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## **Birationally rigid del Pezzo fibrations**

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**Abstract.** In this paper we study del Pezzo fibrations  $z : X \to \mathbb{P}^1$  of degree 1 and 2 such that *X* is smooth, rk Pic(*X*) = 2 and  $K_X^2 \notin \overline{\mathbb{NE}}(X)$ . These are examples of smooth birationally rigid 3-fold Mori fibre spaces. We describe all birational transformations of the 3-fold *X* into elliptic fibrations, fibrations of surfaces of Kodaira dimension zero, and canonical Fano 3-folds.

Let X be a smooth 3-fold<sup>1</sup> of Picard rank 2 and  $\tau : X \to \mathbb{P}^1$  be a flat morphism whose generic fiber is a del Pezzo surface of degree 1 or 2. The following result is proved in [18].

**Theorem 1.** Suppose that  $K_X^2 \notin \text{Int}(\overline{\mathbb{NE}}(X))$ . Then X is not birational to the following:

- a conic bundle;
- a fibration<sup>2</sup> of rational surfaces that is not equivalent<sup>3</sup> to  $\tau$ ;
- a Fano 3-fold of Picard rank 1 having terminal Q-factorial singularities.

The main purpose of this paper is to prove the following result.

**Theorem 2.** Suppose that  $K_X^2 \notin \overline{\mathbb{NE}}(X)$ . Then X is birational neither to a fibration whose generic fiber is a surface of Kodaira dimension zero nor to a Fano 3-fold

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- <sup>1</sup> All varieties are assumed to be projective, normal, and defined over the field  $\mathbb{C}$ .
- <sup>2</sup> For every fibration  $\pi: Y \to Z$  we assume that  $\dim(Y) > \dim(Z) \neq 0$  and  $\pi_*(\mathcal{O}_Y) = \mathcal{O}_Z$ .

<sup>3</sup> Fibrations  $\tau : U \to Z$  and  $\overline{\tau} : \overline{U} \to \overline{Z}$  are called equivalent if there are birational maps  $\alpha : U \to \overline{U}$  and  $\beta : Z \to \overline{Z}$  such that the diagram

$$\begin{array}{c|c} U - - - \stackrel{\alpha}{-} - & > \bar{U} \\ \pi \\ \downarrow \\ Z - - - \stackrel{\beta}{-} - & > \bar{Z} \end{array}$$

is commutative and  $\alpha$  induces the birational isomorphism between the generic fibers of  $\tau$  and  $\overline{\tau}$ .

having canonical singularities. Let  $\rho : X \to Y$  be a birational map, where Y is a 3-fold such that there is an elliptic fibration  $\psi : Y \to \mathbb{P}^2$ . Then there is a rational map  $\alpha : \mathbb{P}^2 \to \mathbb{P}^1$  such that the diagram

$$\begin{array}{c|c} X - - & - & \rho \\ \tau & & & \downarrow \\ \tau & & & \downarrow \\ \mathbb{P}^1 \prec & - & - & \mathbb{P}^2 \end{array}$$

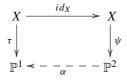
is commutative.

The conditions of Theorem 2 cannot be weakened.

*Example 3.* Let  $\lambda : X \to \mathbb{P}^1 \times \mathbb{P}^2$  be a double cover branched over a smooth divisor of bi-degree (k, 4) for  $k \ge 4$ . Consider the projections  $pr_1 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$  and  $pr_2 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ , and put  $\tau = pr_1 \circ \lambda$ . Then X is smooth,  $\tau : X \to \mathbb{P}^1$  is a del Pezzo fibration of degree 2, and rk  $\operatorname{Pic}(X) = 2$ . Moreover it is easy to see that

 $K_X^2 \notin \overline{\mathbb{NE}}(X) \iff k \ge 5$ 

and  $K_X^2 \in \partial \overline{\mathbb{NE}}(X)$  in the case k = 4. Let  $\psi = pr_2 \circ \lambda$ . Then  $\psi : X \to \mathbb{P}^2$  is an elliptic fibration when k = 4 and there is no rational map  $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$  that makes the diagram



commutative. Let  $\beta : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$  be a projection from a point and  $\gamma = \beta \circ \psi$ . Then the generic fiber of  $\gamma$  is a K3 surface in the case k = 4 (cf. Theorem 27).

