Bir} X = \operatorname{Aut} X.$$

This result gave the first counterexample to the famous Lüroth problem in dimension 3 in the following way. The Quartic theorem implies the non-rationality of each smooth quartic 3-fold. On the other hand some special quartic 3-folds are unirational.

Example. The quartic

$$x_0^4 + x_0x_4^3 + x_1^4 - 6x_1^2x_2^2 + x_2^4 + x_3^4 + x_3^3x_4 = 0$$

is unirational (see [4]).

In 1987, strengthening the method of Iskovskikh and Manin, Pukhlikov proved in [5] the following result.

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Pukhlikov's theorem I. *For an arbitrary smooth quintic 4-fold X in $\mathbb{P}^5$,*

$$\operatorname{Bir} X = \operatorname{Aut} X.$$

In particular, all smooth quintic 4-folds are non-rational. In the 1980s it was shown that the method of Iskovskikh and Manin establishes (implicitly) a much stronger result than the equality of the groups of birational and biregular automorphisms (see [4], [6] and [7]). For example, Pukhlikov's theorem I can be complemented in the following way.

Pukhlikov's theorem II. *A smooth quintic 4-fold X in $\mathbb{P}^5$ is not birationally isomorphic to*

(1) *Mori 4-folds*[1] *that are not isomorphic to X,*
(2) *fibrations by surfaces and 3-folds of Kodaira dimension $-\infty$.*

How, in their turn, can Pukhlikov's Theorems I and II be generalized?

Question 1. What kind of Fano 4-folds with canonical singularities or fibrations by surfaces and 3-folds of Kodaira dimension zero can a smooth quintic 4-fold in $\mathbb{P}^5$ be birationally transformed into?

Note that a smooth quintic 4-fold can be easily birationally transformed into fibrations by Calabi-Yau 3-folds.

Construction I. Take an arbitrary smooth quintic 4-fold $X \subset \mathbb{P}^5$. Let $\psi\colon X \dashrightarrow \mathbb{P}^1$ be a projection from some 3-dimensional linear subspace of $\mathbb{P}^5$. Consider the resolution of the indeterminacies of the map ψ by means of the commutative diagram

$$\begin{array}{ccc} & W & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & \overset{\psi}{\dashrightarrow} & \mathbb{P}^1 \end{array}.$$

Then it is easy to show that g is a fibration by Calabi-Yau 3-folds.

Thus, Question 1 can have no negative answer in principle.

Construction II. In $\mathbb{P}^5$ we choose a smooth quintic 4-fold X containing a 2-dimensional plane P. Let $\psi\colon X \dashrightarrow \mathbb{P}^2$ be a projection from P and let

$$\begin{array}{ccc} & W & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & \overset{\psi}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

be a resolution of the indeterminacies of the map ψ. Then g is a fibration by K3 surfaces.

Important observation. A general smooth quintic 4-fold in $\mathbb{P}^5$ does not contain 2-dimensional planes.

It follows from this important observation and Construction II that the answer to Question 1 depends on the deformation type of the smooth quintic 4-fold.

[1] Mori 4-folds are Fano 4-folds with terminal $\mathbb{Q}$-factorial singularities and Picard group $\mathbb{Z}$.

Agreement. We shall say that *a fibration is induced by a projection in* $\mathbb{P}^5$ if it is birationally equivalent (as a fibration) to one of the fibrations in Constructions I and II.

The aim of this paper is to present an alternative proof of Pukhlikov's theorems I and II and give the following answer to Question 1.

Main theorem. *A smooth quintic* 4*-fold* X *in* $\mathbb{P}^5$ *is not birationally isomorphic to*

(1) *Fano* 4*-folds with canonical singularities that are not biregular to* X,
(2) *fibrations by surfaces and* 3*-folds of Kodaira dimension zero that are not induced by projections in* $\mathbb{P}^5$.

Note that a generalization of the Quartic theorem along the same lines as our main theorem (which generalizes Pukhlikov's theorems I and II) has been performed in [2].

Interesting observation. As the main theorem shows, a smooth quintic 4-fold in $\mathbb{P}^5$ is the first example of a rationally connected 4-fold that is not birationally equivalent to an elliptic fibration.

§ 1. New objects and their properties

In this section we introduce objects that will be used throughout what follows.

Main object. A *movable log pair*

$$(X, M_X) = \left(X, \sum_{i=1}^{n} b_i \mathcal{M}_i\right)$$

is a variety X together with a formal finite linear combination of linear systems $\mathcal{M}_i$ without fixed components such that all $b_i \in \mathbb{Q}_{\geqslant 0}$.

Note that (X, M_X) can be regarded as a usual log pair: it suffices to replace each linear system $\mathcal{M}_i$ by an appropriate weighted sum of sufficiently general divisors in this system.

Observation. The strict transform of M_X is defined in a natural way for each birational map of X.

We shall assume that log canonical divisors of all log pairs under consideration are $\mathbb{Q}$-Cartier divisors. Thus, discrepancies, terminality, canonicalness, log terminality, and log canonicalness can be defined for movable log pairs in a similar way to ordinary ones.

It is easy to verify that an application of the Log Minimal Model Program to a canonical (terminal) movable log pair preserves its canonicalness (its terminality).

Centre of canonical singularities. A proper irreducible and reduced subvariety Y of a variety X is called a *centre of canonical singularities* of a movable log pair (X, M_X) if there exist a birational morphism $f\colon W \to X$ and an f-exceptional divisor $E \subset W$ such that

$$a(X, M_X, E) \leqslant 0 \quad \text{and} \quad f(E) = Y.$$

We shall denote by $\mathrm{CS}(X, M_X)$ the set of all centres of canonical singularities of (X, M_X).

The following example will clarify the nature of the objects just introduced.

Simple example. Consider a movable log pair $(\mathbb{P}^2, bM)$, where M is a linear system of lines in $\mathbb{P}^2$ passing through a fixed point O. Then

$$\mathrm{CS}(\mathbb{P}^2, bM) = \begin{cases} \varnothing & \text{if } b < 1, \\ \{O\} & \text{if } b \geqslant 1. \end{cases}$$

This example is an illustration of the following property of movable log pairs.

Observation. The singularities of a movable log pair coincide with the singularities of the variety outside the base loci of the components of the boundary.

It is natural to ask whether there exists a special model of a movable log pair.

