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On a conjecture of Ciliberto

I. A. Cheltsov

Abstract. We prove that a threefold hypersurface of degree d with at most ordinary double points is factorial if it contains no planes and has at most $(d-1)^2$ singular points.

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§ 1. Introduction

Let X be a normal hypersurface in $\mathbb{P}^4$ of degree $d \geqslant 3$ that has at most isolated singular points. The hypersurface X can be given by an equation

$$f(x, y, z, t, u) = 0 \subset \mathbb{P}^4 \cong \operatorname{Proj}(\mathbb{C}[x, y, z, t, u]),$$

where $f(x, y, z, t, u)$ is a homogeneous polynomial of degree d.

Definition 1.1. The hypersurface X is *factorial* if every Weil divisor on X is a Cartier divisor.

It is well known that the following conditions are equivalent:

– the hypersurface X is factorial;

– each surface $S \subset X$ is cut out on X by a hypersurface in $\mathbb{P}^4$;

– the quotient ring

$$\mathbb{C}[x, y, z, t, u]/\langle f(x, y, z, t, u)\rangle$$

is a unique factorization domain.

Example 1.2. Suppose that the hypersurface X is given by the equation

$$xg(x, y, z, t, u) + yh(x, y, z, t, u) = 0 \subset \mathbb{P}^4 \cong \operatorname{Proj}(\mathbb{C}[x, y, z, t, u]),$$

where g and h are general homogeneous polynomials of degree $d-1$. Then

$$|\operatorname{Sing}(X)| = (d-1)^2,$$

the hypersurface X has at most isolated ordinary double points, X contains the plane $x = y = 0$, but the hypersurface X is not factorial.

Example 1.3. Suppose that the hypersurface X is given by the equation

$$xg(x, y, z, t, u) + (yz + tu)h(x, y, z, t, u) = 0 \subset \mathbb{P}^4 \cong \operatorname{Proj}(\mathbb{C}[x, y, z, t, u]),$$

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where g is a general homogeneous polynomial of degree $d-1$ and h is a general homogeneous polynomial of degree $d-2$. Then

$$|\mathrm{Sing}(X)| = 2(d-1)(d-2),$$

the hypersurface X has at most isolated ordinary double points, X contains the quadric surface $x = yz + tu = 0$, but the hypersurface X is not factorial.

It is natural to expect the following to be true (see [1]).

Conjecture 1.4. The hypersurface X is factorial in the case when

$$|\mathrm{Sing}(X)| \leqslant 2(d-1)(d-2),$$

the hypersurface X has at most isolated ordinary double points, and the hypersurface X contains neither planes nor quadric surfaces.

Currently, the assertion of Conjecture 1.4 has only been proved for $d \leqslant 4$ (see [2], [3]), however, the following weaker version of Conjecture 1.4 holds (see [2] and [4]–[9]).

Theorem 1.5. *The hypersurface X is factorial in the case when*

$$|\mathrm{Sing}(X)| < (d-1)^2$$

and the hypersurface X has only isolated ordinary double points.

Recently Youngho Woo announced the following result.

Theorem 1.6. *The hypersurface X is factorial in the case when*

$$|\mathrm{Sing}(X)| \leqslant (d-1)^2,$$

the hypersurface X has at most isolated ordinary double points and X contains no planes.

The aim of this paper is to give an independent geometric proof of Theorem 1.6, which is based on the results obtained in [8] and [9]. Our paper has the following structure: in § 2 we consider some auxiliary results; in § 3 we prove Theorem 3.1, which is used in the proof of Theorem 1.6; in § 4 we prove Theorem 1.6 omitting the proof of Lemma 4.10; in § 5 we prove Lemma 4.10.

§ 2. Auxiliary results

Let Σ be a finite nonempty subset of $\mathbb{P}^n$, $n \geqslant 2$, and let ξ be a natural number. Then the points of Σ impose independent linear conditions on hypersurfaces in $\mathbb{P}^n$ of degree ξ if and only if for every point $P \in \Sigma$ there exists a hypersurface of degree ξ that contains $\Sigma \setminus P$ and does not contain the point $P \in \Sigma$.

Let us consider Σ as a subscheme of $\mathbb{P}^n$. Then there is an exact sequence of sheaves

$$0 \longrightarrow \mathscr{I}_\Sigma \otimes \mathscr{O}_{\mathbb{P}^n}(\xi) \longrightarrow \mathscr{O}_{\mathbb{P}^n}(\xi) \longrightarrow \mathscr{O}_\Sigma \longrightarrow 0,$$

where $\mathscr{I}_\Sigma$ is the ideal sheaf of the subscheme Σ. Thus Σ imposes independent linear conditions on hypersurfaces of degree ξ if and only if $h^1(\mathscr{I}_\Sigma \otimes \mathscr{O}_{\mathbb{P}^n}(\xi)) = 0$.

Theorem 2.1. *Suppose that the subscheme Σ is a closed subscheme of a zero-dimensional scheme Γ that is a zero-dimensional complete intersection of n hypersurfaces $X_1, \ldots, X_n$ in $\mathbb{P}^n$. Let Λ be a closed subscheme of the scheme Γ such that*

$$\mathscr{I}_\Lambda = \operatorname{Ann}(\mathscr{I}_\Sigma/\mathscr{I}_\Gamma),$$

where $\mathscr{I}_\Lambda$ and $\mathscr{I}_\Gamma$ are the ideal sheaves of the subschemes Λ and Γ, respectively. Then

$$h^1(\mathscr{I}_\Sigma \otimes \mathscr{O}_{\mathbb{P}^n}(\xi)) = h^0\Bigg(\mathscr{I}_\Lambda \otimes \mathscr{O}_{\mathbb{P}^n}\Bigg(\sum_{i=1}^{n} \deg(X_i) - n - 1 - \xi\Bigg)\Bigg) - h^0\Bigg(\mathscr{I}_\Gamma \otimes \mathscr{O}_{\mathbb{P}^n}\Bigg(\sum_{i=1}^{n} \deg(X_i) - n - 1 - \xi\Bigg)\Bigg).$$

This is a consequence of Theorem 3 in [10].

Lemma 2.2. *If $\xi \geqslant 2$ and at most $k\xi + 1$ points of the subset Σ are contained in a linear subspace of dimension k for every $k \in \mathbb{N}$, then the set Σ imposes independent linear conditions on hypersurfaces of degree ξ.*

This is a consequence of Theorem 2 in [11].

Lemma 2.3. *Let P be a point in Σ. Suppose that $n = 2$, the inequality*

$$|\Sigma \setminus P| \leqslant \max\Bigg\{\Bigg\lfloor \frac{\xi+3}{2} \Bigg\rfloor \Bigg(\xi + 3 - \Bigg\lfloor \frac{\xi+3}{2} \Bigg\rfloor\Bigg) - 1,\ \Bigg\lfloor \frac{\xi+3}{2} \Bigg\rfloor^2\Bigg\},$$

holds, $\xi \geqslant 3$ and at most

$$k(\xi + 3 - k) - 2$$

points in $\Sigma \setminus P$ lie on a curve of degree k for every $k \leqslant (\xi+3)/2$. Then there is a curve in $\mathbb{P}^2$ of degree ξ that contains $\Sigma \setminus P$ and does not contain $P \in \Sigma$.

This is a special case of Corollary 4.3 in [12].

Let $\Pi \subset \mathbb{P}^n$ be a linear subspace of dimension $m < n$, let $\Omega \subset \mathbb{P}^n$ be a general linear subspace of dimension $n - m - 1$ and let

$$\psi\colon \mathbb{P}^n \dashrightarrow \Pi \cong \mathbb{P}^m$$

be a linear projection from Ω. Suppose that $m \geqslant 2$. Let λ be a natural number.

Lemma 2.4. *Let $\mathscr{M}$ be a linear system consisting of hypersurfaces in $\mathbb{P}^n$ of degree λ that contain all points of Σ. Then the base locus of the linear system $\mathscr{M}$ is zero-dimensional if*

- *the set Σ is not contained in any irreducible curve of degree λ;*
- *the set $\psi(\Sigma)$ is contained in some irreducible curve of degree λ.*

Proof. We may assume that $m = 2$. Suppose that there is an irreducible curve $Z \subset \mathbb{P}^n$ which is contained in the base locus of the linear system $\mathscr{M}$. Also suppose that

- the set Σ is not contained in an irreducible curve of degree λ;
- the set $\psi(\Sigma)$ is contained in some irreducible curve of degree λ.