*Remark 4.* Suppose that the generic fiber of Z is a del Pezzo surface of degree 2. Let C be a section of the fibration  $\tau$  and  $\mathcal{H}_C \subset |-K_X + \tau^*(\mathcal{O}_{\mathbb{P}^1}(n))|$  be the linear system of surfaces passing through the section C for  $n \gg 0$ . Then the generic fiber of the rational map  $\phi_{\mathcal{H}_C} : X \dashrightarrow Z_C$  is an elliptic curve, the surface  $Z_C$  is rational, a resolution of indeterminacy of the map  $\phi_{\mathcal{H}_C}$  gives an elliptic fibration with a section, and there is a natural rational projection  $\alpha_C : Z_C \dashrightarrow \mathbb{P}^1$  such that  $\alpha_C \circ \phi_{\mathcal{H}_C} = \tau$ .

The fibration  $\tau$  always has a section due to the following result in [16].

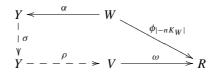
**Theorem 5.** Let Y be a smooth geometrically irreducible surface over a  $C_1$ -field  $\mathbb{F}$  such that Y is geometrically rational. Then the surface Y has a point in  $\mathbb{F}$ .

On the other hand, the following result was proved in [15].

**Theorem 6.** Let Y be a projective variety and  $g : Y \to R$  be a morphism with a section onto a smooth curve R. Suppose that we have a set of closed points  $\{r_1, \ldots, r_k\} \in R$  such that each fiber  $Y_i = g^{-1}(r_i)$  is smooth and separably rationally connected. Then for a set of closed points  $y_i \in Y_i$  there is a section  $C \subset Y$  of the morphism g passing through each point  $y_i$ .

Therefore the fibration  $\tau : X \to \mathbb{P}^1$  has a huge set of sections. Moreover for two general enough sections  $C_1$  and  $C_2$  of  $\tau$  the corresponding rational maps  $\phi_{\mathcal{H}_{C_1}}$ and  $\phi_{\mathcal{H}_{C_2}}$  give two non-equivalent elliptic fibrations. Hence the part of Theorem 2 referring to models as elliptic fibrations cannot be improved, but it can be clarified by means of the following result (see [4]).

**Proposition 7.** Let Y be a smooth del Pezzo surface of degree 1 or 2 defined over a perfect field  $\mathbb{F}$  such that  $\operatorname{rk} \operatorname{Pic}(Y) = 1$ , let  $\rho : Y \dashrightarrow V$  be a birational map and  $\omega : V \to R$  be an elliptic fibration. Then there is a commutative diagram



for  $n \gg 0$ , where  $\sigma$  is a birational map, and  $\alpha$  is a birational morphism such that  $K_W^2 = 0$ , the divisor  $-K_W$  is nef, and the linear system  $|-nK_W|$  is free.

**Corollary 8.** Let Y be a smooth del Pezzo surface of degree 1 of Picard rank 1 defined over a perfect field  $\mathbb{F}$ ,  $\rho : Y \dashrightarrow V$  be a birational map, where V is a smooth surface equipped with a morphism  $\pi : V \to S$  such that  $\pi$  is a relatively minimal elliptic fibration with connected fibers. Then the following holds:

- $\rho$  is a blow up of the surface Y at some  $\mathbb{F}$ -point  $P \in Y$ ;
- there is a curve  $C \in |-K_Y|$  such that  $P \in \hat{C}$ , where  $\hat{C} = C \setminus \text{Sing}(C)$ ;
- the curve  $\hat{C}$  is a group scheme with  $id_{\hat{C}} = O$ , where  $O = Bs | -K_Y |$ ;
- there is  $n \in \mathbb{N}$  such that  $P^n = \mathrm{id}_{\hat{C}}$  in  $\hat{C}$ ;
- $\pi = \phi_{|-nK_V|}$ .

