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Hyperelliptic and trigonal Fano threefolds

V. V. Przyjalkowski, I. A. Cheltsov, and K. A. Shramov

Abstract. We classify Fano 3-folds with canonical Gorenstein singularities whose anticanonical linear system has no base points but does not give an embedding, and we classify anticanonically embedded Fano 3-folds with canonical Gorenstein singularities which are not intersections of quadrics. We also study rationality questions for most of these varieties.

§ 1. Introduction

Consider a Fano threefold X with canonical Gorenstein singularities¹ (see [45], [112], [85], [39], [40]). Suppose that the anticanonical linear system $| -K_X |$ is base point free. It is well known that such varieties are divided into three classes.

1) Hyperelliptic varieties (that is, the morphism $\varphi_{|-K_X|}$ is not an embedding). Then the intersection of two general divisors in $| -K_X |$ is a hyperelliptic curve.

2) Trigonal varieties (that is, the morphism $\varphi_{|-K_X|}$ is an embedding but its image is not an intersection of quadrics). Then the intersection of two general divisors in $| -K_X |$ is a trigonal curve or the canonical image of a smooth plane quintic.

3) Varieties whose image under the embedding $\varphi_{|-K_X|}$ is an intersection of quadrics.

We study varieties of the first two types. Theorems 1.5 and 1.6 give complete classifications of hyperelliptic and trigonal varieties respectively. Proposition 1.10 establishes rationality or non-rationality for most of these varieties.

In the introduction we survey the modern state of the classification problem for Fano threefolds with canonical Gorenstein singularities, including the main results of this paper (Theorems 1.5 and 1.6 and Proposition 1.10). The second section contains various known results that are used in the proofs. In §§ 3 and 4 we prove Theorems 1.5 and 1.6 respectively. In § 5 we study rationality questions for elliptic and trigonal Fano threefolds.

The biregular classification of 3-folds whose curve sections are canonical curves was considered by Fano [72]–[75]. In the smooth case, hyperplane sections of such 3-folds must be K3 surfaces by the adjunction formula. Hence a natural generalization of the problem studied by Fano is the biregular classification of 3-folds containing an ample effective Cartier divisor which is a K3 surface with at most

¹The canonical Gorenstein singularities are exactly the rational Gorenstein singularities (see [70]).

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Du Val singularities. It turns out that, except for generalized cones (ordinary cones in the very ample case) over K3 surfaces, all 3-folds of this class are Fano 3-folds with canonical Gorenstein singularities (see [45], [112], [85], [39], [40]).

A complete classification of smooth Fano 3-folds was obtained in [46], [12], [13], [104], [108], [107], [105], where 105 families were found (see [88]). Moreover, every Fano 3-fold with terminal Gorenstein singularities is a deformation of a smooth one (see [109]). However, there are Fano 3-folds with canonical Gorenstein singularities that cannot be deformed into smooth Fano 3-folds. For example, the weighted projective spaces $\mathbb{P}(1^3, 3)$ and $\mathbb{P}(1^2, 4, 6)$ are Fano 3-folds with canonical Gorenstein singularities (see [68], [76]) but cannot be globally deformed into smooth varieties.

The classification of Fano 3-folds with canonical Gorenstein singularities is as yet far from complete (see [107]). However, four important steps have already been taken. The first is the following result, proved in [45] and [112] (see also [48] and [49]).

Theorem 1.1. *Let X be a Fano 3-fold with canonical Gorenstein singularities. In particular, the anticanonical divisor $-K_X$ is an ample Cartier divisor. Let S be a sufficiently general surface in the complete linear system $|-K_X|$. Then S has at most Du Val singularities.*

The second step is the following result of [107]. It is a natural generalization of the classification of smooth Fano 3-folds with Picard group \mathbb{Z} , which was obtained in [12] and [13].

Theorem 1.2. *Let X be a Fano 3-fold with canonical Gorenstein singularities. Suppose that the linear system $|-K_X|$ has no movable decomposition, that is, the anticanonical divisor $-K_X$ is not rationally equivalent to $A + B$ where A, B are Weil divisors whose complete linear systems $|A|, |B|$ have positive dimension. Then X is one of the following 3-folds.*

- 1) A hypersurface of degree 6 in $\mathbb{P}(1^4, 3)$, $-K_X^3 = 2$.
- 2) A complete intersection of a quadric cone and a quartic hypersurface in $\mathbb{P}(1^5, 2)$, $-K_X^3 = 4$.
- 3) A quartic hypersurface in \mathbb{P}^4 , $-K_X^3 = 4$.
- 4) A complete intersection of a quadric and a cubic in \mathbb{P}^5 , $-K_X^3 = 6$.
- 5) A complete intersection of three quadrics in \mathbb{P}^6 , $-K_X^3 = 8$.
- 6) An intersection of the Grassmannian $G(1, 4) \subset \mathbb{P}^9$ with a linear subspace of codimension 2 and a quadric, $-K_X^3 = 10$.
- 7) An intersection of the orthogonal Grassmannian $OG(5, 10) \subset \mathbb{P}^{15}$ with a linear subspace of codimension 7, $-K_X^3 = 12$.
- 8) An intersection of the Grassmannian $G(2, 6) \subset \mathbb{P}^{14}$ with a linear subspace of codimension 5, $-K_X^3 = 14$.
- 9) An intersection of the symplectic Grassmannian $LG(3, 6) \subset \mathbb{P}^{13}$ with a linear subspace of codimension 3, $-K_X^3 = 16$.
- 10) An intersection of the G_2 -homogeneous space $\Sigma \subset \mathbb{P}^{13}$ with a linear subspace of codimension 2 (see [88], Example 5.2.2), $-K_X^3 = 18$.
- 11) A Mukai–Umemura 3-fold $V_{22} \subset \mathbb{P}^{13}$ (see, for example, [108], [80], [88]), $-K_X^3 = 22$.

The third step is the following boundedness result of [110].

Theorem 1.3. *Let X be a Fano 3-fold with canonical Gorenstein singularities. In particular, the anticanonical divisor $-K_X$ is an ample Cartier divisor. Then $-K_X^3 \leq 72$, and equality implies that either $X \cong \mathbb{P}(1^3, 3)$ or $X \cong \mathbb{P}(1^2, 4, 6)$.*

The fourth step is the following result proved in [84].

Theorem 1.4. *Let X be a Fano 3-fold with canonical Gorenstein singularities. Suppose that the base locus of $|-K_X|$ is non-empty. Then X is one of the following 3-folds.*

- (B_1) *A complete intersection of a quadric cone and a sextic in $\mathbb{P}(1^4, 2, 3)$, $-K_X^3 = 2$.*
- (B_2) *The blow-up of a sextic in $\mathbb{P}(1^3, 2, 3)$ along a curve of arithmetic genus 1, $-K_X^3 = 4$.*
- (B_3) *$S_1 \times \mathbb{P}^1$, where S_1 is a del Pezzo surface of degree 1 with Du Val singularities, $-K_X^3 = 6$.*
- (B_4^m) *The anticanonical model of the blow-up of U_m along a curve Γ_0 , where U_m is a double covering $\pi: U_m \rightarrow \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(m-4) \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{F}(m, m-4, 0)$ such that $-K_{U_m} = \pi^*M$ and U_m has at worst canonical singularities, and Γ_0 is a smooth rational complete intersection contained in the smooth part of U_m such that $\pi(\Gamma_0)$ is a complete intersection of a general divisor in $|M|$ and the (unique) divisor in the linear system $|M - mF|$. Here M is the class of the tautological sheaf on $\mathbb{F}(m, m-4, 0)$, and F is the class of a fibre of the natural projection to \mathbb{P}^1 . We also have $3 \leq m \leq 12$ and $-K_X^3 = 2m - 2$.*

The purpose of this paper is to prove the following two results.

Theorem 1.5. *Let X be a Fano 3-fold with canonical Gorenstein singularities. Suppose that the linear system $|-K_X|$ has no base points but the induced morphism $\varphi_{|-K_X|}$ is not an embedding. Then X is one of the following 47 Fano 3-folds.*

- 1) (H_1) *A hypersurface of degree 6 in $\mathbb{P}(1^3, 2, 3)$, $-K_X^3 = 8$.*
- 2) (H_2) *A hypersurface of degree 6 in $\mathbb{P}(1^4, 3)$, $-K_X^3 = 2$.*
- 3) (H_3) *A complete intersection of a quadric cone and a quartic in $\mathbb{P}(1^5, 2)$, $-K_X^3 = 4$.*
- 4) *An anticanonical model of a “weak Fano 3-fold” V with canonical Gorenstein singularities (that is, $-K_V$ is a numerically effective and big Cartier divisor and $\varphi_{|-rK_V|}(V) = X$ for $r \gg 0$) such that V is a double covering of the rational scroll $\mathbb{F}(d_1, d_2, d_3) = \text{Proj}(\bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^1}(d_i))$ branched over a divisor rationally equivalent to $4M + 2(2 - \sum_{i=1}^3 d_i)L$, where M is the class of the tautological sheaf on $\mathbb{F}(d_1, d_2, d_3)$ and L is the class of a fibre of the natural projection of $\mathbb{F}(d_1, d_2, d_3)$ to \mathbb{P}^1 . Here the following cases are possible:*

- (H_4) $d_1 = 1, d_2 = 1, d_3 = 1, -K_X^3 = 6$;
- (H_5) $d_1 = 2, d_2 = 1, d_3 = 0, -K_X^3 = 6$;
- (H_6) $d_1 = 2, d_2 = 1, d_3 = 1, -K_X^3 = 8$;
- (H_7) $d_1 = 2, d_2 = 2, d_3 = 0, -K_X^3 = 8$;
- (H_8) $d_1 = 2, d_2 = 2, d_3 = 1, -K_X^3 = 10$;
- (H_9) $d_1 = 2, d_2 = 2, d_3 = 2, -K_X^3 = 12$;
- (H_{10}) $d_1 = 3, d_2 = 0, d_3 = 0, -K_X^3 = 6$;
- (H_{11}) $d_1 = 3, d_2 = 1, d_3 = 0, -K_X^3 = 8$;

(H_{12})	$d_1 = 3, d_2 = 1, d_3 = 1, -K_X^3 = 10;$
(H_{13})	$d_1 = 3, d_2 = 2, d_3 = 0, -K_X^3 = 10;$
(H_{14})	$d_1 = 3, d_2 = 2, d_3 = 1, -K_X^3 = 12;$
(H_{15})	$d_1 = 3, d_2 = 3, d_3 = 0, -K_X^3 = 12;$
(H_{16})	$d_1 = 3, d_2 = 3, d_3 = 1, -K_X^3 = 14;$
(H_{17})	$d_1 = 4, d_2 = 0, d_3 = 0, -K_X^3 = 8;$
(H_{18})	$d_1 = 4, d_2 = 1, d_3 = 0, -K_X^3 = 10;$
(H_{19})	$d_1 = 4, d_2 = 2, d_3 = 0, -K_X^3 = 12;$
(H_{20})	$d_1 = 4, d_2 = 2, d_3 = 1, -K_X^3 = 14;$
(H_{21})	$d_1 = 4, d_2 = 3, d_3 = 0, -K_X^3 = 14;$
(H_{22})	$d_1 = 4, d_2 = 3, d_3 = 1, -K_X^3 = 16;$
(H_{23})	$d_1 = 4, d_2 = 4, d_3 = 0, -K_X^3 = 16;$
(H_{24})	$d_1 = 5, d_2 = 1, d_3 = 0, -K_X^3 = 12;$
(H_{25})	$d_1 = 5, d_2 = 2, d_3 = 0, -K_X^3 = 14;$
(H_{26})	$d_1 = 5, d_2 = 3, d_3 = 0, -K_X^3 = 16;$
(H_{27})	$d_1 = 5, d_2 = 3, d_3 = 1, -K_X^3 = 18;$
(H_{28})	$d_1 = 5, d_2 = 4, d_3 = 0, -K_X^3 = 18;$
(H_{29})	$d_1 = 5, d_2 = 4, d_3 = 1, -K_X^3 = 20;$
(H_{30})	$d_1 = 6, d_2 = 2, d_3 = 0, -K_X^3 = 16;$
(H_{31})	$d_1 = 6, d_2 = 3, d_3 = 0, -K_X^3 = 18;$
(H_{32})	$d_1 = 6, d_2 = 4, d_3 = 0, -K_X^3 = 20;$
(H_{33})	$d_1 = 6, d_2 = 4, d_3 = 1, -K_X^3 = 22;$
(H_{34})	$d_1 = 6, d_2 = 5, d_3 = 0, -K_X^3 = 22;$
(H_{35})	$d_1 = 7, d_2 = 3, d_3 = 0, -K_X^3 = 20;$
(H_{36})	$d_1 = 7, d_2 = 4, d_3 = 0, -K_X^3 = 22;$
(H_{37})	$d_1 = 7, d_2 = 5, d_3 = 0, -K_X^3 = 24;$
(H_{38})	$d_1 = 7, d_2 = 5, d_3 = 1, -K_X^3 = 26;$
(H_{39})	$d_1 = 8, d_2 = 4, d_3 = 0, -K_X^3 = 24;$
(H_{40})	$d_1 = 8, d_2 = 5, d_3 = 0, -K_X^3 = 26;$
(H_{41})	$d_1 = 8, d_2 = 6, d_3 = 0, -K_X^3 = 28;$
(H_{42})	$d_1 = 9, d_2 = 5, d_3 = 0, -K_X^3 = 28;$
(H_{43})	$d_1 = 9, d_2 = 6, d_3 = 0, -K_X^3 = 30;$
(H_{44})	$d_1 = 10, d_2 = 6, d_3 = 0, -K_X^3 = 32;$
(H_{45})	$d_1 = 10, d_2 = 7, d_3 = 0, -K_X^3 = 34;$
(H_{46})	$d_1 = 11, d_2 = 7, d_3 = 0, -K_X^3 = 36;$
(H_{47})	$d_1 = 12, d_2 = 8, d_3 = 0, -K_X^3 = 40.$

Theorem 1.6. *Let X be a Fano 3-fold with canonical Gorenstein singularities such that the linear system $| -K_X |$ has no base points and the induced morphism $\varphi_{|-K_X|}: X \rightarrow \mathbb{P}^n$ is an embedding, where $n = -\frac{K_X^3}{2} + 2$. Suppose that the anticanonical image $\varphi_{|-K_X|}(X) \subset \mathbb{P}^n$ is not an intersection of quadrics. Then X is one of the following 69 Fano 3-folds.*

- 1) (T_1) A hypersurface of degree 4 in \mathbb{P}^4 , $-K_X^3 = 4$.
- 2) (T_2) A complete intersection of a quadric and a cubic in \mathbb{P}^5 , $-K_X^3 = 6$.
- 3) (T_3) An anticanonical image of a “weak Fano 3-fold” Y with canonical Gorenstein singularities (that is, $-K_Y$ is a numerically effective and big Cartier divisor

and $\varphi|_{-K_Y}(Y) = X$, where Y is a divisor in the rational scroll $\text{Proj}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2})$ and Y is rationally equivalent to the divisor $2T + F$. Here T is the class of the tautological sheaf on $\text{Proj}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2})$ and F is the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ under the natural projection to \mathbb{P}^2 , $-K_X^3 = 10$.

4) An anticanonical image of a “weak Fano 3-fold” V with canonical Gorenstein singularities (that is, $-K_V$ is a numerically effective and big Cartier divisor and $\varphi|_{-K_V}(V) = X$) such that V is a divisor in $\mathbb{F}(d_1, d_2, d_3, d_4) = \text{Proj}(\bigoplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^1}(d_i))$ rationally equivalent to the divisor $3M + (2 - \sum_{i=1}^4 d_i)L$, where M is the class of the tautological sheaf on $\mathbb{F}(d_1, d_2, d_3, d_4)$ and L is the class of a fibre of the projection of $\mathbb{F}(d_1, d_2, d_3, d_4)$ to \mathbb{P}^1 . Here the following cases are possible:

- (T₄) $d_1 = 1, d_2 = 1, d_3 = 1, d_4 = 0, -K_X^3 = 8;$
- (T₅) $d_1 = 1, d_2 = 1, d_3 = 1, d_4 = 1, -K_X^3 = 10;$
- (T₆) $d_1 = 2, d_2 = 1, d_3 = 0, d_4 = 0, -K_X^3 = 8;$
- (T₇) $d_1 = 2, d_2 = 1, d_3 = 1, d_4 = 0, -K_X^3 = 10;$
- (T₈) $d_1 = 2, d_2 = 1, d_3 = 1, d_4 = 1, -K_X^3 = 12;$
- (T₉) $d_1 = 2, d_2 = 2, d_3 = 0, d_4 = 0, -K_X^3 = 10;$
- (T₁₀) $d_1 = 2, d_2 = 2, d_3 = 1, d_4 = 0, -K_X^3 = 12;$
- (T₁₁) $d_1 = 2, d_2 = 2, d_3 = 1, d_4 = 1, -K_X^3 = 14;$
- (T₁₂) $d_1 = 2, d_2 = 2, d_3 = 2, d_4 = 0, -K_X^3 = 14;$
- (T₁₃) $d_1 = 2, d_2 = 2, d_3 = 2, d_4 = 1, -K_X^3 = 16;$
- (T₁₄) $d_1 = 2, d_2 = 2, d_3 = 2, d_4 = 2, -K_X^3 = 18;$
- (T₁₅) $d_1 = 3, d_2 = 1, d_3 = 0, d_4 = 0, -K_X^3 = 10;$
- (T₁₆) $d_1 = 3, d_2 = 1, d_3 = 1, d_4 = 0, -K_X^3 = 12;$
- (T₁₇) $d_1 = 3, d_2 = 2, d_3 = 0, d_4 = 0, -K_X^3 = 12;$
- (T₁₈) $d_1 = 3, d_2 = 2, d_3 = 1, d_4 = 0, -K_X^3 = 14;$
- (T₁₉) $d_1 = 3, d_2 = 2, d_3 = 1, d_4 = 1, -K_X^3 = 16;$
- (T₂₀) $d_1 = 3, d_2 = 2, d_3 = 2, d_4 = 0, -K_X^3 = 16;$
- (T₂₁) $d_1 = 3, d_2 = 2, d_3 = 2, d_4 = 1, -K_X^3 = 18;$
- (T₂₂) $d_1 = 3, d_2 = 3, d_3 = 1, d_4 = 0, -K_X^3 = 16;$
- (T₂₃) $d_1 = 3, d_2 = 3, d_3 = 2, d_4 = 0, -K_X^3 = 18;$
- (T₂₄) $d_1 = 3, d_2 = 3, d_3 = 2, d_4 = 1, -K_X^3 = 20;$
- (T₂₅) $d_1 = 4, d_2 = 1, d_3 = 0, d_4 = 0, -K_X^3 = 12;$
- (T₂₆) $d_1 = 4, d_2 = 2, d_3 = 0, d_4 = 0, -K_X^3 = 14;$
- (T₂₇) $d_1 = 4, d_2 = 2, d_3 = 1, d_4 = 0, -K_X^3 = 16;$
- (T₂₈) $d_1 = 4, d_2 = 2, d_3 = 1, d_4 = 1, -K_X^3 = 18;$
- (T₂₉) $d_1 = 4, d_2 = 2, d_3 = 2, d_4 = 0, -K_X^3 = 18;$
- (T₃₀) $d_1 = 4, d_2 = 3, d_3 = 1, d_4 = 0, -K_X^3 = 18;$
- (T₃₁) $d_1 = 4, d_2 = 3, d_3 = 2, d_4 = 0, -K_X^3 = 20;$
- (T₃₂) $d_1 = 4, d_2 = 3, d_3 = 2, d_4 = 1, -K_X^3 = 22;$
- (T₃₃) $d_1 = 4, d_2 = 3, d_3 = 3, d_4 = 0, -K_X^3 = 22;$
- (T₃₄) $d_1 = 4, d_2 = 3, d_3 = 3, d_4 = 1, -K_X^3 = 24;$
- (T₃₅) $d_1 = 4, d_2 = 4, d_3 = 2, d_4 = 0, -K_X^3 = 22;$
- (T₃₆) $d_1 = 5, d_2 = 2, d_3 = 0, d_4 = 0, -K_X^3 = 16;$
- (T₃₇) $d_1 = 5, d_2 = 2, d_3 = 1, d_4 = 0, -K_X^3 = 18;$
- (T₃₈) $d_1 = 5, d_2 = 3, d_3 = 1, d_4 = 0, -K_X^3 = 20;$
- (T₃₉) $d_1 = 5, d_2 = 3, d_3 = 2, d_4 = 0, -K_X^3 = 22;$

(T_{40})	$d_1 = 5, d_2 = 3, d_3 = 2, d_4 = 1, -K_X^3 = 24;$
(T_{41})	$d_1 = 5, d_2 = 3, d_3 = 3, d_4 = 0, -K_X^3 = 24;$
(T_{42})	$d_1 = 5, d_2 = 4, d_3 = 2, d_4 = 0, -K_X^3 = 24;$
(T_{43})	$d_1 = 5, d_2 = 4, d_3 = 3, d_4 = 0, -K_X^3 = 26;$
(T_{44})	$d_1 = 5, d_2 = 4, d_3 = 3, d_4 = 1, -K_X^3 = 28;$
(T_{45})	$d_1 = 6, d_2 = 2, d_3 = 0, d_4 = 0, -K_X^3 = 18;$
(T_{46})	$d_1 = 6, d_2 = 3, d_3 = 1, d_4 = 0, -K_X^3 = 22;$
(T_{47})	$d_1 = 6, d_2 = 3, d_3 = 2, d_4 = 0, -K_X^3 = 24;$
(T_{48})	$d_1 = 6, d_2 = 4, d_3 = 2, d_4 = 0, -K_X^3 = 26;$
(T_{49})	$d_1 = 6, d_2 = 4, d_3 = 3, d_4 = 0, -K_X^3 = 28;$
(T_{50})	$d_1 = 6, d_2 = 4, d_3 = 3, d_4 = 1, -K_X^3 = 30;$
(T_{51})	$d_1 = 6, d_2 = 4, d_3 = 4, d_4 = 0, -K_X^3 = 30;$
(T_{52})	$d_1 = 6, d_2 = 5, d_3 = 3, d_4 = 0, -K_X^3 = 30;$
(T_{53})	$d_1 = 7, d_2 = 3, d_3 = 1, d_4 = 0, -K_X^3 = 24;$
(T_{54})	$d_1 = 7, d_2 = 4, d_3 = 2, d_4 = 0, -K_X^3 = 28;$
(T_{55})	$d_1 = 7, d_2 = 4, d_3 = 3, d_4 = 0, -K_X^3 = 30;$
(T_{56})	$d_1 = 7, d_2 = 5, d_3 = 3, d_4 = 0, -K_X^3 = 32;$
(T_{57})	$d_1 = 7, d_2 = 5, d_3 = 4, d_4 = 0, -K_X^3 = 34;$
(T_{58})	$d_1 = 7, d_2 = 5, d_3 = 4, d_4 = 1, -K_X^3 = 36;$
(T_{59})	$d_1 = 8, d_2 = 4, d_3 = 2, d_4 = 0, -K_X^3 = 30;$
(T_{60})	$d_1 = 8, d_2 = 5, d_3 = 3, d_4 = 0, -K_X^3 = 34;$
(T_{61})	$d_1 = 8, d_2 = 5, d_3 = 4, d_4 = 0, -K_X^3 = 36;$
(T_{62})	$d_1 = 8, d_2 = 6, d_3 = 4, d_4 = 0, -K_X^3 = 38;$
(T_{63})	$d_1 = 9, d_2 = 5, d_3 = 3, d_4 = 0, -K_X^3 = 36;$
(T_{64})	$d_1 = 9, d_2 = 6, d_3 = 4, d_4 = 0, -K_X^3 = 40;$
(T_{65})	$d_1 = 9, d_2 = 6, d_3 = 5, d_4 = 0, -K_X^3 = 42;$
(T_{66})	$d_1 = 10, d_2 = 6, d_3 = 4, d_4 = 0, -K_X^3 = 42;$
(T_{67})	$d_1 = 10, d_2 = 7, d_3 = 5, d_4 = 0, -K_X^3 = 46;$
(T_{68})	$d_1 = 11, d_2 = 7, d_3 = 5, d_4 = 0, -K_X^3 = 48;$
(T_{69})	$d_1 = 12, d_2 = 8, d_3 = 6, d_4 = 0, -K_X^3 = 54.$

In the smooth case Theorems 1.4–1.6 were proved by Iskovskikh (see [15], [17]).

Remark 1.7. For any Fano 3-fold X with canonical Gorenstein singularities, there is a birational morphism $f: V \rightarrow X$ (called the *terminal modification* of X) such that $K_V \sim f^*(K_X)$ and V has terminal Gorenstein singularities. The existence of f follows from the Minimal Model Programme and the contraction theorem (see [93]). On the other hand, if V is any “weak Fano 3-fold” (that is, a variety whose anticanonical class $-K_V$ is numerically effective and big) with canonical Gorenstein singularities, then the contraction theorem implies that there is a birational morphism $f: V \rightarrow X$ such that X is a Fano 3-fold with canonical Gorenstein singularities.