Put $\Xi = Z \cap \Sigma$. We may assume that the restriction $\psi|_Z$ is a birational morphism and

$$\psi(Z) \cap \psi(\Sigma \setminus \Xi) = \varnothing$$

because the linear subspace Ω is sufficiently general. In particular, we see that

$$\deg(\psi(Z)) = \deg(Z).$$

Let C be an irreducible curve in Π of degree λ that contains $\psi(\Sigma)$ and let W be a cone in $\mathbb{P}^n$ over C whose vertex is Ω. Then

$$W \in \mathscr{M},$$

which implies that $Z \subset W$. Therefore, we see that $\psi(Z) = C$, which implies that $\Xi = \Sigma$ and $\deg(Z) = \lambda$, giving a contradiction.

Corollary 2.5. *If Σ is not contained in any line, then nor is $\psi(\Sigma)$.*

Lemma 2.6. *Let $\mathscr{M}$ be a linear system consisting of hypersurfaces in $\mathbb{P}^n$ of degree λ that contain the set Σ. Then the base locus of the linear system $\mathscr{M}$ does not contain surfaces if*

- *the set Σ is not contained in any irreducible surface of degree λ;*
- *the set $\psi(\Sigma)$ is contained in some irreducible surface of degree λ;*
- *the inequality $m \geqslant 3$ holds.*

See the proof of Lemma 2.4.

Corollary 2.7. *Suppose that $m \geqslant 3$ and Σ is not contained in any two-dimensional linear subspace. Then $\psi(\Sigma)$ is not contained in any two-dimensional linear subspace, either.*

Lemma 2.8. *Let $\mathscr{M}$ be a linear system consisting of hypersurfaces in Π of degree λ that contain the set $\psi(\Sigma)$. Then the base locus of the linear system $\mathscr{M}$ is zero-dimensional if*

- *the subset Σ is not contained in any irreducible curve of degree λ;*
- *the set $\psi(\Sigma)$ is contained in some irreducible curve of degree λ;*
- *the equality $m = n - 1$ holds and $m \geqslant 3$.*

Proof. Suppose that

- the set Σ is not contained in any irreducible curve of degree λ;
- the set $\psi(\Sigma)$ is contained in some irreducible curve of degree λ;
- $m = n - 1$ and $m \geqslant 3$.

Note that Ω is a point.

Let $\mathscr{Y}$ be the set of all cones in $\mathbb{P}^n$ over all irreducible curves in Π of degree λ that contain all the points in Σ, and let Υ be the set-theoretic intersection of all cones in $\mathscr{Y}$. Then obviously,

$$\Sigma \subseteq \Upsilon \subset \mathbb{P}^n$$

because every cone in $\mathscr{Y}$ contains Σ.

Let C be an irreducible curve in Π of degree λ that contains $\psi(\Sigma)$, and let W be a cone in $\mathbb{P}^n$ over the curve C whose vertex is the point Ω. Then $W \in \mathscr{Y}$, which implies that $\Upsilon \subseteq W$.

We will show that Υ is a finite set.

Suppose that there exists an irreducible curve $Z \subset \Upsilon$. Then the cone W must contain Z. Put $\Xi = Z \cap \Sigma$. We may assume that $\psi|_Z$ is an isomorphism and

$$\psi(Z) \cap \psi(\Sigma \setminus \Xi) = \varnothing$$

because the point Ω is sufficiently general. Then $\psi(Z)$ is a curve of degree $\deg(Z)$. We have

$$\psi(Z) = C,$$

which gives $\Xi = \Sigma$ and $\deg(Z) = \lambda$, which is a contradiction. Hence the set Υ is finite.

Let $\mathscr{S}$ be the set of all irreducible surfaces in $\mathbb{P}^m$ such that

$$S \in \mathscr{S} \quad \iff \quad \exists\, Y \in \mathscr{Y}: \quad \psi(Y) = S,$$

and let Ψ be the set theoretic intersection of all surfaces in $\mathscr{Y}$. Then

$$\psi(\Sigma) \subseteq \psi(\Upsilon) \subseteq \Psi.$$

The set Ψ is a set-theoretic intersection of surfaces of degree at most λ. Each of these surfaces is a set-theoretic intersection of hypersurfaces of degree λ. Thus Ψ is a set-theoretic intersection of surfaces in the linear system $\mathscr{M}$. Hence to finish the proof it is enough to show that Ψ is finite.

Let $W_1, W_2, \dots, W_r$ be irreducible surfaces in $\mathscr{Y}$ such that

$$\Upsilon = \bigcap_{i=1}^{r} W_i$$

and $\psi(W_i) \in \mathscr{S}$ for any i. Put

$$\Theta = \bigcap_{i=1}^{r} \psi(W_i).$$

We will show that $\Theta \subset \mathbb{P}^m$ is a finite set if the point Ω is general enough. Note that if the set Θ is finite, then Ψ is finite because $\Psi \subseteq \Theta$.

Let H be a sufficiently general hypersurface in $\mathbb{P}^n$ that contains the point Ω. Put

$$C_i = W_i \cap H \subset H \cong \mathbb{P}^m$$

for every i. Then $C_1 \cap C_1 \cap \dots \cap C_r = \varnothing$ because Υ is a finite set. But

$$\Theta \cap H = \bigcap_{i=1}^{r} \psi(W_i) \cap H = \bigcap_{i=1}^{r} \psi(C_i)$$

because $\Omega \in H$. Hence to prove that Θ is a finite set it is enough to show that

$$\bigcap_{i=1}^{r} \psi(C_i) = \varnothing.$$

Let Δ be a (possibly empty) subset of H such that

$$P \in \Delta \quad \Longleftrightarrow \quad \exists L \subset H: \ \ P \in L, \ \ L \cap C_i \neq \varnothing \ \ \forall i,$$

where P is a point in H. Then by the definition of Δ

$$\bigcap_{i=1}^{r} \psi(C_i) = \varnothing \quad \Longleftrightarrow \quad \Omega \notin \Delta,$$

but an easy dimension count implies that $\dim(\Delta) \leqslant 2$ because $C_1 \cap C_1 \cap \cdots \cap C_r = \varnothing$. As $m \geqslant 3$, thus $\Delta \neq H$. Hence we may assume that

$$\Omega \in H \setminus \Delta,$$

which implies that Θ is a finite set and completes the proof.

Corollary 2.9. *Suppose that Σ is not contained in an irreducible curve of degree λ, but*

$$|\Sigma| > \lambda^2$$

and $m \geqslant 3$. Then $\psi(\Sigma)$ is not contained in any irreducible curve of degree λ.

Lemma 2.10. *Suppose that Σ is a disjoint union of nonempty finite subsets Λ and Δ such that*

- *there exists a hypersurface in $\mathbb{P}^n$ of degree ζ that passes through all points of the set Λ and does not contain any point of Δ;*
- *the points of the set Λ and the points of Δ impose independent linear conditions on hypersurfaces of degrees ξ and $\xi - \zeta$, respectively,*

where ζ is some natural number such that $\xi \geqslant \zeta$.

Then the points in Σ impose independent linear conditions on hypersurfaces of degree ξ.

Proof. Let P be an arbitrary point in Σ. We must show that there exists a hypersurface of degree ξ that contains the set $\Sigma \setminus P$ and does not contain P.

Note that we may assume that $P \in \Lambda$.

Let F be a homogeneous polynomial of degree ξ that vanishes at every point of the set $\Lambda \setminus P$ and does not vanish at the point P. Put

$$\Delta = \{Q_1, \dots, Q_\delta\},$$

where Q_i is a point. For every Q_i there is a homogeneous polynomial G_i of degree ξ which vanishes at every point of the set $\Sigma \setminus Q_i$ and does not vanish at Q_i. Then

$$F(Q_i) + \mu_i G_i(Q_i) = 0$$

for some $\mu_i \in \mathbb{C}$ because $G_i(Q_i) \neq 0$. Then the hypersurface given by the equation

$$F + \sum_{i=1}^{\delta} \mu_i G_i = 0,$$

contains the set $\Sigma \setminus P$ and does not contain the point P.