Birational transformations into elliptic fibrations were used in [2], [3], and [13] in the proof of the following result.

**Theorem 9.** The set of rational points is potentially dense<sup>4</sup> on all smooth Fano 3-folds defined over a number field  $\mathbb{F}$  with a possible exception of a double cover of  $\mathbb{P}^3$  ramified in a smooth sextic surface.

*Remark 10.* The possible exception appears in Theorem 9 because a smooth sextic double solid is the only smooth Fano 3-fold that is not birationally equivalent to an elliptic fibration (see [4]).

The results of [17], [11], and [12] together with Theorem 1 imply the following result.

**Theorem 11.** Suppose that  $K_X^2 \notin \text{Int}(\overline{\mathbb{NE}}(X))$ . Then the group of birational automorphism of the 3-fold X is generated by Bertini involutions of the generic fiber of  $\tau$  and biregular automorphisms of X.

<sup>&</sup>lt;sup>4</sup> The set of rational points of a variety X defined over a number field  $\mathbb{F}$  is potentially dense if for some finite extension  $\mathbb{K}$  of the field  $\mathbb{F}$  the set of  $\mathbb{K}$ -rational points of X is Zariski dense in X.

**Corollary 12.** In the conditions of Theorem 11 suppose that  $\tau$  is a del Pezzo fibration of degree 1. Then Bir(X) = Aut(X).

In the rest of the paper we prove Theorem 2. We use methods of the paper [18] in the form of Theorems 3.12 and 5.1 of the paper [8]. Suppose that  $K_X^2 \notin \overline{\mathbb{NE}}(X)$  and there is a birational map  $\rho : X \dashrightarrow Y$  such that either

- Y is a Fano 3-fold with canonical singularities (the Fano 3-fold case) or
- $\pi: Y \to S$  is a fibration whose general fiber has a numerically trivial canonical divisor (*the fibration case*).

Let

$$\mathcal{D} = \begin{cases} |-tK_Y| \text{ for } t \gg 0 \text{ in the Fano 3-fold case} \\ |\pi^*(D)| \text{ for some very ample divisor } D \text{ on } S \text{ in the fibration case} \end{cases}$$

and put  $\mathcal{H} = \rho^{-1}(\mathcal{D})$ . Then the linear system  $\mathcal{H}$  has no fixed components and

$$K_X + \frac{1}{n}\mathcal{H} \sim_{\mathbb{Q}} rF$$

for some  $n \in \mathbb{N}$  and  $r \in \mathbb{Q}$ , where *F* is a fiber of  $\tau$ .

**Lemma 13.** *The rational number r is positive.* 

*Proof.* Suppose that  $r \leq 0$ . Then

$$K_X^2 \equiv \frac{1}{n^2} \mathcal{H}^2 - 2r K_X \cdot F \in \overline{\mathbb{NE}}(X),$$

which contradicts our initial assumption.

Consider the movable log pair<sup>5</sup>  $(X, \frac{1}{n}\mathcal{H})$ .

**Proposition 14.** The singularities of the log pair  $(X, \frac{1}{n}\mathcal{H})$  are not terminal.

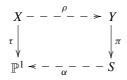
*Proof.* Suppose that the singularities of the log pair  $(X, \frac{1}{n}\mathcal{H})$  are terminal. Then the log pair  $(X, \epsilon\mathcal{H})$  is a canonical model for some  $\epsilon \in \mathbb{Q}_{>\frac{1}{n}}$ . Thus  $\kappa(X, \epsilon\mathcal{H}) = 3$ . However

$$\kappa(X, \epsilon \mathcal{H}) = \kappa(Y, \epsilon \mathcal{D}) \le \dim(S)$$

in the fibration case. In the Fano 3-fold case the log pair  $(Y, \epsilon D)$  is a canonical model and the uniqueness of canonical model (see Theorem 2.9 in [5]) implies  $X \cong Y$ , which is impossible because  $K_X^2 \notin \overline{\mathbb{NE}}(X)$ .  $\Box$ 

<sup>&</sup>lt;sup>5</sup> For basic properties of movable log pairs see [1], [8], and [5].