In what follows we use the symbols B_k and B_4^m , H_i , and T_j to denote the Fano 3-folds described in Theorems 1.4, 1.5, and 1.6 respectively.

Remark 1.8. The 3-fold H_1 is a double covering of a cone over the Veronese surface, and H_2 is a double covering of \mathbb{P}^3 ramified in a sextic surface (which may be singular). The 3-fold H_3 is a double covering of a quadric 3-fold (possibly singular),

and H_4 is a double covering of $\mathbb{P}^1 \times \mathbb{P}^2$ branched over a divisor of bidegree $(2, 4)$. The 3-fold H_6 is the blow-up of a hypersurface of degree 4 in $\mathbb{P}(1^4, 2)$ (or, equivalently, the blow-up of a double covering of \mathbb{P}^3 branched over a quartic surface) along the intersection of two different divisors in the half-anticanonical linear system. The 3-fold H_9 is isomorphic to the product $\mathbb{P}^1 \times S_2$, where S_2 is a del Pezzo surface of degree 2 with at worst Du Val singularities. The 3-fold H_{10} is a hypersurface of degree 10 in $\mathbb{P}(1^2, 3^2, 5)$, H_{17} is a hypersurface of degree 12 in $\mathbb{P}(1^2, 4^2, 6)$, and T_3 is a hypersurface of degree 5 in $\mathbb{P}(1^3, 2^2)$. The 3-fold T_5 is a divisor of bidegree $(1, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^3$. The 3-fold T_8 is obtained by blowing up a plane cubic curve on a cubic 3-fold in \mathbb{P}^4 . The 3-fold T_{14} is the product $\mathbb{P}^1 \times S_3$, where S_3 is a cubic surface in \mathbb{P}^3 with at worst Du Val singularities.

Remark 1.9. The 3-folds $H_1, H_2, H_3, H_4, H_6, H_9, T_1, T_2, T_5, T_8$ and T_{14} are the only 3-folds among the H_i and T_j that can be chosen to be smooth (see [15], [88]).

The 3-folds H_i and T_j are rationally connected (see [95]). Moreover, most of the H_i and T_j must be rational, although some of them are definitely not. For example, it is well known that sufficiently general 3-folds $H_1, H_2, H_3, H_4, H_6, T_1, T_2, T_8$ are non-rational (see [22], [62], [54], [37], [16], [52], [98]). Their non-rationality can also be proved in the smooth case and some singular cases (see [31]–[34], [28], [29], [111], [5], [82], [63], [64], [6], [102], [7], [59]). However, all of these cases include examples of rational singular 3-folds H_i and T_j , even when the singularities are isolated ordinary double points (see [90]). The 3-folds H_9, T_5 and T_{14} are always rational by Remark 1.8. In this paper we prove the following result.

Proposition 1.10. *The 3-folds H_i and T_j are rational for $i \in \{8, 9, 22, 26, 27, 28, 29, 31, \dots, 47\}$ and $j \in \{5, 10, 11, 12, 13, 14, 17, \dots, 69\}$. On the other hand, sufficiently general 3-folds H_i and T_j are non-rational for $i \leq 7$ and $j \in \{1, 2, 3, 4, 6, 7, 8, 9\}$.*

There are birational relations between some of the 3-folds H_i, T_j, B_k and B_4^m . The simplest example is a projection from a cDV-point: the anticanonical model of the blow-up of a cDV-point on any of the 3-folds H_i or T_j of anticanonical degree $d \geq 4$ must be one of the 3-folds H_i, T_j, B_k or B_4^m of anticanonical degree $d - 2$. For example, B_4^4 is birationally isomorphic to H_{17} , H_5 is birationally isomorphic to H_1 with a cDV-point (see [5], Lemma 3.4), and the 3-folds T_1 and T_2 having a cDV-point are birationally isomorphic to the singular 3-folds H_2 and T_1 respectively. Moreover, there are many non-obvious birational transformations of the 3-folds H_i and T_j .

Example 1.11. In the notation of Theorem 1.6, let X be a sufficiently general 3-fold T_7 , and let V be the corresponding weak Fano 3-fold $V \subset \mathbb{F}(2, 1, 1, 0)$. Then V is smooth (see the proof of Theorem 1.6) and $-K_V$ has trivial intersection with only one rational curve $Y_4 \subset \mathbb{F}(2, 1, 1, 0)$ (see Corollary 2.20). It follows that the birational morphism $\varphi_{|-K_V|}: V \rightarrow X$ contracts the curve Y_4 to an ordinary double point of X . Let $f: V \dashrightarrow \tilde{V}$ be a flop in the curve Y_4 . Then one can find a birational morphism $g: \tilde{V} \rightarrow Y$ such that there is a double covering $\pi: Y \rightarrow \mathbb{P}^3$ branched over a smooth hypersurface of degree 4, that is, Y is a double space of index 2 (see [60], [61]). Moreover, the birational morphism g is a blow-up of a smooth rational curve

$C \subset Y$ with $-K_Y \cdot C = 2$. All these constructions of birational maps are easily seen to be reversible. More precisely, let $\pi: Y \rightarrow \mathbb{P}^3$ be any double covering branched over a smooth quartic surface, and let $C \subset Y$ be any non-singular rational curve with $-K_Y \cdot C = 2$. Then one can always construct the corresponding Fano 3-fold T_7 (see [56], § 4.4.1).

There are only two Fano 3-folds with canonical Gorenstein singularities whose anticanonical divisor is divisible (in the Picard group) by an integer greater than 2. These are \mathbb{P}^3 and a quartic 3-fold $Q \subset \mathbb{P}^4$. Fano 3-folds with canonical Gorenstein singularities whose anticanonical divisor is divisible by 2 are called *del Pezzo 3-folds* (see [88], Theorem 3.3.1). It is easy to prove explicitly that H_1 is the only del Pezzo 3-fold among the 3-folds H_i and T_j . This is also confirmed by the classification of del Pezzo 3-folds (see [78], [79], [57], [117]).

Remark 1.12. The 3-folds H_i and T_j are naturally birationally isomorphic to del Pezzo fibrations of degrees 2 and 3 respectively, except for the following cases: H_1, H_2, H_3, T_1, T_2 and T_3 . On the other hand, sufficiently general 3-folds H_1, H_2, H_3, T_1 and T_2 are not birationally isomorphic to any del Pezzo fibration of degree 2 or 3 (see [22], [16], [29], [63], [6], [7]).

Remark 1.13. It is well known that 3-folds with a pencil of del Pezzo surfaces of degree 2 or 3 are unirational (see [23]–[25]). Therefore the 3-folds H_i and T_j are unirational for $i \geq 4$ and $j \geq 4$. The 3-fold H_3 is also known to be unirational (see [115], [16]). The proof of Proposition 5.5 below implies that the 3-fold T_3 is unirational since it is birationally equivalent to a conic bundle with a rational multisection. The 3-fold T_2 is also unirational (see [71], [115], [36], [16]). However, it is still unknown whether a general quartic 3-fold T_1 is unirational or not, despite several examples of smooth unirational quartic 3-folds (see [118], [22], [16], [100]). Unfortunately, nothing is known about the unirationality of general 3-folds H_1 and H_2 . It is expected that general 3-folds H_2 are not unirational (see [97], Conjecture 4.1.6).

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§ 2. Preliminaries

In what follows all varieties are assumed to be projective, normal, and defined over \mathbb{C} .

Proposition 2.1 ([96], Proposition 3.1.6). *Suppose that $\rho: V \rightarrow X$ is a finite morphism, D_X is an effective \mathbb{Q} -divisor on X , and $D_V = \rho^*(D_X) - K_{V/X}$, that is, $K_V + D_V = \rho^*(K_X + D_X)$. The singularities of the log pair (V, D_V) are Kawamata log terminal (see [93], [96]) if and only if the singularities of the log pair (X, D_X) are Kawamata log terminal.*

We note that Kawamata log terminal singularities are canonical if the canonical divisor is a Cartier divisor. Hence Proposition 2.1 yields the following result.

Corollary 2.2. *Let X be a smooth variety, and $\rho: V \rightarrow X$ a double covering branched over a reduced effective divisor $D \subset X$. The singularities of V are canonical if and only if the singularities of the log pair $(X, \frac{1}{2}D)$ are Kawamata log terminal.*

Theorem 2.3 ([96], Theorem 4.5.1). *Let X be a smooth variety, \mathcal{H} a linear system on X whose base locus has codimension at least 2, and D a sufficiently general divisor in \mathcal{H} . Suppose that for every point $x \in X$ there is a divisor $H \in \mathcal{H}$ such that the singularities of (X, H) are canonical in the neighbourhood of x . Then the singularities of the log pair (X, D) are canonical.*

The next result is Theorem 7.9 of [96], which was proved in [119].

Theorem 2.4. *Let X be a normal variety such that ω_X is locally free, and let $S \subset X$ be an effective Cartier divisor on X . Then S has canonical singularities if and only if the singularities of the log pair (X, S) are canonical.*

Theorems 2.3 and 2.4 yield the following result.

Corollary 2.5. *Let X be a smooth variety, \mathcal{H} a linear system on X whose base locus has codimension at least 2, and D a sufficiently general divisor in \mathcal{H} . Suppose that for every point $x \in X$ there is a divisor $H \in \mathcal{H}$ whose singularities are canonical in a neighbourhood of x . Then D has canonical singularities.*

The following result is implied by Theorem 4.8 of [96].

Theorem 2.6. *Let X be a smooth variety, \mathcal{H} a linear system on X , D a sufficiently general divisor in \mathcal{H} , and $\lambda \in \mathbb{Q} \cap [0, 1)$. Suppose that for every point $x \in X$ there is $H \in \mathcal{H}$ such that the log pair $(X, \lambda H)$ has Kawamata log terminal singularities in the neighbourhood of x . Then the singularities of $(X, \lambda D)$ are Kawamata log terminal.*

Corollary 2.2 and Theorem 2.6 imply the following result.

Corollary 2.7. *Let X be a smooth variety, \mathcal{H} a linear system on X , and D a sufficiently general divisor in \mathcal{H} . Suppose that for every point $O \in X$ there is an effective reduced divisor $H \in \mathcal{H}$ such that there is a double covering $\beta: Y \rightarrow X$ branched over $H \subset X$ with Y having canonical singularities in the neighbourhood of $\beta^{-1}(O)$. Let $\rho: V \rightarrow X$ be the double covering branched over $D \subset X$. Then V has canonical singularities.*

Remark 2.8. It is also easy to deduce Corollary 2.7 from Corollary 2.5. Indeed, in the notation of Corollary 2.7, let $B \subset X$ be a divisor with $D \sim 2B$. We put $U = \text{Proj}(\mathcal{O}_X \oplus \mathcal{O}_X(B))$. Let M be the tautological line bundle on U , and let $f: U \rightarrow X$ be the natural projection. Then Y may be regarded as a divisor on U in the linear system $|2M|$ such that $\beta = f|_Y$. We may assume that $\mathcal{H} = |H|$ without loss of generality. Hence we can identify V with a sufficiently general divisor in the linear system $|2M|$. The base locus of $|2M|$ is contained in $Y \cap f^{-1}(H)$ because $2S + f^{-1}(H) \sim 2M$ and $S \cap Y = \emptyset$, where $S \sim M - f^*(B)$ is a negative section of $f: U \rightarrow X$. Therefore the base locus of $|2M|$ has codimension at least 2, and the singularities of $V \in |2M|$ are canonical by Corollary 2.5.

We recall the following classical result (see [53], [1], [114]).

Theorem 2.9. *Suppose that X is a normal algebraic surface and $O \in X$ is an isolated singular point such that the singularities of X are canonical in the neighbourhood of O , that is, O is a Du Val singular point on X . Then $O \in X$ is a hypersurface quasi-homogeneous singularity and is locally isomorphic to the singularity $(0, 0, 0) \in \mathbb{C}^3 \cong \text{Spec}(\mathbb{C}[x, y, z])$ of one of the following types:*

- (\mathbb{A}_n) $x^2 + y^2 + z^{n+1} = 0$, $\text{wt}(x) = n + 1$, $\text{wt}(y) = n + 1$, $\text{wt}(z) = 2$, $n \geq 1$;
- (\mathbb{D}_n) $x^2 + y^2 z + z^{n-1} = 0$, $\text{wt}(x) = n - 1$, $\text{wt}(y) = n - 2$, $\text{wt}(z) = 2$, $n \geq 4$;
- (\mathbb{E}_6) $x^2 + y^3 + z^4 = 0$, $\text{wt}(x) = 6$, $\text{wt}(y) = 4$, $\text{wt}(z) = 3$;
- (\mathbb{E}_7) $x^2 + y^3 + yz^3 = 0$, $\text{wt}(x) = 9$, $\text{wt}(y) = 6$, $\text{wt}(z) = 4$;
- (\mathbb{E}_8) $x^2 + y^3 + z^5 = 0$, $\text{wt}(x) = 15$, $\text{wt}(y) = 10$, $\text{wt}(z) = 6$.

The following result is proved in § 12.3, § 12.6 and § 13.1 of [1].

Theorem 2.10. *Let $X \subset \mathbb{C}^3 \cong \text{Spec}(\mathbb{C}[x, y, z])$ be a hypersurface $f(x, y, z) = 0$ such that the origin $O \in \mathbb{C}^3$ is an isolated singular point of X . Write*

$$f(x, y, z) = f_d(x, y, z) + f_{d+1}(x, y, z) + \dots,$$

where $f_i(x, y, z)$ is a quasi-homogeneous polynomial of quasi-homogeneous degree $i \geq 2$ with respect to positive integer weights $\text{wt}(x) = a$, $\text{wt}(y) = b$, $\text{wt}(z) = c$. Suppose that the origin $O \in \mathbb{C}^3$ is an isolated singular point of the hypersurface $f_d(x, y, z) = 0$, where $2a \leq d$, $2b \leq d$, $2c \leq d$ and $a + b + c > d$.

- 1) If $(a, b, c) = (n + 1, n + 1, 2)$, then $O \in X$ is a singularity of type \mathbb{A}_n .
- 2) If $(a, b, c) = (n - 1, n - 2, 2)$, then $O \in X$ is a singularity of type \mathbb{D}_n .
- 3) If $(a, b, c) = (6, 4, 3)$, then $O \in X$ is a singularity of type \mathbb{E}_6 .
- 4) If $(a, b, c) = (9, 6, 4)$, then $O \in X$ is a singularity of type \mathbb{E}_7 .
- 5) If $(a, b, c) = (15, 10, 6)$, then $O \in X$ is a singularity of type \mathbb{E}_8 .

The following result is due to Enriques (see [77], [15], [69], [17]).

Theorem 2.11. *Let $X \subset \mathbb{P}^n$ be a variety of degree $n - \dim(X) + 1$ such that X is not contained in any hyperplane. Then X is one of the following varieties:*

- 1) a projective space \mathbb{P}^n ;
- 2) a quadric hypersurface in \mathbb{P}^n ;
- 3) the image of a rational scroll $\mathbb{F}(d_1, \dots, d_k) = \text{Proj}(\bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^1}(d_i))$ under the map given by the tautological line bundle, where $0 \neq d_1 \geq \dots \geq d_k \geq 0$ and $n + 1 = \sum_{i=1}^k (d_i + 1)$;
- 4) a Veronese surface in \mathbb{P}^5 when $n = 5$;
- 5) a cone in \mathbb{P}^n over a Veronese surface in \mathbb{P}^5 .

It is easy to see that the varieties in Theorem 2.11 have the smallest possible degree among all varieties of the same dimension in \mathbb{P}^n .

Using the Kawamata–Viehweg vanishing theorem (see [91], [120]) and elementary properties of linear systems on K3 surfaces (see [116]), we get the following well-known result (see [15], [17], [88]).

Theorem 2.12. *Let X be a Fano 3-fold with canonical Gorenstein singularities such that the linear system $| -K_X |$ has no base points but the anticanonical divisor $-K_X$ is not very ample. Then $\varphi_{|-K_X|}: X \rightarrow V \subset \mathbb{P}^n$ is a double covering and $V \subset \mathbb{P}^n$ is a subvariety of minimal degree, that is, $\deg(V) = n - 2$, where $n = -\frac{1}{2}K_X^3 + 2$.*

The following result is a theorem of Noether–Enriques–Petri (see [44], [81]).

Theorem 2.13. *Let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded smooth irreducible curve whose genus $g(C)$ is at least 3. Then the following assertions hold.*

- 1) *The curve $C \subset \mathbb{P}^{g-1}$ is projectively normal.*
- 2) *If $g(C) = 3$, then C is a plane quartic curve.*
- 3) *If $g(C) \geq 4$, then the graded ideal I_C of the curve $C \subset \mathbb{P}^{g-1}$ is generated by the components of degree 2 and 3, that is, the curve $C \subset \mathbb{P}^{g-1}$ is cut out by quadrics and cubics in \mathbb{P}^{g-1} in the scheme-theoretic sense.*
- 4) *If $g(C) \geq 4$, then the graded ideal I_C of the curve $C \subset \mathbb{P}^{g-1}$ is generated by the component of degree 2 except in the following two cases:
the curve C is trigonal, that is, there is a map $\psi: C \rightarrow \mathbb{P}^1$ of degree 3;
the curve C is isomorphic to a smooth plane quintic (in particular, $g(C) = 6$).*
- 5) *In the trigonal case, quadrics through C in \mathbb{P}^{g-1} cut out either an irreducible (possibly singular) quadric surface when $g(C) = 4$, or a smooth irreducible surface of degree $g - 2$ which is the image of $\text{Proj}(\mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2))$ under the map given by the tautological line bundle, where $d_1 \geq d_2 > 0$ and $g = d_1 + d_2 + 2$.*
- 6) *If C is isomorphic to a smooth plane quintic, then quadrics through C in \mathbb{P}^5 cut out a Veronese surface.*

The following result is a corollary of Theorem 2.13 (see [15], [17], [88]).

Theorem 2.14. *Let $X \subset \mathbb{P}^n$ be an anticanonically embedded Fano 3-fold with canonical singularities, that is, $-K_X \sim \mathcal{O}_{\mathbb{P}^n}(1)|_X$ and $n = -\frac{1}{2}K_X^3 + 2$. Then the following assertions hold.*

- 1) *The 3-fold X is projectively normal in \mathbb{P}^n .*
- 2) *If $-K_X^3 = 4$, then X is a quartic 3-fold in \mathbb{P}^4 .*
- 3) *If $-K_X^3 \geq 6$, then the graded ideal I_X of the 3-fold $X \subset \mathbb{P}^n$ is generated by the components of degree 2 and 3.*
- 4) *If $-K_X^3 \geq 6$, then the graded ideal I_X of the 3-fold $X \subset \mathbb{P}^n$ is generated by the component of degree 2 except in the case when, for a general linear subspace $\Pi \subset \mathbb{P}^n$ of codimension 2, the curve $X \cap \Pi$ is either a canonically embedded smooth trigonal curve or a canonically embedded smooth plane quintic curve and $\deg(X \subset \mathbb{P}^n) = 10$.*
- 5) *In the trigonal case, quadrics through X in \mathbb{P}^n cut out either an irreducible (possibly singular) quadric 4-fold when $-K_X^3 = 6$, or a 4-fold of degree $n - 3$ which is the image of a rational scroll $\text{Proj}(\bigoplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^1}(d_i))$ under the map given by the tautological line bundle, where $0 \neq d_1 \geq \dots \geq d_4 \geq 0$ and $n + 1 = \sum_{i=1}^4 (d_i + 1)$.*
- 6) *If $X \cap \Pi$ is a canonically embedded plane quintic, then quadrics through X in \mathbb{P}^7 cut out a 4-dimensional cone over a Veronese surface.*

Proposition 2.15 ([110], Claim 6.9). *Let X be a 3-fold with composite Du Val (cDV) singularities, $\Gamma \subset \text{Sing}(X)$ a smooth curve regarded as a reduced subscheme of X , and $f: V \rightarrow X$ the blow-up of Γ . Then V has at most cDV-singularities and $K_V \sim f^*(K_X)$, that is, the map f is crepant.*

The following result is proved in [47] and is a special case of a conjectured rationality criterion for standard 3-dimensional conic bundles (see [86], [87], [18], [19]).

Theorem 2.16. *Suppose that Y is a smooth 3-fold, Z is either \mathbb{P}^2 or a minimal rational ruled surface \mathbb{F}_r , and $\xi: Y \rightarrow Z$ is a conic bundle with $\text{Pic}(Y/Z) = \mathbb{Z}$ and*

$|2K_Z + \Delta| \neq \emptyset$, where $\Delta \subset Z$ is the degeneration divisor of $\xi: Y \rightarrow Z$. Then Y is non-rational.

Remark 2.17. In the notation of Theorem 2.16, the hypothesis $|2K_Z + \Delta| = \emptyset$ implies that the 3-fold Y is rational except in the case when there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \chi \downarrow & & \downarrow \xi \\ \mathbb{P}^2 & \xrightarrow{\beta} & Z \end{array}$$

where α and β are birational maps, X is a smooth 3-fold, and $\chi: X \rightarrow \mathbb{P}^2$ is a conic bundle with $\text{Pic}(X/\mathbb{P}^2) = \mathbb{Z}$ whose degeneration divisor $D \subset \mathbb{P}^2$ is a quintic curve and the double covering $\psi: \tilde{D} \rightarrow D$ induced by χ corresponds to even θ -characteristic (see [18]).

The following result was proved in [101]. It is a particular case of a more general result in [95] (see also [94], [98]), which generalizes the standard degeneration technique (see [21]).

Theorem 2.18. *Let $\xi: Y \rightarrow Z$ be a flat proper morphism with irreducible and reduced geometric fibres. Then there are countably many closed subsets $Z_i \subset Z$ such that the fibre $\xi^{-1}(s)$ over a closed point $s \in Z$ is ruled if and only if $s \in \bigcup Z_i$.*

Proposition 2.19 [114]. *Let V be a rational scroll $\text{Proj}(\sum_{i=1}^k \mathcal{O}_{\mathbb{P}^1}(d_i))$ and let $f: V \rightarrow \mathbb{P}^1$ be the natural projection. Then $\text{Pic}(V) \cong \mathbb{Z}M \oplus \mathbb{Z}L$, where M is the tautological line bundle on V and L is the class of a fibre of f . Let $(t_1 : t_2)$ be homogeneous coordinates on the base \mathbb{P}^1 , and let $(x_1 : \dots : x_k)$ be homogeneous coordinates (corresponding to the coordinates on $\sum_{i=1}^k \mathcal{O}_{\mathbb{P}^1}(d_i)$) on the fibre of f which is isomorphic to \mathbb{P}^{k-1} . Then $|aM + bL|$ is generated by bihomogeneous coordinates*

$$c_{i_1, \dots, i_k} x_1^{i_1} x_2^{i_2} \dots x_k^{i_k},$$

where $\sum_{j=1}^k i_j = a$, $i_j \geq 0$, and $c_{i_1, \dots, i_k} = c_{i_1, \dots, i_k}(t_0 : t_2)$ is a homogeneous polynomial of degree $b + \sum_{j=1}^k i_j d_j$.

Proposition 2.19 implies the following result, which is known as a lemma of Reid.

Corollary 2.20. *Let V be a k -dimensional rational scroll $\text{Proj}(\sum_{i=1}^k \mathcal{O}_{\mathbb{P}^1}(d_i))$ with $d_1 \geq \dots \geq d_k \geq 0$, and let $Y_j \subset V$ be the “negative rational subscroll” $\text{Proj}(\bigoplus_{i=j}^k \mathcal{O}_{\mathbb{P}^1}(d_i))$, which corresponds to the natural projection*

$$\bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^1}(d_i) \rightarrow \bigoplus_{i=j}^k \mathcal{O}_{\mathbb{P}^1}(d_i).$$

Take an effective divisor $D \subset V$ that is rationally equivalent to $aM + bL$, where M is the tautological line bundle on V , L is a fibre of the natural projection to \mathbb{P}^1 , and $a, b \in \mathbb{Z}$. We have $\text{mult}_{Y_j}(D) \geq q$ for $q \in \mathbb{N}$ if and only if $ad_j + b + (d_1 - d_j)(q - 1) < 0$.

The following result is implied by the Riemann–Roch theorem (see [15], [17], [88]), the Kawamata–Viehweg vanishing theorem (see [91], [120]), the rationality of canonical singularities (see [93], [96]), and the global-to-local spectral sequence.

Proposition 2.21. *Let X be a Fano 3-fold with canonical Gorenstein singularities. Then*

$$h^0(\mathcal{O}_X(-mK_X)) = \frac{m(m+1)(2m+1)}{12} (-K_X)^3 + 2m + 1.$$

The following two results are well known. Their proof can be found in [8], [23], [24], [9], [10], [11], [14]. For a modern proof see [103] and [20].