§ 3. Points in projective spaces

Let Σ be a finite subset of $\mathbb{P}^n$, $n \geqslant 2$. Let d and ε be natural numbers such that $d \geqslant 3$ and $\varepsilon < d$. In this section we prove the following result.

Theorem 3.1. *The set Σ imposes independent linear conditions on hypersurfaces of degree $2d-4-\varepsilon$ if the strict inequality*

$$|\Sigma| < (d-1)(d-\varepsilon)$$

holds and no curve in $\mathbb{P}^n$ of degree k contains more than $k(d-1)$ points of the set Σ for every $k \leqslant d-\varepsilon-1$.

Proof. Note that the assertion of Theorem 3.1 obviously holds for $\varepsilon = d-1$, and, as follows from [9], Theorem 1.1, the assertion of Theorem 3.1 obviously holds for $\varepsilon = 1$. Hence we may suppose that

$$|\Sigma| \leqslant (d-1)(d-\varepsilon) - 1,$$

at most $k(d-1)$ points of the subset Σ are contained in a curve in $\mathbb{P}^n$ of degree k for every natural number $k \leqslant d-\varepsilon-1$, and $2 \leqslant \varepsilon \leqslant d-2$.

Suppose that Theorem 3.1 fails. Then points of Σ impose dependent linear conditions on hypersurfaces of degree $2d-4-\varepsilon$.

Lemma 3.2. *The inequality $\varepsilon \leqslant d-3$ holds.*

Proof. Suppose that $\varepsilon = d-2$. Then $2d-4-\varepsilon = d-2$. But

$$|\Sigma| \leqslant 2d-3,$$

and at most $d-1$ points of Σ are contained on a line in $\mathbb{P}^n$. By Lemma 2.2 the points of the set Σ impose independent linear conditions on hypersurfaces of degree $2d-4-\varepsilon$, which is a contradiction.

There exists a point $P \in \Sigma$ such that each hypersurface in $\mathbb{P}^n$ of degree $2d-4-\varepsilon$ that contains the set $\Sigma \setminus P$ must also contain the point $P \in \Sigma$. Note that $d \geqslant 5$.

Lemma 3.3. *The inequality $n \neq 2$ holds.*

Proof. Suppose that $n = 2$. Put $\xi = 2d-4-\varepsilon$. Then $\xi \geqslant 3$ and

$$|\Sigma \setminus P| \leqslant \max\biggl\{\biggl\lfloor\frac{\xi+3}{2}\biggr\rfloor\biggl(\xi+3-\biggl\lfloor\frac{\xi+3}{2}\biggr\rfloor\biggr)-1,\ \biggl\lfloor\frac{\xi+3}{2}\biggr\rfloor^2\biggr\}$$

because $|\Sigma| \leqslant (d-1)(d-\varepsilon)-1$.

Let us show that at most $k(\xi+3-k)-2$ points of the set $\Sigma \setminus P$ lie on a curve of degree k for every natural number $k \leqslant (\xi+3)/2$. We must show that

$$k(2d-1-\varepsilon-k)-2 \geqslant k(d-1)$$

for every $k \leqslant (\xi+3)/2$. However, we only need prove this for natural numbers $k \geqslant 1$ such that

$$k(2d-1-\varepsilon-k)-2 < |\Sigma \setminus P| \leqslant (d-1)(d-\varepsilon)-2.$$

We may assume that $k < d - \varepsilon$ because otherwise

$$k(2d-1-\varepsilon-k)-2 \geqslant (d-\varepsilon)(2d-1-\varepsilon-d+\varepsilon)-2 = (d-1)(d-\varepsilon)-2 \geqslant |\Sigma \setminus P|.$$

We may assume that $k \neq 1$ because $\varepsilon \leqslant d-3$ and at most

$$d-1 \leqslant \xi = 2d-4-\varepsilon$$

points of the set $\Sigma \setminus P$ lie on a line. Then

$$k(2d-1-\varepsilon-k)-2 \geqslant k(d-1) \iff k(d-\varepsilon-k) \geqslant 2 \iff d-\varepsilon > k,$$

which immediately implies that at most $k(\xi+3-k)-2$ points of the subset $\Sigma \setminus P$ are contained on a curve of degree k for every natural number $k \leqslant (\xi+3)/2$.

By Lemma 2.3 there is a curve in $\mathbb{P}^2$ of degree $2d-4-\varepsilon$ that contains $\Sigma \setminus P$ and does not contain the point $P \in \Sigma$, which is a contradiction.

By Lemma 2.4 and Corollary 2.9, to complete the proof of Theorem 3.1 we may assume that $n = 3$. Let Π be a sufficiently general plane in $\mathbb{P}^3$ and let

$$\psi\colon \mathbb{P}^3 \dashrightarrow \Pi \cong \mathbb{P}^2$$

be a projection from a sufficiently general point $O \in \mathbb{P}^3$. Put $\Sigma' = \psi(\Sigma)$ and $P' = \psi(P)$.

Lemma 3.4. *There exists a curve $C \subset \Pi$ of degree $k \leqslant d-\varepsilon-1$ such that*

$$|C \cap \Sigma'| \geqslant k(d-1)+1.$$

Proof. Suppose that no curve of degree k contains $k(d-1)+1$ points of the subset Σ' for every $k \leqslant d-\varepsilon-1$. Arguing as in the proof of Lemma 3.3, we see that there is a curve

$$Z \subset \Pi \cong \mathbb{P}^2$$

of degree $2d-4-\varepsilon$ that contains the set $\Sigma' \setminus P'$ and does not contain the point $P' \in \Sigma'$.

A cone in $\mathbb{P}^3$ over Z whose vertex is O is a surface of degree $2d-4-\varepsilon$ that contains $\Sigma \setminus P$ and does not contain the point $P \in \Sigma$, which is a contradiction.

We may assume that k is the smallest natural number such that at least $k(d-1)+1$ points of the set Σ' are contained in an irreducible curve in $\Pi \cong \mathbb{P}^2$ of degree k. We see that there is a disjoint union of sets

$$\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i \subset \Sigma$$

such that $|\Lambda_j^i| \geqslant j(d-1)+1$, all points of $\psi(\Lambda_j^i)$ are contained in an irreducible curve of degree j, and at most $\zeta(d-1)$ points of the subset

$$\psi\left(\Sigma \setminus \left(\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i\right)\right) \subsetneq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

can lie on a curve in $\Pi \cong \mathbb{P}^2$ of degree ζ for every natural number ζ. Put

$$\Lambda = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i.$$

Let Ξ_j^i be the base locus of the linear subsystem of $|\mathscr{O}_{\mathbb{P}^3}(j)|$ that contains all surfaces that pass through all points of the subset Λ_j^i. Put

$$\Delta = \Sigma \cap \left(\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Xi_j^i \right).$$

The set Ξ_j^i is finite by Lemma 2.4. On the other hand we have

$$|\Sigma \setminus \Lambda| \leqslant (d-1)\left(d - \varepsilon - \sum_{i=k}^{l} c_i i\right) - 2. \tag{$*$}$$

Corollary 3.5. *The inequality* $\sum_{i=k}^{l} ic_i \leqslant d - \varepsilon - 1$ *holds.*

Note that $\Lambda \subseteq \Delta \subseteq \Sigma$. We have $k \geqslant 2$ by Corollary 2.5.

Lemma 3.6. *The points of the set* Δ *impose independent linear conditions on hypersurfaces of degree* $2d - \varepsilon - 4$.

Proof. Suppose that the points of the set Δ impose dependent linear conditions on hypersurfaces of degree $2d - \varepsilon - 4$. Let us consider Δ as a zero-dimensional subscheme of $\mathbb{P}^3$. Then

$$h^1\big(\mathscr{I}_\Delta \otimes \mathscr{O}_{\mathbb{P}^3}(2d - \varepsilon - 4)\big) \neq 0,$$

where $\mathscr{I}_\Delta$ is the ideal sheaf of the subscheme Δ.