Canonical model. We say that a movable log pair (V, M_V) is a *canonical model* of a movable log pair (X, M_X) if there exists a birational map $\psi\colon X \dashrightarrow V$ such that

$$(V, M_V) = (V, \psi(M_X)),$$

the divisor $K_V + M_V$ is ample, and (V, M_V) has canonical singularities.

This definition of a canonical model is justified by the following important property.

Uniqueness theorem. *A canonical model is unique, once it exists.*

To prove the uniqueness theorem one merely has to write "canonical" in place of "log canonical" throughout [8].

For an arbitrary movable log pair (X, M_X) we consider a birational morphism $f\colon W \to X$ such that the movable log pair

$$(W, M_W) = (W, f^{-1}(M_X))$$

has canonical singularities.

Iitaka map and Kodaira dimension. If the linear system $|n(K_W + M_W)|$ is not empty for $n \gg 0$, then the map

$$I(X, M_X) = \varphi_{|n(K_W+M_W)|} \circ f^{-1} \quad \text{for } n \gg 0$$

is called the *Iitaka map* of (X, M_X) and

$$\varkappa(X, M_X) = \dim(I(X, M_X)(X))$$

is called the *Kodaira dimension* of (X, M_X). Otherwise $I(X, M_X)$ is considered to be nowhere defined on X and $\varkappa(X, M_X) = -\infty$.

One can prove the following result.

Consistency theorem. *The map $I(X, M_X)$ and the quantity $\varkappa(X, M_X)$ do not depend on the choice of the morphism f.*

Note that the Iitaka map and the Kodaira dimension of a movable log pair depend a priori on the positive integer $n \gg 0$ used in their definition. One can show that the Kodaira dimension does not depend on this quantity. Moreover, it follows from the Log Abundance (see [1]) that the Iitaka map also depends only on the properties of the movable log pair. Unfortunately, the Log Abundance is proved only in dimensions 2 and 3. Nevertheless, one can see a posteriori that the Iitaka maps of all movable log pairs in this paper do not depend on $n \gg 0$.

We use mostly[2] movable log pairs, and we shall call them simply log pairs.

§ 2. Log Calabi-Yau structures on a quintic 4-fold

We now show the relation between the objects introduced in the previous section and the main theorem.

Main assumption. In what follows X is a smooth quintic 4-fold in $\mathbb{P}^5$.

The adjunction formula and Lefschetz's theorem yield the relations

$$\operatorname{Pic} X = \mathbb{Z}K_X \quad \text{and} \quad K_X \sim \mathcal{O}_{\mathbb{P}^5}(-1)|_X.$$

We fix a log pair (X, M_X) and choose $\lambda \in \mathbb{Q}_{>0} \cup \{+\infty\}$ such that

$$K_X + \lambda M_X \sim_{\mathbb{Q}} 0,$$

where $\lambda = +\infty$ for $M_X = \varnothing$.

Agreement. In the case $\lambda = 1$ we call (X, M_X) a *log Calabi-Yau quintic* 4*-fold*.

This means simply that the log canonical divisor $K_X + M_X$ is $\mathbb{Q}$-rationally trivial.

Core theorem. *Let (X, M_X) be a log Calabi–Yau quintic* 4*-fold. Then (X, M_X) is canonical*, $\varkappa(X, M_X) = 0$, *and*

$$\operatorname{CS}(X, M_X) = \begin{cases} \varnothing, & \\ \{S\} & \text{for a linear subspace } S \subset X \text{ of dimension } 2, \\ \{X \cap H\} & \text{for a linear subspace } H \subset \mathbb{P}^5 \text{ of dimension } 3. \end{cases}$$

One may notice a similarity between the core theorem and the main theorem. In the next two sections we shall deduce the main theorem from the core theorem. Unfortunately, the core theorem says nothing about the structure of the boundary M_X. Nevertheless, we can obtain a rather precise description of M_X on the basis of the core theorem in the case $\operatorname{CS}(X, M_X) \neq \varnothing$.

[2] Except in § 5.

Refinement of the core theorem. *Under the assumptions of the core theorem*

$$M_X = \psi^{-1}(M_Y),$$

where the rational map $\psi\colon X \dashrightarrow Y$ *is a projection from* $\mathrm{CS}(X, M_X)$.[3]

Proof. Let S be the union of all elements of $\mathrm{CS}(X, M_X)$. It follows from the core theorem that S is a surface lying in a linear subspace $T \subset \mathbb{P}^5$ of dimension 3.

We consider a linear system $\mathcal{H}_T$ of hyperplane sections of X containing S. Next we choose a birational morphism $f\colon W \to X$ such that the linear system $f^{-1}(\mathcal{H}_T)$ is free and the variety W is smooth.

We may assume that f is an isomorphism outside S and W contains a single f-exceptional divisor lying over the generic point of each irreducible component of S. In the case when S is a linear subspace of $\mathbb{P}^5$ we set f to be a blow up of S.

Let

$$g = \varphi_{\mathcal{H}_T} \circ f \quad \text{and} \quad E = f^{-1}(S).$$

We take a sufficiently general divisor D in the linear system $f^{-1}(\mathcal{H}_T)$. Then

$$D \sim f^*(-K_X) - E - \sum_{i=1}^{k} a_i F_i,$$

where $a_i \in \mathbb{N}$ and

$$\dim f(F_i) \leqslant 1.$$

On the other hand,

$$f^{-1}(M_X)|_D \sim_{\mathbb{Q}} \begin{cases} \alpha D|_D \text{ for some } \alpha \in \mathbb{Q}_{>0} & \text{if } S \text{ is a 2-dimensional plane in } \mathbb{P}^5, \\ \sum\limits_{i=1}^{k} c_i F_i|_D \text{ for some } c_i \in \mathbb{Q} & \text{otherwise.} \end{cases}$$

These equivalences show that the boundary $f^{-1}(M_X)$ lies in the fibres of g. It is now easy to demonstrate that the map $g \circ f^{-1}$ is a projection from the locus $\mathrm{CS}(X, M_X)$.

To illustrate the result just proved consider the following example.