Theorem 2.22. *Let W be a smooth minimal geometrically irreducible and geometrically rational surface defined over a perfect field \mathbb{F} . This means that no curve on W can be contracted to a smooth point over \mathbb{F} and W is irreducible and rational over $\overline{\mathbb{F}}$. Then either $\text{Pic}(W) \cong \mathbb{Z}$ and W is a smooth del Pezzo surface or $\text{Pic}(W) \cong \mathbb{Z} \oplus \mathbb{Z}$ and W is a conic bundle $\pi: W \rightarrow Z$.*

Theorem 2.23. *Let W be a smooth minimal geometrically irreducible and geometrically rational surface defined over a perfect field \mathbb{F} . The surface W is rational over \mathbb{F} if and only if W has an \mathbb{F} -point and $K_W^2 \geq 5$.*

Theorem 2.24 [23]. *Let W be a smooth geometrically irreducible and geometrically rational surface defined over a C_1 -field \mathbb{F} , say, over $\mathbb{F} = \mathbb{C}(x)$. Then W has an \mathbb{F} -point.*

Theorem 2.25 [95]. *Let Y be a projective variety, and let $g: Y \rightarrow R$ be a morphism with a section onto a smooth curve R . Suppose that there is a set $\{r_1, \dots, r_k\} \subset R$ of closed points such that each fibre $Y_i = g^{-1}(r_i)$ is smooth and separably rationally connected. Then for every set of closed points $y_i \in Y_i$ there is a section $C \subset Y$ of the morphism g passing through each point y_i .*

§ 3. Proof of Theorem 1.5

Let X be a Fano 3-fold with canonical Gorenstein singularities such that the linear system $|-K_X|$ has no base points but the induced morphism $\varphi_{|-K_X|}: X \rightarrow Y \subset \mathbb{P}^n$ is not an embedding. Then $\varphi_{|-K_X|}: X \rightarrow Y \subset \mathbb{P}^n$ is a double covering and $\deg(Y \subset \mathbb{P}^n) = n - 2$, where $n = -\frac{1}{2}K_X^3 + 2$.

Remark 3.1. If $-K_X^3 = 2$, then the 3-fold Y is nothing but \mathbb{P}^3 and $\varphi_{|-K_X|}$ is a double covering ramified in a sextic surface (possibly singular). In this case, X may be regarded as a hypersurface of degree 6 in $\mathbb{P}(1^4, 3)$. The birational geometry of such varieties X was studied in [54], [37], [16], [27], [111], [82], [42], [59].

Remark 3.2. If $-K_X^3 = 4$, then Y is a quadric (possibly singular) in \mathbb{P}^4 and $\varphi_{|-K_X|}$ is a double covering branched over a surface cut out on Y by a quartic hypersurface in \mathbb{P}^4 . In this case, X may be regarded as a complete intersection of a quadric cone and a quartic in $\mathbb{P}(1^5, 2)$. The birational geometry of such varieties X was studied in [54], [37], [16], [27], [3], [4].

Thus we may assume that $-K_X^3 \geq 6$. Hence Theorem 2.11 implies that either $-K_X^3 = 8$ and $Y \subset \mathbb{P}^6$ is a cone over a Veronese surface $F_4 \subset \mathbb{P}^5$ or Y is the image of a rational scroll $\mathbb{F}(d_1, d_2, d_3) = \text{Proj}(\bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^1}(d_i))$ under the map given by the tautological line bundle, where $0 \neq d_1 \geq \dots \geq d_3 \geq 0$ and $-K_X^3 = 2(d_1 + d_2 + d_3)$.

Lemma 3.3. *Suppose that Y is a cone over a Veronese surface F_4 with vertex O . Then X is a hypersurface of degree 6 in $\mathbb{P}(1^3, 2, 3)$.*

Proof. We have $Y \cong \mathbb{P}(1^3, 2)$. The double covering $\varphi_{|-K_X|}$ is branched over the vertex O because O is not a Gorenstein point of Y . On the other hand, the equation $-K_X^3 = 8$ implies that the double covering $\varphi_{|-K_X|}$ is branched over a divisor $D \subset Y$ such that $D \sim \mathcal{O}_Y(6)$. As in the smooth case (see [15]), it follows (see [117]) that X is a hypersurface of degree 6 in $\mathbb{P}(1^3, 2, 3)$.

Hence we may assume that there is a birational morphism $f: U \rightarrow Y$ for some $U = \text{Proj}(\bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^1}(d_i))$, and we have $f = \varphi_{|M|}$, where M is the tautological line bundle on U , $0 \neq d_1 \geq \dots \geq d_3 \geq 0$ and $-K_X^3 = 2(d_1 + d_2 + d_3) \geq 6$.

Lemma 3.4. *Suppose that $d_2 = d_3 = 0$. Then X is either a hypersurface of degree 10 in $\mathbb{P}(1^2, 3^2, 5)$ or a hypersurface of degree 12 in $\mathbb{P}(1^2, 4^2, 6)$.*

Proof. Take a sufficiently general divisor $H \in |-K_X|$. Then H is a K3 surface with Du Val singularities and $f(H)$ is a cone in \mathbb{P}^{n-1} over a rational normal curve. Moreover, the restriction map

$$H^0(\mathcal{O}_X(-K_X)) \rightarrow H^0(\mathcal{O}_H(-K_X|_H))$$

is surjective since $H^1(\mathcal{O}_X) = 0$. Hence the equations $d_1 = d_2 = 0$ imply that $-K_X^3 \leq 8$ by [116].

Thus there are two possible cases: $d_1 = 3$ and $d_1 = 4$. We have $Y \cong \mathbb{P}(1^2, 3^2)$ in the first case and $Y \cong \mathbb{P}(1^2, 4^2)$ in the second. To get the desired result, we now proceed as in the proof of Lemma 3.3.

Remark 3.5. One can use basic properties of hypersurfaces in weighted projective spaces (see [76]) to prove the existence of a hypersurface of degree 10 in $\mathbb{P}(1^2, 3^2, 5)$ and a hypersurface of degree 12 in $\mathbb{P}(1^2, 4^2, 6)$ having only canonical Gorenstein singularities. However, we shall prove this in a different and more geometric way together with the proof for other possible cases.

Let V be the normalization of the fibred product $X \times_Y U$, $\pi: V \rightarrow U$ the double covering induced by $\varphi_{|-K_X|}: X \rightarrow Y$, and $h: V \rightarrow X$ the birational morphism induced by $f: U \rightarrow Y$.

Lemma 3.6. *The 3-fold V has canonical Gorenstein singularities, the anticanonical divisor $-K_V$ is numerically effective and big, and $K_V \sim h^*(K_X)$, that is, the map h is crepant.*

Proof. If $d_2 \neq 0$, then the 3-folds X and V are isomorphic in codimension 2, which easily yields the lemma (compare [92]).

Thus we may assume that $d_2 = 0$. Then $f: U \rightarrow Y$ contracts a divisor $D \subset U$ to a curve $C \cong \mathbb{P}^1$, and Lemma 3.4 implies that either $d_1 = 3$ or $d_1 = 4$. In both cases $\varphi_{|-K_X|}$ must be ramified in the curve C since Y is non-Gorenstein at a general point of C .

Let $R \subset U$ be the ramification divisor of $\pi: V \rightarrow U$, M the tautological line bundle on U , and L a fibre of the natural projection of U to \mathbb{P}^1 . Then the equivalences

$$-K_X \sim \varphi_{|-K_X|}^*(\mathcal{O}_{\mathbb{P}^n}(1)|_Y), \quad M \sim f^*(\mathcal{O}_{\mathbb{P}^n}(1)|_Y)$$

imply that $R \sim 4M - 2(d_1 - 2)L + aD$ for some $a \in \mathbb{Z}$. Moreover, we have $a \geq 0$ since the singularities of X are canonical. Suppose that $a > 0$. Then Corollary 2.20 shows that $R = 2D \cup S$, where S is an effective divisor on U , since $D \sim M - d_1L$. This contradicts the normality of V . Thus $a = 0$, $R \sim 4M - 2(d_1 - 2)L$, and $-K_V \sim f^*(M) \sim h^*(K_X)$, which easily yields the desired assertion.

Let $D \subset U$ be the ramification divisor of $\pi: V \rightarrow U$, M the tautological line bundle on U , and L a fibre of the natural projection of U to \mathbb{P}^1 . Then $-K_V \sim \pi^*(M)$ by construction. Hence,

$$D \sim 4M - 2(d_1 + d_2 + d_3 - 2)L.$$

Let $Y_2 \subset V$ and $Y_3 \subset V$ be the subscrolls corresponding to the natural projections

$$\bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^1}(d_i) \rightarrow \mathcal{O}_{\mathbb{P}^1}(d_2) \oplus \mathcal{O}_{\mathbb{P}^1}(d_3), \quad \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^1}(d_i) \rightarrow \mathcal{O}_{\mathbb{P}^1}(d_3).$$

Then $Y_2 \cong \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(d_2) \oplus \mathcal{O}_{\mathbb{P}^1}(d_3))$ and $Y_3 \cong \mathbb{P}^1$.

Lemma 3.7. *We have $\text{mult}_{Y_2}(D) \leq 1$ and $\text{mult}_{Y_3}(D) \leq 3$.*

Proof. The first inequality follows from the normality of V . Suppose that $d = \text{mult}_{Y_3}(D) \geq 2$. Then the local equation of V in the neighbourhood of a generic point of the curve $C = \pi^{-1}(Y_3)$ is

$$\omega^2 = f_d(x, y) + f_{d+1}(x, y) + \cdots \subset \text{Spec}(\mathbb{C}[x, y, z, \omega]),$$

where $x = y = 0$ are the local equations of the curve C , and $f_i(x, y)$ is a homogeneous polynomial of degree i . On the other hand, the singularities of V at the general point of C must be locally isomorphic to one of the following types of singularities: $\mathbb{C} \times \mathbb{A}_n$, $\mathbb{C} \times \mathbb{D}_n$, $\mathbb{C} \times \mathbb{E}_6$, $\mathbb{C} \times \mathbb{E}_7$, $\mathbb{C} \times \mathbb{E}_8$. Then Theorem 2.9 implies that $d \leq 3$.

Therefore Corollary 2.20 implies that

$$d_1 - d_3 - 2d_2 + 4 \geq 0, \quad d_2 - d_1 - 2d_3 + 4 \geq 0.$$

We also have $0 \neq d_1 \geq \cdots \geq d_3 \geq 0$ and $d_1 + d_2 + d_3 \geq 3$ by assumption. The resulting inequalities determine 44 different rational scrolls

$$\mathbb{F}(d_1, d_2, d_3) = \text{Proj}\left(\bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^1}(d_i)\right)$$

with the ramification divisor

$$D \sim 4M - 2(d_1 + d_2 + d_3 - 2)L,$$

where M is the tautological line bundle on $\mathbb{F}(d_1, d_2, d_3)$ and L is a fibre of the natural projection to \mathbb{P}^1 .

Remark 3.8. The 3-fold X is an anticanonical model of the 3-fold V , that is, $X \cong \varphi_{|-rK_V|}(V)$ for $r \gg 0$. Thus the contraction theorem (see [93]) implies that X is a Fano 3-fold with canonical Gorenstein singularities if and only if V has canonical Gorenstein singularities and $-K_V$ is numerically effective and big. On the other hand, V is uniquely determined by the rational scroll $\mathbb{F}(d_1, d_2, d_3)$ and the ramification divisor $D \in |4M - 2(d_1 + d_2 + d_3 - 2)L|$. The only trouble is that the linear system $|4M - 2(d_1 + d_2 + d_3 - 2)L|$ may contain no divisor D such that the corresponding double covering V has canonical singularities.

In the rest of this section we show explicitly that, in each of the cases obtained, the linear system $|4M - 2(d_1 + d_2 + d_3 - 2)L|$ contains a divisor D such that V has canonical singularities. We shall use Corollary 2.7 together with Proposition 2.19. This verification will complete the proof of Theorem 1.5.

Remark 3.9. The same ideas were actually used in the classification of smooth hyperelliptic Fano 3-folds (see [15], [17]). In the smooth case, the corresponding inequalities are much stronger and the calculations much shorter. This method was also used in [41] to find an effective bound on the degree of hyperelliptic Fano 3-folds with canonical Gorenstein singularities, but there is a gap in the proof of Lemma 3.2. Namely, the stronger inequality $\text{mult}_{Y_3}(D) \leq 2$ was used there instead of the inequality $\text{mult}_{Y_3}(D) \leq 3$. This gave a wrong bound $-K_X^3 \leq 16$ instead of the right one $-K_X^3 \leq 40$, which is *a posteriori* seen to be sharp. Nevertheless, one can use the estimate $-K_X^3 \leq 40$ to prove the main result of [41], which is now obsolete because of [110].

Let us consider one of the possible cases in full detail.

Example 3.10. Let $\pi: V \rightarrow \mathbb{F}(6, 2, 0)$ be a double covering branched over a sufficiently general divisor $D \subset \mathbb{F}(6, 2, 0)$ such that $D \sim 4M - 12L$, where M is the class of the tautological line bundle on $\mathbb{F}(6, 2, 0)$ and L is the class of a fibre of the projection of $\mathbb{F}(6, 2, 0)$ to \mathbb{P}^1 . We must show that the 3-fold V has canonical singularities.

By Proposition 2.19, the divisor D is given by the zeros of the bihomogeneous polynomial

$$\begin{aligned} & \alpha_{12}(t_1, t_2) x_1^4 + \alpha_8(t_1, t_2) x_1^3 x_2 + \alpha_6(t_1, t_2) x_1^3 x_3 + \alpha_4(t_1, t_2) x_1^2 x_2^2 \\ & + \alpha_2(t_1, t_2) x_1^2 x_2 x_3 + \alpha_0^1(t_1, t_2) x_1^2 x_3^2 + \alpha_0^2(t_1, t_2) x_1 x_2^3, \end{aligned}$$

where $\alpha_d(t_1, t_2)$ (or $\alpha_d^i(t_1, t_2)$) is an arbitrary form of degree d . We define a surface $E \subset \mathbb{F}(6, 2, 0)$ and a curve $C \subset \mathbb{F}(6, 2, 0)$ by the equations $x_1 = 0$ and $x_1 = x_2 = 0$ respectively. The base locus of the linear system $|4M - 12L|$ equals E . (In particular, $D \setminus E$ and $V \setminus \pi^{-1}(E)$ are smooth by Bertini's theorem.) The automorphism group of $E \cong \mathbb{F}(4, 0)$ acts transitively on $E \setminus C$, that is, all points of $E \setminus C$ are mapped to each other by changes of the coordinates t_1, t_2, x_4, x_5 . By Lemma 3.7, the divisor D has multiplicity 1 at a general point of E . Hence for every point of $E \setminus C$ there is a divisor D' whose multiplicity at this point is equal to 1: it suffices to make an appropriate change of coordinates in the equation of D . The singularities of a general divisor D on $E \setminus C$ are canonical by Corollary 2.7, and it suffices to prove that for every point p of C there is a divisor D such that

the corresponding variety V has a canonical singularity in the neighbourhood of the point $\pi^{-1}(p) \in \pi^{-1}(C)$.

Let Y be a fibre of the projection of $\mathbb{F}(6, 2, 0)$ to \mathbb{P}^1 over a sufficiently general point $P \in \mathbb{P}^1$. We put $Z = \pi^{-1}(Y)$. Then Z is a del Pezzo surface of degree 2. Moreover, the only possible singular point of Z is $O = \pi^{-1}(C \cap Y)$. Let us prove that O is a Du Val point on Z . This already implies that the singularities of Y are canonical.

Suppose that the point $P \in \mathbb{P}^1$ has homogeneous coordinates $(\gamma : \delta)$. Then Z may be given as a hypersurface

$$\omega^2 = \alpha_{12}x_1^4 + \alpha_8x_1^3x_2 + \alpha_6x_1^3x_3 + \alpha_4x_1^2x_2^2 + \alpha_2x_1^2x_2x_3 + \alpha_0^1x_1^2x_3^2 + \alpha_0^2x_1x_2^3$$

in $\mathbb{P}(1, 1, 1, 2) \cong \text{Proj}(\mathbb{C}[x_1, x_2, x_3, \omega])$, where $\alpha_d^i = \alpha_d^i(\gamma, \delta)$ (respectively, $\alpha_d = \alpha_d(\gamma, \delta)$). Since Y is general, we have $\alpha_d^i \neq 0$ (resp. $\alpha_d \neq 0$) for all d and i . Therefore we may assume for convenience that $\alpha_d^i = 1$ (resp. $\alpha_d = 1$) for all d and i .

Let $\omega = x$, $x_1 = y$, $x_2 = z$ and $x_3 = 1$. Then the local equation of Z in a neighbourhood of O is given by

$$x^2 + y^4 + y^3z + y^3 + y^2z^2 + y^2z + y^2 + yz^3 = 0.$$

Let $\text{wt}(x) = 3$, $\text{wt}(y) = 3$ and $\text{wt}(z) = 1$. Then $\text{wt}(x^2 + y^2 + yz^3) = 6$, $\text{wt}(y^4) = 12$, $\text{wt}(y^3z) = 10$, $\text{wt}(y^3) = 9$, $\text{wt}(y^2z^2) = 8$ and $\text{wt}(y^2z) = 7$. Moreover, the equation $x^2 + y^2 + yz^3 = 0$ determines an isolated point. Hence Theorem 2.10 implies that the singularity of Z at O is locally isomorphic to a Du Val singularity of type \mathbb{A}_5 . Hence the 3-fold V has a singularity of type $\mathbb{A}_5 \times \mathbb{C}$ at a general point of the curve $\pi^{-1}(C)$.

Thanks to the generality in the choice of D , we may actually assume that the point $P \in \mathbb{P}^1$ is not just a general point but an arbitrary point of \mathbb{P}^1 . In other words, given any point $P \in \mathbb{P}^1$, one can find homogeneous polynomials α_d such that $\alpha_d(P) \neq 0$ and repeat all the above arguments in the neighbourhood of the corresponding point $O = \pi^{-1}(C \cap Y)$. Hence the singularities of V are canonical by Corollary 2.7.

In the rest of the section we consider the other possible cases following the pattern of Example 3.10. Differences appear only in the numerical characteristics of the varieties, their equations, types of singularities and so on. They are collected in Table 1.

Table 1 is organized as follows. The first column contains the labels of the varieties V in the notation of Theorem 1.5. The second column gives a triple (d_1, d_2, d_3) such that there is a double covering $\pi: V \rightarrow \mathbb{F}(d_1, d_2, d_3)$ which is branched over a divisor D . The third column displays the number b such that $|D| = |4M + bL|$. The corresponding linear system appears to be base point free in the cases H_4 , H_6 and H_9 . Then the divisor D is non-singular by Bertini's theorem and, therefore, V is non-singular (and we do not need the information in the other columns). In all other cases, the set $\text{Bs}|D|$ of base points is either the curve $C = Y_3$ given by $x_1 = x_2 = 0$ (and then Bertini's theorem shows that D and V are smooth outside C and $\pi^{-1}(C)$ respectively, so it suffices to study the singularities of V

over C only) or the surface $E = Y_2$ given by $x_1 = 0$ (and then the divisor D has multiplicity 1 at a general point of E and, since the automorphism group of $E \cong \mathbb{F}(d_2, d_3)$ is transitive on $E \setminus C$ (compare Example 3.10), Corollary 2.7 shows that it again suffices to study the singularities of V over C only).

The fourth column contains the equations of general divisors D in the linear system $|4M + bL|$, and the fifth column yields an equation of the fibre Z of the projection $V \rightarrow \mathbb{P}^1$ over a general point of \mathbb{P}^1 in the neighbourhood of a general point of $\pi^{-1}(C)$ after a change of coordinates $\omega = x$, $x_1 = y$, $x_2 = z$, $x_3 = 1$ (see Example 3.10). The same equation determines V locally if we regard it as an equation in t, x, y, z . The corresponding point appears to be non-singular in the cases H_5 , H_{12} and H_{17} . In the other cases we attribute new weights $\text{wt}(x) = w_x$, $\text{wt}(y) = w_y$, $\text{wt}(z) = w_z$ (listed in the sixth column) to the variables x, y, z and see that the terms of lowest weight determine an isolated Du Val singularity. We notice that the weights $\text{wt}(x)$, $\text{wt}(y)$, $\text{wt}(z)$ coincide with the weights of the Du Val singularity indicated in the seventh column. Hence the singularity of Z in the chosen neighbourhood is Du Val of this type (by Theorem 2.10), and the singularity of V is locally isomorphic to the product of \mathbb{C} and the corresponding Du Val singularity. In any case, V has canonical singularities in this neighbourhood and, by Corollary 2.7, all singularities of V are canonical.

TABLE 1

H_i	(d_1, d_2, d_3)	b	Equation of D	Local equation of V	Weights	Singularity
H_4	(1, 1, 1)	-2	—	—	—	—
H_5	(2, 1, 0)	-2	$\alpha_6 x_1^4 + \alpha_5 x_1^3 x_2$ $+ \alpha_4^1 x_1^3 x_3 + \alpha_4^2 x_1^2 x_2^2$ $+ \alpha_3^1 x_1^2 x_2 x_3 + \alpha_2^1 x_1^2 x_3^2$ $+ \alpha_3^2 x_1 x_2^3 + \alpha_2^2 x_1 x_2^2 x_3$ $+ \alpha_1^1 x_1 x_2 x_3^2 + \alpha_0^1 x_1 x_3^3$ $+ \alpha_2^3 x_2^4 + \alpha_1^2 x_2^3 x_3$ $+ \alpha_0^2 x_2^2 x_3^2 = 0$	$x^2 + y^3 z + y^3$ $+ y^2 z^2 + y^2 z$ $+ y^2 + y z^3$ $+ y z^2 + y z$ $+ y + z^4 + z^3$ $+ z^2 = 0$	—	Non-singular point
H_6	(2, 1, 1)	-4	—	—	—	—
H_7	(2, 2, 0)	-4	$\alpha_4^1 x_1^4 + \alpha_4^2 x_1^3 x_2$ $+ \alpha_4^3 x_1^2 x_2^2 + \alpha_4^4 x_1 x_2^3$ $+ \alpha_4^5 x_2^4 + \alpha_2^1 x_1^3 x_3$ $+ \alpha_2^2 x_1^2 x_2 x_3$ $+ \alpha_2^3 x_1 x_2^2 x_3 + \alpha_2^4 x_2^3 x_3$ $+ \alpha_0^1 x_1^2 x_3^2 + \alpha_0^2 x_1 x_2 x_3^2$ $+ \alpha_0^3 x_2^2 x_3^2 = 0$	$x^2 + P_2(y, z)$ $+ P_3(y, z)$ $+ P_4(y, z) = 0$ $(P_i \text{ is a homogeneous polynomial of degree } i)$	$w_x = 1$ $w_y = 1$ $w_z = 1$	\mathbb{A}_1

TABLE 1 CONTINUED

H_8	$(2, 2, 1)$	-6	$\alpha_2^1 x_1^4 + \alpha_2^2 x_1^3 x_2$ $+ \alpha_3^2 x_1^2 x_2^2 + \alpha_4^2 x_1 x_2^3$ $+ \alpha_2^5 x_2^4 + \alpha_1^1 x_1^3 x_3$ $+ \alpha_1^2 x_1^2 x_2 x_3$ $+ \alpha_1^3 x_1 x_2^2 x_3 + \alpha_1^4 x_2^3 x_3$ $+ \alpha_0^1 x_1^2 x_3^2 + \alpha_0^2 x_1 x_2 x_3^2$ $+ \alpha_0^3 x_2^2 x_3^2 = 0$	$x^2 + P_2(y, z)$ $+ P_3(y, z)$ $+ P_4(y, z) = 0$ $(P_i \text{ is a homo-}$ geneous poly- nomial of $\text{degree } i)$	$w_x = 1$ $w_y = 1$ $w_z = 1$	\mathbb{A}_1
H_9	$(2, 2, 2)$	-8	—	—	—	—
H_{10}	$(3, 0, 0)$	-2	$\alpha_{10} x_1^4 + \alpha_7^1 x_1^3 x_2$ $+ \alpha_7^2 x_1^2 x_3 + \alpha_4^1 x_1^2 x_2^2$ $+ \alpha_4^2 x_1^2 x_2 x_3 + \alpha_4^3 x_1^2 x_3^2$ $+ \alpha_1^1 x_1 x_2^3 + \alpha_1^2 x_1 x_2^2 x_3$ $+ \alpha_1^3 x_1 x_2 x_3^2$ $+ \alpha_1^4 x_1 x_3^3 = 0$	$x^2 + y^3 z + y^3$ $+ y^2 z^2 + y^2 z$ $+ y^2 + y z^3$ $+ y z^2 + y z$ $+ y = 0$	—	Non-singular point
H_{11}	$(3, 1, 0)$	-4	$\alpha_8 x_1^4 + \alpha_6 x_1^3 x_2$ $+ \alpha_5 x_1^3 x_3 + \alpha_4 x_1^2 x_2^2$ $+ \alpha_3 x_1^2 x_2 x_3 + \alpha_2^1 x_1^2 x_3^2$ $+ \alpha_0^1 x_1 x_2 x_3^2$ $+ \alpha_2^2 x_1 x_2^3 + \alpha_1 x_1 x_2^2 x_3$ $+ \alpha_0^2 x_2^4 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y^2 z + y^2$ $+ y z^3 + y z^2$ $+ y z + z^4 = 0$	$w_x = 1$ $w_y = 1$ $w_z = 1$	\mathbb{A}_1
H_{12}	$(3, 1, 1)$	-6	$\alpha_6 x_1^4 + \alpha_4^1 x_1^3 x_2$ $+ \alpha_4^2 x_1^3 x_3 + \alpha_2^1 x_1^2 x_2^2$ $+ \alpha_2^2 x_1^2 x_2 x_3 + \alpha_2^3 x_1^2 x_3^2$ $+ \alpha_0^1 x_1 x_2^3 + \alpha_0^2 x_1 x_2^2 x_3$ $+ \alpha_0^3 x_1 x_2 x_3^2$ $+ \alpha_0^4 x_1 x_3^3 = 0$	$x^2 + y^3 z + y^3$ $+ y^2 z^2 + y^2 z$ $+ y^2 + y z^3$ $+ y z^2 + y z$ $+ y = 0$	—	Non-singular point
H_{13}	$(3, 2, 0)$	-6	$\alpha_6 x_1^4 + \alpha_5 x_1^3 x_2$ $+ \alpha_3^1 x_1^3 x_3 + \alpha_4 x_1^2 x_2^2$ $+ \alpha_2^1 x_1^2 x_2 x_3 + \alpha_0^1 x_1^2 x_3^2$ $+ \alpha_3^2 x_1 x_2^3 + \alpha_1 x_1 x_2^2 x_3$ $+ \alpha_2^2 x_2^4 + \alpha_0^2 x_2^3 x_3 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y^2 z + y^2$ $+ y z^3 + y z^2$ $+ z^4 + z^3 = 0$	$w_x = 3$ $w_y = 3$ $w_z = 2$	\mathbb{A}_2