Let $\mathscr{M}$ be the linear subsystem of the linear system $|\mathscr{O}_{\mathbb{P}^3}(d-\varepsilon-1)|$ that contains all surfaces that pass through Δ. Then the base locus of the linear system $\mathscr{M}$ is zero-dimensional since $\sum_{i=k}^{l} ic_i \leqslant d - \varepsilon - 1$ and

$$\Delta \subseteq \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Xi_j^i,$$

whilst Ξ_j^i is a zero-dimensional base locus of the system $|\mathscr{O}_{\mathbb{P}^3}(j)|$. Put

$$\Gamma = M_1 \cdot M_2 \cdot M_3,$$

where M_1, M_2, M_3 are general enough surfaces in $\mathscr{M}$. Then Γ is a closed zero-dimensional subscheme of $\mathbb{P}^3$ and Δ is a closed subscheme of the scheme Γ.

Let Υ be a closed subscheme of the scheme Γ such that

$$\mathscr{I}_\Upsilon = \operatorname{Ann}(\mathscr{I}_\Delta / \mathscr{I}_\Gamma),$$

where $\mathscr{I}_\Upsilon$ and $\mathscr{I}_\Gamma$ are ideal sheaves of the subschemes Υ and Γ, respectively. Then

$$0 \neq h^1\big(\mathscr{O}_{\mathbb{P}^3}(2d-\varepsilon-4) \otimes \mathscr{I}_\Delta\big) = h^0\big(\mathscr{O}_{\mathbb{P}^3}(d-2\varepsilon-3) \otimes \mathscr{I}_\Upsilon\big) - h^0\big(\mathscr{O}_{\mathbb{P}^3}(d-2\varepsilon-3) \otimes \mathscr{I}_\Gamma\big)$$

by Theorem 2.1. Hence there is a surface $F \in |\mathscr{O}_{\mathbb{P}^3}(d-2\varepsilon-3)\otimes\mathscr{I}_{\Upsilon}|$. Then

$$(d-2\varepsilon-3)(d-\varepsilon-1)^2 = F\cdot M_1\cdot M_2 \geqslant h^0(\mathscr{O}_{\Upsilon}) = h^0(\mathscr{O}_{\Gamma})-h^0(\mathscr{O}_{\Delta}) = (d-\varepsilon-1)^3-|\Delta|,$$

which implies that $|\Delta| \geqslant (\varepsilon+2)(d-\varepsilon-1)^2$. Thus we see that

$$(d-1)(d-\varepsilon)-1 \geqslant |\Sigma| \geqslant |\Delta| \geqslant (\varepsilon+2)(d-\varepsilon-1)^2,$$

which easily leads to a contradiction. The proof is complete.

Put $\Gamma = \Sigma\setminus\Delta$, $\Gamma' = \psi(\Gamma)$ and $\xi = 2d-\varepsilon-4-\sum_{i=k}^{l} ic_i$.

Lemma 3.7. *The inequality $\xi \geqslant 3$ holds.*

Proof. Suppose that $\xi \leqslant 2$. Then it follows from Corollary 3.5 that

$$2 \geqslant \xi = 2d-\varepsilon-4-\sum_{i=k}^{l} ic_i \geqslant d-3,$$

which gives $d \leqslant 5$. Then $d=5$ and $\varepsilon=2$ because $2\leqslant\varepsilon\leqslant d-3$. We have $|\Sigma|\leqslant 11$.

By Lemma 2.2 the points of the set Σ impose independent linear conditions on hypersurfaces of degree $2d-\varepsilon-4$ if at most 9 points of the set Σ are contained in a plane $\mathbb{P}^3$. This implies that there exists a plane $\Upsilon\subset\mathbb{P}^3$ such that $|\Upsilon\cup\Sigma|\geqslant 10$.

It follows from Lemma 3.4 that $|\Upsilon\cup\Sigma| = 10$. Note that $P\in\Upsilon$.

Arguing as in the proof of Lemma 3.3 we see that there is a curve

$$Z\subset\Upsilon\cong\mathbb{P}^2$$

of degree $2d-\varepsilon-4$ that contains the set $\Upsilon\setminus P$ and does not contain the point $P\in\Sigma$.

A cone in $\mathbb{P}^3$ over Z whose vertex is $\Sigma\setminus\Upsilon$ is a surface of degree $2d-\varepsilon-4$ that contains $\Sigma\setminus P$ and does not contain the point $P\in\Sigma$, which is a contradiction.

It easily follows from inequality ($*$) that

$$|\Gamma'| \leqslant \max\left\{\left\lfloor\frac{\xi+3}{2}\right\rfloor\left(\xi+3-\left\lfloor\frac{\xi+3}{2}\right\rfloor\right)-1,\ \left\lfloor\frac{\xi+3}{2}\right\rfloor^2\right\}.$$

Lemma 3.8. *At most ξ points of the set Γ are contained in a line.*

Proof. Suppose that $\xi+1$ points of the set Γ are contained in some line. Then

$$d-1 \geqslant \xi+1 = 2d-\varepsilon-4-\sum_{i=k}^{l} ic_i,$$

because at most $d-1$ points of the set Γ are contained in a line in $\mathbb{P}^3$. Then

$$d-\varepsilon-1 \geqslant \sum_{i=k}^{l} c_i i \geqslant d-\varepsilon-2$$

by Corollary 3.5. We see that either $\sum_{i=k}^{l} c_i i = d-\varepsilon-2$ or $\sum_{i=k}^{l} c_i i = d-\varepsilon-1$.

Suppose that $\sum_{i=k}^{l} c_i i = d - \varepsilon - 2$. Then

$$|\Gamma| \leqslant |\Sigma \setminus \Lambda| \leqslant (d-1)\left(d - \varepsilon - \sum_{i=k}^{l} c_i i\right) - 2 = 2d - 4,$$

so by Lemma 2.2 the points of the set Γ impose independent linear conditions on hypersurfaces of degree $d-2$. The points of the set Σ impose independent linear conditions on hypersurfaces of degree $2d - \varepsilon - 4$ by Lemma 2.10, which is a contradiction.

We see that $\sum_{i=k}^{l} c_i i = d - \varepsilon - 1$. Then

$$|\Gamma| \leqslant |\Sigma \setminus \Lambda| \leqslant (d-1)\left(d - \varepsilon - \sum_{i=k}^{l} c_i i\right) - 2 = d - 3,$$

which implies that the points of the set Γ impose independent linear conditions on hypersurfaces of degree $\xi = d - 3$. By Lemma 2.10 the points of the set Σ impose independent linear conditions on hypersurfaces of degree $2d - \varepsilon - 4$, which is a contradiction.

It follows from Corollary 2.5 that at most ξ points of the set Γ' are contained in a line.

Lemma 3.9. *For every $t \leqslant (\xi+3)/2$ at most*

$$t(\xi + 3 - t) - 2$$

points of the set Γ' are contained in a curve in $\Pi \cong \mathbb{P}^2$ of degree t.

Proof. At most $t(d-1)$ points of the subset Γ' are contained in a curve of degree t. Thus by Lemma 3.8, we need to show that

$$t(\xi + 3 - t) - 2 \geqslant t(d-1)$$

for every $t \leqslant (\xi+3)/2$ such that $t(\xi + 3 - t) - 2 < |\Gamma'|$ and $t > 1$. But

$$t(\xi + 3 - t) - 2 \geqslant t(d-1) \quad \iff \quad d - \varepsilon - \sum_{i=k}^{l} c_i i > t$$

because $t > 1$. Therefore, we may assume that $t(\xi + 3 - t) - 2 < |\Gamma'|$ and

$$d - \varepsilon - \sum_{i=k}^{l} c_i i \leqslant t \leqslant \frac{\xi + 3}{2}.$$

Put $g(x) = x(\xi + 3 - x) - 2$. Then

$$g(t) \geqslant g\left(d - \varepsilon - \sum_{i=k}^{l} c_i i\right)$$

because $g(x)$ is an increasing function for $x < (\xi+3)/2$. We have

$$(d-1)\left(d-\varepsilon-\sum_{i=k}^{l} ic_i\right)-2 \geqslant |\Gamma'| > g(t) \geqslant g\left(d-\varepsilon-\sum_{i=k}^{l} c_i i\right)$$
$$= (d-1)\left(d-\varepsilon-\sum_{i=k}^{l} ic_i\right)-2,$$

which is a contradiction.

The points of the set Γ impose independent linear conditions on hypersurfaces of degree ξ, because the points of the set Γ' impose independent linear conditions on hypersurfaces of degree ξ by Lemma 2.3. Hence the points of the set Σ impose independent linear conditions on hypersurfaces of degree $2d-\varepsilon-4$ by Lemma 2.10, which is a contradiction.