Example. Let $(X, b\mathcal{M})$ be a log Calabi-Yau quintic 4-fold such that the linear system $\mathcal{M}$ has an irreducible general member and $b \in \mathbb{Q}_{>0}$. Let

$$\mathrm{CS}(X, M_X) = \{X \cap H\},$$

where H is a 3-dimensional linear subspace of $\mathbb{P}^5$. Then it follows from the refinement of the core theorem that $\mathcal{M}$ is a pencil of hyperplane sections of X passing through H.

We prove the core theorem in §§ 5 and 6.

[3]We set ψ to be the identity map if $\mathrm{CS}(X, M_X) = \varnothing$.

§ 3. Iitaka maps and the quintic 4-fold

We use the notation and the assumptions of the previous section. The assertions of the core theorem impose fairly strong structural constraints on log pairs on X.

Corollary to the core theorem. *The following relations hold*:

$$\begin{aligned}\lambda = 1 &\iff \varkappa(X, M_X) = 0,\\ \lambda < 1 &\iff \varkappa(X, M_X) > 0,\\ \lambda > 1 &\iff \varkappa(X, M_X) = -\infty.\end{aligned}$$

Proof. All these relations are easy to prove. We shall show that $\lambda < 1$ implies $\varkappa(X, M_X) > 0$. Assume that $\lambda < 1$ and $\varkappa(X, M_X) \leqslant 0$. Then

$$0 = \varkappa(X, \lambda M_X) \leqslant \varkappa(X, M_X) \leqslant 0.$$

Hence $\varkappa(X, \lambda M_X) = \varkappa(X, M_X) = 0$. The last equality is easily seen to contradict the definition of the Kodaira dimension and the movability of the log pair (X, M_X).

Log pairs with $\varkappa(X, M_X) \in [1, 3]$ can be described rather explicitly.

Description theorem I. *Let* $\varkappa(X, M_X) \in [1, 3]$. *Then* (X, M_X) *is not canonical and*

$$I(X, M_X) = \begin{cases} \text{the restriction of a projection from} \\ \text{a 2-dimensional linear subspace } S \subset X, \\ \text{the restriction of a projection from} \\ \text{a 3-dimensional linear subspace } H \subset \mathbb{P}^5. \end{cases}$$

Moreover,

$$M_X = I(X, M_X)^{-1}(M_Y),$$

where $Y = I(X, M_X)(X)$.

Proof. The fact that $(X, \lambda M_X)$ is canonical is a consequence of the core theorem. Thus,

$$\varkappa(X, M_X) \geqslant \varkappa(X, \lambda M_X) = 0.$$

Assume now that $(X, \lambda M_X)$ is terminal. We select $\delta \in \mathbb{Q} \cap (\lambda, 1)$ such that $(X, \delta M_X)$ is still terminal. Then

$$4 = \varkappa(X, \delta M_X) \leqslant \varkappa(X, M_X) < 4.$$

Hence

$$\mathrm{CS}(X, \lambda M_X) \neq \varnothing.$$

The assertion now follows from the refinement of the core theorem.

Log pairs of Kodaira dimension $-\infty$ cannot be described so neatly.

Description theorem II. *Let* $\varkappa(X, M_X) = -\infty$. *Then* $\mathrm{CS}(X, M_X) = \varnothing$.

Proof. The log pair $(X, \lambda M_X)$ is canonical by the core theorem and the required result follows from the inequality $\lambda > 1$.

We shall see in the next chapter that Pukhlikov's theorems I and II follow from Description theorem II.

§ 4. Birational geometry of a quintic 4-fold

It is now time to prove the main theorem and Pukhlikov's theorems I and II using the results of the previous section, the core theorem and the refinement of the core theorem. In view of the main assumption, X is a smooth quintic 4-fold in $\mathbb{P}^5$.

Theorem A. *X is not birationally isomorphic to any fibration with general fibre of Kodaira dimension $-\infty$.*

Proof. Assume that ρ is a birational transformation of the quintic 4-fold X into a fibration $\tau\colon Y \to Z$ such that the general fibre of τ has Kodaira dimension $-\infty$. For example, τ can be a conic bundle or a del Pezzo fibration.

We take a 'sufficiently big' very ample divisor H on Z and choose $\mu \in \mathbb{Q}_{>0}$ such that

$$(X, M_X) = (X, \mu\rho^{-1}(|\tau^*(H)|))$$

is a log Calabi-Yau quintic 4-fold. By construction,

$$\varkappa(X, M_X) = -\infty;$$

but this contradicts the core theorem.

Note, that Theorem A covers one-half of Pukhlikov's theorem II. Our next result completes the proofs of Pukhlikov's theorems I and II and proves the first part of the main theorem.

Theorem B. $\operatorname{Bir} X = \operatorname{Aut} X$ *and X is not birational to any Fano 4-fold with canonical singularities that is not biregularly equivalent to X.*

Proof. We shall establish a slightly stronger result. Assume that we have a birational map $\rho\colon X \dashrightarrow Y$ such that Y is a weak Fano 4-fold with canonical singularities and the divisor $-K_Y$ is nef and big. We claim that ρ is an isomorphism.

For $n \gg 0$ the linear system $|-nK_Y|$ is well known to be free. Consider the log pairs

$$(Y, M_Y) = \left(Y, \frac{1}{n}\,|-nK_Y|\right) \quad \text{and} \quad (X, M_X) = (X, \rho^{-1}(M_Y)).$$

The corollary to the core theorem shows that (X, M_X) is a log Calabi-Yau quintic 4-fold, and the refinement of the core theorem ensures the terminality of the log pair (X, M_X). Thus, we can find $\zeta \in \mathbb{Q}_{>1}$ such that both log pairs $(X, \zeta M_X)$ and $(Y, \zeta M_Y)$ are canonical models. The uniqueness theorem now shows that ρ is an isomorphism.

The next result completes the proof of the main theorem.

Theorem C. *All fibrations birational to X and with general fibre of Kodaira dimension zero are induced by projections in $\mathbb{P}^5$.*

Proof. Let ρ be a birational transformation of the quintic X into a fibration $\tau\colon Y \to Z$ such that the Kodaira dimension of the general fibre of τ is zero. We take a 'sufficiently big' very ample divisor H on Z. The equality

$$\varkappa(X, \rho^{-1}(|\tau^*(H)|)) = \dim Z$$

and Description theorem I now ensure the required result.

§ 5. Proof of the core theorem, part I

In this section we prove one half of the core theorem. We use ideas based on Corti's ideas and results [9]. Fix a log Calabi-Yau 4-fold (X, M_X), where X is a smooth quintic 4-fold. The main result of this section is as follows.