TABLE 1 CONTINUED

H_{14}	$(3, 2, 1)$	-8	$\alpha_4 x_1^4 + \alpha_3 x_1^3 x_2$ $+ \alpha_2^1 x_1^3 x_3 + \alpha_2^2 x_1^2 x_2^2$ $+ \alpha_1^1 x_1^2 x_2 x_3 + \alpha_0^1 x_1^2 x_3^2$ $+ \alpha_1^2 x_1 x_2^3 + \alpha_0^2 x_1 x_2^2 x_3$ $+ \alpha_0^3 x_2^4 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y^2 z + y^2$ $+ y z^3 + y z^2$ $+ z^4 = 0$	$w_x = 2$ $w_y = 2$ $w_z = 1$	\mathbb{A}_3
H_{15}	$(3, 3, 0)$	-8	$\alpha_4^1 x_1^4 + \alpha_4^2 x_1^3 x_2$ $+ \alpha_4^3 x_1^2 x_2^2 + \alpha_4^4 x_1 x_2^3$ $+ \alpha_4^5 x_2^4 + \alpha_1^1 x_1^3 x_3$ $+ \alpha_1^2 x_1^2 x_2 x_3$ $+ \alpha_1^3 x_1 x_2^2 x_3$ $+ \alpha_1^4 x_2^3 x_3 = 0$	$x^2 + P_3(y, z)$ $+ P_4(y, z) = 0$ $(P_i \text{ is a homo-}$ geneous poly- nomial of $\text{degree } i)$	$w_x = 3$ $w_y = 2$ $w_z = 2$	\mathbb{D}_4
H_{16}	$(3, 3, 1)$	-10	$\alpha_2^1 x_1^4 + \alpha_2^2 x_1^3 x_2$ $+ \alpha_2^3 x_1^2 x_2^2 + \alpha_2^4 x_1 x_2^3$ $+ \alpha_2^5 x_2^4 + \alpha_0^1 x_1^3 x_3$ $+ \alpha_0^2 x_1^2 x_2 x_3$ $+ \alpha_0^3 x_1 x_2^2 x_3$ $+ \alpha_0^4 x_2^3 x_3 = 0$	$x^2 + P_3(y, z)$ $+ P_4(y, z) = 0$ $(P_i \text{ is a homo-}$ geneous poly- nomial of $\text{degree } i)$	$w_x = 3$ $w_y = 2$ $w_z = 2$	\mathbb{D}_4
H_{17}	$(4, 0, 0)$	-4	$\alpha_{12} x_1^4 + \alpha_8^1 x_1^3 x_2$ $+ \alpha_8^2 x_1^3 x_3 + \alpha_4^1 x_1^2 x_2^2$ $+ \alpha_4^2 x_1^2 x_2 x_3 + \alpha_4^3 x_1^2 x_3^2$ $+ \alpha_0^1 x_1 x_2^3 + \alpha_0^2 x_1 x_2^2 x_3$ $+ \alpha_0^3 x_1 x_2 x_3^2$ $+ \alpha_0^4 x_1 x_3^3 = 0$	$x^2 + y^3 z + y^3$ $+ y^2 z^2 + y^2 z$ $+ y^2 + y z^3$ $+ y z^2 + y z$ $+ y = 0$	$-$	Non-singular point
H_{18}	$(4, 1, 0)$	-6	$\alpha_{10} x_1^4 + \alpha_8 x_1^3 x_2$ $+ \alpha_6 x_1^3 x_3 + \alpha_4 x_1^2 x_2^2$ $+ \alpha_3 x_1^2 x_2 x_3$ $+ \alpha_2 x_1^2 x_3^2 + \alpha_1 x_1 x_2^3$ $+ \alpha_0 x_1 x_2^2 x_3 = 0$	$x^2 + y^4$ $+ y^3 z + y^3$ $+ y^2 z^2 + y^2 z$ $+ y^2 + y z^3$ $+ y z^2 = 0$	$w_x = 2$ $w_y = 2$ $w_z = 1$	\mathbb{A}_3
H_{19}	$(4, 2, 0)$	-8	$\alpha_8 x_1^4 + \alpha_6 x_1^3 x_2$ $+ \alpha_4 x_1^3 x_3 + \alpha_4 x_1^2 x_2^2$ $+ \alpha_2 x_1^2 x_2 x_3 + \alpha_0 x_1^2 x_3^2$ $+ \alpha_2 x_1 x_2^3 + \alpha_0 x_1 x_2^2 x_3$ $+ \alpha_0 x_2^4 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y^2 z + y^2$ $+ y z^3 + y z^2$ $+ z^4 = 0$	$w_x = 2$ $w_y = 2$ $w_z = 1$	\mathbb{A}_3

TABLE 1 CONTINUED

H_{20}	$(4, 2, 1)$	-10	$\alpha_6 x_1^4 + \alpha_4 x_1^3 x_2$ $+ \alpha_3 x_1^3 x_3 + \alpha_2 x_1^2 x_2^2$ $+ \alpha_1 x_1^2 x_2 x_3 + \alpha_0^1 x_1 x_2^3$ $+ \alpha_0^2 x_1^2 x_3^2 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y^2 z + y z^3$ $+ y^2 = 0$	$w_x = 3$ $w_y = 3$ $w_z = 1$	\mathbb{A}_5
H_{21}	$(4, 3, 0)$	-10	$\alpha_6 x_1^4 + \alpha_5 x_1^3 x_2$ $+ \alpha_4 x_1^2 x_2^2 + \alpha_3 x_1 x_2^3$ $+ \alpha_2^1 x_2^4 + \alpha_2^2 x_1^3 x_3$ $+ \alpha_1 x_1^2 x_2 x_3$ $+ \alpha_0 x_1 x_2^2 x_3 = 0$	$x^2 + y^4 + y^3 z$ $+ y^2 z^2 + y z^3$ $+ z^4 + y^3 + y^2 z$ $+ y z^2 = 0$	$w_x = 3$ $w_y = 2$ $w_z = 2$	\mathbb{D}_4
H_{22}	$(4, 3, 1)$	-12	$\alpha_4 x_1^4 + \alpha_3^1 x_1^3 x_2$ $+ \alpha_1^1 x_1^3 x_3 + \alpha_2 x_1^2 x_2^2$ $+ \alpha_0^1 x_1^2 x_2 x_3 + \alpha_1^2 x_1 x_2^3$ $+ \alpha_0^2 x_2^4 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y^2 z + y z^3$ $+ z^4 = 0$	$w_x = 4$ $w_y = 3$ $w_z = 2$	\mathbb{D}_5
H_{23}	$(4, 4, 0)$	-12	$\alpha_4^1 x_1^4 + \alpha_4^2 x_1^3 x_2$ $+ \alpha_4^3 x_1^2 x_2^2 + \alpha_4^4 x_1 x_2^3$ $+ \alpha_4^5 x_2^4 + \alpha_0^1 x_1^3 x_3$ $+ \alpha_0^2 x_1^2 x_2 x_3$ $+ \alpha_0^3 x_1 x_2^2 x_3$ $+ \alpha_0^4 x_2^3 x_3 = 0$	$x^2 + P_3(y, z)$ $+ P_4(y, z) = 0$ $(P_i \text{ is a homo-}$ geneous poly- nomial of $\text{degree } i)$	$w_x = 3$ $w_y = 2$ $w_z = 2$	\mathbb{D}_4
H_{24}	$(5, 1, 0)$	-8	$\alpha_{12} x_1^4 + \alpha_8 x_1^3 x_2$ $+ \alpha_7 x_1^3 x_3 + \alpha_4 x_1^2 x_2^2$ $+ \alpha_3 x_1^2 x_2 x_3 + \alpha_0 x_1 x_2^3$ $+ \alpha_2 x_1^2 x_3^2 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y^2 z + y z^3$ $+ y^2 = 0$	$w_x = 3$ $w_y = 3$ $w_z = 1$	\mathbb{A}_5
H_{25}	$(5, 2, 0)$	-10	$\alpha_{10} x_1^4 + \alpha_7 x_1^3 x_2$ $+ \alpha_5 x_1^3 x_3 + \alpha_4 x_1^2 x_2^2$ $+ \alpha_0^1 x_1^2 x_2 x_3 + \alpha_1 x_1 x_2^3$ $+ \alpha_0^2 x_1^2 x_3^2 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y^2 z + y z^3$ $+ y^2 = 0$	$w_x = 3$ $w_y = 3$ $w_z = 1$	\mathbb{A}_5
H_{26}	$(5, 3, 0)$	-12	$\alpha_8 x_1^4 + \alpha_6 x_1^3 x_2$ $+ \alpha_3 x_1^3 x_3 + \alpha_4 x_1^2 x_2^2$ $+ \alpha_1^1 x_1^2 x_2 x_3 + \alpha_1^2 x_1 x_2^3$ $+ \alpha_0 x_2^4 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y^2 z + y z^3$ $+ z^4 = 0$	$w_x = 4$ $w_y = 3$ $w_z = 2$	\mathbb{D}_5

TABLE 1 CONTINUED

H_{27}	$(5, 3, 1)$	-14	$\alpha_6 x_1^4 + \alpha_4 x_1^3 x_2$ $+ \alpha_2^1 x_1^3 x_3 + \alpha_2^2 x_1^2 x_2^2$ $+ \alpha_0^1 x_1^2 x_2 x_3$ $+ \alpha_0^2 x_1 x_2^3 = 0$	$x^2 + y^4$ $+ y^3 z + y^3$ $+ y^2 z^2 + y^2 z$ $+ y z^3 = 0$	$w_x = 5$ $w_y = 4$ $w_z = 2$	\mathbb{D}_6
H_{28}	$(5, 4, 0)$	-14	$\alpha_6 x_1^4 + \alpha_5 x_1^3 x_2$ $+ \alpha_1 x_1^3 x_3 + \alpha_4 x_1^2 x_2^2$ $+ \alpha_0 x_1^2 x_2 x_3 + \alpha_3 x_1 x_2^3$ $+ \alpha_2 x_2^4 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y^2 z + y z^3$ $+ z^4 = 0$	$w_x = 4$ $w_y = 3$ $w_z = 2$	\mathbb{D}_5
H_{29}	$(5, 4, 1)$	-16	$\alpha_4 x_1^4 + \alpha_3 x_1^3 x_2$ $+ \alpha_0^1 x_1^3 x_3 + \alpha_2 x_1^2 x_2^2$ $+ \alpha_1 x_1 x_2^3 + \alpha_0^2 x_2^4 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y z^3 + z^4 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
H_{30}	$(6, 2, 0)$	-12	$\alpha_{12} x_1^4 + \alpha_8 x_1^3 x_2$ $+ \alpha_6 x_1^3 x_3 + \alpha_4 x_1^2 x_2^2$ $+ \alpha_2 x_1^2 x_2 x_3 + \alpha_0^1 x_1^2 x_3^2$ $+ \alpha_0^2 x_1 x_2^3 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y^2 z + y^2$ $+ y z^3 = 0$	$w_x = 3$ $w_y = 3$ $w_z = 1$	\mathbb{A}_5
H_{31}	$(6, 3, 0)$	-14	$\alpha_{10} x_1^4 + \alpha_7 x_1^3 x_2$ $+ \alpha_4 x_1^3 x_3 + \alpha_2 x_1^2 x_2^2$ $+ \alpha_0 x_1^2 x_2 x_3$ $+ \alpha_1 x_1 x_2^3 = 0$	$x^2 + y^4$ $+ y^3 z + y^3$ $+ y^2 z^2 + y^2 z$ $+ y z^3 = 0$	$w_x = 5$ $w_y = 4$ $w_z = 2$	\mathbb{D}_6
H_{32}	$(6, 4, 0)$	-16	$\alpha_8 x_1^4 + \alpha_6 x_1^3 x_2$ $+ \alpha_2 x_1^3 x_3 + \alpha_4 x_1^2 x_2^2$ $+ \alpha_0^1 x_1^2 x_2 x_3$ $+ \alpha_0^2 x_2^4 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y^2 z + z^4 = 0$	$w_x = 4$ $w_y = 3$ $w_z = 2$	\mathbb{D}_5
H_{33}	$(6, 4, 1)$	-18	$\alpha_6 x_1^4 + \alpha_4 x_1^3 x_2$ $+ \alpha_0^1 x_1^3 x_3 + \alpha_2 x_1^2 x_2^2$ $+ \alpha_0^2 x_1 x_2^3 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y z^3 = 0$	$w_x = 9$ $w_y = 6$ $w_z = 4$	\mathbb{E}_7
H_{34}	$(6, 5, 0)$	-18	$\alpha_6 x_1^4 + \alpha_5 x_1^3 x_2$ $+ \alpha_0 x_1^3 x_3 + \alpha_4 x_1^2 x_2^2$ $+ \alpha_3 x_1 x_2^3 + \alpha_2 x_2^4 = 0$	$x^2 + y^4 + y^3 z$ $+ y^3 + y^2 z^2$ $+ y z^3 + z^4 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
H_{35}	$(7, 3, 0)$	-16	$\alpha_{12} x_1^4 + \alpha_8 x_1^3 x_2$ $+ \alpha_5 x_1^3 x_3 + \alpha_4 x_1^2 x_2^2$ $+ \alpha_1 x_1^2 x_2 x_3$ $+ \alpha_0 x_1 x_2^3 = 0$	$x^2 + y^4$ $+ y^3 z + y^3$ $+ y^2 z^2 + y^2 z$ $+ y z^3 = 0$	$w_x = 5$ $w_y = 4$ $w_z = 2$	\mathbb{D}_6

TABLE 1 CONTINUED

H_{36}	$(7, 4, 0)$	-18	$\alpha_{10}x_1^4 + \alpha_7x_1^3x_2$ $+ \alpha_3x_1^3x_3 + \alpha_4x_1^2x_2^2$ $+ \alpha_0x_1^2x_2x_3$ $+ \alpha_1x_1x_2^3 = 0$	$x^2 + y^4$ $+ y^3z + y^3$ $+ y^2z^2 + y^2z$ $+ yz^3 = 0$	$w_x = 5$ $w_y = 4$ $w_z = 2$	\mathbb{D}_6
H_{37}	$(7, 5, 0)$	-20	$\alpha_8x_1^4 + \alpha_6x_1^3x_2$ $+ \alpha_1x_1^3x_3 + \alpha_4x_1^2x_2^2$ $+ \alpha_2x_1x_2^3 + \alpha_0x_2^4 = 0$	$x^2 + y^4 + y^3z$ $+ y^3 + y^2z^2$ $+ yz^3 + z^4 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
H_{38}	$(7, 5, 1)$	-22	$\alpha_6x_1^4 + \alpha_4x_1^3x_2$ $+ \alpha_0^1x_1^3x_3 + \alpha_2x_1^2x_2^2$ $+ \alpha_0^2x_1x_2^3 = 0$	$x^2 + y^4 + y^3z$ $+ y^3 + y^2z^2$ $+ yz^3 = 0$	$w_x = 9$ $w_y = 6$ $w_z = 4$	\mathbb{E}_7
H_{39}	$(8, 4, 0)$	-20	$\alpha_{12}x_1^4 + \alpha_8x_1^3x_2$ $+ \alpha_4^1x_1^3x_3 + \alpha_4^2x_1^2x_2^2$ $+ \alpha_0^1x_1^2x_2x_3$ $+ \alpha_0^2x_1x_2^3 = 0$	$x^2 + y^4$ $+ y^3z + y^3$ $+ y^2z^2 + y^2z$ $+ yz^3 = 0$	$w_x = 5$ $w_y = 4$ $w_z = 2$	\mathbb{D}_6
H_{40}	$(8, 5, 0)$	-22	$\alpha_{10}x_1^4 + \alpha_7x_1^3x_2$ $+ \alpha_2x_1^3x_3 + \alpha_4x_1^2x_2^2$ $+ \alpha_1x_1x_2^3 = 0$	$x^2 + y^4 + y^3z$ $+ y^3 + y^2z^2$ $+ yz^3 = 0$	$w_x = 9$ $w_y = 6$ $w_z = 4$	\mathbb{E}_7
H_{41}	$(8, 6, 0)$	-24	$\alpha_8x_1^4 + \alpha_6x_1^3x_2$ $+ \alpha_0^1x_1^3x_3 + \alpha_4x_1^2x_2^2$ $+ \alpha_2x_1x_2^3 + \alpha_0^2x_2^4 = 0$	$x^2 + y^4 + y^3z$ $+ y^3 + y^2z^2$ $+ yz^3 + z^4 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
H_{42}	$(9, 5, 0)$	-24	$\alpha_{12}x_1^4 + \alpha_8x_1^3x_2$ $+ \alpha_3x_1^3x_3 + \alpha_4x_1^2x_2^2$ $+ \alpha_0x_1x_2^3 = 0$	$x^2 + y^4 + y^3z$ $+ y^3 + y^2z^2$ $+ yz^3 = 0$	$w_x = 9$ $w_y = 6$ $w_z = 4$	\mathbb{E}_7
H_{43}	$(9, 6, 0)$	-26	$\alpha_{10}x_1^4 + \alpha_7x_1^3x_2$ $+ \alpha_1^1x_1^3x_3 + \alpha_4x_1^2x_2^2$ $+ \alpha_1^2x_1x_2^3 = 0$	$x^2 + y^4 + y^3z$ $+ y^3 + y^2z^2$ $+ yz^3 = 0$	$w_x = 9$ $w_y = 6$ $w_z = 4$	\mathbb{E}_7
H_{44}	$(10, 6, 0)$	-28	$\alpha_{12}x_1^4 + \alpha_8x_1^3x_2$ $+ \alpha_2x_1^3x_3 + \alpha_4x_1^2x_2^2$ $+ \alpha_0x_1x_2^3 = 0$	$x^2 + y^4 + y^3z$ $+ y^3 + y^2z^2$ $+ yz^3 = 0$	$w_x = 9$ $w_y = 6$ $w_z = 4$	\mathbb{E}_7
H_{45}	$(10, 7, 0)$	-30	$\alpha_{10}x_1^4 + \alpha_7x_1^3x_2$ $+ \alpha_0x_1^3x_3 + \alpha_4x_1^2x_2^2$ $+ \alpha_1x_1x_2^3 = 0$	$x^2 + y^4 + y^3z$ $+ y^3 + y^2z^2$ $+ yz^3 = 0$	$w_x = 9$ $w_y = 6$ $w_z = 4$	\mathbb{E}_7

TABLE 1 CONTINUED

H_{46}	$(11, 7, 0)$	-32	$\alpha_{12}x_1^4 + \alpha_8x_1^3x_2$ $+ \alpha_1x_1^3x_3 + \alpha_4x_1^2x_2^2$ $+ \alpha_0x_1x_2^3 = 0$	$x^2 + y^4 + y^3z$ $+ y^3 + y^2z^2$ $+ yz^3 = 0$	$w_x = 9$ $w_y = 6$ $w_z = 4$	\mathbb{E}_7
H_{47}	$(12, 8, 0)$	-36	$\alpha_{12}x_1^4 + \alpha_8x_1^3x_2$ $+ \alpha_0x_1^3x_3 + \alpha_4x_1^2x_2^2$ $+ \alpha_0x_1x_2^3 = 0$	$x^2 + y^4 + y^3z$ $+ y^3 + y^2z^2$ $+ yz^3 = 0$	$w_x = 9$ $w_y = 6$ $w_z = 4$	\mathbb{E}_7

Thus Theorem 1.5 is proved.

Remark 3.11. The proof of Theorem 1.5 can also be used to describe the singularities of all 3-folds H_i . For example, sufficiently general 3-folds H_i are smooth for $i \in \{1, 2, 3, 4, 6, 9\}$. A sufficiently general 3-fold H_5 has a single isolated ordinary double point and is not \mathbb{Q} -factorial. The singularities of H_i are non-isolated for $i \in \{8, 10, 12, 14, 16, 17, 18, 20, 22, 27, 29, 33, 38\}$. In all other cases, a sufficiently general 3-fold H_i has a single isolated non-cDV singular point.

Remark 3.12. One can simplify the proof of Theorem 1.5 by arguing as follows. If X is a del Pezzo surface of degree 2 over some field with a non-Du Val singular point defined over this field, then the ramification divisor of the double covering $X \rightarrow \mathbb{P}^2$ is a union of four lines. This condition can easily be checked in terms of the numbers d_i . However, the authors did not use this approach, having in mind future applications to rationality questions for the H_i : it is sometimes useful to know the type of singularity or even the explicit local equations of X (see § 5).

§ 4. Proof of Theorem 1.6

Let X be a Fano 3-fold with canonical Gorenstein singularities such that the linear system $|-K_X|$ has no base points and the induced morphism $\varphi_{|-K_X|}$ is an embedding, but the anticanonical image $\varphi_{|-K_X|}(X) \subset \mathbb{P}^n$ is not an intersection of quadrics. (Here $n = -\frac{K_X^3}{2} + 2$.)

Remark 4.1. If $-K_X^3 = 4$, then X is a quartic (possibly singular) in \mathbb{P}^4 . The birational geometry of such 3-folds was studied in [22], [54], [37], [16], [28], [63], [64], [102], [58].

Remark 4.2. If $-K_X^3 = 6$, then X is a complete intersection (possibly singular) of a quadric and a cubic in \mathbb{P}^5 . This follows easily from either Theorem 2.14 or Proposition 2.21. The birational geometry of such 3-folds was studied in [54], [37], [16], [29], [89], [63].

Thus we may assume that $-K_X^3 \geq 8$. Hence Theorem 2.14 implies that X is projectively normal in \mathbb{P}^n and the quadrics through X in \mathbb{P}^n cut out a 4-fold $Y \subset \mathbb{P}^n$ of degree $n - 3$. Moreover, if $\Pi \subset \mathbb{P}^n$ is a general linear subspace of codimension 2, then the curve $X \cap \Pi$ is either a canonically embedded smooth trigonal curve or a canonically embedded smooth plane quintic curve, and $\deg(X \subset \mathbb{P}^n) = 10$. In the former case, the 4-fold Y is the image of a rational scroll $\text{Proj}(\bigoplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^1}(d_i))$ under

the map given by the tautological line bundle over \mathbb{P}^1 , where $0 \neq d_1 \geq \dots \geq d_4 \geq 0$ and $-K_X^3 = 2(d_1 + d_2 + d_3 + d_4) + 2$. In the latter case we have $n = 7$ and Y is a cone over the Veronese surface $v_2(\mathbb{P}^2)$.

Remark 4.3. The cone over a Veronese surface in \mathbb{P}^7 is isomorphic to $\mathbb{P}(1^3, 2, 2)$. Therefore, if Y is a cone over a Veronese surface, then the 3-fold X is a hypersurface in $\mathbb{P}(1^3, 2^2)$ of degree 5 because $-K_X^3 = 10$.

Lemma 4.4. *Let Y be a cone over a Veronese surface in \mathbb{P}^7 whose vertex is a line $L \subset \mathbb{P}^7$. Take a resolution of singularities $f: U \rightarrow Y$, where $U = \text{Proj}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2})$. Put $T = f^*(\mathcal{O}_{\mathbb{P}^7}(1)|_Y)$ and let F be the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ under the natural projection of U to \mathbb{P}^2 . Put $V = f^{-1}(X) \subset U$. Then V has canonical Gorenstein singularities, $-K_V$ is big and numerically effective, and we have $V \sim 2T + F$ on U and $X = \varphi_{|-rK_V|}(V)$ for $r \gg 0$. In particular, the birational morphism $f|_V: V \rightarrow X$ is crepant.*

Proof. The line L is contained in X by Remark 4.3. On the other hand, X is singular along $L \subset X$. Indeed, let $O \in L$ be a point, $\Pi \subset \mathbb{P}^7$ a sufficiently general linear subspace of codimension 2 through O , and $C = \Pi \cap X$. Suppose that O is smooth on X . Then C is a smooth anticanonically embedded plane quintic curve. Hence quadrics through C in $\Pi \cong \mathbb{P}^5$ cut out a smooth Veronese surface by Theorem 2.13. On the other hand, quadrics through C in $\Pi \cong \mathbb{P}^5$ cut out $Y \cap \Pi$. However, the surface $Y \cap \Pi$ must be singular because Y is singular at O by hypothesis.