The assertion of Theorem 3.1 is proved.

§ 4. The main result

The goal of this section is to prove Theorem 1.6. Let X be hypersurface in $\mathbb{P}^4$ of degree d with at most isolated ordinary double points.

Lemma 4.1. *Let C be a curve in $\mathbb{P}^4$ of degree λ. Then*

$$|\mathrm{Supp}(C)\cap \mathrm{Sing}(X)| \leqslant \lambda(d-1),$$

and if $|\mathrm{Supp}(C)\cap \mathrm{Sing}(X)| = \lambda(d-1)$, *then*

$$\mathrm{Sing}(C)\cap \mathrm{Sing}(X) = \varnothing.$$

See the proof in [8], Lemma 29.

It follows from [13] that the following conditions are equivalent:

- the hypersurface X is factorial;
- the points of the set $\mathrm{Sing}(X)$ impose independent linear conditions on hypersurfaces in $\mathbb{P}^4$ of degree $2d-5$.

Suppose that

$$|\mathrm{Sing}(X)| \leqslant (d-1)^2$$

and the hypersurface X contains no planes. Let $\Sigma = \mathrm{Sing}(X)$.

Lemma 4.2. *Suppose that $|\Sigma| < (d-1)^2$. Then X is factorial.*

Proof. By Theorem 3.1 the points of Σ impose independent linear conditions on hypersurfaces of degree $2d-5$, which implies that X is factorial.

Let $|\Sigma| = (d-1)^2$, but assume that points of Σ impose dependent linear conditions on hypersurfaces of degree $2d-5$. We shall show this leads to a contradiction.

Lemma 4.3. *Let $\Pi \subset \mathbb{P}^4$ be a plane. Then* $|\Pi\cap\Sigma| \leqslant d-1$.

Proof. It easily follows from [6], Lemma 2.9 that

$$|\Pi \cap \Sigma| \leqslant \frac{d(d-1)}{2} \leqslant (d-1)^2 - 1$$

since X does not contain planes. Then the points of the set $\Pi \cap \Sigma$ impose independent linear conditions on hypersurfaces of degree $2d-5$ by Theorem 3.1.

Suppose that $|\Pi \cap \Sigma| \geqslant d-1$. Let H be a general hyperplane in $\mathbb{P}^4$ containing Π. Then $H \cap \Sigma = \Pi \cap \Sigma$. On the other hand we have

$$|\Sigma \setminus (\Pi \cap \Sigma)| \leqslant (d-1)^2 - d = (d-1)(d-2) - 1,$$

which implies that the points of the set $\Sigma \setminus (\Pi \cap \Sigma)$ impose independent linear conditions on hypersurfaces of degree $2d-6$ by Theorem 3.1. Then Σ imposes independent linear conditions on hypersurfaces of degree $2d-5$ by Lemma 2.10, which is a contradiction.

Corollary 4.4. *At most $d-2$ points of the set Σ lie on a line.*

The assertion of Theorem 1.6 is proved in [2] for $d \leqslant 4$. Thus, we have shown that $d \geqslant 5$.

Lemma 4.5. *The inequality $d \geqslant 6$ holds.*

Proof. Suppose that $d = 5$. By Lemmas 2.2 and 4.3 the points of Σ impose independent linear conditions on hypersurfaces of degree $2d-5$, which is a contradiction.

Lemma 4.6. *Let C be a curve in $\mathbb{P}^4$ of degree $\lambda \leqslant d-2$. Then*

$$|C \cap \Sigma| \leqslant \lambda(d-1) - 1.$$

Proof. We may assume that C is irreducible. Suppose that $|C \cap \Sigma| = \lambda(d-1)$. Then

$$|\Sigma \setminus (C \cap \Sigma)| = (d-1)(d-\lambda-1) \geqslant 5$$

by Lemma 4.5. Moreover, it follows from Corollary 4.4 that $\lambda \neq 1$.

Let P and Q be two distinct points in the set $\Sigma \setminus (C \cap \Sigma)$. Let Y_P and Y_Q be the cones in $\mathbb{P}^4$ over the curve C whose vertices are at the points P and Q, respectively. Then Y_P and Y_Q are irreducible. Let us show that $Y_P \neq Y_Q$. Suppose that $Y_P = Y_Q$. Let L be the line in $\mathbb{P}^4$ that contains P and Q. Then Y_P is a cone over the curve C whose vertex is on the line L. Therefore, the surface Y_P must be a plane, which is impossible by Lemma 4.3. Hence we see that $Y_P \neq Y_Q$.

Let O be a point on the surface Y_P such that $O \notin Y_Q$, and let Y_O be the cone over the curve C whose vertex is the point O. Then $Q \notin Y_O$ because $O \notin Y_Q$. The cone Y_O is a set-theoretic intersection of hypersurfaces of degree λ, which implies that there is a hypersurface $F \subset \mathbb{P}^4$ of degree λ such that

$$F \cap \Sigma = Y_O \cap \Sigma,$$

which implies that $Q \notin F$. Thus, the points of the set $F \cap \Sigma$ impose independent linear conditions on hypersurfaces of degree $2d-5$ by Theorem 3.1. On the other hand we have

$$|\Sigma \setminus (\Pi \cap \Sigma)| \leqslant (d-1)(d-1-\lambda) - 1,$$

which implies that the points of the set $\Sigma \setminus (F \cap \Sigma)$ impose independent linear conditions on hypersurfaces of degree $2d-5-\lambda$ by Theorem 3.1. Then Σ imposes independent linear conditions on hypersurfaces of degree $2d-5$ by Lemma 2.10, which is a contradiction.

Lemma 4.7. *Let C be a curve in $\mathbb{P}^4$ of degree $d-1$. Then*

$$|C \cap \Sigma| \leqslant (d-1)^2 - 1.$$

Proof. Suppose that $|C \cap \Sigma| = (d-1)^2$. Then $\Sigma \subset C$, where C is irreducible by Lemma 4.6, and C is not contained in a two-dimensional linear subspace by Lemma 4.3.

We have to consider the following two mutually exclusive cases:

– the curve C is contained in some three-dimensional linear subspace of $\mathbb{P}^4$,

– the curve C is not contained in any three-dimensional linear subspace of $\mathbb{P}^4$.

Suppose that C is contained in some three-dimensional linear subspace $H \subset \mathbb{P}^4$. Then $H \cong \mathbb{P}^3$ and we may consider Σ as a zero-dimensional subscheme of $\mathbb{P}^3$. Then

$$h^1(\mathscr{I}_\Sigma \otimes \mathscr{O}_{\mathbb{P}^3}(2d-5)) \neq 0,$$

where $\mathscr{I}_\Sigma$ is the ideal sheaf of the subscheme Σ.

Taking into account the linear projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from a sufficiently general point of C we see that there exist two different irreducible surfaces F_1 and F_2 in the linear system $|\mathscr{O}_{\mathbb{P}^3}(d-2)|$ such that $C \subset F_1 \cap F_2$.

Let $\mathscr{M}$ be a linear subsystem in $|\mathscr{O}_{\mathbb{P}^3}(d-1)|$ that contains all surfaces that pass through the set Σ. Then the base locus of the linear system $\mathscr{M}$ is zero-dimensional. Put

$$\Gamma = M \cdot F_1 \cdot F_2,$$

where M is a general surface in the linear system $\mathscr{M}$. Then Γ is a closed zero-dimensional subscheme of $\mathbb{P}^3$ and Σ is a closed subscheme of the scheme Γ.

Let Υ be a closed subscheme of the scheme Γ such that

$$\mathscr{I}_\Upsilon = \operatorname{Ann}(\mathscr{I}_\Sigma / \mathscr{I}_\Gamma),$$

where $\mathscr{I}_\Upsilon$ and $\mathscr{I}_\Gamma$ are the ideal sheaves of the subschemes Υ and Γ, respectively. Then

$$0 \neq h^1\big(\mathscr{O}_{\mathbb{P}^3}(2d-5) \otimes \mathscr{I}_\Sigma\big) = h^0\big(\mathscr{O}_{\mathbb{P}^3}(d-4) \otimes \mathscr{I}_\Upsilon\big) - h^0\big(\mathscr{O}_{\mathbb{P}^3}(d-4) \otimes \mathscr{I}_\Gamma\big)$$

by Theorem 2.1. Thus, there exists a surface $G \in |\mathscr{O}_{\mathbb{P}^3}(d-4) \otimes \mathscr{I}_\Upsilon|$. Then

$$(d-4)(d-2)(d-1) = G \cdot F_1 \cdot M \geqslant h^0(\mathscr{O}_\Upsilon) = h^0(\mathscr{O}_\Gamma) - h^0(\mathscr{O}_\Sigma) = (d-1)(d-2)^2 - |\Sigma|,$$

which implies that $(d-1)^2 = |\Sigma| \geqslant 2(d-2)(d-1)$, which is a contradiction.