Non-existence theorem. $\mathrm{CS}(X, M_X)$ *contains no points.*

We shall prove the non-existence theorem in several steps; meanwhile, we explain its importance.

Important observation. As shown in [6], Pukhlikov's theorems I and II are consequences of the non-existence theorem.

First, we outline the scheme of the proof of the non-existence theorem.

Global strategy: (1) Assume the existence of a point in $\mathrm{CS}(X, M_X)$; (2) replace (X, M_X) by a new log pair that contains the above-mentioned point in $\mathrm{CS}(X, M_X)$ as a centre of log canonical singularities (LCS); (3) reduce the non-existence theorem to a 3-fold problem.

Assume that $\mathrm{CS}(X, M_X)$ contains a point O and consider the log pair

$$(X, B_X) = (X, H_X + M_X),$$

where H_X is a sufficiently general hyperplane section of X passing through O.

Note that the log pair (X, B_X) is neither a movable nor an ordinary log pair. Nevertheless, we may handle it as an ordinary log pair.

Observation. $O \in \mathrm{LCS}(X, B_X)$.

Let $f\colon W \to X$ be a blow up of O and let $E = f^{-1}(O)$. Then

$$a(X, B_X, E) = a(X, M_X, E) - 1.$$

Note that, in principle, the f-exceptional divisor E can realize the point O as a centre of canonical singularities of the log pair (X, M_X) and as a centre of log canonical singularities of the log pair (X, B_X).

Lemma 5.1. *The following inequalities hold*:

$$a(X, B_X, E) > -1 \quad \textit{and} \quad a(X, M_X, E) > 0.$$

Proof. Assume the contrary. Then

$$\mathrm{mult}_O(M_X) \geqslant 3$$

and

$$5 = (-K_X)^2 \cdot M_X^2 \geqslant \mathrm{mult}_O(M_X^2) \geqslant \mathrm{mult}_O^2(M_X) \geqslant 9.$$

Consider the log pair

$$(W, B^W) = (W, (\mathrm{mult}_O(M_X) - 2)E + B_W) = (W, (\mathrm{mult}_O(M_X) - 2)E + H_W + M_W),$$

where $H_W = f^{-1}(H_X)$ and $M_W = f^{-1}(M_X)$.

Observation. By construction,

$$K_W + B^W \sim_{\mathbb{Q}} f^*(K_X + B_X).$$

The following result is a consequence of Lemma 5.1.

Lemma 5.2. $\mathrm{LCS}(W, B^W)$ *contains a proper irreducible subvariety of* E *not lying in* H_W.

Proof. The equivalence

$$(\mathrm{mult}_O(M_X) - 3)E + M_W \sim_{\mathbb{Q}} f^*(K_X + M_X)$$

and Lemma 5.1 yield

$$S \in \mathrm{CS}(W, (\mathrm{mult}_O(M_X) - 3)E + M_W),$$

where S is a proper irreducible subvariety of the f-exceptional divisor E. Hence

$$S \in \mathrm{LCS}(W, (\mathrm{mult}_O(M_X) - 2)E + M_W)$$

and the assumption that H_X is general completes the proof.

We consider now a subvariety of W that plays a rather important role in the proof of the non-existence theorem.

New object. Let S be an element of maximum dimension of $\mathrm{LCS}(W, B^W)$ such that S is a proper subvariety of E and $S \not\subset H_W$.

Note that S can be a point, a curve, or a surface.

Local strategy: (1) prove that S is not a surface; (2) deduce from Shokurov's connectedness theorem that S is not a point; (3) show that S is a 'line' in $E \cong \mathbb{P}^3$.

We shall now use a result of Corti's — more precisely, Theorem 3.1 of [9].

Corti's lemma. *Let* P *be a smooth point on a surface* H *and assume that for some non-negative rational numbers* a_1 *and* a_2,

$$P \in \mathrm{LCS}(H, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H),$$

where the boundary M_H *is movable and the irreducible reduced curves* Δ_1 *and* Δ_2 *have a normal intersection at* P. *Then*

$$\mathrm{mult}_P(M_H^2) \geqslant \begin{cases} 4a_1a_2 & \text{if } a_1 \leqslant 1 \text{ or } a_2 \leqslant 1, \\ 4(a_1 + a_2 - 1) & \text{if } a_1 > 1 \text{ and } a_2 > 1. \end{cases}$$

Note that Corti's lemma here differs slightly, but not significantly, from its original form.

Lemma 5.3. *S is not a surface.*

Proof. Assume the contrary. The fact that

$$S \in \mathrm{LCS}(W, (\mathrm{mult}_O(M_X) - 2)E + M_W)$$

allows us to apply Corti's lemma. This gives us the inequality

$$\mathrm{mult}_S(M_W^2) \geqslant 4(3 - \mathrm{mult}_O(M_X)).$$

Thus,

$$\mathrm{mult}_O(M_X^2) \geqslant \mathrm{mult}_O^2(M_X) + \mathrm{mult}_S(M_W^2) \geqslant \mathrm{mult}_O^2(M_X) + 4(3 - \mathrm{mult}_O(M_X)).$$

Hence

$$5 = (-K_X)^2 \cdot M_X^2 \geqslant \mathrm{mult}_O(M_X^2) \geqslant (\mathrm{mult}_O(M_X) - 2)^2 + 8.$$

The next result is a special case of Shokurov's connectedness theorem (see [10]).

Connectedness theorem. $\mathrm{LCS}(W, B^W)$ *is connected in the neighbourhood of* E.

In particular, we have the following result.

Lemma 5.4. *S is not a point.*

Thus, S is a curve in E. Corollary 3.6 in [9] shows that S is a 'line' in $E \cong \mathbb{P}^3$. Nevertheless, to keep this paper self-contained we prove this result as the following lemma.