Therefore the morphism $f|_V: V \rightarrow X$ is crepant at the general point of the line $L \subset X$ by Proposition 2.15. It follows that V contains no fibres of f and $V \sim 2T + F$ on U . Hence the 3-fold V is normal (see [83], Proposition 8.23). Therefore V has canonical Gorenstein singularities and $-K_V$ is a crepant pullback of $-K_X$.

Remark 4.5. Lemma 4.4 does not *a priori* imply the existence of a corresponding Fano 3-fold $X \subset Y$. However, this existence is easily seen. In the notation of Lemma 4.4, the linear system $|2T + L|$ on the 4-fold $U = \text{Proj}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2})$ is free. In particular, sufficiently general divisors in this system are smooth. Let D be a divisor in $|2T + L|$ with canonical singularities. Then the adjunction formula implies that $-K_D \sim T$. Therefore D is a weak Fano 3-fold, that is, the divisor $-K_D$ is numerically effective and big. The vanishing theorem (see [91], [120]) implies that $\varphi_{|-K_D|} = \varphi_{|T|}|_D$. In particular, the 3-fold $\varphi_{|-K_D|}(D)$ is a Fano 3-fold with canonical Gorenstein singularities.

We may thus assume in what follows that $X \cap \Pi$ is a canonically embedded smooth trigonal curve for any general linear subspace $\Pi \subset \mathbb{P}^n$ of codimension 2. Therefore quadrics through X in \mathbb{P}^n cut out a 4-fold Y which is the image of a rational scroll $\text{Proj}(\bigoplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^1}(d_i))$ under the map given by the tautological line bundle, where $0 \neq d_1 \geq \dots \geq d_4 \geq 0$ and $-K_X^3 = 2(d_1 + d_2 + d_3 + d_4) + 2$.

Lemma 4.6. *We have an inclusion $\text{Sing}(Y) \cap X \subset \text{Sing}(X)$.*

Proof. Let O be a singular point on Y such that $O \in X$ and the 3-fold X is non-singular at O . Take a sufficiently general linear subspace $\Pi \subset \mathbb{P}^n$ of codimension 2 passing through O . Put $C = \Pi \cap X$. Then the curve $C \subset \Pi \cong \mathbb{P}^{n-2}$ is a smooth anticanonically embedded trigonal curve. Therefore quadrics through C in $\Pi \cong \mathbb{P}^5$

cut out a smooth surface by Theorem 2.13. On the other hand, quadrics through C in $\Pi \cong \mathbb{P}^{n-2}$ cut out the surface $Y \cap \Pi$, which must be singular at O because the 4-fold Y is singular at O by assumption.

Let $f: U \rightarrow Y$ be the birational morphism $f = \varphi_{|M|}$, where $U = \text{Proj}(\bigoplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^1}(d_i))$, $0 \neq d_1 \geq \dots \geq d_4 \geq 0$, $-K_X^3 = 2(d_1 + d_2 + d_3 + d_4) + 2 \geq 8$ and M is the tautological line bundle on U . We put $V = f^{-1}(X) \subset U$ and $h = f|_V: V \rightarrow X$.

Lemma 4.7. *The 3-fold V has canonical Gorenstein singularities, the anticanonical divisor $-K_V$ is numerically effective and big, and $K_V \sim h^*(K_X)$, that is, the morphism h is crepant.*

Proof. The 3-fold V is normal if and only if it is smooth in codimension 1 (see [83], Proposition 8.23). On the other hand, if $d_3 \neq 0$, then X and V are isomorphic in codimension 2. This immediately yields the claim (compare [92]).

We may thus assume that $d_3 = d_4 = 0$. Put $Z = \text{Sing}(Y)$. Then $\dim(Z) \leq 2$, and the equation $\dim(Z) = 2$ holds if and only if $d_2 = 0$. Moreover, if $\dim(Z \cap X) = 0$, then the 3-folds X and V are isomorphic in codimension 2, which implies the claim. On the other hand, we have $Z \cap X \subset \text{Sing}(X)$ by Lemma 4.6. Since X is normal, it follows that $\dim(Z \cap X) \leq \dim(\text{Sing}(X)) \leq 1$. We may thus assume that the intersection $Z \cap X$ consists of finitely many curves and X is singular along each of them.

The canonicity of singularities of X and Proposition 2.15 imply that g is crepant at the general point of every curve in $Z \cap X$ and the singularities of V are canonical Gorenstein over the general point of every curve in $Z \cap X$. This proves the assertion of the lemma for the complement of a subset of codimension 2 in V . It follows that V is normal (see [83], Proposition 8.23). Hence $V \subset U$ is a divisor on a smooth 4-fold, and V is a normal 3-fold with canonical Gorenstein singularities in codimension 2. It follows that singularities of V are canonical Gorenstein and $K_V \sim h^*(K_X)$.

Let M be the tautological line bundle on U , and let L be a fibre of the natural projection of U to \mathbb{P}^1 . Then $-K_V \sim M|_V$ by construction.

Remark 4.8. The contraction theorem (see [93]) does not *a priori* imply that X is an anticanonical model of V . However, the vanishing theorem (see [91], [120]) implies that $|-K_V| = |M|_V|$. Therefore we see *a posteriori* that X is an anticanonical image of V , that is, $X = \varphi_{|-K_V|}(V)$.

The adjunction formula implies that $V \sim 3M - (d_1 + d_2 + d_3 + d_4 - 2)L$ on U . Let $Y_j \subset V$ be the subscroll induced by the natural projection

$$\bigoplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^1}(d_i) \rightarrow \bigoplus_{i=j}^4 \mathcal{O}_{\mathbb{P}^1}(d_i).$$

In particular, Y_4 is a curve, Y_3 is a surface, and Y_2 is a 3-fold.

Lemma 4.9. *The following inequalities hold:*

$$\text{mult}_{Y_2}(V) = 0, \quad \text{mult}_{Y_3}(V) \leq 1, \quad \text{mult}_{Y_4}(V) \leq 2.$$

Proof. The first is obvious, the second follows since V is normal, and the third follows from the canonicity of V at a general point of Y_4 by Theorem 2.9.

Therefore Corollary 2.20 yield the inequalities

$$2d_2 - d_1 - d_3 - d_4 + 2 \geq 0, \quad d_3 - d_2 - d_4 + 2 \geq 0, \quad 2 - d_2 - d_3 + d_1 \geq 0.$$

We also have $0 \neq d_1 \geq \dots \geq d_4 \geq 0$ and $d_1 + d_2 + d_3 + d_4 \geq 3$ by assumption. These inequalities determine exactly 66 different rational scrolls

$$\mathbb{F}(d_1, d_2, d_3, d_4) = \text{Proj} \left(\bigoplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^1}(d_i) \right).$$

Remark 4.10. X is an anticanonical model of V , that is, $X = \varphi_{|-rK_V|}(V)$ for $r \gg 0$. Thus the contraction theorem (see [93]) implies that X is a Fano 3-fold with canonical Gorenstein singularities if and only if V has canonical Gorenstein singularities and $-K_V$ is numerically effective and big. On the other hand, V is uniquely determined by the rational scroll $\mathbb{F}(d_1, d_2, d_3, d_4)$ and the class $3M - (d_1 + d_2 + d_3 + d_4 - 2)L$ in the Picard group of $\mathbb{F}(d_1, d_2, d_3, d_4)$. However, the linear system $|3M - (d_1 + d_2 + d_3 + d_4 - 2)L|$ may *a priori* contain no divisor with canonical singularities.

In the rest of the section we show explicitly that, in each of the 66 possible cases, the linear system $|3M - (d_1 + d_2 + d_3 + d_4 - 2)L|$ contains a divisor with canonical singularities. To do this, we use Corollary 2.5 and Proposition 2.19. This will complete the proof of Theorem 1.6.

Remark 4.11. The same idea was used in the classification of smooth trigonal Fano 3-folds (see [15], [17]) and in [41] to prove the effective boundedness of degree for trigonal Fano 3-folds with canonical singularities. The maximal value of the degree is attained by Theorem 1.6.

We start by considering two different possibilities in full detail.

Example 4.12. Let $V \subset \mathbb{F} = \mathbb{F}(6, 5, 3, 0)$ be a sufficiently general divisor in the linear system $|3M - 12L|$. Let us show that V has canonical singularities.

By Proposition 2.19, V is given by the zeros of a bihomogeneous polynomial

$$\begin{aligned} & \alpha_6(t_1, t_2) x_1^3 + \alpha_5(t_1, t_2) x_1^2 x_2 + \alpha_3^1(t_1, t_2) x_1^2 x_3 + \alpha_0^1(t_1, t_2) x_1^2 x_4 + \alpha_4(t_1, t_2) x_1 x_2^2 \\ & + \alpha_2(t_1, t_2) x_1 x_2 x_3 + \alpha_0^2(t_1, t_2) x_1 x_3^2 + \alpha_3^2(t_1, t_2) x_2^3 + \alpha_1(t_1, t_2) x_2^2 x_3, \end{aligned}$$

where $\alpha_d(t_1, t_2)$ (or $\alpha_d^i(t_1, t_2)$) is a form of degree d . Let E be the surface $x_1 = x_2 = 0$, and let C be the curve $x_1 = x_2 = x_3 = 0$. We note that the base locus of $|3M - 12L|$ is equal to E . Since the automorphism group of $E \cong \mathbb{F}(3, 0)$ acts transitively on $E \setminus C$ and V has multiplicity 1 at a general point of E , we see from Corollary 2.5 that it suffices to prove that for any point P on C there is a divisor V with canonical singularities in a neighbourhood of P (compare Example 3.10).

Let Y be the fibre of $V \subset \mathbb{F}$ over a sufficiently general point $P \in \mathbb{P}^1$. We put $O = C \cap Y$. As above, it suffices to prove that V has at most canonical singularities in a neighbourhood of O . Hence it suffices to prove that O is a Du Val point on Y .

Let $(\gamma : \delta)$ be the homogeneous coordinates of the point $P \in \mathbb{P}^1$. Then Y may be presented as the hypersurface

$$\begin{aligned} & \alpha_6 x_1^3 + \alpha_5 x_1^2 x_2 + \alpha_3^1 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4 + \alpha_4 x_1 x_2^2 \\ & + \alpha_2 x_1 x_2 x_3 + \alpha_0^2 x_1 x_3^2 + \alpha_3^2 x_2^3 + \alpha_1 x_2^2 x_3 = 0 \end{aligned}$$

in \mathbb{P}^4 , where $\alpha_i = \alpha_i(\gamma, \delta)$ (resp. $\alpha_i^j = \alpha_i^j(\gamma, \delta)$) and $O = (0 : 0 : 0 : 1)$. Since P is general, we have $\alpha_i \neq 0$ (resp. $\alpha_i^j \neq 0$) for all i . Hence we may assume for convenience that $\alpha_i = 1$ (resp. $\alpha_i^j = 1$) for all i and j .

Let $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = 1$. Then the local equation of Y in a neighbourhood of O is

$$x^3 + x^2 y + x^2 z + x^2 + x y^2 + x y z + x z^2 + y^3 + y^2 z = 0.$$

We put $\text{wt}(x) = 4$, $\text{wt}(y) = 3$ and $\text{wt}(z) = 2$. Then $\text{wt}(x^2 + xz^2 + y^2 z) = 8$, $\text{wt}(x^3) = 12$, $\text{wt}(x^2 y) = 11$, $\text{wt}(xy^2) = 10$, $\text{wt}(y^3) = 9$, $\text{wt}(x^2 z) = 10$ and $\text{wt}(xyz) = 9$. Moreover, the singularity given by the equation $x^2 + xz^2 + y^2 z = 0$ is isolated. Therefore the singularity of Y at O is locally isomorphic to a Du Val singularity of type \mathbb{D}_5 . In particular, V has a singularity of type $\mathbb{D}_5 \times \mathbb{C}$ at a general point of C . Using the generality in the choice of V , we may actually assume that P is an arbitrary point of the curve C (compare Example 3.10). Thus, for any given point P of C , the linear system $|3M - 12L|$ contains a divisor with at most canonical singularities in the neighbourhood of P . Hence the singularities of V are canonical by Corollary 2.5.

Example 4.13. Let $V \subset \mathbb{F} = \mathbb{F}(7, 3, 1, 0)$ be a general divisor in the linear system $|3M - 9L|$. Let us show that V has canonical singularities.

By Proposition 2.19, V is given by the zeros of a bihomogeneous polynomial

$$\begin{aligned} & \alpha_{12}(t_1, t_2) x_1^3 + \alpha_8(t_1, t_2) x_1^2 x_2 + \alpha_6(t_1, t_2) x_1^2 x_3 + \alpha_5(t_1, t_2) x_1^2 x_4 \\ & + \alpha_4(t_1, t_2) x_1 x_2^2 + \alpha_2(t_1, t_2) x_1 x_2 x_3 + \alpha_1(t_1, t_2) x_1 x_2 x_4 \\ & + \alpha_0^1(t_1, t_2) x_1 x_3^2 + \alpha_0^2(t_1, t_2) x_2^3, \end{aligned}$$

where $\alpha_d(t_1, t_2)$ (or $\alpha_d^i(t_1, t_2)$) is a form of degree d . Let E be the surface $x_1 = x_2 = 0$, and let C be the curve $x_1 = x_2 = x_3 = 0$. We note that the base locus of $|3M - 9L|$ is equal to E . Since the automorphism group of $E \cong \mathbb{F}(1, 0)$ acts transitively on $E \setminus C$ and V has multiplicity 1 at a general point of E , Corollary 2.5 implies that V has canonical singularities on $E \setminus C$, and it remains to verify that V has canonical singularities at points of C (compare Example 3.10).

Let Y be the fibre of $V \subset \mathbb{F}$ over a general point $P \in \mathbb{P}^1$. We put $O = C \cap Y$. As above, it suffices to prove that V has canonical singularities at O . Hence it suffices to prove that O is a Du Val point on Y .

Let $(\gamma : \delta)$ be the homogeneous coordinates of $P \in \mathbb{P}^1$. Then Y may be presented as a hypersurface corresponding to the polynomial

$$\begin{aligned} &\alpha_{12}x_1^3 + \alpha_8x_1^2x_2 + \alpha_6x_1^2x_3 + \alpha_5x_1^2x_4 + \alpha_4x_1x_2^2 \\ &+ \alpha_2x_1x_2x_3 + \alpha_1x_1x_2x_4 + \alpha_0^1x_1x_3^2 + \alpha_0^2x_2^3 \end{aligned}$$

on \mathbb{P}^4 , where $\alpha_i = \alpha_i(\gamma, \delta)$ (resp. $\alpha_i^j = \alpha_i^j(\gamma, \delta)$) and $O = (0 : 0 : 0 : 1)$. Since P is general, we have $\alpha_i \neq 0$ (resp. $\alpha_i^j \neq 0$) for all i . Therefore we may assume for convenience that $\alpha_i = 1$ (resp. $\alpha_i^j = 1$) for all i and j .

We put $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = 1$. Then the local equation of Y in a neighbourhood of O is

$$x^3 + x^2y + x^2z + x^2 + xy^2 + xyz + xy + xz^2 + y^3 = 0.$$

It is easy to prove that one cannot choose “good” weights for this polynomial. Hence we cannot use Theorem 2.10 as before and we must explicitly resolve singularities in a neighbourhood of O . It is easy to see that, resolving the singularity at O , we can skip monomials whose weight is larger than at least one of the others. (This will be clear from the forthcoming blow-ups.) Hence it suffices to consider the polynomial

$$x^2 + xy + xz^2 + y^3.$$

This polynomial determines an isolated singular point at O . We blow up this point. The formulae in the local charts are as follows.

1) $x \neq 0$: the change of coordinates $x = x$, $y = xy$, $z = xz$ brings the local equation (after dividing by x^2) to the form

$$1 + y + xz^2 + xy^3 = 0,$$

so our surface is smooth in this chart.

2) $y \neq 0$: the change of coordinates $x = xy$, $y = y$, $z = yz$ yields (after dividing by y^2)

$$x^2 + x + xyz^2 + y = 0,$$

and our surface is smooth in this chart.

3) $z \neq 0$: the change of coordinates $x = xz$, $y = yz$, $z = z$ yields (after dividing by z^2)

$$x^2 + xy + xz + y^3z = 0,$$

and we have two extremal (-2) -curves with the only singular point $(0, 0, 0)$ near $z = 0$.

Therefore more blow-ups are necessary. We must study the singularities given by the local equation

$$x^2 + xy + xz + y^3z = 0.$$

We blow-up the point $(0, 0, 0)$. Here are the formulae in the local charts.

1) $x \neq 0$: the change of coordinates $x = x$, $y = xy$, $z = xz$ yields (after dividing by x^2)

$$1 + y + z + x^2 y^3 z = 0,$$

and the surface is smooth in this chart.

2) $y \neq 0$: the change of coordinates $x = xy$, $y = y$, $z = yz$ yields (after dividing by y^2)

$$x^2 + x + xz + y^2 z = x^2 + x(z + 1) + y^2(z + 1) - y^2 = x^2 + y'^2 + z'^2 - y'^2 z',$$

where $y' = iy$ and $z' = z + 1$. Thus the point $(0, 0, -1)$ is Du Val of type \mathbb{A}_1 .

3) $z \neq 0$: the change of coordinates $x = xz$, $y = yz$, $z = z$ yields (after dividing by z^2)

$$x^2 + xy + x + y^3 z^2 = 0.$$

We can see that the point $(0, -1, 0)$ is singular. It coincides with the singular point in the chart $y \neq 0$.

Summarizing, we have two (-2) -curves after the first blow-up, two (-2) -curves after the second blow-up and one point of type \mathbb{A}_1 on one of these curves. Hence the graph of resolution of the original singularity corresponds to a Du Val singularity of type \mathbb{A}_5 . In particular, the 3-fold V has a singularity of type $\mathbb{A}_5 \times \mathbb{C}$ at the general point of C . As in Example 4.12, it follows that the singularities of V are canonical by Corollary 2.5.

We state the results of some easy calculations, which will be used in the remaining part of the proof of Theorem 1.6.

Lemma 4.14. 1. *A surface singularity given by*

$$x^3 + x^2 y + x^2 z + x^2 + xy^2 + xyz + xy + xz^2 + xz + y^3 = 0$$

is Du Val of type \mathbb{A}_2 .

2. *A surface singularity given by*

$$x^3 + x^2 y + x^2 z + x^2 + xy^2 + xyz + xy + xz^2 + xz + y^3 + y^2 z,$$

is Du Val of type \mathbb{A}_3 .

3. *A surface singularity given by*

$$x^3 + x^2 y + x^2 z + x^2 + xy^2 + xyz + xy + xz^2 + xz + y^3 + y^2 z + yz^2 + z^3,$$

is Du Val of type \mathbb{A}_3 .

4. *A surface singularity given by*

$$x^3 + x^2 y + x^2 z + x^2 + xy^2 + xyz + xy + xz^2 + y^3 + y^2 z,$$

is Du Val of type \mathbb{A}_4 .

5. A surface singularity given by

$$x^3 + x^2y + x^2z + x^2 + xy^2 + xyz + xy + xz^2 + y^3,$$

is Du Val of type \mathbb{A}_5 .

Proof. In case 1, changing coordinates to $x' = x$, $y' = y$, $z' = x + y + z$, we get the equation

$$x'z' + y'^3 + Q(x', y', z') = 0,$$

where Q consists of the terms whose weight (with respect to any choice of weights) is larger than that of either $x'z'$ or y'^3 . Hence this singularity is Du Val of type \mathbb{A}_2 .

In the remaining cases, it is easy to see that the corresponding equation describes an isolated singularity. Since it is impossible to use Theorem 2.5, we shall explicitly resolve these singularities. Case 5 has already been discussed in Example 4.13, along with details of the calculations. In cases 2 and 3, a single blow-up yields two exceptional (-2) -curves and one point of type \mathbb{A}_1 on one of them. This means that the original singularities are Du Val of type \mathbb{A}_3 . In case 4, two blow-ups yield a smooth surface, and the exceptional curves form a Dynkin diagram of type \mathbb{A}_4 .

In the rest of this section we consider all possible cases, following the pattern of Examples 4.12 and 4.13 and using Lemma 4.14 when necessary. Differences appear only in the numerical characteristics, equations, types of singularities and so on. They are collected in Table 2.

Table 2 is organized as follows. The first column contains the labels of the varieties V in the notation of Theorem 1.6. The second column gives a quadruple (d_1, d_2, d_3, d_4) such that V is a divisor on $\mathbb{F}(d_1, d_2, d_3, d_4)$. The third column gives the number b such that $|V| = |3M + bL|$. In cases T_5 , T_8 and T_{14} , this linear system appears to be base point free, whence V is smooth by Bertini's theorem. Then we do not need the information contained in the other columns. In the other cases, $\text{Bs}|V|$ is either the curve $C = Y_4$ given by $x_1 = x_2 = x_3 = 0$ (and then V is non-singular outside C , so it suffices to verify that V has only canonical singularities at points of C) or the surface $E = Y_3$ given by $x_1 = x_2 = 0$ (and then V has multiplicity 1 at a general point of E and, since the automorphism group of $E \cong \mathbb{F}(d_3, d_4)$ acts transitively on $E \setminus C$ (compare Example 3.10), we see from Corollary 2.5 that it suffices to study the singularities of V at points of C only).

The fourth column contains an equation of a general divisor V in the linear system $|3M + bL|$. The fifth column contains the equation of the fibre Y of the projection $V \rightarrow \mathbb{P}^1$ over a general point of \mathbb{P}^1 in the neighbourhood of a general point of C after the change of coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = 1$ (see Example 4.12). The same equation describes V locally as an equation in t, x, y, z .

For the 3-folds T_j with $j \in \{4, 6, 7, 9, 11, 15, 16, 17, 19, 25, 26, 28, 36, 45\}$, the corresponding point appears to be non-singular. For T_j with $j \in \{10, 12, 13, 18, 22, 23, 24, 30, 31, 33, 34, 35, 41, 43, 44, 49, 50, 51, 52, 55, 56, 57, 58, 60, \dots, 69\}$, we attribute new weights $\text{wt}(x) = w_x$, $\text{wt}(y) = w_y$, $\text{wt}(z) = w_z$ (listed in the sixth column) to the variables x, y, z and see that the terms of the lowest weight describe an isolated Du Val singularity. We note that the weights $\text{wt}(x)$, $\text{wt}(y)$, $\text{wt}(z)$ coincide with those of a Du Val singularity of the type given in the seventh column. Hence the singularity of Y in the chosen neighbourhood is Du Val of this type by Theorem 2.10. Unfortunately, it is impossible to find such weights in the remaining cases. But the corresponding equations have already been considered in Lemma 4.14: the case of T_{37} is exactly case 1, the case of T_{27} is case 2, the cases of T_{20} , T_{21} , T_{29} fall into case 3, the cases of T_{32} , T_{38} , T_{39} , T_{42} and T_{48} correspond to case 4, and the cases of T_{40} , T_{46} , T_{47} , T_{53} , T_{54} , T_{59} correspond to case 5 of Lemma 4.14. In every case, the singularity of V is Du Val of the type given in the seventh column.

The singularity of V is locally isomorphic to the product of \mathbb{C} and the corresponding Du Val singularity. Hence V has canonical singularities in the chosen neighbourhood and, by Corollary 2.5, V has canonical singularities.