We see that C is not contained in any three-dimensional linear subspace of $\mathbb{P}^4$. It should be pointed out that $C \subset X$ because otherwise we have

$$d(d-1) = \deg(C)\deg(X) \geqslant 2(d-1)^2,$$

which is a contradiction because $d \geqslant 6$.

Let O be a sufficiently general point of C and let

$$\psi\colon \mathbb{P}^4 \dashrightarrow \Pi$$

be a projection from the point O, where Π is a three-dimensional linear subspace of $\mathbb{P}^4$. Then ψ induces a birational morphism $C \dashrightarrow \psi(C)$. Put $Z = \psi(C)$. Then the degree of the curve Z is $d-2$.

Let Y be a cone in $\mathbb{P}^4$ over the curve Z whose vertex is O. Then

$$C \subset Y \not\subset X$$

since O is a sufficiently general point because X is not a secant variety of the curve C.

Since O is sufficiently general, we may assume that O is not contained in a three-dimensional linear subspace that is tangent to X at some point of the curve C because C is not contained in a three-dimensional linear subspace of $\mathbb{P}^4$. Then the cycle $X \cdot Y$ is reduced at a general point on the curve C. Put

$$X \cdot Y = C + R,$$

where R is a curve of degree $d^2 - 3d + 1$ such that $C \not\subseteq \mathrm{Supp}(R)$. By Lemma 4.1, since O is sufficiently general, we have

$$C \cap \Sigma \subset Y \setminus \mathrm{Sing}(Y).$$

Let $\alpha\colon \overline{Z} \to Z$ be a normalization of the curve Z. Then there is a commutative diagram

$$\begin{tikzcd}
\overline{Y} \arrow[r, "\beta"] \arrow[d, "\pi"'] & Y \arrow[d, dashed, "\psi|_Y"] \\
\overline{Z} \arrow[r, "\alpha"'] & Z
\end{tikzcd}$$

where $\overline{Y}$ is a smooth surface, β is a birational morphism, and π is a morphism with connected fibres that is a $\mathbb{P}^1$-bundle.

Let L and E be a fibre and a section of π such that $\beta(E) = O$, respectively. Then

$$E^2 = -d + 2$$

on the surface $\overline{Y}$. Let $\overline{C}$ and $\overline{R}$ be curves on $\overline{Y}$ such that $\alpha(\overline{C}) = C$, the equality

$$\overline{R} \cdot \alpha^*(\mathscr{O}_{\mathbb{P}^4}(1)|_Y) = d^2 - 3d + 1$$

holds and $\alpha(\overline{R}) = R$. Then

$$\overline{R} \equiv (d-2)E + (d^2 - 3d + 1)L$$

on the surface $\overline{Y}$ and similarly $\overline{C} \equiv E + (d-1)L$. Put $s = (d-1)^2$ and

$$\Sigma = \{Q_1, Q_2, \dots, Q_s\},$$

where Q_i is a point of the set Σ. For every point Q_i there is a point $\overline{Q}_i \in \overline{Y}$ such that

$$\overline{Q}_i \in \operatorname{Supp}(\overline{C} \cdot \overline{R})$$

and $\beta(\overline{Q}_i) = Q_i$. Therefore, we have

$$(d-1)^2 - 2 = \overline{C} \cdot \overline{R} \geqslant \sum_{i=1}^{s} \operatorname{mult}_{Q_i}(\overline{C} \cdot \overline{R}) \geqslant (d-1)^2,$$

which is a contradiction.

Corollary 4.8. *Let C be a curve in $\mathbb{P}^4$ of degree λ. Then*

$$|C \cap \Sigma| \leqslant \lambda(d-1) - 1.$$

Let $\eta\colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ be a general linear projection. Put $\Xi = \eta(\Sigma)$. Then it follows from Corollaries 2.9 and 2.7 that the set Ξ has the following properties:

- $|\Xi| = (d-1)^2$;
- at most $\lambda(d-1) - 1$ points in the set Ξ are contained in a curve of degree $\lambda \leqslant d-2$;
- at most $d-1$ points of the set Ξ are contained in a plane.

However, the points of Ξ impose dependent linear conditions on hypersurfaces of degree $2d-5$. Let us consider Ξ as a subscheme of $\mathbb{P}^3$. Then

$$h^1(\mathscr{I}_{\Xi} \otimes \mathscr{O}_{\mathbb{P}^3}(2d-5)) \neq 0,$$

where $\mathscr{I}_{\Xi}$ is the ideal sheaf of the subscheme Ξ.

Lemma 4.9. *Let C be a curve in $\mathbb{P}^3$ of degree $d-1$. Then $|C \cap \Xi| \leqslant (d-1)^2 - 1$.*

Proof. Suppose that $|C \cap \Xi| = (d-1)^2$. Then C is an irreducible curve not contained in a plane. Arguing as in the proof of Lemma 4.7 and using Lemma 2.8 we get a contradiction.

Thus we have shown that the set Ξ has the following properties:

- $|\Xi| = (d-1)^2$;
- at most $\lambda(d-1) - 1$ points of the set Ξ are contained in a curve of degree $\lambda \leqslant d-2$;
- at most $d-1$ points of the set Ξ are contained in a plane;
- there is a point $Q \in \Xi$ such that every hypersurface in $\mathbb{P}^3$ of degree $2d-5$ that contains the set $\Xi \setminus Q$ must also contain $Q \in \Xi$.

Lemma 4.10. *Let $\mathscr{M}$ be a linear subsystem in $|\mathscr{O}_{\mathbb{P}^3}(d-1)|$ consisting of all surfaces that contain Ξ. Then the base locus of the linear system $\mathscr{M}$ contains a curve.*

See the proof in § 5.

Let $\Pi \subset \mathbb{P}^3$ be a general plane and let

$$\psi\colon \mathbb{P}^3 \dashrightarrow \Pi \cong \mathbb{P}^2$$

be a linear projection from a sufficiently general point $O \in \mathbb{P}^3$. Put $\Xi' = \psi(\Xi)$ and $Q' = \psi(Q)$.

Lemma 4.11. *Suppose that no more than $\lambda(d-1)$ points of the set Ξ' are contained in a curve of degree λ for every $\lambda \leqslant d-2$. Then Ξ' is not contained in a curve of degree $d-1$.*

Proof. Suppose that Ξ' is contained in a curve $C \subset \mathbb{P}^2$ of degree $d-1$. We claim that this contradicts Lemma 4.10.

Let $\mathscr{M}$ be a linear subsystem of the linear system $|\mathscr{O}_{\mathbb{P}^3}(d-1)|$ consisting of all surfaces that contains Ξ. Then the base locus of $\mathscr{M}$ contains an irreducible $Z \subset \mathbb{P}^3$ by Lemma 4.10.

The curve C is reducible by Lemma 2.4. Put

$$C = \sum_{i=1}^{s} C_i,$$

where C_i is an irreducible curve of degree d_i. Then $|C_i \cap \Xi'| = d_i(d-1)$.

Let Ξ_i be a subset in Ξ such that $|\Xi_i| = d_i(d-1)$ and $\psi(\Xi_i) \subset C_i$, and let $\mathscr{M}_i$ be a linear system consisting of all surfaces of degree d_i that contain the subset Ξ_i. Then, by Lemma 4.10 and Corollary 4.8, the base locus of the linear system $\mathscr{M}_i$ does not contain any curves.

Let M_i be a surface in $\mathscr{M}_i$ that does not contain the curve Z. Then

$$\sum_{i=1}^{s} M_i \in \mathscr{M},$$

which is a contradiction, since Z is contained in the base locus of the linear system $\mathscr{M}$.

