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Three-dimensional algebraic manifolds having a divisor with a numerically trivial canonical class

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All manifolds are assumed projective and defined over $\mathbb{C}$.

Lemma 1. *Let $\mathcal{G}$ be a reflexive sheaf on a normal manifold X with an abundant divisor H. Then $H^i(\mathcal{G}\otimes\mathcal{O}_X(-nH))=0$ for $i=0,1$ and $n\gg 0$.*

Proof. We consider the exact sequences

$$\begin{gathered} 0\longrightarrow \mathcal{G}\longrightarrow \mathcal{E}\longrightarrow \mathcal{F}\longrightarrow 0,\\ 0\longrightarrow H^0(\mathcal{G}\otimes\mathcal{O}_X(-nH))\longrightarrow H^0(\mathcal{E}\otimes\mathcal{O}_X(-nH))\longrightarrow H^0(\mathcal{F}\otimes\mathcal{O}_X(-nH))\longrightarrow\cdots, \end{gathered}\tag{1}$$

where $\mathcal{E}$ is a locally free sheaf, and $\mathcal{F}$ is a torsion-free sheaf. It is easy to see that $H^0(\mathcal{E}\otimes\mathcal{O}_X(-nH))=H^0(\mathcal{F}\otimes\mathcal{O}_X(-nH))=0$; $H^1(\mathcal{E}\otimes\mathcal{O}_X(-nH))=0$, since X is normal; (1) implies that $H^i(\mathcal{G}\otimes\mathcal{O}_X(-nH))=0$ for $i=0,1$.

Lemma 2. *Let a normal manifold X contain an abundant Cartier divisor H and a Weil divisor D, such that $\dim\{x\in X\mid D$ is not a Cartier divisor in $x\}=0$ and $D|_Y\sim 0$ for a general $Y\in|nH|$ when $n\gg 0$. Then $D\sim 0$.*

Proof. We consider the exact sequence

$$H^0(\mathcal{O}_X(D)\otimes\mathcal{O}_X(-nH))\longrightarrow H^0(\mathcal{O}_X(D))\longrightarrow H^0(\mathcal{O}_Y)\longrightarrow H^1(\mathcal{O}_X(D)\otimes\mathcal{O}_X(-nH)).\tag{2}$$

$\mathcal{O}_X(D)$ is reflexive (see [2], 1.6) and $H^i(\mathcal{O}_X(D)\otimes\mathcal{O}_X(-nH))=0$ for $i=0,1$ and $n\gg 0$ by Lemma 1. From $H^0(\mathcal{O}_Y)=\mathbb{C}$ and from (2) we obtain $H^0(\mathcal{O}_X(D))=\mathbb{C}$, and thus $D\sim 0$.

Theorem. *Suppose that the normal three-dimensional manifold X possesses an abundant effective Cartier divisor H with Du Val singularities and $K_H\equiv 0$. Then $-K_X\sim_{\mathbb{Q}} H$ and either X is a retraction of the section $\mathbb{P}(\mathcal{O}_H\oplus\mathcal{O}_H(H|_H))$ or X possesses canonical singularities and H is a surface of type $K3$ or an Enriques surface.*

Proof. In a neighbourhood of H the singularities of X are canonical Gorenstein (see [3], 7-2-4) and the non-canonical singularities of X are isolated. We consider a general element $Y\in|nH|$ for $n\gg 0$. Let $f\colon X_{\mathrm{can}}\dashrightarrow X$ be a canonical modification, and let $D=12(K_X+H)$, $\widehat{H}=f^{-1}(H)$, $\widehat{Y}=f^{-1}(Y)$, $C=Y\cap H$. We have $12K_H\sim 0$ (see [1]). Consequently, $D|_H\sim 0$, $D|_Y H|_Y=D|_H nH|_H=DC=0$, but $H|_Y$ is abundant on Y. By the index theorem, $D|_Y D|_Y<0$ or $D|_Y\equiv 0$. The map f is an isomorphism along $H\cup Y$, and

$$D|_Y D|_Y=f^{-1}(D)|_{\widehat{Y}}f^{-1}(D)|_{\widehat{Y}}=nf^{-1}(D)|_{\widehat{H}}f^{-1}(D)|_{\widehat{H}}=nD|_H D|_H=0.$$