**Lemma 15.** Suppose that the singularities of the log pair  $(X, \frac{1}{n}\mathcal{H})$  are canonical. Then the Fano 3-fold case is impossible, in the fibration case the generic fiber of  $\pi$  is an elliptic curve, the surface S is rational and the following diagram



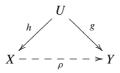
commutates, where  $\alpha$  is a dominant rational map.

Proof. In the Fano 3-fold case

$$1 = \kappa(X, \frac{1}{n}\mathcal{H}) = \kappa(Y, \frac{1}{n}\mathcal{D}) \in \{-\infty, 0, 3\}$$

by construction of the linear system  $\mathcal{D}$ , which is a contradiction.

In the fibration case consider the commutative diagram



with U is a smooth 3-fold, and h and g birational morphisms. Then

$$K_U + \frac{1}{n} \mathcal{R} \sim_{\mathbb{Q}} h^*(rF) + \sum_{i=1}^k a_i E_i,$$

where  $E_i$  is a *h*-exceptional divisor,  $\mathcal{R}$  is the proper transform of  $\mathcal{D}$  on U, and each  $a_i$  is a non-negative rational number. Let C be the proper transform on U of a sufficiently general curve lying in a fiber of  $\pi$ . Then  $K_U \cdot C = 0$  and  $\mathcal{R} \cdot C = 0$ . Thus, we have

$$(K_U + \frac{1}{n}\mathcal{R}) \cdot C = h^*(\epsilon F) \cdot C + \sum_{i=1}^k a_i E_i \cdot C \ge rF \cdot h(C),$$

which implies  $F \cdot C = 0$ . Therefore the curve h(C) lies in a fiber of  $\tau$ . Hence there is a rational map  $\alpha : S \longrightarrow \mathbb{P}^1$  such that  $\alpha \circ \pi \circ \rho = \tau$  and  $\pi$  is an elliptic fibration.  $\Box$ 

The following result is due to [18] (see Step 1 in the proof of Theorem 5.1 in [8]).

**Lemma 16.** Let *C* be a curve contained in a fiber of  $\tau$ . Then  $\operatorname{mult}_C(\mathcal{H}) \leq n$ .

**Lemma 17.** The singularities of the log pair  $(X, \frac{1}{n}\mathcal{H})$  are canonical in codimension one in the case when the generic fiber of the fibration  $\tau$  is a del Pezzo surface of degree 1.

*Proof.* Suppose that  $\operatorname{mult}_C(\mathcal{H}) > n$  for some curve  $C \subset X$ . Let F be a general enough fiber of the morphism  $\tau$ . Then

$$n^2 = \mathcal{H}^2 \cdot F \ge \operatorname{mult}_C(\mathcal{H}^2)F \cdot C > n^2,$$

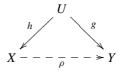
because  $F \cdot C \neq 0$  by Lemma 16, which is a contradiction.  $\Box$ 

The following result is classical and due to [17] (see also [7] and [18]).

**Proposition 18.** Suppose that  $\tau$  is a del Pezzo fibration of degree 2. Then there is a composition  $\sigma$  of Bertini involutions of the generic fiber of  $\tau$  such that the singularities of the movable log pair  $(X, \frac{1}{n}\sigma(\mathcal{H}))$  are canonical in codimension one, where  $\bar{n} \in \mathbb{N}$  such that  $K_X + \frac{1}{n}\sigma(\mathcal{H}) \sim_{\mathbb{Q}} \bar{r} F$  for  $\bar{r} \in \mathbb{Q}_{>0}$ .

*Remark 19.* In the following we may assume that the singularities of the log pair  $(X, \frac{1}{n}\mathcal{H})$  are canonical in codimension one. Indeed in the case when  $\tau$  is a del Pezzo fibration of degree 2 we may substitute  $\rho$  by  $\rho \circ \sigma^{-1}$  and use the fact that  $\tau \circ \sigma = \tau$ , where  $\sigma$  is the composition  $\sigma$  of Bertini involutions of the generic fiber of  $\tau$  from Proposition 18.