Lemma 5.5. *S is a 'line' in $E \cong \mathbb{P}^3$.*

Proof. We know that the set

$$\mathrm{LCS}(W, (\mathrm{mult}_O(M_X) - 2)E + H_W + M_W)$$

is connected in the neighbourhood of E and we have much freedom in our choice of H_X. Combined with the adjunction formula this shows that

$$\mathrm{LCS}(H_W, (\mathrm{mult}_O(M_X) - 2)E|_{H_W} + M_W|_{H_W})$$

contains only points and

$$\{S \cap H_W\} \subset \mathrm{LCS}(H_W, (\mathrm{mult}_O(M_X) - 2)E|_{H_W} + M_W|_{H_W}).$$

Applying Shokurov's connectedness theorem to

$$\mathrm{LCS}(H_W, (\mathrm{mult}_O(M_X) - 2)E|_{H_W} + M_W|_{H_W})$$

and the morphism $f|_{H_W}$ we obtain the connectedness of

$$\mathrm{LCS}(H_W, (\mathrm{mult}_O(M_X) - 2)E|_{H_W} + M_W|_{H_W})$$

in the neighbourhood of $E|_{H_W}$. Hence $S \cap H_W$ consists of one point.

We now summarize what we have established so far.

Summary. $\mathrm{LCS}(W, B^W)$ *contains an irreducible curve* $S \not\subset H_W$ *such that* S *is a 'line' in* $E \cong \mathbb{P}^3$.

We now come down from a 4-fold to a 3-fold.

Local strategy: (1) restrict 'everything' to a special hyperplane section passing through the point O; (2) use a result of Lemma 5.3 type to show that the quintic X contains a plane passing through the point O.

Note that we have, incidentally, shown that

$$O \in \mathrm{LCS}(H_X, M_X|_{H_X}).$$

Unfortunately, this is not sufficient for the proof of the non-existence theorem. We require a stronger result.

Let Y be a sufficiently general hyperplane section of X passing through O such that

$$S \subset f^{-1}(Y).$$

We set

$$(Y, M_Y) = (Y, M_X|_Y).$$

Warning. Y may be singular and the log pair (Y, M_Y) may no longer be movable!

Nevertheless, we can rather freely handle the log pair (Y, M_Y) in the neighbourhood of the point O.

Remark. The point O is smooth on Y and

$$O \in \mathrm{LCS}(Y, M_Y).$$

Let $g\colon V \to Y$ be a blow up of O and let $F = g^{-1}(O)$. By construction

$$S \subset F, \quad E|_V = F, \quad \mathrm{mult}_O(M_Y) = \mathrm{mult}_O(M_X), \quad g^{-1}(M_Y) = M_W|_V.$$

Consider now the log pair

$$(V, M^V) = (V, (\mathrm{mult}_O(M_Y) - 2)F + M_V),$$

where $M_V = g^{-1}(M_Y)$. Then

$$K_V + M^V \sim_{\mathbb{Q}} f^*(K_Y + M_Y).$$

Most important property of Y. $S \in \mathrm{LCS}(V, M^V)$.

Proof. By construction $S \subset V$ and

$$S \in \mathrm{LCS}(W, B^W).$$

By the adjunction formula and Shokurov's connectedness theorem the log pair (V, M^V) is not log terminal in the neighbourhood of the curve S.

Assume that $\mathrm{LCS}(V, M^V)$ does not contain the curve S. Then $\mathrm{LCS}(V, M^V)$ contains a point on the curve S. Moreover, applying Shokurov's connectedness theorem to the log pair (V, M^V) again, we conclude that $\mathrm{LCS}(V, M^V)$ contains precisely one point on S.

Let $h\colon U \to W$ be a blow up of S and let $G = h^{-1}(S)$. Consider the log pair

$$(U, B^U) = (U, h^{-1}(B^W) + (\mathrm{mult}_S(B^W) - 2)G).$$

Then

$$K_U + B^U \sim_{\mathbb{Q}} h^*(K_W + B^W).$$

The adjunction formula and Shokurov's connectedness theorem ensure (see the proof of Lemma 5.1) the existence of a curve $\widetilde{S} \subset G$ in $\mathrm{LCS}(U, B^U)$ such that $\widetilde{S}$ is a section of $h|_G$ and either $\widetilde{S}$ lies in the 3-fold $h^{-1}(V)$ or the intersection of $\widetilde{S}$ and $h^{-1}(V)$ consists of one point.

Note that everything here is local with respect to X. Hence, applying the Kawamata–Viehweg vanishing theorem (see [1]) to the divisor $h^{-1}(V)-G$ we obtain the surjectivity

$$H^0(h^{-1}(V)) \to H^0(h^{-1}(V)|_G) \to 0.$$

On the other hand, the linear system $\left|h^{-1}(V)|_G\right|$ is free and, therefore, in view of the generality in our choice of Y, the curve $\widetilde{S}$ does not lie in $h^{-1}(V)$.

Direct calculations show that the divisor $h^{-1}(V)|_G$ is nef and big on G. Moreover, its intersection with each section of the morphism $h|_G$ is either trivial or contains more than one point. However, we have already proved that the intersection of $\widetilde{S}$ and the 3-fold $h^{-1}(V)$ consists of a single point.

How can we use the most important property of Y?

Observation. We must apply arguments similar to the proof of Lemma 5.3 to S and Y to arrive at a contradiction.

Recall that the log pair (Y, M_Y) is not necessarily movable.

Lemma 5.6. *The log pair (Y, M_Y) is not movable.*

Proof. If (Y, M_Y) is movable, then we can repeat the proof of Lemma 5.3 word for word and obtain a contradiction.

Thus, it would be nice to adjust the proof of Lemma 5.3 so that one can apply it to the non-movable log pair (Y, M_Y).

Observation. X contains a 2-dimensional plane P such that the curve S lies in $f^{-1}(P)$, because otherwise (Y, M_Y) is movable.

Note that the multiplicity of P in M_Y is equal to the multiplicity of P in the boundary M_X.

Decomposition. We have

$$M_Y = \operatorname{mult}_P(M_X)P + R_Y,$$

where the log pair (Y, R_Y) is movable.

What shall we do now?

Local strategy: (1) use Corti's lemma to obtain a lower bound for $\operatorname{mult}_O(R_Y^2)$; (2) use the properties of the embedding of Y in $\mathbb{P}^4$ to obtain an upper bound for $\operatorname{mult}_O(R_Y^2)$.

Note that our proof of Lemma 5.3 is based on Corti's lemma for two normally intersecting prime divisors and one movable boundary.

Observation. The divisors F and $g^{-1}(P)$ intersect normally at S.