By summing we obtain $D|_Y\equiv 0$, $D|_C\sim 0$. We consider the exact sequence

$$0\longrightarrow H^0(\mathcal{O}_Y(D|_Y))\longrightarrow H^0(\mathcal{O}_C)\longrightarrow H^1(\mathcal{O}_Y((D-H)|_Y)).\tag{3}$$

By the Kleiman abundance criterion, $(H-D)|_Y$ is abundant on Y, and by the vanishing theorem (see [3], 1-2-5) $H^1(\mathcal{O}_Y((D-H)|_Y))=0$. From $H^0(\mathcal{O}_C)=\mathbb{C}$ and from (3) we obtain $H^0(\mathcal{O}_Y(D|_Y))=\mathbb{C}$ and $D|_Y\sim 0$. It follows from Lemma 2 that $D=12(K_X+H)\sim 0$.

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We assume that the singularities of X are not canonical. Then $K_{X_{\mathrm{can}}} \sim_{\mathbb{Q}} -\widehat{H} - B$, where B is an effective non-zero $\mathbb{Q}$-divisor (see [4], 2.18). There exists a one-dimensional face $R \in \mathbf{NE}(X)$ such that $-BR < 0$. Since $\widehat{H}$ is numerically effective, $K_{X_{\mathrm{can}}} R < 0$, and R is an extremal ray. Let $g\colon X_{\mathrm{can}} \to Z$ be a retraction of R. Then either g is birational or $\dim(Z) = 1, 2$. For a curve $C \in R$ the inequality $K_{X_{\mathrm{can}}} C < -1$ is satisfied, since $-BC < 0$, $-\widehat{H}C < 0$ and $-\widehat{H}C \in \mathbb{Z}$. Hence it follows that g is not a small retraction and not a retraction of a divisor to a curve (see [5], 2.3.2). If g is a retraction of a divisor E to a point, then E does not lie in the fibres of f and the effective 1-cycle $C = \widehat{Y} \cap E$ is contained in R, which contradicts the equality $BC = 0$. If $\dim(Z) = 1$, then for any curve l lying in the fibre $g|_F$, where F is an exclusive divisor for f, we have $K_{X_{\mathrm{can}}} l < 0$, which contradicts the f-abundance of $K_{X_{\mathrm{can}}}$. Therefore, $\dim(Z) = 2$. Let C be a general fibre of g. Then $\widehat{H}C + BC = 2$, $\widehat{H}C \geq 1$, $BC > 0$. Consequently, $\widehat{H}C = 1$, $BC = 1$ and $Z \cong \widehat{H}$. From $R^1 g_*(\mathcal{O}_{X_{\mathrm{can}}}) = 0$ we obtain the exact sequence

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow g_*(\mathcal{O}_{X_{\mathrm{can}}}(\widehat{H})) \longrightarrow \mathcal{O}_{\widehat{H}}(\widehat{H}|_{\widehat{H}}) \longrightarrow 0. \tag{4}$$

$g_*(\mathcal{O}_{X_{\mathrm{can}}}(\widehat{H}))$ is a locally free sheaf of rank two, and $\alpha\colon g^* g_* \mathcal{O}_{X_{\mathrm{can}}}(\widehat{H}) \to \mathcal{O}_{X_{\mathrm{can}}}(\widehat{H})$ is surjective and defines an isomorphism $X_{\mathrm{can}} \cong \mathbb{P}(g_* \mathcal{O}_{X_{\mathrm{can}}}(\widehat{H}))$, since $\widehat{H}$ is g-abundant and is a section of g. By the vanishing theorem,

$$\mathrm{Ext}^1(g_*(\mathcal{O}_{\widehat{H}}(\widehat{H}|_{\widehat{H}})), \mathcal{O}_Z) = \mathrm{Ext}^1(\mathcal{O}_{\widehat{H}}(\widehat{H}|_{\widehat{H}}), \mathcal{O}_{\widehat{H}}) = H^1(\mathcal{O}_{\widehat{H}}(-\widehat{H}|_{\widehat{H}})) = 0.$$

Consequently, (4) splits and X is a retraction of $\mathbb{P}(\mathcal{O}_H \oplus \mathcal{O}_H(H|_H))$.

We assume that the singularities of X are canonical. It is easy to show that in this case $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X(-H)) = 0$. It follows from the exact sequence

$$H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_H) \longrightarrow H^2(\mathcal{O}_X(-H))$$

that $H^1(\mathcal{O}_H) = 0$, and H is a surface of type K3 or an Enriques surface.

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