Consider a commutative diagram



with U is smooth, and h and g birational morphisms. Let  $\mathcal{R}$  be the proper transform of the linear system  $\mathcal{H}$  on U. Then

$$K_U + \frac{1}{n}\mathcal{R} \sim_{\mathbb{Q}} h^*(rF) + \sum_{i=1}^k a_i E_i,$$

where  $E_i$  is an *h*-exceptional divisor and  $a_i \in \mathbb{Q}$ . Now let us consider a (finite or empty) subset  $\mathcal{J} \subset \mathbb{P}^1$  that is defined as follows:

 $\mathcal{J} = \{\lambda \in \mathbb{P}^1 \mid h(E_i) \text{ is a point on } \tau^{-1}(\lambda) \text{ for some } E_i \text{ with } a_i < 0\}.$ 

For every point  $\lambda$  in  $\mathcal{J}$  we have

$$h^*(F_{\lambda}) \sim h^{-1}(F_{\lambda}) + \sum_{j=1}^{k_{\lambda}} b_j E_j,$$

where  $b_i \in \mathbb{N}$  and  $F_{\lambda}$  is a fiber of  $\tau$  over  $\lambda$ . For every  $\lambda \in \mathcal{J}$  let  $\mathcal{I}_{\lambda} \subset \{1, \ldots, k\}$  be a finite subset such that  $i \in \mathcal{I}_{\lambda}$  if and only if  $a_i < 0$  and  $h(E_i) \in F_{\lambda}$  is a point. Put  $\mathcal{I} = \bigcup_{\lambda \in \mathcal{J}} \mathcal{I}_{\lambda}$ . In the following we assume that either *Y* is a Fano 3-fold with canonical singularities or there is no rational map  $\alpha : S \longrightarrow \mathbb{P}^1$  that makes the diagram

$$\begin{array}{c|c} X - - - \stackrel{\rho}{-} - & \Rightarrow & Y \\ \tau \\ \downarrow & & & \downarrow \\ \mathbb{P}^{1} \ll - -\alpha^{-} - - S \end{array}$$

commutative. Let us show that this assumption leads to a contradiction.

**Corollary 20.** *The set*  $\mathcal{J}$  *is not empty.* 

The following result is known as the existence of *the super-maximal singularity* in the notations of the paper [19].

**Proposition 21.** The inequality

$$r + \sum_{\lambda \in \mathcal{J}} \min\{\frac{a_i}{b_i} \mid h(E_i) \in F_\lambda \text{ and } a_i < 0\} \le 0$$

holds.

*Proof.* Suppose the claim is false. Then there are positive rationals  $\epsilon$  and  $c_{\lambda}$  such that

$$c_{\lambda} + \min\{\frac{a_i}{b_i} \mid h(E_i) \in F_{\lambda} \text{ and } a_i < 0\} > 0$$

and  $r = \epsilon + \sum_{\lambda \in \mathcal{J}} c_{\lambda}$ . Then

$$K_U + \frac{1}{n}\mathcal{R} \sim_{\mathbb{Q}} h^*(\epsilon F) + \sum_{\lambda \in \mathcal{J}} \left( h^*(c_\lambda F_\lambda) + \sum_{i \in \mathcal{I}_\lambda} a_i E_i \right) + \sum_{i \notin \mathcal{I}} a_i E_i$$

and the divisor

$$h^*(c_\lambda F_\lambda) + \sum_{i\in\mathcal{I}_\lambda} a_i E_i$$

is effective for  $\lambda \in \mathcal{J}$  by the choice of  $c_{\lambda}$ . The divisor  $\sum_{i \notin \mathcal{I}} a_i E_i$  is effective because the singularities of  $(X, \frac{1}{n}\mathcal{H})$  are canonical in codimension one. Thus  $\kappa(U, \frac{1}{n}\mathcal{R}) = 1$ .