Thus, we can apply Corti's lemma to the log pair

$$(V, M^V) = (V, (\operatorname{mult}_O(R_Y) + \operatorname{mult}_P(M_X) - 2)F + \operatorname{mult}_P(M_X)g^{-1}(P) + R_V),$$

where $R_V = g^{-1}(R_Y)$. This gives us

$$\operatorname{mult}_S(R_V^2) \geqslant 4(3 - \operatorname{mult}_O(R_Y) - \operatorname{mult}_P(M_X))(1 - \operatorname{mult}_P(M_X)).$$

Combining this with the relation

$$\mathrm{mult}_O(R_Y^2) \geqslant \mathrm{mult}_O^2(R_Y) + \mathrm{mult}_S(R_V^2)$$

we obtain the following inequality.

Important inequality I:

$$\mathrm{mult}_O(R_Y^2) \geqslant (\mathrm{mult}_O(R_Y) - 1 + \mathrm{mult}_P(M_X))^2 + 8(1 - \mathrm{mult}_P(M_X)).$$

We shall now find an upper estimate of $\mathrm{mult}_O(R_Y^2)$. Let Z be a sufficiently general hyperplane section of Y passing through the point O.

Observation. Z is a smooth quintic surface in $\mathbb{P}^3$.

On the one hand

$$(R_Y|_Z)^2 \geqslant \mathrm{mult}_O(R_Y^2).$$

On the other hand

$$(R_Y|_Z)^2 = ((Z - \mathrm{mult}_P(M_X)P)|_Z)^2 = 5 - 2\,\mathrm{mult}_P(M_X) - 3\,\mathrm{mult}_P^2(M_X).$$

Hence we obtain the following inequality.

Important inequality II:

$$\mathrm{mult}_O(R_Y^2) \leqslant 5 - 2\,\mathrm{mult}_P(M_X) - 3\,\mathrm{mult}_P^2(M_X).$$

Proof of the non-existence theorem. Important inequalities I and II yield

$$\mathrm{mult}_P(M_X) = 1 \quad \text{and} \quad \mathrm{mult}_O(R_Y) = 0.$$

Thus,

$$\mathrm{mult}_O(M_X) = \mathrm{mult}_O(M_Y) = \mathrm{mult}_P(M_X) + \mathrm{mult}_O(R_Y) = 1.$$

Hence

$$O \notin \mathrm{CS}(X, M_X).$$

§ 6. Proof of the core theorem, part II

This section completes the proof of the core theorem.

As in the previous section, let (X, M_X) be a smooth log Calabi-Yau quintic 4-fold. We may assume that $\mathrm{CS}(X, M_X) \neq \varnothing$. Hence the non-existence theorem ensures the existence of a variety S with $\dim S \neq 0$ such that

$$S \in \mathrm{CS}(X, M_X).$$

In [6] Pukhlikov proved the following result.

Pukhlikov's lemma. *The following equality holds*:

$$\mathrm{mult}_S(M_X) = 1.$$

Observation. It follows from the non-existence theorem and Pukhlikov's lemma that the log pair (X, M_X) is canonical.

Global strategy: (1) obtain restrictions on the dimension and the degree of S; (2) show that S lies in a 3-dimensional linear subspace; (3) use the properties of the embedding of X in $\mathbb{P}^5$ to complete the proof of the core theorem.

Note that S is either a surface or a curve.

Lemma 6.1. *S is a surface.*

Proof. Assume that S is a curve. Let $f\colon W \to X$ be a blow up of a generic point in the curve S.[4] We have

$$a(X, M_X, E) = 1.$$

Hence there exists a proper subvariety T of E such that

$$T \in \mathrm{CS}(W, f^{-1}(M_X) - E)$$

and the morphism $f|_T\colon T \to S$ is surjective. Thus,

$$\mathrm{mult}_S(M_X) \geqslant \mathrm{mult}_T(g^{-1}(M_X)) > 1.$$

Lemma 6.2. $\deg S \leqslant 5$.

Proof. The required result is a consequence of the inequality

$$5 = (-K_X)^2 \cdot M_X^2 \geqslant \mathrm{mult}_S(M_X^2) \deg S \geqslant \deg S.$$

Lemma 6.3. *S lies in a 3-dimensional linear subspace of $\mathbb{P}^5$.*

Proof. Assume the contrary. Taking the intersection of X with a general hyperplane section we can assume that X is a quintic in $\mathbb{P}^4$ containing the curve S. Using the method of [6] we shall show that the curve S is contained in a 2-dimensional linear subspace of $\mathbb{P}^4$.

Consider a sufficiently general cone R_S over the curve S. Then

$$R_S \cap X = S \cup \widetilde{S}$$

and $\deg \widetilde{S} = 4 \deg S$. Let

$$Z = \mathrm{Supp}\biggl(\bigcup_{i=1}^{n} \mathrm{Bs}\,\mathcal{M}_i\biggr).$$

[4] We may assume that W is a quasiprojective variety.

Then $S \subset Z$; but since R_S is general, it follows that $\widetilde{S} \not\subset Z$. As Pukhlikov showed in [6], the generality of R_S means that the curves S and $\widetilde{S}$ intersect transversally at $4 \deg S$ distinct points.

On the other hand,

$$4 \deg S = \deg \widetilde{S} = \deg M_X|_{\widetilde{S}} \geqslant 4 \deg S \operatorname{mult}_S(M_X) = 4 \deg S.$$

Thus,

$$\widetilde{S} \cap M_X = S \cap \widetilde{S}.$$

Note that the general secant of the curve S intersects X at precisely 5 points, because otherwise it lies in X and must be a component of Z. Consider now the divisor

$$D = \sum_{i=1}^{n} b_i M_i,$$

where M_i is a general member of the linear system $\mathcal{M}_i$. By assumption,

$$\operatorname{mult}_S(D) = 1.$$

We choose two general points P_S and P_D in the curve S and the divisor D, respectively. Let L be a line passing through P_S and P_D, and let P be a sufficiently general point in L. We denote by $R_{S,P}$ the cone over the curve S with vertex P, and let

$$R_{S,P} \cap X = S \cup \widetilde{S}_P.$$

We already showed before that the divisor D either contains the curve $\widetilde{S}_P$ or intersects it only at the points in $S \cap \widetilde{S}_P$. By construction,

$$P_D \in \widetilde{S}_P \cap D \quad \text{and} \quad P_D \notin S.$$

Hence $\widetilde{S}_P \subset D$ and, in particular,

$$L \cap X \subset D.$$

The last condition is closed, and we can assume that

$$P_D \in S \setminus P_S.$$

Hence, in view of the generality of D, a general secant of the curve S intersects Z at 5 distinct points.