TABLE 2

T_i	(d_1, d_2, d_3, d_4)	b	Equation of V	Local equation of V	Weights	Singularity
T_4	$(1, 1, 1, 0)$	-1	$\alpha_2^1 x_1^3 + \alpha_2^2 x_1^2 x_2$ $+ \alpha_2^3 x_1^2 x_3 + \alpha_2^4 x_1^2 x_2$ $+ \alpha_2^5 x_1 x_2 x_3 + \alpha_2^6 x_1 x_3^2$ $+ \alpha_2^7 x_2^3 + \alpha_2^8 x_2^2 x_3$ $+ \alpha_2^9 x_2 x_3^2 + \alpha_2^{10} x_3^3$ $+ \alpha_1^1 x_1^2 x_4 + \alpha_1^2 x_1 x_2 x_4$ $+ \alpha_1^3 x_1 x_3 x_4 + \alpha_1^4 x_2^2 x_4$ $+ \alpha_1^5 x_2 x_3 x_4 + \alpha_1^6 x_3^2 x_4$ $+ \alpha_0^1 x_1 x_4^2 + \alpha_0^2 x_2 x_4^2$ $+ \alpha_0^3 x_3 x_4^2 = 0$	$P_1(x, y, z)$ $+ P_2(x, y, z)$ $+ P_3(x, y, z) = 0$ $(P_i \text{ is a (general) homogeneous polynomial of degree } i)$	—	Non-singular point
T_5	$(1, 1, 1, 1)$	-2	—	—	—	—

TABLE 2 CONTINUED

T_6	$(2, 1, 0, 0)$	-1	$\alpha_5 x_1^3 + \alpha_4 x_1^2 x_2$ $+ \alpha_3^1 x_1^2 x_3 + \alpha_3^2 x_1^2 x_4$ $+ \alpha_3^3 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3$ $+ \alpha_2^2 x_1 x_2 x_4 + \alpha_1^1 x_1 x_3^2$ $+ \alpha_1^2 x_1 x_3 x_4$ $+ \alpha_1^3 x_1 x_4^2 + \alpha_2^3 x_2^3$ $+ \alpha_1^4 x_2^2 x_3 + \alpha_1^5 x_2^2 x_4$ $+ \alpha_0^1 x_2 x_3^2 + \alpha_0^2 x_2 x_3 x_4$ $+ \alpha_0^3 x_2 x_4^2 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + xz$ $+ x + y^3 + y^2 z$ $+ y^2 + yz^2 + yz$ $+ y = 0$	$-$	Non-singular point
T_7	$(2, 1, 1, 0)$	-2	$\alpha_4 x_1^3 + \alpha_3^1 x_1^2 x_2$ $+ \alpha_3^2 x_1^2 x_3 + \alpha_2^1 x_1^2 x_4$ $+ \alpha_2^2 x_1 x_2^2 + \alpha_2^3 x_1 x_2 x_3$ $+ \alpha_1^1 x_1 x_2 x_4 + \alpha_2^4 x_1 x_3^2$ $+ \alpha_1^2 x_1 x_3 x_4 + \alpha_0^1 x_1 x_4^2$ $+ \alpha_1^3 x_2^3 + \alpha_1^4 x_2^2 x_3$ $+ \alpha_0^2 x_2^2 x_4 + \alpha_1^5 x_2 x_3^2$ $+ \alpha_0^3 x_2 x_3 x_4 + \alpha_1^6 x_3^3$ $+ \alpha_0^4 x_3^2 x_4 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + xz + x$ $+ y^3 + y^2 z + y^2$ $+ yz^2 + yz + z^3$ $+ z^2 = 0$	$-$	Non-singular point
T_8	$(2, 1, 1, 1)$	-3	$-$	$-$	$-$	$-$
T_9	$(2, 2, 0, 0)$	-2	$\alpha_4^1 x_1^3 + \alpha_4^2 x_1^2 x_2$ $+ \alpha_2^1 x_1^2 x_3 + \alpha_2^2 x_1^2 x_4$ $+ \alpha_4^3 x_1 x_2^2 + \alpha_2^3 x_1 x_2 x_3$ $+ \alpha_2^4 x_1 x_2 x_4 + \alpha_0^1 x_1 x_3^2$ $+ \alpha_0^2 x_1 x_3 x_4$ $+ \alpha_0^3 x_1 x_4^2 + \alpha_4^4 x_2^3$ $+ \alpha_2^5 x_2^2 x_3 + \alpha_2^6 x_2^2 x_4$ $+ \alpha_0^4 x_2 x_3^2 + \alpha_0^5 x_2 x_3 x_4$ $+ \alpha_0^6 x_2 x_4^2 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + xz$ $+ x + y^3 + y^2 z$ $+ y^2 + yz^2 + yz$ $+ y = 0$	$-$	Non-singular point

TABLE 2 CONTINUED

T_{10}	$(2, 2, 1, 0)$	-3	$\alpha_3^1 x_1^3 + \alpha_3^2 x_1^2 x_2$ $+ \alpha_2^1 x_1^2 x_3 + \alpha_1^1 x_1^2 x_4$ $+ \alpha_3^3 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_1^2 x_1 x_2 x_4 + \alpha_3^3 x_1 x_3^2$ $+ \alpha_0^1 x_1 x_3 x_4 + \alpha_3^4 x_2^3$ $+ \alpha_2^3 x_2^2 x_3 + \alpha_1^4 x_2^2 x_4$ $+ \alpha_1^5 x_2 x_3^2 + \alpha_0^2 x_2 x_3 x_4$ $+ \alpha_0^3 x_3^3 = 0$	$P_3(x, y)$ $+ P_2^1(x, y)z$ $+ P_2^2(x, y)$ $+ P_1(x, y)z = 0$ $(P_i, P_i^j \text{ are}$ homogeneous polynomials of degree i)	$w_x = 1$ $w_y = 1$ $w_z = 1$	\mathbb{A}_1
T_{11}	$(2, 2, 1, 1)$	-4	$\alpha_2^1 x_1^3 + \alpha_2^2 x_1^2 x_2$ $+ \alpha_1^1 x_1^2 x_3 + \alpha_1^2 x_1^2 x_4$ $+ \alpha_2^3 x_1 x_2^2 + \alpha_1^3 x_1 x_2 x_3$ $+ \alpha_1^4 x_1 x_2 x_4 + \alpha_0^1 x_1 x_3^2$ $+ \alpha_0^2 x_1 x_3 x_4$ $+ \alpha_0^3 x_1 x_4^2 + \alpha_2^4 x_2^3$ $+ \alpha_1^5 x_2^2 x_3 + \alpha_1^6 x_2^2 x_4$ $+ \alpha_0^4 x_2 x_3^2 + \alpha_0^5 x_2 x_3 x_4$ $+ \alpha_0^6 x_2 x_4^2 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + xz$ $+ x + y^3 + y^2 z$ $+ y^2 + yz^2 + yz$ $+ y = 0$	$-$	Non-singular point
T_{12}	$(2, 2, 2, 0)$	-4	$\alpha_2^1 x_1^3 + \alpha_2^2 x_1^2 x_2$ $+ \alpha_2^3 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4$ $+ \alpha_2^4 x_1 x_2^2 + \alpha_2^5 x_1 x_2 x_3$ $+ \alpha_0^2 x_1 x_2 x_4 + \alpha_2^6 x_1 x_3^2$ $+ \alpha_0^3 x_1 x_3 x_4 + \alpha_2^7 x_2^3$ $+ \alpha_2^8 x_2^2 x_3 + \alpha_0^4 x_2^2 x_4$ $+ \alpha_2^9 x_2 x_3^2 + \alpha_0^5 x_2 x_3 x_4$ $+ \alpha_2^{10} x_3^3 + \alpha_0^6 x_3^2 x_4 = 0$	$P_3(x, y, z)$ $+ P_2(x, y, z) = 0$ $(P_i \text{ is a homo-}$ geneous poly- nomial of degree i)	$w_x = 1$ $w_y = 1$ $w_z = 1$	\mathbb{A}_1
T_{13}	$(2, 2, 2, 1)$	-5	$\alpha_1^1 x_1^3 + \alpha_1^2 x_1^2 x_2$ $+ \alpha_1^3 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4$ $+ \alpha_1^4 x_1 x_2^2 + \alpha_1^5 x_1 x_2 x_3$ $+ \alpha_0^2 x_1 x_2 x_4 + \alpha_1^6 x_1 x_3^2$ $+ \alpha_0^3 x_1 x_3 x_4 + \alpha_1^7 x_2^3$ $+ \alpha_1^8 x_2^2 x_3 + \alpha_0^4 x_2^2 x_4$ $+ \alpha_1^9 x_2 x_3^2 + \alpha_0^5 x_2 x_3 x_4$ $+ \alpha_1^{10} x_3^3 + \alpha_0^6 x_3^2 x_4 = 0$	$P_3(x, y, z)$ $+ P_2(x, y, z) = 0$ $(P_i \text{ is a homo-}$ geneous poly- nomial of degree i)	$w_x = 1$ $w_y = 1$ $w_z = 1$	\mathbb{A}_1

TABLE 2 CONTINUED

T_{14}	$(2, 2, 2, 2)$	-6	—	—	—	—
T_{15}	$(3, 1, 0, 0)$	-2	$\alpha_7 x_1^3 + \alpha_5 x_1^2 x_2$ $+ \alpha_4^1 x_1^2 x_3 + \alpha_4^2 x_1^2 x_4$ $+ \alpha_3 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3$ $+ \alpha_2^2 x_1 x_2 x_4 + \alpha_1^1 x_1 x_3^2$ $+ \alpha_1^2 x_1 x_3 x_4 + \alpha_1^3 x_1 x_4^2$ $+ \alpha_1^4 x_2^3 + \alpha_0^1 x_2^2 x_3$ $+ \alpha_0^2 x_2^2 x_4 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + x y^2$ $+ x y z + x y$ $+ x z^2 + x z$ $+ x + y^3 + y^2 z$ $+ y^2 = 0$	—	Non-singular point
T_{16}	$(3, 1, 1, 0)$	-3	$\alpha_6 x_1^3 + \alpha_4^1 x_1^2 x_2$ $+ \alpha_4^2 x_1^2 x_3 + \alpha_3 x_1^2 x_4$ $+ \alpha_2^1 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_1^1 x_1 x_2 x_4 + \alpha_2^3 x_1 x_3^2$ $+ \alpha_2^2 x_1 x_3 x_4 + \alpha_0^1 x_1 x_4^2$ $+ \alpha_0^2 x_2^3 + \alpha_0^3 x_2^2 x_3$ $+ \alpha_0^4 x_2 x_3^2 + \alpha_0^5 x_3^3 = 0$	$xQ(x, y, z)$ $+ P_3(y, z) = 0$ $(Q(0) \neq 0, P_3$ is a homoge- neous poly- nomial of degree 3)	—	Non-singular point
T_{17}	$(3, 2, 0, 0)$	-3	$\alpha_6 x_1^3 + \alpha_5 x_1^2 x_2$ $+ \alpha_3^1 x_1^2 x_3 + \alpha_3^2 x_1^2 x_4$ $+ \alpha_4 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3$ $+ \alpha_2^2 x_1 x_2 x_4 + \alpha_0^1 x_1 x_3^2$ $+ \alpha_0^2 x_1 x_3 x_4 + \alpha_0^3 x_1 x_4^2$ $+ \alpha_3^3 x_2^3 + \alpha_1^1 x_2^2 x_3$ $+ \alpha_1^2 x_2^2 x_4 = 0$	$xQ_1(x, y, z)$ $+ y^2 Q_2(y, z) = 0$ $(Q_1(0) \neq 0)$	—	Non-singular point
T_{18}	$(3, 2, 1, 0)$	-4	$\alpha_5 x_1^3 + \alpha_4 x_1^2 x_2$ $+ \alpha_3^1 x_1^2 x_3 + \alpha_2^1 x_1^2 x_4$ $+ \alpha_3^2 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_1^1 x_1 x_2 x_4 + \alpha_1^2 x_1 x_3^2$ $+ \alpha_0^1 x_1 x_3 x_4 + \alpha_2^3 x_2^3$ $+ \alpha_1^3 x_2^2 x_3 + \alpha_0^2 x_2^2 x_4$ $+ \alpha_0^3 x_2 x_3^2 = 0$	$x^2 Q_1(x, y, z)$ $+ x y Q_2(x, y, z)$ $+ x z Q_3(x, y, z)$ $+ y^2 Q_4(x, y, z)$ $+ y z^2 = 0$ $(Q_i(0) \neq 0)$	$w_x = 1$ $w_y = 1$ $w_z = 1$	\mathbb{A}_1

TABLE 2 CONTINUED

T_{19}	$(3, 2, 1, 1)$	-5	$\alpha_4 x_1^3 + \alpha_3 x_1^2 x_2$ $+ \alpha_2^1 x_1^2 x_3 + \alpha_2^2 x_1^2 x_4$ $+ \alpha_2^3 x_1 x_2^2 + \alpha_1^1 x_1 x_2 x_3$ $+ \alpha_1^2 x_1 x_2 x_4 + \alpha_0^1 x_1 x_3^2$ $+ \alpha_0^2 x_1 x_3 x_4 + \alpha_0^3 x_1 x_4^2$ $+ \alpha_1^3 x_2^3 + \alpha_0^4 x_2^2 x_3$ $+ \alpha_0^5 x_2^2 x_4 = 0$	$xQ_1(x, y, z)$ $+ Q_2(y, z) = 0$ $(Q_1(0) \neq 0)$	$-$	Non-singular point
T_{20}	$(3, 2, 2, 0)$	-5	$\alpha_4 x_1^3 + \alpha_3^1 x_1^2 x_2$ $+ \alpha_3^2 x_1^2 x_3 + \alpha_1^1 x_1^2 x_4$ $+ \alpha_2^1 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_2 x_4 + \alpha_3^3 x_1 x_3^2$ $+ \alpha_0^2 x_1 x_3 x_4 + \alpha_1^2 x_2^3$ $+ \alpha_1^3 x_2^2 x_3 + \alpha_1^4 x_2 x_3^2$ $+ \alpha_1^5 x_3^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + xz$ $+ y^3 + y^2 z$ $+ yz^2 + z^3 = 0$	$-$	\mathbb{A}_3
T_{21}	$(3, 2, 2, 1)$	-6	$\alpha_3 x_1^3 + \alpha_2^1 x_1^2 x_2$ $+ \alpha_2^2 x_1^2 x_3 + \alpha_1^1 x_1^2 x_4$ $+ \alpha_1^2 x_1 x_2^2 + \alpha_1^3 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_2 x_4 + \alpha_1^4 x_1 x_3^2$ $+ \alpha_0^2 x_1 x_3 x_4 + \alpha_0^3 x_2^3$ $+ \alpha_0^4 x_2^2 x_3 + \alpha_0^5 x_2 x_3^2$ $+ \alpha_0^6 x_3^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + xz$ $+ y^3 + y^2 z$ $+ yz^2 + z^3 = 0$	$-$	\mathbb{A}_3
T_{22}	$(3, 3, 1, 0)$	-5	$\alpha_4^1 x_1^3 + \alpha_4^2 x_1^2 x_2$ $+ \alpha_2^1 x_1^2 x_3 + \alpha_1^1 x_1^2 x_4$ $+ \alpha_4^3 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_1^2 x_1 x_2 x_4$ $+ \alpha_0^1 x_1 x_3^2 + \alpha_4^4 x_2^3$ $+ \alpha_2^3 x_2^2 x_3 + \alpha_1^3 x_2^2 x_4$ $+ \alpha_0^2 x_2 x_3^2 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3$ $+ y^2 z + y^2$ $+ yz^2 = 0$	$w_x = 2$ $w_y = 2$ $w_z = 1$	\mathbb{A}_3

TABLE 2 CONTINUED

T_{23}	$(3, 3, 2, 0)$	-6	$\alpha_3^1 x_1^3 + \alpha_3^2 x_1^2 x_2$ $+ \alpha_2^1 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4$ $+ \alpha_3^3 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_0^2 x_1 x_2 x_4 + \alpha_1^1 x_1 x_3^2$ $+ \alpha_3^4 x_2^3 + \alpha_2^3 x_2^2 x_3$ $+ \alpha_0^3 x_2^2 x_4 + \alpha_1^2 x_2 x_3^2$ $+ \alpha_0^4 x_3^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3$ $+ y^2 z + y^2$ $+ yz^2 + z^3 = 0$	$w_x = 3$ $w_y = 3$ $w_z = 2$	\mathbb{A}_2
T_{24}	$(3, 3, 2, 1)$	-7	$\alpha_2^1 x_1^3 + \alpha_2^2 x_1^2 x_2$ $+ \alpha_1^1 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4$ $+ \alpha_2^3 x_1 x_2^2 + \alpha_1^2 x_1 x_2 x_3$ $+ \alpha_0^2 x_1 x_2 x_4$ $+ \alpha_0^3 x_1 x_3^2 + \alpha_2^4 x_2^3$ $+ \alpha_1^3 x_2^2 x_3 + \alpha_0^4 x_2^2 x_4$ $+ \alpha_0^5 x_2 x_3^2 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3$ $+ y^2 z + y^2$ $+ yz^2 = 0$	$w_x = 2$ $w_y = 2$ $w_z = 1$	\mathbb{A}_3
T_{25}	$(4, 1, 0, 0)$	-3	$\alpha_9 x_1^3 + \alpha_6 x_1^2 x_2$ $+ \alpha_5^1 x_1^2 x_3 + \alpha_5^2 x_1^2 x_4$ $+ \alpha_3 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3$ $+ \alpha_2^2 x_1 x_2 x_4 + \alpha_1^1 x_1 x_3^2$ $+ \alpha_1^2 x_1 x_3 x_4 + \alpha_1^3 x_1 x_4^2$ $+ \alpha_0 x_2^3 = 0$	$xQ(x, y, z)$ $+ y^3 = 0$ $(Q(0) \neq 0)$	$-$	Non-singular point
T_{26}	$(4, 2, 0, 0)$	-4	$\alpha_8 x_1^3 + \alpha_6 x_1^2 x_2$ $+ \alpha_4^1 x_1^2 x_3 + \alpha_4^2 x_1^2 x_4$ $+ \alpha_4^3 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3$ $+ \alpha_2^2 x_1 x_2 x_4 + \alpha_0^1 x_1 x_3^2$ $+ \alpha_0^2 x_1 x_3 x_4 + \alpha_0^3 x_1 x_4^2$ $+ \alpha_2^3 x_2^3 + \alpha_0^4 x_2^2 x_3$ $+ \alpha_0^5 x_2^2 x_4 = 0$	$xQ(x, y, z)$ $+ y^2 + y^3 = 0$ $(Q(0) \neq 0)$	$-$	Non-singular point
T_{27}	$(4, 2, 1, 0)$	-5	$\alpha_7 x_1^3 + \alpha_5 x_1^2 x_2$ $+ \alpha_4 x_1^2 x_3 + \alpha_3^1 x_1^2 x_4$ $+ \alpha_3^2 x_1 x_2^2 + \alpha_2 x_1 x_2 x_3$ $+ \alpha_1^1 x_1 x_2 x_4 + \alpha_1^2 x_1 x_3^2$ $+ \alpha_0^1 x_1 x_3 x_4 + \alpha_1^3 x_2^3$ $+ \alpha_0^2 x_2^2 x_3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + xz$ $+ y^3 + y^2 z = 0$	$-$	\mathbb{A}_3

TABLE 2 CONTINUED

T_{28}	$(4, 2, 1, 1)$	-6	$\alpha_6 x_1^3 + \alpha_4 x_1^2 x_2$ $+ \alpha_3 x_1^2 x_3 + \alpha_3^2 x_1^2 x_4$ $+ \alpha_2 x_1 x_2^2 + \alpha_1^2 x_1 x_2 x_3$ $+ \alpha_1^2 x_1 x_2 x_4 + \alpha_0^1 x_1 x_3^2$ $+ \alpha_0^2 x_1 x_3 x_4 + \alpha_0^3 x_1 x_4^2$ $+ \alpha_0^4 x_2^3 = 0$	$xQ(x, y, z)$ $+ y^3 = 0$ $(Q(0) \neq 0)$	$-$	Non-singular point
T_{29}	$(4, 2, 2, 0)$	-6	$\alpha_6 x_1^3 + \alpha_4 x_1^2 x_2$ $+ \alpha_4^2 x_1^2 x_3 + \alpha_2^1 x_1^2 x_4$ $+ \alpha_2^2 x_1 x_2^2 + \alpha_2^3 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_2 x_4 + \alpha_2^4 x_1 x_3^2$ $+ \alpha_0^2 x_1 x_3 x_4 + \alpha_0^3 x_2^3$ $+ \alpha_0^4 x_2^2 x_3 + \alpha_0^5 x_2 x_3^2$ $+ \alpha_0^6 x_3^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + xz$ $+ y^3 + y^2 z$ $+ yz^2 + z^3 = 0$	$-$	\mathbb{A}_3
T_{30}	$(4, 3, 1, 0)$	-6	$\alpha_6 x_1^3 + \alpha_5 x_1^2 x_2$ $+ \alpha_3^1 x_1^2 x_3 + \alpha_2^1 x_1^2 x_4$ $+ \alpha_4 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_1^1 x_1 x_2 x_4 + \alpha_0^1 x_1 x_3^2$ $+ \alpha_3^2 x_2^3 + \alpha_1^2 x_2^2 x_3$ $+ \alpha_0^2 x_2^2 x_4 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3$ $+ y^2 z + y^2 = 0$	$w_x = 2$ $w_y = 2$ $w_z = 1$	\mathbb{A}_3
T_{31}	$(4, 3, 2, 0)$	-7	$\alpha_5 x_1^3 + \alpha_4 x_1^2 x_2$ $+ \alpha_3^1 x_1^2 x_3 + \alpha_1^1 x_1^2 x_4$ $+ \alpha_2^2 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_2 x_4 + \alpha_1^2 x_1 x_3^2$ $+ \alpha_2^2 x_2^3 + \alpha_1^3 x_2^2 x_3$ $+ \alpha_0^2 x_2 x_3^2 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3$ $+ y^2 z + yz^2$ $= 0$	$w_x = 2$ $w_y = 2$ $w_z = 1$	\mathbb{A}_3
T_{32}	$(4, 3, 2, 1)$	-8	$\alpha_4 x_1^3 + \alpha_3 x_1^2 x_2$ $+ \alpha_2^1 x_1^2 x_3 + \alpha_1^1 x_1^2 x_4$ $+ \alpha_2^2 x_1 x_2^2 + \alpha_1^2 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_2 x_4 + \alpha_0^2 x_1 x_3^2$ $+ \alpha_1^3 x_2^3 + \alpha_0^3 x_2^2 x_3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3$ $+ y^2 z = 0$	$-$	\mathbb{A}_4

TABLE 2 CONTINUED

T_{33}	$(4, 3, 3, 0)$	-8	$\alpha_4 x_1^3 + \alpha_3 x_1^2 x_2$ $+ \alpha_3^2 x_1^2 x_3 + \alpha_0 x_1^2 x_4$ $+ \alpha_2^1 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_2^3 x_1 x_3^2 + \alpha_1^1 x_2^3$ $+ \alpha_1^2 x_2^2 x_3 + \alpha_1^3 x_2 x_3^2$ $+ \alpha_1^4 x_3^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 + y^2 z$ $+ yz^2 + z^3 = 0$	$w_x = 3$ $w_y = 2$ $w_z = 2$	\mathbb{D}_4
T_{34}	$(4, 3, 3, 1)$	-9	$\alpha_3 x_1^3 + \alpha_2^1 x_1^2 x_2$ $+ \alpha_2^2 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4$ $+ \alpha_1^1 x_1 x_2^2 + \alpha_1^2 x_1 x_2 x_3$ $+ \alpha_1^3 x_1 x_3^2 + \alpha_0^2 x_2^3$ $+ \alpha_0^3 x_2^2 x_3 + \alpha_0^4 x_2 x_3^2$ $+ \alpha_0^5 x_3^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 + y^2 z$ $+ yz^2 + z^3 = 0$	$w_x = 3$ $w_y = 2$ $w_z = 2$	\mathbb{D}_4
T_{35}	$(4, 4, 2, 0)$	-8	$\alpha_4^1 x_1^3 + \alpha_4^2 x_1^2 x_2$ $+ \alpha_2^1 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4$ $+ \alpha_4^3 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_0^2 x_1 x_2 x_4$ $+ \alpha_0^3 x_1 x_3^2 + \alpha_4^4 x_2^3$ $+ \alpha_2^3 x_2^2 x_3 + \alpha_0^4 x_2 x_4^2$ $+ \alpha_0^5 x_2 x_3^2 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3$ $+ y^2 z + y^2$ $+ yz^2 = 0$	$w_x = 2$ $w_y = 2$ $w_z = 1$	\mathbb{A}_3
T_{36}	$(5, 2, 0, 0)$	-5	$\alpha_{10} x_1^3 + \alpha_7 x_1^2 x_2$ $+ \alpha_5^1 x_1^2 x_3 + \alpha_5^2 x_1^2 x_4$ $+ \alpha_4 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3$ $+ \alpha_2^2 x_1 x_2 x_4 + \alpha_0^1 x_1 x_3^2$ $+ \alpha_0^2 x_1 x_3 x_4 + \alpha_0^3 x_1 x_4^2$ $+ \alpha_1 x_2^3 = 0$	$xQ(x, y, z)$ $+ y^3 = 0$ $(Q(0) \neq 0)$	$-$	Non-singular point
T_{37}	$(5, 2, 1, 0)$	-6	$\alpha_9 x_1^3 + \alpha_6 x_1^2 x_2$ $+ \alpha_5 x_1^2 x_3 + \alpha_4 x_1^2 x_4$ $+ \alpha_3 x_1 x_2^2 + \alpha_2 x_1 x_2 x_3$ $+ \alpha_1^1 x_1 x_2 x_4$ $+ \alpha_1^2 x_1 x_3^2 + \alpha_0^1 x_1 x_3 x_4$ $+ \alpha_0^2 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + xz$ $+ y^3 = 0$	$-$	\mathbb{A}_2