Lemma 4.12. *There exists a curve $C \subset \Pi$ of degree $k \leqslant d-2$ such that*

$$|C \cap \Xi'| > k(d-1).$$

Proof. We will prove the required assertion by reductio ad absurdum. Suppose that every curve in Π of degree k contains at most $k(d-1)$ points of the set Ξ' for every $k \leqslant d-2$. Suppose further that there is no curve in $\mathbb{P}^2$ of degree $d-1$ which contains the whole set Ξ'.

Put $\xi = 2d-5$. Then $\xi \geqslant 7$ because $d \geqslant 6$.

Suppose that no more than $k(\xi+3-k)-2$ points of the subset $\Xi' \setminus Q'$ are contained in a curve of degree k for every $k \leqslant (\xi+3)/2$. By Lemma 2.3 there exists a curve

$$Z \subset \mathbb{P}^2$$

of degree $2d-5$ that contains $\Xi' \setminus Q'$ and does not contain Q'. Let S be a cone in $\mathbb{P}^3$ over the curve Z whose vertex is the point O. Then S is a surface in $\mathbb{P}^3$ of degree $2d-5$ that contains $\Xi \setminus Q$ and does not contain the point Q, which is a contradiction.

Hence we see that there exists a curve $R \subset \mathbb{P}^2$ of degree $k \leqslant d-1$ that contains at least $k(\xi+3-k)-1$ points of the set $\Xi' \setminus Q'$.

Suppose that $k = d-1$. Then the curve R contains at least

$$k(\xi+3-k)-1 = k(2d-2-k)-1 = (d-1)^2-1$$

points of the set $\Xi' \setminus Q'$. Then $Q' \notin R$ because there is no curve of degree $d-1$ containing the whole of Ξ'. The cone in $\mathbb{P}^3$ over R whose vertex is the point O is a surface of degree $2d-5$ that contains $\Xi \setminus Q$ and does not contain the point $Q \in \Xi$, which is a contradiction.

Hence we see that $k \leqslant d-2$. Then $k(2d-2-k)-1 \leqslant k(d-1)$.

Suppose that $k=1$. Then $2d-4 \leqslant d-1$, which is impossible because $d \geqslant 6$. Hence we see that $k \neq 1$. Then

$$k(2d-2-k)-1 \leqslant k(d-1) \quad \Longleftrightarrow \quad k(d-1-k) \leqslant 1 \quad \Longleftrightarrow \quad k \geqslant d-1,$$

which is a contradiction because $k \leqslant d-2$.

Without loss of generality we may assume that the number k is the smallest natural number with this property. Then the curve C is irreducible.

Lemma 4.13. *The curve C contains the set Ξ'.*

Proof. Suppose that $\Xi' \not\subset C$. Let S be a cone in $\mathbb{P}^3$ over C whose vertex is O. Then $\Xi \not\subset S$ and

$$|\Xi \setminus (S \cap \Xi)| \leqslant (d-1)(d-1-k)-1.$$

Thus, the set $\Xi \setminus (S \cap \Xi)$ imposes independent linear conditions on hypersurfaces of degree $2d-5-k$ by Theorem 3.1. Then the set Ξ imposes independent linear conditions on hypersurfaces of degree $2d-5$ by Lemma 2.10, which is a contradiction.

Let us consider Ξ as a subscheme of $\mathbb{P}^3$ with ideal sheaf $\mathscr{I}_\Xi$. Then

$$h^1(\mathscr{I}_\Xi \otimes \mathscr{O}_{\mathbb{P}^3}(2d-5)) \neq 0.$$

Let $\mathscr{D}$ be a linear subsystem of the linear system $|\mathscr{O}_{\mathbb{P}^3}(d-2)|$ consisting of all surfaces that contain the set Ξ. Then its base locus is zero-dimensional by Lemma 2.4. Put

$$\Gamma = M_1 \cdot M_2 \cdot M_3,$$

where M_1, M_2 and M_3 are general surfaces in the linear system $\mathscr{D}$. Then Γ is a closed zero-dimensional subscheme of $\mathbb{P}^3$, and Ξ is closed subscheme of the scheme Γ.

Let Υ be a closed subscheme of the scheme Γ such that

$$\mathscr{I}_\Upsilon = \mathrm{Ann}(\mathscr{I}_\Xi / \mathscr{I}_\Gamma),$$

where $\mathscr{I}_\Upsilon$ and $\mathscr{I}_\Gamma$ are the ideal sheaves of the subschemes Υ and Γ, respectively. Then

$$0 \neq h^1\bigl(\mathscr{O}_{\mathbb{P}^3}(2d-5) \otimes \mathscr{I}_\Xi\bigr) = h^0\bigl(\mathscr{O}_{\mathbb{P}^3}(d-5) \otimes \mathscr{I}_\Upsilon\bigr) - h^0\bigl(\mathscr{O}_{\mathbb{P}^3}(d-5) \otimes \mathscr{I}_\Gamma\bigr)$$

by Theorem 2.1. Thus there exists a surface $F \in |\mathscr{O}_{\mathbb{P}^3}(d-5) \otimes \mathscr{I}_\Upsilon|$. Then

$$(d-5)(d-2)^2 = F \cdot M_1 \cdot M_2 \geqslant h^0(\mathscr{O}_\Upsilon) = h^0(\mathscr{O}_\Gamma) - h^0(\mathscr{O}_\Xi) = (d-2)^3 - |\Xi|,$$

which implies that $(d-1)^2 = |\Xi| \geqslant 3(d-2)^2$, which is a contradiction.

The assertion of Theorem 1.6 is proved.

§ 5. A special projection

The purpose of this section is to prove Lemma 4.10.

Let Ξ be a finite subset in $\mathbb{P}^3$, let P be a point in Ξ, and let d be a natural number such that $d \geqslant 6$, Suppose that Ξ has the following properties:
- $|\Xi| = (d-1)^2$;
- at most $\lambda(d-1)-1$ points of Ξ are contained in a curve of degree λ for any $\lambda \in \mathbb{N}$;
- at most $d-1$ points of the set Ξ are contained in a plane;
- each surface in $\mathbb{P}^3$ of degree $2d-5$ that contains $\Xi \setminus P$ passes through $P \in \Xi$.

Lemma 5.1. *Let S be a surface in $\mathbb{P}^3$ of degree μ such that $|S \cap \Xi| \geqslant (d-1)\mu+1$. Then*

$$\Xi \subset S.$$

Proof. Suppose that $|S \cap \Xi| \geqslant (d-1)\mu+1$, but $\Xi \not\subset S$. Then

$$|\Xi \setminus (S \cap \Xi)| \leqslant (d-1)^2 - (d-1)\mu + 1 = (d-1)(d-1-\mu) - 1,$$

which implies that the subset $\Xi \setminus (S \cap \Xi)$ imposes independent linear conditions on hypersurfaces of degree $2d-5-\mu$ by Theorem 3.1. Then Ξ imposes independent linear conditions on hypersurfaces of degree $2d-5$ by Lemma 2.10, which is a contradiction.

Let $\mathscr{M}$ be a linear system consisting of all surfaces of degree $d-1$ that contain the set Ξ. To prove Lemma 4.10 we must show that the base locus of $\mathscr{M}$ contains a curve. Suppose that this base locus is zero-dimensional. We shall derive a contradiction.

Lemma 5.2. *The set $\Xi \subset \mathbb{P}^3$ contains two different point Q_1 and Q_2 such that*
- *the line that passes through Q_1 and Q_2 does not contain the point $P \in \Xi$;*
- *the line that passes through Q_1 and Q_2 contains at most $d-3$ points of the set Ξ.*

This assertion is obvious.

Let L be a line in $\mathbb{P}^3$ that passes through the points Q_1 and Q_2, let O be a sufficiently general point in the line L, let Π be a plane in $\mathbb{P}^3$ such that $L \not\subset \Pi$, and let

$$\psi\colon \mathbb{P}^3 \dashrightarrow \Pi \cong \mathbb{P}^2$$

be a projection from $O \in \mathbb{P}^3$. Put $\Xi' = \psi(\Xi)$ and $P' = \psi(P)$. Then ψ induces a bijection

$$\Xi \setminus (\Xi \cap L) \longleftrightarrow \Xi' \setminus \psi(L)$$

and $(d-1)(d-2) < |\Xi'| < (d-1)^2$.