On the other hand, let A be a set of 3 distinct collinear points in a general hyperplane section of Z. Then one can show that A lies in some plane component of Z.

We now summarize what we have already proved.

Summary. $\mathrm{CS}(X, M_X)$ *contains a surface* S *with* $\deg S \in [1,5]$ *such that* S *lies in a* 3*-dimensional linear subspace* $T \subset \mathbb{P}^5$.

Note that we have not used anywhere the irreducibility of the surface S, although it has been convenient to keep this property in mind.

Important remark. We can assume that S is the union of all elements of $\mathrm{CS}(X, M_X)$.

This apparently simple remark brings us to the following useful observation.

Observation. Either of the equalities $\deg S = 1$ or $\deg S = 5$ establishes the core theorem!

Thus, we can assume that $\deg S \in [2,4]$. Let $\mathcal{H}_T$ be a pencil on X cut by the hyperplane sections containing T. Then

$$X \cap T = S \cup \sum_{i=1}^{r} S_i,$$

where the S_i are irreducible surfaces.

Observation. All the surfaces S_i are reduced.

To prove the core theorem we merely need to show that $\mathrm{CS}(X, M_X)$ contains all the S_i.

Reduction. As in the proof of Lemma 6.3, by taking the intersection of X with the general hyperplane section we may assume that X is a quintic 3-fold, T is 2-dimensional linear subspace, and S and all the S_i are curves.

What can we do now?

Local strategy: Consider the intersection form of the curves S_i on a hyperplane section of X containing T.

Consider a surface D in the pencil $\mathcal{H}_T$ that is smooth at the points of intersection of the curve S with the curves S_i. On D we have

$$\left(\sum_{i=1}^{r} S_i\right) \cdot S_j = (D|_D - S) \cdot S_j = \deg S_j - S \cdot S_j.$$

On the plane T,

$$\deg S_j - S \cdot S_j = \deg S_j - \deg S \deg S_j < 0.$$

Thus,

$$(S \cdot S_j)_D = (S \cdot S_j)_T$$

because the surface D is smooth at the points of intersection of S with the curves S_i. It follows from [11] that the intersection form of the curves S_i on the surface D is negative-definite.

Proof of the core theorem. The divisor

$$M_X|_D - S - \sum_{i=1}^{r} \mathrm{mult}_{S_i}(M_X) S_i$$

is nef on the surface D.

On the other hand, on the surface D we have

$$M_X|_D - S - \sum_{i=1}^{r} \mathrm{mult}_{S_i}(M_X)S_i \sim_{\mathbb{Q}} \sum_{i=1}^{r}(1 - \mathrm{mult}_{S_i}(M_X))S_i$$

and

$$\sum_{i=1}^{r}(1 - \mathrm{mult}_{S_i}(M_X))S_i \cdot S_j \geqslant 0, \qquad j = 1, \dots, r.$$

Since the intersection form of the curves S_i on D is negative-definite, it follows that

$$\mathrm{mult}_{S_i}(M_X) \geqslant 1$$

for all S_i.

Addendum

We fix a smooth hypersurface X of degree n in $\mathbb{P}^n$, where $n = 6$, 7 or 8. In this section we shall generalize our main theorem in the following way.

Hypersurface theorem. $\mathrm{Bir}\, X = \mathrm{Aut}\, X$ *and the hypersurface* X *is not birationally isomorphic to the following varieties*: *Fano varieties with canonical singularities that are not biregular to* X, *fibrations with general fibre of Kodaira dimension* $-\infty$, *fibrations with general fibre of Kodaira dimension zero distinct from fibrations induced by projections from an* $(n-2)$-*dimensional linear subspace of* $\mathbb{P}^n$.

Fix a log pair (X, M_X) such that

$$K_X + M_X \sim_{\mathbb{Q}} 0.$$

It follows from the proof of the main theorem that the hypersurface theorem is a consequence of the following generalization of the core theorem.

Core theorem I. (X, M_X) *is canonical*, $\varkappa(X, M_X) = 0$, *and*

$$\mathrm{CS}(X, M_X) = \begin{cases} \varnothing, & \\ \{X \cap H\} & \text{for a linear subspace} \\ & H \subset \mathbb{P}^n \text{ of dimension } n-2. \end{cases}$$

Moreover, it is clear in its turn from the proof of the core theorem that, in place of core theorem I, it suffices to demonstrate the following analogue of the non-existence theorem.

Statement. $\mathrm{CS}(X, M_X)$ *contains no points.*

Assume that $\mathrm{CS}(X, M_X)$ contains a point O. Consider the log pair

$$(X, B_X) = (X, H_X + M_X),$$

where

$$H_X = \sum_{i=1}^{n-4} H_i$$

and the divisors H_i are general hyperplane sections of X passing through O.

Observation. The log pair (X, B_X) is not log canonical at the point O.

Let $f\colon W \to X$ be a blow up of O and let $E = f^{-1}(O)$. Then

$$a(X, B_X, E) = a(X, M_X, E) - n + 4$$

and the proof of Lemma 5.1 gives us the following inequalities:

$$a(X, B_X, E) > -1, \qquad a(X, M_X, E) > 0.$$

Consider now the log pair

$$(W, B^W) = (W, (\mathrm{mult}_O(M_X) - 2)E + B_W) = (W, (\mathrm{mult}_O(M_X) - 2)E + H_W + M_W),$$

where $H_W = f^{-1}(H_X)$ and $M_W = f^{-1}(M_X)$. By construction,

$$K_W + B^W \sim_{\mathbb{Q}} f^*(K_X + B_X).$$

Arguments similar to the proof of Lemma 5.2 show that the set $\mathrm{LCS}(W, B^W)$ contains a proper irreducible subvariety $S \subset E$ such that S does not lie in H_W and the log pair (W, B^W) is not log canonical at the generic point of S.

Let S be an element of maximum dimension among all elements of $\mathrm{LCS}(W, B^W)$ possessing the above properties. Then the proof of Lemma 5.3 shows that the codimension of S is larger than 2, while the proof of Lemma 5.4 shows that S is not a point.