TABLE 2 CONTINUED

T_{38}	$(5, 3, 1, 0)$	-7	$\alpha_8 x_1^3 + \alpha_6 x_1^2 x_2$ $+ \alpha_4^1 x_1^2 x_3 + \alpha_3 x_1^2 x_4$ $+ \alpha_4^2 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3$ $+ \alpha_1 x_1 x_2 x_4 + \alpha_0^1 x_1 x_3^2$ $+ \alpha_2^2 x_2^3 + \alpha_0^2 x_2^2 x_3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3$ $+ y^2 z = 0$	$-$	\mathbb{A}_4
T_{39}	$(5, 3, 2, 0)$	-8	$\alpha_7 x_1^3 + \alpha_5 x_1^2 x_2$ $+ \alpha_4 x_1^2 x_3 + \alpha_2^1 x_1^2 x_4$ $+ \alpha_3 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_2 x_4 + \alpha_1^1 x_1 x_3^2$ $+ \alpha_1^2 x_2^3 + \alpha_0^2 x_2^2 x_3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3$ $+ y^2 z = 0$	$-$	\mathbb{A}_4
T_{40}	$(5, 3, 2, 1)$	-9	$\alpha_6 x_1^3 + \alpha_4 x_1^2 x_2$ $+ \alpha_3 x_1^2 x_3 + \alpha_2^1 x_1^2 x_4$ $+ \alpha_2^2 x_1 x_2^2 + \alpha_1 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_2 x_4 + \alpha_0^2 x_1 x_3^2$ $+ \alpha_0^3 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3 = 0$	$-$	\mathbb{A}_5
T_{41}	$(5, 3, 3, 0)$	-9	$\alpha_6 x_1^3 + \alpha_4^1 x_1^2 x_2$ $+ \alpha_4^2 x_1^2 x_3 + \alpha_1 x_1^2 x_4$ $+ \alpha_2^1 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_2^3 x_1 x_3^2 + \alpha_0^1 x_2^3$ $+ \alpha_0^2 x_2^2 x_3 + \alpha_0^3 x_2 x_3^2$ $+ \alpha_0^4 x_3^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 + y^2 z$ $+ yz^2 + z^3 = 0$	$w_x = 3$ $w_y = 2$ $w_z = 2$	\mathbb{D}_4
T_{42}	$(5, 4, 2, 0)$	-9	$\alpha_6 x_1^3 + \alpha_5 x_1^2 x_2$ $+ \alpha_3^1 x_1^2 x_3 + \alpha_1^1 x_1^2 x_4$ $+ \alpha_4 x_1 x_2^2 + \alpha_2 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_2 x_4 + \alpha_0^2 x_1 x_3^2$ $+ \alpha_3^2 x_2^3 + \alpha_1^2 x_2^2 x_3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3$ $+ y^2 z = 0$	$-$	\mathbb{A}_4
T_{43}	$(5, 4, 3, 0)$	-10	$\alpha_5 x_1^3 + \alpha_4 x_1^2 x_2$ $+ \alpha_3^1 x_1^2 x_3$ $+ \alpha_0^1 x_1^2 x_4 + \alpha_3^2 x_1 x_2^2$ $+ \alpha_2^1 x_1 x_2 x_3 + \alpha_1^1 x_1 x_3^2$ $+ \alpha_2^2 x_2^3 + \alpha_1^2 x_2^2 x_3$ $+ \alpha_0^2 x_2 x_3^2 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 + y^2 z$ $+ yz^2 = 0$	$w_x = 3$ $w_y = 2$ $w_z = 2$	\mathbb{D}_4

TABLE 2 CONTINUED

T_{44}	$(5, 4, 3, 1)$	-11	$\alpha_4 x_1^3 + \alpha_3 x_1^2 x_2$ $+ \alpha_2^1 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4$ $+ \alpha_2^2 x_1 x_2^2 + \alpha_1^1 x_1 x_2 x_3$ $+ \alpha_0^2 x_1 x_3^2 + \alpha_1^2 x_2^3$ $+ \alpha_0^3 x_2^2 x_3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 + y^2 z = 0$	$w_x = 4$ $w_y = 3$ $w_z = 2$	\mathbb{D}_5
T_{45}	$(6, 2, 0, 0)$	-6	$\alpha_{12} x_1^3 + \alpha_8 x_1^2 x_2$ $+ \alpha_6^1 x_1^2 x_3 + \alpha_6^2 x_1^2 x_4$ $+ \alpha_4 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3$ $+ \alpha_2^2 x_1 x_2 x_4 + \alpha_0^1 x_1 x_3^2$ $+ \alpha_0^2 x_1 x_3 x_4 + \alpha_0^3 x_1 x_4^2$ $+ \alpha_0^4 x_2^3 = 0$	$xQ(x, y, z)$ $+ y^3 = 0$ $(Q(0) \neq 0)$	$-$	Non-singular point
T_{46}	$(6, 3, 1, 0)$	-8	$\alpha_{10} x_1^3 + \alpha_7 x_1^2 x_2$ $+ \alpha_5 x_1^2 x_3 + \alpha_4^1 x_1^2 x_4$ $+ \alpha_4^2 x_1 x_2^2 + \alpha_2 x_1 x_2 x_3$ $+ \alpha_1^1 x_1 x_2 x_4 + \alpha_0 x_1 x_3^2$ $+ \alpha_1^2 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3 = 0$	$-$	\mathbb{A}_5
T_{47}	$(6, 3, 2, 0)$	-9	$\alpha_9 x_1^3 + \alpha_6 x_1^2 x_2$ $+ \alpha_5 x_1^2 x_3 + \alpha_3^1 x_1^2 x_4$ $+ \alpha_3^2 x_1 x_2^2 + \alpha_2 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_2 x_4 + \alpha_1 x_1 x_3^2$ $+ \alpha_0^2 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3 = 0$	$-$	\mathbb{A}_5
T_{48}	$(6, 4, 2, 0)$	-10	$\alpha_8 x_1^3 + \alpha_6 x_1^2 x_2$ $+ \alpha_4^1 x_1^2 x_3 + \alpha_2^1 x_1^2 x_4$ $+ \alpha_4^2 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_2 x_4 + \alpha_0^2 x_1 x_3^2$ $+ \alpha_2^3 x_2^3 + \alpha_0^3 x_2^2 x_3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3$ $+ y^2 z = 0$	$-$	\mathbb{A}_4
T_{49}	$(6, 4, 3, 0)$	-11	$\alpha_7 x_1^3 + \alpha_5 x_1^2 x_2$ $+ \alpha_4 x_1^2 x_3 + \alpha_1^1 x_1^2 x_4$ $+ \alpha_3 x_1 x_2^2 + \alpha_2 x_1 x_2 x_3$ $+ \alpha_2^2 x_1 x_3^2 + \alpha_1^3 x_2^3$ $+ \alpha_0 x_2^2 x_3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 + y^2 z = 0$	$w_x = 4$ $w_y = 3$ $w_z = 2$	\mathbb{D}_5

TABLE 2 CONTINUED

T_{50}	$(6, 4, 3, 1)$	-12	$\alpha_6 x_1^3 + \alpha_4 x_1^2 x_2$ $+ \alpha_3 x_1^2 x_3 + \alpha_1^1 x_1^2 x_4$ $+ \alpha_2 x_1 x_2^2 + \alpha_1^2 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_3^2 + \alpha_0^2 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
T_{51}	$(6, 4, 4, 0)$	-12	$\alpha_6 x_1^3 + \alpha_4^1 x_1^2 x_2$ $+ \alpha_4^2 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4$ $+ \alpha_2^1 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_2^3 x_1 x_3^2 + \alpha_0^1 x_2^3$ $+ \alpha_0^2 x_2^2 x_3 + \alpha_0^3 x_2 x_3^2$ $+ \alpha_0^4 x_3^3 = 0$	$x^2 + P_3(y, z)$ $+ xP_2(y, z) = 0$ $(P_i \text{ is a homo-}$ geneous poly- nomial of $\text{degree } i)$	$w_x = 3$ $w_y = 2$ $w_z = 2$	\mathbb{D}_4
T_{52}	$(6, 5, 3, 0)$	-12	$\alpha_6 x_1^3 + \alpha_5 x_1^2 x_2$ $+ \alpha_3^1 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4$ $+ \alpha_4 x_1 x_2^2 + \alpha_2 x_1 x_2 x_3$ $+ \alpha_0^2 x_1 x_3^2 + \alpha_3^2 x_2^3$ $+ \alpha_1 x_2^2 x_3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 + y^2 z = 0$	$w_x = 4$ $w_y = 3$ $w_z = 2$	\mathbb{D}_5
T_{53}	$(7, 3, 1, 0)$	-9	$\alpha_{12} x_1^3 + \alpha_8 x_1^2 x_2$ $+ \alpha_6 x_1^2 x_3 + \alpha_5 x_1^2 x_4$ $+ \alpha_4 x_1 x_2^2 + \alpha_2 x_1 x_2 x_3$ $+ \alpha_1 x_1 x_2 x_4 + \alpha_0^1 x_1 x_3^2$ $+ \alpha_0^2 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3 = 0$	$-$	\mathbb{A}_5
T_{54}	$(7, 4, 2, 0)$	-11	$\alpha_{10} x_1^3 + \alpha_7 x_1^2 x_2$ $+ \alpha_5 x_1^2 x_3 + \alpha_3 x_1^2 x_4$ $+ \alpha_4 x_1 x_2^2 + \alpha_2 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_2 x_4 + \alpha_0^2 x_1 x_3^2$ $+ \alpha_1 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3 = 0$	$-$	\mathbb{A}_5
T_{55}	$(7, 4, 3, 0)$	-12	$\alpha_9 x_1^3 + \alpha_6 x_1^2 x_2$ $+ \alpha_5 x_1^2 x_3 + \alpha_2^1 x_1^2 x_4$ $+ \alpha_3 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_1 x_1 x_3^2 + \alpha_0 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
T_{56}	$(7, 5, 3, 0)$	-13	$\alpha_8 x_1^3 + \alpha_6 x_1^2 x_2$ $+ \alpha_4^1 x_1^2 x_3 + \alpha_1^1 x_1^2 x_4$ $+ \alpha_4^2 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3$ $+ \alpha_0 x_1 x_3^2 + \alpha_2^2 x_2^3$ $+ \alpha_1^2 x_2^2 x_3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 + y^2 z = 0$	$w_x = 4$ $w_y = 3$ $w_z = 2$	\mathbb{D}_5

TABLE 2 CONTINUED

T_{57}	$(7, 5, 4, 0)$	-14	$\alpha_7 x_1^3 + \alpha_5 x_1^2 x_2$ $+ \alpha_4 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4$ $+ \alpha_3 x_1 x_2^2 + \alpha_2 x_1 x_2 x_3$ $+ \alpha_1^1 x_1 x_3^2 + \alpha_1^2 x_2^3$ $+ \alpha_0^2 x_2^2 x_3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 + y^2 z = 0$	$w_x = 4$ $w_y = 3$ $w_z = 2$	\mathbb{D}_5
T_{58}	$(7, 5, 4, 1)$	-15	$\alpha_6 x_1^3 + \alpha_4 x_1^2 x_2$ $+ \alpha_3 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4$ $+ \alpha_2 x_1 x_2^2 + \alpha_1 x_1 x_2 x_3$ $+ \alpha_0^2 x_1 x_3^2 + \alpha_0^3 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
T_{59}	$(8, 4, 2, 0)$	-12	$\alpha_{12} x_1^3 + \alpha_8 x_1^2 x_2$ $+ \alpha_6 x_1^2 x_3 + \alpha_4^1 x_1^2 x_4$ $+ \alpha_4^2 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3$ $+ \alpha_2^2 x_1 x_2 x_4 + \alpha_0^1 x_1 x_3^2$ $+ \alpha_0^2 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xy$ $+ xz^2 + y^3 = 0$	$-$	\mathbb{A}_5
T_{60}	$(8, 5, 3, 0)$	-14	$\alpha_{10} x_1^3 + \alpha_7 x_1^2 x_2$ $+ \alpha_5 x_1^2 x_3 + \alpha_2^1 x_1^2 x_4$ $+ \alpha_4 x_1 x_2^2 + \alpha_2^2 x_1 x_2 x_3$ $+ \alpha_0 x_1 x_3^2 + \alpha_1 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
T_{61}	$(8, 5, 4, 0)$	-15	$\alpha_9 x_1^3 + \alpha_6 x_1^2 x_2$ $+ \alpha_5 x_1^2 x_3 + \alpha_1^1 x_1^2 x_4$ $+ \alpha_3 x_1 x_2^2 + \alpha_2 x_1 x_2 x_3$ $+ \alpha_1^2 x_1 x_3^2 + \alpha_0 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
T_{62}	$(8, 6, 4, 0)$	-16	$\alpha_8 x_1^3 + \alpha_6 x_1^2 x_2$ $+ \alpha_4^1 x_1^2 x_3 + \alpha_0^1 x_1^2 x_4$ $+ \alpha_4^2 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3$ $+ \alpha_0^2 x_1 x_3^2 + \alpha_2^2 x_2^3$ $+ \alpha_0^3 x_2^2 x_3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 + y^2 z = 0$	$w_x = 4$ $w_y = 3$ $w_z = 2$	\mathbb{D}_5
T_{63}	$(9, 5, 3, 0)$	-15	$\alpha_{12} x_1^3 + \alpha_8 x_1^2 x_2$ $+ \alpha_6 x_1^2 x_3 + \alpha_3 x_1^2 x_4$ $+ \alpha_4 x_1 x_2^2 + \alpha_2 x_1 x_2 x_3$ $+ \alpha_0^1 x_1 x_3^2 + \alpha_0^2 x_2^3 = 0$	$x^3 + x^2 y + x^2 z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6

TABLE 2 CONTINUED

T_{64}	$(9, 6, 4, 0)$	-17	$\alpha_{10}x_1^3 + \alpha_7x_1^2x_2$ $+ \alpha_5x_1^2x_3 + \alpha_1^1x_1^2x_4$ $+ \alpha_4x_1x_2^2 + \alpha_2x_1x_2x_3$ $+ \alpha_0x_1x_3^2 + \alpha_1^2x_2^3 = 0$	$x^3 + x^2y + x^2z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
T_{65}	$(9, 6, 5, 0)$	-18	$\alpha_9x_1^3 + \alpha_6x_1^2x_2$ $+ \alpha_5x_1^2x_3 + \alpha_6^1x_1^2x_4$ $+ \alpha_3x_1x_2^2 + \alpha_2x_1x_2x_3$ $+ \alpha_1x_1x_3^2 + \alpha_0^2x_2^3 = 0$	$x^3 + x^2y + x^2z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
T_{66}	$(10, 6, 4, 0)$	-18	$\alpha_{12}x_1^3 + \alpha_8x_1^2x_2$ $+ \alpha_6x_1^2x_3 + \alpha_2^1x_1^2x_4$ $+ \alpha_4x_1x_2^2 + \alpha_2^2x_1x_2x_3$ $+ \alpha_0^1x_1x_3^2 + \alpha_0^2x_2^3 = 0$	$x^3 + x^2y + x^2z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
T_{67}	$(10, 7, 5, 0)$	-20	$\alpha_{10}x_1^3 + \alpha_7x_1^2x_2$ $+ \alpha_5x_1^2x_3 + \alpha_6^1x_1^2x_4$ $+ \alpha_4x_1x_2^2 + \alpha_2x_1x_2x_3$ $+ \alpha_0^2x_1x_3^2 + \alpha_1x_2^3 = 0$	$x^3 + x^2y + x^2z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
T_{68}	$(11, 7, 5, 0)$	-21	$\alpha_{12}x_1^3 + \alpha_8x_1^2x_2$ $+ \alpha_6x_1^2x_3 + \alpha_1^1x_1^2x_4$ $+ \alpha_4x_1x_2^2 + \alpha_2x_1x_2x_3$ $+ \alpha_0x_1x_3^2 + \alpha_1^2x_2^3 = 0$	$x^3 + x^2y + x^2z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6
T_{69}	$(12, 8, 6, 0)$	-24	$\alpha_{12}x_1^3 + \alpha_8x_1^2x_2$ $+ \alpha_6x_1^2x_3 + \alpha_6^1x_1^2x_4$ $+ \alpha_4x_1x_2^2 + \alpha_2x_1x_2x_3$ $+ \alpha_0^2x_1x_3^2 + \alpha_0^3x_2^3 = 0$	$x^3 + x^2y + x^2z$ $+ x^2 + xy^2$ $+ xyz + xz^2$ $+ y^3 = 0$	$w_x = 6$ $w_y = 4$ $w_z = 3$	\mathbb{E}_6

Thus Theorem 1.6 is proved.

Remark 4.15. The proof of Theorem 1.6 gives a description of the possible singularities of the 3-folds T_j . For example, sufficiently general 3-folds T_j are smooth for $j \in \{1, 2, 5, 8, 14\}$ and have only isolated ordinary double points for $j \in \{4, 7, 11, 16, 19\}$. The smooth trigonal 3-folds T_j are well known (see [15], [88]). On the other hand, the 3-fold T_j always has non-isolated singularities for $j \in \{6, 9, 13, 15, 17, 21, 24, 25, 26, 28, 32, 34, 36, 40, 41, 44, 45, 50, 58\}$. In all other cases, the 3-fold T_j has at least one non-cDV-point.

Remark 4.16. In the case of T_{15} the variety V is always singular along the curve $x_1 = x_2 = \alpha_1^1x_3^2 + \alpha_1^2x_3x_4 + \alpha_1^3x_4^2 = 0$. In the case of T_{17} it is singular along the curve $x_1 = x_2 = \alpha_0^1x_3^2 + \alpha_0^2x_3x_4 + \alpha_0^3x_4^2 = 0$. In the case of T_{19} it is singular along

the curve $x_1 = x_2 = \alpha_0^1 x_3^2 + \alpha_0^2 x_3 x_4 + \alpha_0^3 x_4^2 = 0$. In the case of T_{25} it is singular along the curve $x_1 = x_2 = \alpha_1^1 x_3^2 + \alpha_1^2 x_3 x_4 + \alpha_1^3 x_4^2 = 0$. In the case of T_{26} it is singular along the curve $x_1 = x_2 = \alpha_0^1 x_3^2 + \alpha_0^2 x_3 x_4 + \alpha_0^3 x_4^2 = 0$. In the case of T_{28} it is singular along the curve $x_1 = x_2 = \alpha_0^1 x_3^2 + \alpha_0^2 x_3 x_4 + \alpha_0^3 x_4^2 = 0$. In the case of T_{36} it is singular along the curve $x_1 = x_2 = \alpha_0^1 x_3^2 + \alpha_0^2 x_3 x_4 + \alpha_0^3 x_4^2 = 0$. In the case of T_{45} it is singular along the curve $x_1 = x_2 = \alpha_0^1 x_3^2 + \alpha_0^2 x_3 x_4 + \alpha_0^3 x_4^2 = 0$. All of these curves are bisections of the corresponding projections $\varphi: \mathbb{F}(d_1, d_2, d_3, d_4) \rightarrow \mathbb{P}^1$. They are reducible in the cases of T_{17} , T_{19} , T_{26} , T_{28} , T_{36} and T_{45} . This simple observation will enable us to apply Lemma 5.2 to these varieties and prove their rationality.

Remark 4.17. In the case of T_7 , the linear system $|M - L|$ determines a birational map $\psi: V \dashrightarrow \mathbb{P}^3$, which may be factorized as $\psi = \omega \circ \gamma \circ \beta$. Here β flops the curve C , γ contracts the strict transform of the surface with equation $x_1 = 0$ (on V) onto a smooth rational curve whose image on \mathbb{P}^3 is a line, and ω is a double covering of \mathbb{P}^3 branched over a non-singular quartic surface. In particular, V is birationally isomorphic to a hypersurface of degree 4 in $\mathbb{P}(1^4, 2)$. The latter variety is also known as a double space of index two. It was studied in [54], [31], [32], [37], [33], [60] and [34].

Remark 4.18. In the cases of T_4 and T_6 , the variety $X \subset \mathbb{P}^6$ is an anticanonically embedded Fano variety with canonical Gorenstein singularities and with $(-K_X)^3 = 8$ (compare [88], Statement 4.1.12).

Remark 4.19. One can simplify the proof of Theorem 1.6 by arguing as follows. If X is a del Pezzo surface of degree 3 over some field with a non-Du Val singular point defined over this field, then X is a cone. The authors did not use this approach for the reasons pointed out in Remark 3.12.

§ 5. Rationality and non-rationality

In this section we prove Proposition 1.10. Let H_i and T_j be the Fano 3-folds in Theorems 1.5 and 1.6 respectively. The non-rationality of sufficiently general 3-folds $H_1, H_2, H_3, H_4, H_6, T_1, T_2, T_7, T_8$ certainly follows (see Remark 1.8 and Example 1.11) from the results of [22], [62], [54], [37], [16], [31]–[33], [60], [38], [52], [34], [28], [29], [61], [111], [5], [82], [63], [98], [64], [6], [102], [7], [59]. On the other hand, it is clear that the 3-folds H_9, T_5 and T_{14} are always rational (see Remark 1.8).

We may thus assume that $i \notin \{1, 2, 3, 4, 6, 9\}$ and $j \notin \{1, 2, 5, 7, 8, 14\}$. Then the 3-fold H_i is naturally birationally equivalent to a del Pezzo fibration $\tau: Y_i \rightarrow \mathbb{P}^1$ of degree 2 (see Theorem 1.5) with canonical Gorenstein singularities, and the 3-fold T_j is naturally birationally equivalent to a del Pezzo fibration $\psi: V_j \rightarrow \mathbb{P}^1$ of degree 3 (see Theorem 1.6) with canonical Gorenstein singularities. Let \overline{Y}_i and \overline{V}_j be generic fibres of τ and ψ respectively. Then \overline{Y}_i and \overline{V}_j are del Pezzo surfaces with Du Val singularities defined over the field $\mathbb{C}(x)$.

Remark 5.1. The rationality of the surfaces \overline{Y}_i and \overline{V}_j over $\mathbb{C}(x)$ implies that of the 3-folds Y_i and V_j respectively.

The del Pezzo surfaces \overline{Y}_i and \overline{V}_j always have a $\mathbb{C}(x)$ -point by Theorem 2.24. Moreover, the sets of their $\mathbb{C}(x)$ -points are huge by Theorem 2.25.

Lemma 5.2. *Let S be a del Pezzo surface of degree 3 with canonical singularities defined over an arbitrary perfect field \mathbb{F} . Suppose that the set $\text{Sing}(S)$ contains an \mathbb{F} -point $O \in S$. Then S is rational over \mathbb{F} .*

Proof. The surface S is a cubic hypersurface in \mathbb{P}^3 (see [23], [25], [20], [95]). Thus the projection from O gives a birational map to \mathbb{P}^2 .

Therefore the proof of Theorem 1.6 along with Lemma 5.2 immediately yields the rationality of the 3-fold T_j for $j \in \{10, 12, 13, 17, \dots, 24, 26, \dots, 69\}$.

Lemma 5.3. *Let S be a del Pezzo surface of degree 2 with at most Du Val singularities and defined over an arbitrary perfect field \mathbb{F} . Suppose that the singularity set $\text{Sing}(S)$ contains an \mathbb{F} -point $O \in S$ which is locally isomorphic to one of the following Du Val points: \mathbb{E}_6 , \mathbb{E}_7 , \mathbb{D}_n or \mathbb{A}_k for $n \geq 5$ and $k \geq 7$. Then S is rational over \mathbb{F} .*

Proof. Let $f: W \rightarrow S$ be a minimal resolution of singularities of S , and let $E = f^{-1}(O) \subset W$ be a connected curve defined over \mathbb{F} . Then $K_W \sim f^*(K_S)$. In particular, W is a weak del Pezzo surface (see [67]) of degree 2, the curve E is $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ -invariant, and all the irreducible components of E that are defined over $\overline{\mathbb{F}}$ must split into disjoint $\text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ -orbits. However, the irreducible components of E form a graph of type \mathbb{E}_6 , \mathbb{E}_7 , \mathbb{D}_n or \mathbb{A}_k for $n \geq 5$ and $k \geq 7$. Therefore the curve E splits into at least 4 (possibly reducible) curves defined over \mathbb{F} . Since the intersection form of irreducible components of E is negative (see [51]), it follows that the rank of $\text{Pic}(W)$ is at least 5.

There is a birational morphism $g: W \rightarrow U$ defined over \mathbb{F} such that the surface U is minimal (see [14], [103], [20]), that is, no curve on U can be contracted to a smooth point. Moreover, the rank of $\text{Pic}(U)$ does not exceed 2 by Theorem 2.22. Therefore $K_U^2 \geq K_W^2 + 3 = 5$. Thus the surface U is rational over \mathbb{F} by Theorem 2.23.

Lemma 5.3 and the proof of Theorem 1.5 imply that the hyperelliptic 3-folds H_i are rational for $i \in \{22, 26, 27, 28, 29, 31, \dots, 47\}$.

Remark 5.4. The non-rationality of the surfaces \overline{Y}_i and \overline{V}_j over $\mathbb{C}(x)$ does not imply the non-rationality of the 3-folds H_i and T_j respectively. However we believe that the rough method used above can be also applied to prove the non-rationality of H_i in many of the remaining cases. For example, one can try to use the proofs of Theorems 1.5 and 1.6 to describe the geometry of the surfaces \overline{Y}_i and \overline{V}_j in more detail and then use the results of [26], [35] and [20].