Lemma 5.3. *Let λ be a natural number and let Λ be a subset of the set Ξ such that*

$$|\psi(\Lambda)| \geqslant \lambda(d-1)+1.$$

Suppose that there exists a curve C of degree λ such that

$$\psi(\Lambda) \subset C \subset \Pi \cong \mathbb{P}^2.$$

Let $\mathscr{D}$ be a linear subsystem of $|\mathscr{O}_{\mathbb{P}^3}(\lambda)|$ consisting of all surfaces of degree λ that contain Λ. Then the base locus of the linear system $\mathscr{D}$ is contained in the union of the line L and some finite set.

Proof. Suppose that there exists an irreducible curve $Z \subset \mathbb{P}^3$ that is contained in the base locus of the linear system $\mathscr{D}$. We must show that $Z = L$.

We suppose that $Z \neq L$ and show this leads to contradiction. We may assume that $O \notin Z$. Then $\psi(Z)$ is an irreducible curve.

For every point $Q \in \Lambda$ let Y_Q be a cone in $\mathbb{P}^3$ over Z whose vertex is Q. Then

$$L \subset Y_Q \iff Q \in L,$$

which implies that we may assume that $O \notin Y_Q$ if $Q \notin L$ because $O \in L$ is sufficiently general. Put $\Theta = Z \cap \Lambda$ and $\Omega = L \cap \Lambda$. Then

$$\psi(Z) \cap \psi(\Lambda \setminus (\Xi \cup \Omega)) = \varnothing.$$

As $O \in L$ is a general point, we may assume that $|\Lambda \setminus \Omega| = |\psi(\Lambda \setminus \Omega)|$.

Let C be an irreducible curve in Π of degree λ that contains the set $\psi(\Sigma)$, and let W be a cone in $\mathbb{P}^3$ over the curve C whose vertex is our point O. Then $W \in \mathscr{D}$, which implies that $Z \subset W$. Then $\psi(Z) = C$. Thus, we have $\Lambda \setminus (\Xi \cup \Omega) \subset Z$.

Let B be any smooth point of the curve Z such that B is not contained in the line L, and let H be a plane in $\mathbb{P}^3$ that passes through the line L and the point B.

If $Z \subset H$, then $H \cap \Pi = Z$, which gives $\lambda = 1$, a contradiction.

Thus we have shown that $Z \not\subset H$, so the intersection $H \cap Z$ is a finite set containing the point B. In particular, there exists a line $L' \subset H$ such that

$$L' \cap Z = B$$

and L' is not tangent to Z at the point B. If $O = L \cap L'$, then the morphism

$$\psi\big|_Z \colon Z \longrightarrow C$$

is birational, which implies that $\deg(Z) = \lambda$. Thus, as $O \in L$ is a general point, we may assume that $\deg(Z) = \lambda$.

We see that Z is an irreducible curve in $\mathbb{P}^3$ of degree λ that contains $\Lambda \setminus \Omega$. But

$$|\Lambda \setminus \Omega| = |\psi(\Lambda)| - |\psi(\Omega)| \geqslant |\psi(\Lambda)| - 1 \geqslant \lambda(d-1)$$

because $\psi(\Omega) = \psi(L)$ of $\Omega \neq \emptyset$. But at most $\lambda(d-1) - 1$ points of the set Ξ are contained in any curve of degree λ, which is a contradiction.

Lemma 5.4. *There exists a curve $C \subset \Pi$ of degree $k \leqslant d-2$ such that*

$$|C \cap \Xi'| > k(d-1).$$

Proof. Suppose that at most $k(d-1)$ points of the set Ξ' are contained in a curve of degree k for every $k \leqslant d-2$. Put $\xi = 2d-5$. Then $\xi \geqslant 7$ because $d \geqslant 6$.

Suppose that at most $k(\xi+3-k) - 2$ points of the set $\Xi' \setminus P'$ are contained in any curve of degree k for every $k \leqslant (\xi+3)/2$. By Lemma 2.3, there exists a curve

$$Z \subset \mathbb{P}^2$$

of degree $2d-5$ that contains $\Xi'\setminus P'$ and does not contain P'. Let S be a cone in $\mathbb{P}^3$ over the curve Z whose vertex is the point O. Then S is a surface of degree $2d-5$ that contains all points of the set $\Xi\setminus P$ and does not contain the point P, which is a contradiction.

Thus, we see that there exists some curve $R\subset\mathbb{P}^2$ of degree $k\leqslant d-1$ such that R contains at least $k(\xi+3-k)-1$ points of the set $\Xi'\setminus P'$.

If $k=d-1$, then the curve R contains at least

$$k(\xi+3-k)-1=k(2d-2-k)-1=(d-1)^2-1$$

points of the set $\Xi'\setminus P'$. But the set $\Xi'\setminus P'$ consists of at most $(d-1)^2-2$ points.

We see that $k\leqslant d-2$. Then $k(2d-2-k)-1\leqslant k(d-1)$.

If $k=1$, then $2d-4\leqslant d-1$, which is impossible since $d\geqslant 6$. We see that $k\neq 1$. Then

$$k(2d-2-k)-1\leqslant k(d-1)\quad\iff\quad k(d-1-k)\leqslant 1\quad\iff\quad k\geqslant d-1,$$

which is a contradiction because $k\leqslant d-2$.

Without loss of generality, we may assume that k is the smallest natural number such that there is a curve in Π of degree $k\leqslant d-2$ that contains at least $k(d-1)+1$ points of the set Ξ', which implies that the curve C is irreducible. Let S be a cone in $\mathbb{P}^3$ over the curve C whose vertex is the point O. Then

$$|S\cap\Xi|\geqslant k(d-1)+1,$$

which implies that $\Xi\subset S$ by Lemma 5.1. Then $\Xi'\subset C$.

Let us consider Ξ as a closed zero-dimensional subscheme of $\mathbb{P}^3$. Then

$$h^1(\mathscr{I}_\Xi\otimes\mathscr{O}_{\mathbb{P}^3}(2d-5))\neq 0,$$

where $\mathscr{I}_\Xi$ is the ideal sheaf of the subscheme Ξ.

Let $\mathscr{R}$ be the linear subsystem of the linear system $|\mathscr{O}_{\mathbb{P}^3}(d-2)|$ consisting of all surfaces that pass through Ξ. By Lemma 5.3 the base locus of the linear system $\mathscr{R}$ is contained in the union of the line L with some finite set. Put

$$\Gamma=R_1\cdot R_2\cdot M,$$

where R_1 and R_2 are general surfaces in the linear system $\mathscr{R}$ and M is a general surface in the linear system $\mathscr{M}$. Then Γ is a zero-dimensional scheme in $\mathbb{P}^3$ and Ξ is its closed subscheme.

Let Υ be a closed subscheme of the scheme Γ such that

$$\mathscr{I}_\Upsilon=\mathrm{Ann}(\mathscr{I}_\Xi/\mathscr{I}_\Gamma),$$

where $\mathscr{I}_\Upsilon$ and $\mathscr{I}_\Gamma$ are the ideal sheaves of the subschemes Υ and Γ, respectively. Then

$$0\neq h^1\big(\mathscr{O}_{\mathbb{P}^3}(2d-5)\otimes\mathscr{I}_\Xi\big)=h^0\big(\mathscr{O}_{\mathbb{P}^3}(d-4)\otimes\mathscr{I}_\Upsilon\big)-h^0\big(\mathscr{O}_{\mathbb{P}^3}(d-4)\otimes\mathscr{I}_\Gamma\big)$$

by Theorem 2.1. Thus there exists a surface $F \in |\mathscr{O}_{\mathbb{P}^3}(d-4) \otimes \mathscr{I}_\Upsilon|$. Then

$$(d-4)(d-1)(d-2) = F \cdot R_1 \cdot M \geqslant h^0(\mathscr{O}_\Upsilon) = h^0(\mathscr{O}_\Gamma) - h^0(\mathscr{O}_\Xi) = (d-2)^2(d-1) - |\Xi|,$$

which implies that $|\Xi| \geqslant 2(d-2)(d-1)$. Therefore, we see that

$$(d-1)^2 = |\Xi| \geqslant 2(d-2)(d-1),$$

which is a contradiction because $d \geqslant 4$.

The assertion of Lemma 4.10 is proved.

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I. A. Cheltsov
University of Edinburgh, UK
E-mail: cheltsov@yahoo.com

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