Reduction to a 3*-fold.* Let $Z = \bigcap_{i=1}^{n-4} H_i$. Consider the log pair

$$(Z, M_Z) = (Z, M_X|_Z).$$

In view of the generality of H_X, Z is smooth and the log pair (Z, M_Z) is movable. Note that the adjunction formula yields

$$K_Z + M_Z \sim_{\mathbb{Q}} (K_X + B_X)|_Z.$$

What properties does the log pair (Z, M_Z) inherit from (X, M_X)?

Lemma. *The log pair (Z, M_Z) is not log canonical at the point O.*

Proof. This follows from the repeated use of the adjunction formula and Shokurov's connectedness theorem (see [10]).

Let $h\colon U \to Z$ be a blow up of the point O and let $G = h^{-1}(O)$. Then, by construction,

$$E|_U = G, \quad \mathrm{mult}_O(M_Z) = \mathrm{mult}_O(M_X), \quad h^{-1}(M_Z) = M_W|_U.$$

Consider now the log pair

$$(U, M^U) = (U, (\mathrm{mult}_O(M_Z) - 2)G + M_U),$$

where $M_U = h^{-1}(M_Z)$. Then the adjunction formula yields

$$K_U + M^U \sim_{\mathbb{Q}} f^*(K_Z + M_Z) \sim_{\mathbb{Q}} (K_W + B^W)|_U.$$

Observation. In view of the generality of H_X,

$$\mathrm{LCS}(U, M^U) = \mathrm{LCS}(W, (\mathrm{mult}_O(M_X) - 2)E + M_W) \cap U.$$

Corollary 3.6 of [9] implies the following property of Z, which we prove here to keep our paper self-contained.

Main property of Z. $\mathrm{LCS}(U, M^U)$ *consists of one point and the log pair (U, M^U) is not log canonical at this point.*

Proof. It follows from the last observation that $\mathrm{LCS}(U, M^U)$ contains only points. Shokurov's connectedness theorem (see [10]) shows that $\mathrm{LCS}(U, M^U)$ contains precisely one point, and by the lemma the log pair (U, M^U) is not log canonical at this point.

The main property of Z and the last observation ensure that S is a linear subspace of $E \cong \mathbb{P}^{n-2}$ of dimension $n - 4$. Let Y be a sufficiently general hyperplane section of X passing through the point O such that

$$S \subset f^{-1}(Y).$$

We set

$$(Y, M_Y) = (Y, M_X|_Y).$$

Warning. Y may be singular.

Nevertheless, we can still handle the log pair (Y, M_Y) in the neighbourhood of the point O because O is smooth on Y.

Observation. The log pair (Y, M_Y) is movable.

Let $g\colon V \to Y$ be a blow up of O and let $F = g^{-1}(O)$. Then, by construction,

$$S \subset F, \quad E|_V = F, \quad \mathrm{mult}_O(M_Y) = \mathrm{mult}_O(M_X), \quad g^{-1}(M_Y) = M_W|_V.$$

Consider the log pair

$$(V, (\mathrm{mult}_O(M_Y) - 2)F + M_V),$$

where $M_V = g^{-1}(M_Y)$.

Most important property of Y. *The log pair*

$$(V, (\mathrm{mult}_O(M_Y) - 2)F + M_V)$$

is not log canonical at the generic point in S.

Proof. Repeatedly restricting 'everything' to the general hyperplane section of X passing through the point O and applying the adjunction formula we can assume that W is a 4-fold, V is a 3-fold, and S is a curve. We can now use the arguments from the proof of the analogous result in § 5.

Hence Corti's lemma yields the relation

$$\mathrm{mult}_S(M_V^2) > 4(3 - \mathrm{mult}_O(M_Y)).$$

Thus,

$$\mathrm{mult}_O(M_Y^2) \geqslant \mathrm{mult}_O^2(M_Y) + \mathrm{mult}_S(M_V^2) > \mathrm{mult}_O^2(M_Y) + 4(3 - \mathrm{mult}_O(M_Y)).$$

Finally,

$$n = (-K_X)^{n-4} \cdot M_Y^2 \geqslant \mathrm{mult}_O(M_Y^2) > (\mathrm{mult}_O(M_Y) - 2)^2 + 8.$$

Bibliography

[1] Y. Kawamata, K. Matsuda, and K. Matsuki, "Introduction to the minimal model problem", *Adv. Stud. Pure Math.* **10** (1987), 283–360.

[2] I. A. Cheltsov, "Log models of birationally rigid varieties", *J. Math. Sci.* (2000) (to appear).

[3] V. A. Iskovskikh and Yu. I. Manin, "Three-dimensional quartics and counterexamples to the Lüroth problem", *Mat. Sb.* **86 (128)**:1 (1971), 140–166; English transl. in *Math. USSR-Sb.* **15** (1971).

[4] V. A. Iskovskikh, "Birational automorphisms of three-dimensional algebraic varieties", *Itogi Nauki i Tekhniki, Ser. Sovrem. Probl. Mat.* **12** (1978), 159–236; English transl. in *J. Soviet Math.* **13** (1980).

[5] A. V. Pukhlikov, "Birational automorphisms of four-dimensional quintic", *Invent. Math.* **87** (1987), 303–329.

[6] A. V. Pukhlikov, "Notes on a theorem of V. A. Iskovskikh and Yu. I. Manin about the 3-fold quartic", *Trudy Mat. Inst. Steklov.* **208** (1995), 278–289; English transl. in *Proc. Steklov Inst. Math.* **208** (1995).

[7] A. Corti, "Factorizing birational maps of threefolds after Sarkisov", *J. Algebraic Geom.* **4** (1995), 223–254.

[8] V. V. Shokurov, "3-fold log models", *J. Math. Sci.* **81** (1996), 2667–2699.

[9] A. Corti, "Singularities of linear systems and 3-fold birational geometry", Preprint 11, Warwick, 1999.

[10] V. V. Shokurov, "Semistable 3-fold flips", *Izv. Ross. Akad. Nauk Ser. Mat.* **57** (1993), 165-222; English transl. in *Russian Acad. Sci. Izv. Math.* **42**:2 (1994).

[11] M. Artin, "On isolated rational singularities of surfaces", *Amer. J. Math.* **88** (1966), 129–136.

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