Proposition 5.5. *Let X be a sufficiently general² Fano 3-fold T_3 in Theorem 1.6. Then X is non-rational.*

Proof. Suppose that $U = \text{Proj}(\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2})$, $f: U \rightarrow \mathbb{P}^2$ is the natural projection, T is the tautological line bundle on U , and $F = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Then X is an anticanonical image of a sufficiently general divisor $V \in |2T + F|$. The 3-fold V is smooth by Bertini's theorem. Moreover, Lefschetz' theorem (see [55], [50]) implies that $\text{Pic}(V) \cong \mathbb{Z} \oplus \mathbb{Z}$.

²Here and in what follows we always understand “general” as “belonging to a Zariski open subset of the moduli space” unless otherwise specified.

Let $g: V \rightarrow \mathbb{P}^2$ be the restriction of the projection $f: U \rightarrow \mathbb{P}^2$. Then g is a conic bundle. Let Δ be the degeneration divisor of g , and let Y be a sufficiently general surface in the linear system $|g^*(\mathcal{O}_{\mathbb{P}^2}(1))|$. Then Y is smooth and $K_Y^2 = 1$ by the adjunction formula. Therefore the conic bundle $g|_Y$ has 7 reducible fibres. Thus the degree of the divisor $\Delta \subset \mathbb{P}^2$ is equal to 7 (see [56], § 3.5), and V is non-rational by Theorem 2.16.

Proposition 5.6. *Let X be a sufficiently general³ 3-fold H_5 in Theorem 1.5. Then X is non-rational.*

Proof. The 3-fold X is an anticanonical model of a smooth weak Fano 3-fold V , which may be described as a double covering $\pi: V \rightarrow U = \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$ branched over a divisor $D \in |4M - 2L|$, where M is the tautological line bundle on U and L is a fibre of the natural projection of U to \mathbb{P}^1 . The divisor D may be given in the bihomogeneous coordinates (see Proposition 2.19) by the zeros of the bihomogeneous polynomial

$$\begin{aligned} & \alpha_6 x_1^4 + \alpha_5 x_1^3 x_2 + \alpha_4 x_1^3 x_3 + \alpha_4^2 x_1^2 x_2^2 + \alpha_3 x_1^2 x_2 x_3 + \alpha_2 x_1^2 x_3^2 + \alpha_3^2 x_1 x_2^3 \\ & + \alpha_2^2 x_1 x_2^2 x_3 + \alpha_1 x_1 x_2 x_3^2 + \alpha_6^2 x_1 x_3^3 + \alpha_2^3 x_2^4 + \alpha_1^2 x_2^3 x_3 + \alpha_6^2 x_2^2 x_3^2, \end{aligned}$$

where $\alpha_d^i = \alpha_d^i(t_1, t_2)$ is a homogeneous polynomial of degree d .

Consider a double covering $\chi: Y \rightarrow U$ branched over a sufficiently general divisor $\Delta \subset U$ which is given by the zeros of the same bihomogeneous polynomial as D with the only difference being that $\alpha_6^1 = 0$. Then Y is not smooth because Δ has singularities along the curve $Y_3 \subset U$ given by $x_1 = x_2 = 0$. The curve Y_3 is the smallest negative subscroll of U (see Proposition 2.19). We may assume that $\Delta \subset U$ is a sufficiently general element of the linear subsystem of the system $|4M - 2L|$ consisting of all divisors with singularities along Y_3 . The divisor Δ is smooth outside Y_3 by Bertini's theorem.

Put $C = \chi^{-1}(Y_3)$. Then the 3-fold Y has singularities of type $\mathbb{A}_1 \times \mathbb{C}$ at the general point of the curve C . Moreover, the singularities of Y at other points of C are locally isomorphic to the singularity

$$x^2 + y^2 + z^2 t = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x, y, z, t]),$$

where the curve C is given locally by $x = y = z = 0$. It follows that one can resolve the singularities of Y by one blow-up $f: \tilde{Y} \rightarrow Y$ of the curve C .

Let $g: \tilde{U} \rightarrow U$ be the blow-up of the curve $Y_3 \subset U$. Then the diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\chi}} & \tilde{U} \\ f \downarrow & & \downarrow g \\ Y & \xrightarrow{\chi} & U \end{array}$$

is commutative, where $\tilde{\chi}: \tilde{Y} \rightarrow \tilde{U}$ is a double covering. Let E be the exceptional divisor of g . Then $\tilde{\chi}$ is branched over the divisor $g^{-1}(\Delta) \sim g^*(4M - 2L) - 2E$.

³The complement of a countable union of Zariski-closed subsets in the moduli space.

In the case when the divisor $g^{-1}(\Delta)$ is ample on \tilde{U} , Lefschetz' theorem (see [55], [50], [121]) implies that $\text{Pic}(\tilde{Y}) \cong \text{Pic}(\tilde{U}) \cong \mathbb{Z}^3$ (see [60], [65], [66]). However, the divisor $g^{-1}(\Delta)$ is not ample, although it is numerically effective and big. Indeed, the linear system $|g^*(M - L) - E|$ is free and the linear system $|g^*(M) - E|$ gives a \mathbb{P}^1 -bundle

$$\tau: \tilde{U} \rightarrow \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathbb{F}_1.$$

Therefore the divisor $g^{-1}(\Delta) \sim g^*(4M - 2L) - 2E$ is numerically effective and big. Hence we can replace Lefschetz' theorem by the first part of the proof of Proposition 32 in [56] to get $\text{Pic}(\tilde{Y}) \cong \text{Pic}(\tilde{U}) \cong \mathbb{Z}^3$.

Let $Y_2 \subset U$ be the largest negative subscroll (see Proposition 2.19). The surface Y_2 is given by the equation $x_1 = 0$ in the bihomogeneous coordinates on U . Moreover, $Y_2 \cong \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$. Put $S = g^{-1}(Y_2)$. Then $S \cong Y_2$ and the morphism τ contracts the surface S to the exceptional section of \mathbb{F}_1 .

By construction, the \mathbb{P}^1 -bundle τ induces a conic bundle $\tilde{\tau} = \tau \circ \tilde{\chi}: \tilde{Y} \rightarrow \mathbb{F}_1$. Put $\tilde{S} = \tilde{\chi}^{-1}(S)$, and let $Z \subset \tilde{Y}$ be a general fibre of the natural projection of \tilde{Y} to \mathbb{P}^1 . Then Z is a smooth weak del Pezzo surface of degree 2, that is, $-K_Z$ is numerically effective and big and $K_Z^2 = 2$. Moreover, the morphism $g \circ \tilde{\chi}|_{\tilde{S}}: \tilde{S} \rightarrow Y_2$ is a double covering branched over a divisor with the following equation in bihomogeneous coordinates:

$$\alpha_2^3(t_0, t_1) x_2^2 + \alpha_1^2(t_0, t_1) x_2 x_3 + \alpha_0^2(t_0, t_1) x_3^2 = 0,$$

where $\alpha_d^i(t_1, t_2)$ is the homogeneous polynomial of degree d in the bihomogeneous equation of Δ .

Let $\Xi \subset \mathbb{F}_1$ be the degeneration divisor of the conic bundle $\tilde{\tau}$. Then $\Xi \sim 6s_\infty + al$, where s_∞ is the exceptional section of \mathbb{F}_1 , l is a fibre of the projection of \mathbb{F}_1 to \mathbb{P}^1 , and $a \in \mathbb{Z}$. The structure of the morphism $g \circ \tilde{\chi}|_{\tilde{S}}$ implies that $s_\infty \not\subset \Xi$. Moreover, the intersection $s_\infty \cdot \Xi$ is equal to the number of reducible fibres of the induced conic bundle $\tilde{\tau}|_{\tilde{S}}$. This number can easily be calculated from the bihomogeneous equation of the ramification divisor of $g \circ \tilde{\chi}|_{\tilde{S}}$. More precisely, the reducible fibres of $\tilde{\tau}|_{\tilde{S}}$ correspond to zeros of the discriminant $(\alpha_1^2)^2 - 4\alpha_0^2\alpha_2^3$, whence $s_\infty \cdot \Xi = 2$. Therefore $a = 8$. Thus Y is non-rational by Theorem 2.16.

The 3-fold Y is rationally connected (see [95]). Thus the non-rationality of Y implies that Y is non-ruled as well. Therefore the 3-fold V is non-ruled by Theorem 2.18 because we assumed V to be sufficiently general. Hence X is non-rational.

Proposition 5.7. *Let X be a sufficiently general 3-fold H_7 from Theorem 1.5. Then X is non-rational.*

Proof. The 3-fold X is an anticanonical model of a weak Fano 3-fold V such that there is a double covering

$$\pi: V \rightarrow U = \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}),$$

branched over a divisor $D \in |4M - 4L|$, where M is the tautological line bundle on U and L is a fibre of the natural projection of U to \mathbb{P}^1 . The divisor D may be given in bihomogeneous coordinates (see Proposition 2.19) by the zeros of a

bihomogeneous polynomial

$$\begin{aligned} & \alpha_4^1 x_1^4 + \alpha_4^2 x_1^3 x_2 + \alpha_4^3 x_1^2 x_2^2 + \alpha_4^4 x_1 x_2^3 + \alpha_4^5 x_2^4 + \alpha_2^1 x_1^3 x_3 + \alpha_2^2 x_1^2 x_2 x_3 \\ & + \alpha_2^3 x_1 x_2^2 x_3 + \alpha_2^4 x_2^3 x_3 + \alpha_0^1 x_1^2 x_3^2 + \alpha_0^2 x_1 x_2 x_3^2 + \alpha_0^3 x_2^2 x_3^2, \end{aligned}$$

where $\alpha_d^i = \alpha_d^i(t_1, t_2)$ is a homogeneous polynomial of degree d .

The divisor D has singularities along the curve $Y_3 \subset U$ given by $x_1 = x_2 = 0$. Since X is general, the divisor $D \subset U$ is a sufficiently general element of the linear system $|4M - 4L|$. In particular, D is smooth outside Y_3 by Bertini's theorem. The 3-fold V has singularities of the type $\mathbb{A}_1 \times \mathbb{C}$ at a general point of the curve $C = \chi^{-1}(Y_3)$. Moreover, one can resolve the singularities of V by one blow-up $f: \tilde{V} \rightarrow V$ of the curve C .

Let $g: \tilde{U} \rightarrow U$ be the blow-up of the curve $Y_3 \subset U$. Then the diagram

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\tilde{\pi}} & \tilde{U} \\ f \downarrow & & \downarrow g \\ V & \xrightarrow{\pi} & U \end{array}$$

is commutative, where the morphism $\tilde{\pi}: \tilde{V} \rightarrow \tilde{U}$ is a double covering. Let E be the exceptional divisor of g . Then $\tilde{\pi}$ is branched over the divisor $g^{-1}(D) \sim g^*(4M - 4L) - 2E$. On the other hand, the linear system $|g^*(M - 2L) - E|$ is a free pencil whose image on U is generated by the divisors $x_1 = 0$ and $x_2 = 0$. In particular, the divisor $g^{-1}(D) \sim g^*(4M - 4L) - 2E$ is numerically effective and big on \tilde{U} . Then the first part of the proof of Proposition 32 in [56] (a stronger version of Lefschetz' theorem) implies that

$$\text{Pic}(\tilde{V}) \cong \text{Pic}(\tilde{U}) \cong \mathbb{Z}^3.$$

The linear system $|g^*(M - L) - E|$ is also free and gives a \mathbb{P}^1 -bundle

$$\tau: \tilde{U} \rightarrow \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \cong \mathbb{F}_0.$$

The rational map $\tau \circ g^{-1}$ is given in bihomogeneous coordinates by the linear system on U spanned by $\beta_1(t_0, t_2)x_1 + \beta_2(t_0, t_2)x_2$, where $\beta_i(t_0, t_2)$ is a homogeneous polynomial of degree 1.

The \mathbb{P}^1 -bundle τ induces a conic bundle $\tilde{\tau} = \tau \circ \tilde{\pi}: \tilde{V} \rightarrow \mathbb{F}_0$. Let $\Delta \subset \mathbb{F}_0$ be the degeneration divisor of $\tilde{\tau}$, and let L_1, L_2 be fibres of the two projections of \mathbb{F}_0 to \mathbb{P}^1 such that $\tau^*(L_1) \sim g^*(L)$ and $\tau^*(L_2) \sim g^*(M - 2L) - E$. Then $\Delta \sim nL_1 + 6L_2$ for some $n \in \mathbb{Z}$. Moreover, we have $n = 4$ by elementary calculations (see the proof of Proposition 5.6). Hence V is non-rational by Theorem 2.16.

Proposition 5.8. *Let X be a 3-fold H_8 in Theorem 1.5. Then X is rational.*

Proof. Arguing as in the proof of Proposition 5.7, we get a conic bundle $\tilde{\tau} = \tau \circ \tilde{\pi}: \tilde{V} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$, where \tilde{V} is birationally isomorphic to X . Moreover, this case is simpler since the proof of rationality of \tilde{V} does not require a proof that the conic bundle $\tilde{\tau}$ is standard, that is, that $\text{Pic}(\tilde{V}) \cong \mathbb{Z}^3$. Simple calculations show that the degeneration divisor $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ of $\tilde{\tau}$ has bidegree $(6, 2)$. Now the rationality of X follows immediately from Theorems 2.24, 2.22 and 2.23.

Proposition 5.9. *Let X be a sufficiently general 3-fold T_4 in Theorem 1.6. Then X is non-rational.*

Proof. The 3-fold X is an anticanonical image of a sufficiently general divisor

$$V \subset U = \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$$

belonging to the linear system $|3M - L|$, where M is the tautological line bundle on U , and L is a fibre of the natural projection of U to \mathbb{P}^1 . The divisor D is given in bihomogeneous coordinates on U by

$$\begin{aligned} & \alpha_2^1 x_1^3 + \alpha_2^2 x_1^2 x_2 + \alpha_2^3 x_1^2 x_3 + \alpha_2^4 x_1^2 x_4 + \alpha_2^5 x_1 x_2 x_3 + \alpha_2^6 x_1 x_2 x_4 + \alpha_2^7 x_2^3 \\ & + \alpha_2^8 x_2^2 x_3 + \alpha_2^9 x_2^2 x_4 + \alpha_2^{10} x_3^3 + \alpha_1^1 x_1^2 x_4 + \alpha_1^2 x_1 x_2 x_4 + \alpha_1^3 x_1 x_3 x_4 \\ & + \alpha_1^4 x_1^2 x_4 + \alpha_1^5 x_2 x_3 x_4 + \alpha_1^6 x_3^2 x_4 + \alpha_0^1 x_1 x_4^2 + \alpha_0^2 x_2 x_4^2 + \alpha_0^3 x_3 x_4^2 = 0, \end{aligned}$$

where $\alpha_d^i = \alpha_d^i(t_0, t_1)$ is a homogeneous polynomial of degree d . Since X is general, V is smooth. Moreover, the anticanonical morphism $\varphi_{|-K_V|}$ contracts a single curve $C \subset V$ (given by $x_1 = x_2 = x_3 = 0$) to an ordinary double point O on X . The corresponding birational morphism $\varphi_{|M|}$ maps the rational scroll U to the cone \overline{U} over $\mathbb{P}^1 \times \mathbb{P}^2$ with vertex O .

The 3-fold X and the 4-fold \overline{U} are not \mathbb{Q} -factorial. Moreover, the birational morphisms $\varphi_{|-K_V|}$ and $\varphi_{|M|}$ may be regarded as \mathbb{Q} -factorializations of X and \overline{U} respectively (see [92]). There are also other ways to \mathbb{Q} -factorialize X and \overline{U} . Namely, one can find a scroll $\tilde{U} = \text{Proj}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ and a birational morphism $\varphi_{|T|}: \tilde{U} \rightarrow \overline{U}$, where T is the tautological line bundle on \tilde{U} . Moreover, the birational map $\varphi_{|T|}^{-1} \circ \varphi_{|M|}$ is an antiflip (see [93], [99]) in the curve $C \subset U$.

Let $Y \subset \tilde{U}$ be the proper transform of X on the 4-fold \tilde{U} . Then Y is a smooth weak Fano 3-fold and $Y \sim 2T + F$ for $F = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$, where f is the natural projection of \tilde{U} to \mathbb{P}^2 . The original 3-fold X is an anticanonical image of the 3-fold Y , and the birational map $\varphi_{|-K_Y|}^{-1} \circ \varphi_{|-K_V|}$ is a simple flop in the curve $C \subset V$ induced by the antiflip $\varphi_{|T|}^{-1} \circ \varphi_{|M|}$. Lefschetz' theorem implies that $\text{Pic}(Y) \cong \mathbb{Z} \oplus \mathbb{Z}$.

The restriction $g: Y \rightarrow \mathbb{P}^2$ of the projection $f: \tilde{U} \rightarrow \mathbb{P}^2$ is a conic bundle. Let Δ be the degeneration divisor of g . Simple calculations (see the proof of Proposition 5.5) imply that $\Delta \sim \mathcal{O}_{\mathbb{P}^2}(7)$ (see [56], §4.4.1). Therefore the 3-fold Y is non-rational by Theorem 2.16 (see [54], [37]).

Proposition 5.10. *Let X be a sufficiently general 3-fold T_6 in Theorem 1.6. Then X is non-rational.*

Proof. The 3-fold X is an anticanonical image of a weak Fano 3-fold V , which may be regarded as a sufficiently general divisor on the rational scroll $U = \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$ lying in the linear system $|3M - L|$, where M is the tautological line bundle on U and L is a fibre of the natural projection of U to \mathbb{P}^1 . Thus the 3-fold V is given in bihomogeneous coordinates on U by

$$\begin{aligned} & \alpha_5 x_1^3 + \alpha_4 x_1^2 x_2 + \alpha_3^1 x_1^2 x_3 + \alpha_3^2 x_1^2 x_4 + \alpha_3^3 x_1 x_2^2 + \alpha_2^1 x_1 x_2 x_3 \\ & + \alpha_2^2 x_1 x_2 x_4 + \alpha_1^1 x_1 x_3^2 + \alpha_1^2 x_1 x_3 x_4 + \alpha_1^3 x_1 x_4^2 + \alpha_2^3 x_2^3 \\ & + \alpha_1^4 x_2^2 x_3 + \alpha_1^5 x_2^2 x_4 + \alpha_0^1 x_2 x_3^2 + \alpha_0^2 x_2 x_3 x_4 + \alpha_0^3 x_2 x_4^2 = 0, \end{aligned}$$

where $\alpha_d^i = \alpha_d^i(t_0, t_1)$ is a homogeneous polynomial of degree d . The 3-fold V contains a surface $Y_3 \cong \mathbb{P}^1 \times \mathbb{P}^1$ given by $x_1 = x_2 = 0$. This surface is the base locus of the linear system $|3M - L|$. However, V is smooth at the general point of Y_3 . On the other hand, V is always singular at the points where

$$x_1 = x_2 = \alpha_1^1 x_3^2 + \alpha_1^2 x_3 x_4 + \alpha_1^3 x_4^2 = \alpha_0^1 x_3^2 + \alpha_0^2 x_3 x_4 + \alpha_0^3 x_4^2 = 0.$$

Since V is general, it follows that these points are ordinary double points on V and V is smooth outside them.

Let $g: \tilde{U} \rightarrow U$ be the blow-up of $Y_3 \subset U$, E the exceptional divisor of g , and $\tilde{V} = g^{-1}(V) \subset \tilde{U}$. Then $\tilde{V} \sim g^*(3M - L) - E$, \tilde{V} is smooth, and $g|_{\tilde{V}}$ is a small resolution of the 3-fold V . On the other hand, the linear system $|g^*(M - L) - E|$ is free. Therefore the divisor \tilde{V} is numerically effective and big on \tilde{U} . In the case when \tilde{V} is ample, Lefschetz' theorem implies that $\text{Pic}(\tilde{V}) \cong \text{Pic}(\tilde{U}) \cong \mathbb{Z}^3$. However, the divisor \tilde{V} is not ample. Nevertheless, we can replace Lefschetz' theorem by the arguments in the first part of the proof of Proposition 32 in [56] to get $\text{Pic}(\tilde{V}) \cong \text{Pic}(\tilde{U}) \cong \mathbb{Z}^3$.

The linear system $|g^*(M) - E|$ is free and determines a \mathbb{P}^2 -bundle

$$\tau: \tilde{U} \rightarrow \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathbb{F}_1.$$

The rational map $\tau \circ g^{-1}$ is given in bihomogeneous coordinates by a linear system on U spanned by $\beta_1(t_0, t_2)x_1 + \beta_2(t_0, t_2)x_2$, where $\beta_i(t_0, t_2)$ is a homogeneous polynomial of degree 1.

The \mathbb{P}^2 -bundle τ induces a conic bundle $\tilde{\tau} = \tau|_{\tilde{V}}: \tilde{V} \rightarrow \mathbb{F}_1$. Let $\Delta \subset \mathbb{F}_1$ be the degeneration divisor of $\tilde{\tau}$. It follows from the construction that $\Delta \sim 5s_\infty + al$, where s_∞ is the exceptional section of \mathbb{F}_1 and l is a fibre of the natural projection of \mathbb{F}_1 to \mathbb{P}^1 .

Let s_0 be a sufficiently general section of \mathbb{F}_1 such that $s_0 \cap s_\infty = \emptyset$. We put $S = \tilde{\tau}^{-1}(s_0)$ and $B = \tau^{-1}(s_0)$. Then $S = B \cap \tilde{V} \subset B$, and the divisor B is naturally isomorphic to the scroll

$$\text{Proj}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}).$$

Moreover, $g(B) \cong B$ and $g(B) \cap V = g(S) \cup Y_3$. However, the surface Y_3 is determined by the equation $x_1 = 0$ on the scroll $g(B)$, while $g(B)$ is a general divisor in the linear system $|M - L|$. Therefore we have $S \sim 2T + F$ on the scroll B , where T is the tautological line bundle on B and F is a fibre of the natural projection of B to \mathbb{P}^1 . It follows that $K_S^2 = 1$, $s_0 \cdot \Delta = 7$ and $a = 7$. Hence \tilde{V} is non-rational by Theorem 2.16.

Proposition 5.11. *Let X be a 3-fold T_{25} in Theorem 1.6. Then X is rational.*

Proof. We can repeat the construction of the conic bundle in Proposition 5.10 to get a conic bundle $\tilde{\tau}: \tilde{V} \rightarrow \mathbb{F}_3$, where \tilde{V} is birationally equivalent to X . However, we do not need the condition $\text{Pic}(\tilde{V}) \cong \mathbb{Z}^3$ or the smoothness of \tilde{V} . Let $\Delta \subset \mathbb{F}_3$ be the degeneration divisor of $\tilde{\tau}$. Then elementary calculations imply that $\Delta \cdot s_0 = 1$, where s_0 is a sufficiently general section on \mathbb{F}_3 which is disjoint from the exceptional section of \mathbb{F}_3 . Therefore the 3-fold \tilde{V} is rational by Theorems 2.24, 2.22 and 2.23.

Proposition 5.12. *Let X be a sufficiently general 3-fold T_9 in Theorem 1.6. Then X is non-rational.*

Proof. We can repeat the construction of the conic bundle in the proof of Proposition 5.10 to get a conic bundle $\tilde{\tau} = \tilde{V} \rightarrow \mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that \tilde{V} is birationally equivalent to X , \tilde{V} is smooth, and $\text{Pic}(\tilde{V}) \cong \mathbb{Z}^3$. Let $\Delta \subset \mathbb{F}_0$ be the degeneration divisor of $\tilde{\tau}$. Then elementary calculations (see the proof of Proposition 5.10) imply that the divisor $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ has bidegree $(5, 4)$. Therefore \tilde{V} is non-rational by Theorem 2.16.

Proposition 5.13. *Let X be a 3-fold T_{11} in Theorem 1.6. Then X is rational.*

Proof. We can repeat the construction of the conic bundle in the proof of Proposition 5.10 to get a conic bundle $\tilde{\tau} = \tilde{V} \rightarrow \mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ such that \tilde{V} is birationally isomorphic to X . However we do not need to prove that \tilde{V} is smooth or that $\text{Pic}(\tilde{V}) \cong \mathbb{Z}^3$. Let $\Delta \subset \mathbb{F}_0$ be the degeneration divisor of $\tilde{\tau}$. Then elementary calculations imply that the divisor $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ has bidegree $(5, 2)$. Hence we can consider the composite $\theta: \tilde{V} \rightarrow \mathbb{P}^1$ of the conic bundle $\tilde{\tau}$ and one of the projections of $\mathbb{P}^1 \times \mathbb{P}^1$ onto \mathbb{P}^1 such that a sufficiently general fibre of θ is a surface S with $K_S^2 = 6$. Then the rationality of \tilde{V} follows from Theorems 2.24, 2.22 and 2.23.






Thus Proposition 1.10 is proved. The approach to proving the non-rationality of H_i and T_j together with the standard degeneration technique (see [54], [37], [94]) can be used as a pattern to prove non-rationality of many 3-folds fibred into del Pezzo surfaces of degree 2 and 3 (see [52], [30], [56], [43]).

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