In the Fano 3-fold case we have

$$\kappa(Y,\frac{1}{n}\mathcal{D}) \in \{-\infty,0,3\},\$$

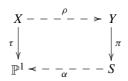
which is a contradiction. Therefore there is a fibration  $\pi : Y \to S$  whose sufficiently general fiber has a numerically trivial canonical divisor. Let  $C \subset U$  be the proper transform of a sufficiently general curve lying in a fiber of the fibration  $\pi$ . Then

$$K_U \cdot C = \mathcal{R} \cdot C = 0$$

and we have

$$(K_U + \frac{1}{n}\mathcal{R}) \cdot C = h^*(\epsilon F) \cdot C + \sum_{\lambda \in \mathcal{J}} \left( h^*(c_\lambda F_\lambda) + \sum_{i \in \mathcal{I}_\lambda} a_i E_i \right) \cdot C + \sum_{i \notin \mathcal{I}} a_i E_i \cdot C \ge \epsilon F \cdot h(C),$$

which implies that the curve h(C) lies in a fiber of the fibration  $\tau$ . Therefore  $\pi$  is an elliptic fibration and there is a rational map  $\alpha : S \longrightarrow \mathbb{P}^1$  such that the diagram



is commutative, which is impossible by our assumption.  $\Box$ 

**Corollary 22.** There are positive rational numbers  $c_{\lambda}$  such that  $\sum_{\lambda \in \mathcal{J}} c_{\lambda} = r$  and

$$\mathbb{CS}(X,\frac{1}{n}\mathcal{H}-c_{\lambda}F_{\lambda})\cap F_{\lambda}\neq\emptyset,$$

where  $\mathbb{CS}$  stands for a set of centers of canonical singularities of a log pair (see [5]).

For every  $\lambda \in \mathcal{J}$  the set  $\mathbb{CS}(X, \frac{1}{n}\mathcal{H} - c_{\lambda}F_{\lambda}) \cap F_{\lambda}$  consists of finite number of points due to inclusion  $\mathbb{CS}(X, \frac{1}{n}\mathcal{H}) \subset \mathbb{CS}(X, \frac{1}{n}\mathcal{H} - c_{\lambda}F_{\lambda})$  and Remark 19.

**Lemma 23.** For every  $\lambda \in \mathcal{J}$  the points in  $\mathbb{CS}(X, \frac{1}{n}\mathcal{H} - c_{\lambda}F_{\lambda}) \cap F_{\lambda}$  are smooth on  $F_{\lambda}$ .

*Proof.* Let *O* be a point in  $\mathbb{CS}(X, \frac{1}{n}\mathcal{H} - c_{\lambda}F_{\lambda}) \cap F_{\lambda}$  that is singular on  $F_{\lambda}$ . Then there is a pencil  $\mathcal{L} \subset |-\frac{2}{d}K_{F_{\lambda}}|$  of curves singular at the point *O*. Therefore we have

 $2n = \mathcal{H} \cdot \mathcal{L} \ge \operatorname{mult}_{O}(\mathcal{H}) \operatorname{mult}_{O}(\mathcal{L}) \ge 2 \operatorname{mult}_{O}(\mathcal{H}),$ 

but the inequality  $c_{\lambda} > 0$  implies  $\text{mult}_{O}(\mathcal{H}) > n$ . This contradiction implies the claim.  $\Box$ 

Consider two sufficiently general divisors  $D_1$  and  $D_2$  in  $\mathcal{H}$ . Put  $d = K_X^2 \cdot F$ and

$$\frac{1}{n^2}D_1 \cdot D_2 = Z + \sum_{\lambda \in \mathbb{P}^1} C_\lambda,$$

where *Z* is an effective cycle whose components do not lie in fibers of  $\tau$  and  $C_{\lambda}$  is an effective cycle contained in the fiber  $F_{\lambda}$  of  $\tau$  over the point  $\lambda \in \mathbb{P}^1$ . Let *C* be a curve in a fiber of  $\tau$  with  $-K_X \cdot C = 1$ . Then  $C_{\lambda} \equiv \beta_{\lambda}C$  for some  $\beta_{\lambda} \in \mathbb{Q}_{\geq 0}$ . Put  $\beta = \sum_{\lambda \in \mathbb{P}^1} \beta_{\lambda}$ . **Lemma 24.** The inequality  $\beta \leq 2rd$  holds.

*Proof.* Suppose that  $\beta > 2rd$ . Then the equivalence

$$Z + \beta C \equiv Z + \sum_{\lambda \in \mathbb{P}^1} C_{\lambda} \equiv K_X^2 + 2rF \cdot K_X \equiv K_X^2 + 2rdC$$

implies  $K_X^2 \equiv Z + (\beta - 2rd)C \in \overline{\mathbb{NE}}(X)$ , which is a contradiction.  $\Box$ 

For every  $\lambda \in \mathcal{J}$  let  $O_{\lambda}$  be a point in  $\mathbb{CS}(X, \frac{1}{n}\mathcal{H} - c_{\lambda}F_{\lambda}) \cap F_{\lambda}$ . Then

 $\operatorname{mult}_{O_{\lambda}}(Z) + \operatorname{mult}_{O_{\lambda}}(C_{\lambda}) \geq 4$ 

due to [14] (see Corollary 7.3 in [19] or Theorem 3.1 in [8]), but Theorem 3.12 in [8] implies the existence of a rational number  $t_{\lambda} \in [0, 1]$  such that the inequality

$$\operatorname{mult}_{O_{\lambda}}(Z) + t_{\lambda} \operatorname{mult}_{O_{\lambda}}(C_{\lambda}) \ge 4(1 + c_{\lambda}t_{\lambda})$$

holds. However  $\operatorname{mult}_{O_{\lambda}}(Z) \leq Z \cdot F_{\lambda} = d \leq 2$ . In particular,  $t_{\lambda} \neq 0$  and the inequalities

$$\operatorname{mult}_{O_{\lambda}}(C_{\lambda}) \geq \frac{2 + 4c_{\lambda}t_{\lambda}}{t_{\lambda}} > 4c_{\lambda}$$

hold. On the other hand,  $\operatorname{mult}_{O_{\lambda}}(C_{\lambda}) \leq \frac{2}{d}\beta_{\lambda}$ . Therefore, we have

$$2rd \ge \beta = \sum_{\lambda \in \mathbb{P}^1} \beta_\lambda \ge \sum_{\lambda \in \mathcal{J}} \beta_\lambda > 2d \sum_{\lambda \in \mathcal{J}} c_\lambda = 2dr,$$

which is a contradiction. Hence, Theorem 2 is proved.

*Remark* 25. There is a fibration  $\tau : X \to \mathbb{P}^1$  such that the generic fiber of  $\tau$  is a del Pezzo surface of degree 1 or 2, the 3-fold X is smooth, rk  $\operatorname{Pic}(X) = 2$ , and  $K_X^2 \in \overline{\mathbb{NE}}(X)$ , but all claims of Theorem 2 hold for X.

**Theorem 26.** Let  $\lambda : X \to \operatorname{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$  be a double cover ramified in a smooth divisor  $R \sim 4M + 2L$  and  $\tau : X \to \mathbb{P}^1$  be the natural projection, where *M* is the tautological line bundle and *L* is a fiber of the projection to  $\mathbb{P}^1$ . Then the following holds:

- the 3-fold X is smooth;
- the generic fiber of  $\tau$  is a del Pezzo surface of degree 2,
- rk Pic(X) = 2 and  $K_X^2 \in \overline{\mathbb{NE}}(X)$ ,
- the 3-fold X is not birational to a Fano 3-fold with canonical singularities;
- the 3-fold X is not birational to a fibration on surfaces of Kodaira dimension zero;
- any dominant rational map  $\gamma : X \longrightarrow \mathbb{P}^2$  whose generic fiber is an elliptic curve is induced by the corresponding rational map of the generic fiber of  $\tau$ .

*Proof.* Every step of the proof of Theorem 2 is valid in this case except for Lemma 13, but the proof of Lemma 13 gives  $r \ge 0$ . However in the case r = 0 every divisor in  $\mathcal{H}$  has a negative intersection with curves whose images on V are contracted by  $\phi_{|M|}$ . The latter implies that  $\mathcal{H}$  has a fixed component in the case r = 0. In the case when r > 0 we can proceed as in the proof of Theorem 2.  $\Box$ 

Our technique can be applied to del Pezzo fibrations studied in [9] and [10].

**Theorem 27.** In the conditions of Example 3, let k = 4. Then every claim of Theorem 2 holds for the del Pezzo fibration  $\tau : X \to \mathbb{P}^1$  with the only exceptions of the unique elliptic fibration and fibrations of K3 surfaces described in Example 3 up to the action of Bir(X).

*Proof.* Every step of the proof of Theorem 2 is valid in this case except for Lemma 13, but the proof of Lemma 13 gives  $r \ge 0$ . Moreover

$$r=0 \iff \mathcal{H} \subset |-nK_X|,$$

but the linear system  $|-K_X|$  is free and the morphism  $\phi_{|-K_X|}$  is the elliptic fibration described in Example 3. Thus in the case r = 0 the linear system  $\mathcal{H}$  lies in fibers of the fibration  $\phi_{|-K_X|}$ , which implies the claim. In the case when r > 0 we can proceed as in the proof of Theorem 2.  $\Box$ 

**Theorem 28.** Let  $\lambda : X \to \operatorname{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$  be a double cover ramified in a smooth divisor  $R \sim 4M + 2L$  and  $\tau : X \to \mathbb{P}^1$  be the natural projection, where M is the tautological line bundle and L is a fiber of the projection to  $\mathbb{P}^1$ . Then the following holds:

- the 3-fold X is smooth;
- the generic fiber of  $\tau$  is a del Pezzo surface;
- rk  $\operatorname{Pic}(X) = 2$  and  $K_X^2 \in \overline{\mathbb{NE}}(X)$ ,
- the 3-fold X is not birational to a Fano 3-fold with canonical singularities;
- the 3-fold X is not birational to a fibration on surfaces of Kodaira dimension zero with the unique (up to the action of the group Bir(X)) exception of the fibration of surfaces of Kodaira dimension zero given by the map  $\phi_{|-K_X|}$ ;
- any dominant rational map  $\gamma : X \dashrightarrow \mathbb{P}^2$  whose generic fiber is an elliptic curve is induced by the corresponding rational map of the generic fiber of  $\tau$ .

*Proof.* Every step of the proof of Theorem 2 is valid in this case except for Lemma 13, but the proof of Lemma 13 gives  $r \ge 0$ . Moreover r = 0 if and only if  $\mathcal{H} \sim -nK_X$ .

Let *T* be the reduced curve on *X* such that  $\lambda(T)$  is contracted by  $\phi_{|M|}$  and *C* be a curve lying in a fiber of  $\tau$  such that  $-K_X \cdot C = 1$ . Then  $K_X^2 = T$  and

$$\mathbb{NE}(X) = \mathbb{R}_{>0}T \oplus \mathbb{R}_{>0}C,$$

which implies  $\text{Supp}(\mathcal{H}^2) = T$  in the case r = 0.

Suppose that r = 0 and  $S \ncong \mathbb{P}^1$ . Then the linear system  $\mathcal{H}$  is not composed from a pencil. There is a point  $P \in X \setminus T$  such that the subsystem  $\mathcal{H}_P \subset \mathcal{H}$  of

surfaces passing through the point *P* has no fixed components. For two general surfaces *A* and *B* in the linear system  $\mathcal{H}_P$  we have

$$P \in A \cap B \subset T$$

in the set-theoretical sense, which is a contradiction.

In the case when r = 0 and  $S \cong \mathbb{P}^1$  the previous arguments give  $\pi \circ \rho = \phi_{|-K_X|}$ . In the case of r > 0 we can proceed as in the proof of Theorem 2 to get a contradiction.  $\Box$ 

In the proof of Theorem 2 we use the method of [8]. Nevertheless we can use the method of [18] as well. The latter way is longer than the former one. However there are cases when the method of [8] fails, but the method of [18] succeeds (see [6]).

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