ON FACTORIALITY OF NODAL THREEFOLDS

IVAN CHELTSOV

Abstract
We prove the \( \mathbb{Q} \)-factoriality of a nodal hypersurface in \( \mathbb{P}^4 \) of degree \( n \) with at most \( \frac{(n-1)^2}{4} \) nodes and the \( \mathbb{Q} \)-factoriality of a double cover of \( \mathbb{P}^3 \) branched over a nodal surface of degree \( 2r \) with at most \( \frac{(2r-1)^3}{3} \) nodes.

1. Introduction

Nodal 3-folds\(^1\) arise naturally in many different topics of algebraic geometry. For example, the non-rationality of many smooth rationally connected 3-folds is proved via the degeneration to nodal 3-folds (see [10], [5]). However, the geometry can be very different in smooth and nodal cases: every surface in a smooth hypersurface in \( \mathbb{P}^4 \) is a complete intersection by the Lefschetz theorem, which is not the case if the hypersurface is nodal; the birational automorphisms of a smooth quartic 3-fold in \( \mathbb{P}^4 \) form a finite group consisting of projective automorphisms (see [20]), but for any non-smooth nodal quartic 3-fold this group is always infinite (see [24]). The simplest examples of nodal 3-folds are nodal hypersurfaces in \( \mathbb{P}^4 \) and double covers of \( \mathbb{P}^3 \) branched over a nodal surface. The latter are called double solids (see [9]).

For a given nodal 3-fold \( X \), one of the substantial questions is whether \( X \) is \( \mathbb{Q} \)-factorial\(^2\) or not. The global topological condition \( \text{rk} H^2(X,Z) = \text{rk} H_4(X,Z) \) is equivalent to the \( \mathbb{Q} \)-factoriality of \( X \) when it is a hypersurface or a double solid. On the other hand, a three-dimensional ordinary double point admits two small resolutions that differ by a simple flop (see [31]). Thus

\(^1\) A 3-fold is called nodal if all its singular points are ordinary double points.
\(^2\) A variety is called \( \mathbb{Q} \)-factorial if a multiple of every Weil divisor on the variety is a Cartier divisor.
a nodal 3-fold with $k$ nodes has $2^k$ small resolutions. In particular, the $\mathbb{Q}$-factoriality of a nodal 3-fold implies that it has no projective small resolutions.

**Remark 1.** The $\mathbb{Q}$-factoriality of a nodal 3-fold imposes strong geometrical restrictions on its birational geometry. For example, $\mathbb{Q}$-factorial nodal quartic 3-folds and nodal sextic double solids are non-rational, but there are rational non-$\mathbb{Q}$-factorial ones (see [24], [7]).

Consider a double cover $\pi : X \to \mathbb{P}^3$ branched over a nodal hypersurface $S \subset \mathbb{P}^3$ of degree $2r$ and a nodal hypersurface $V \subset \mathbb{P}^4$ of degree $n$. The proof of the following result is due to [9], [31], [15], [13].

**Proposition 2.** The 3-folds $X$ and $V$ are $\mathbb{Q}$-factorial if and only if their nodes impose independent linear conditions on homogeneous forms of degree $3r - 4$ and $2n - 5$ respectively. In particular, $X$ and $V$ are $\mathbb{Q}$-factorial if $|\text{Sing}(X)| \leq 3r - 3$ and $|\text{Sing}(V)| \leq 2n - 4$ respectively. The $\mathbb{Q}$-factoriality of $X$ and $V$ implies

\[
\text{Cl}(X) \otimes \mathbb{Q} \cong \text{Pic}(X) \otimes \mathbb{Q} \cong \text{Cl}(V) \otimes \mathbb{Q} \cong \text{Pic}(V) \otimes \mathbb{Q} \cong \mathbb{Q}
\]
due to the Lefschetz theorem and [9]. Moreover, the groups Pic(X) and Pic(V) have no torsion due to the Lefschetz theorem and [9]. On the other hand, the local class group of an ordinary double point is $\mathbb{Z}$ (see [25]). Therefore, the groups Cl(X) and Cl(V) have no torsion as well. Hence, the $\mathbb{Q}$-factoriality of $X$ and $V$ is equivalent to the following two conditions respectively:

- $\text{Cl}(X)$ and $\text{Pic}(X)$ are generated by $\pi^*(H)$, where $H$ is a hyperplane in $\mathbb{P}^3$;
- $\text{Cl}(V)$ and $\text{Pic}(V)$ are generated by the class of a hyperplane section.

The main purpose of this paper is to prove the following two results.

**Theorem 3.** Suppose that $|\text{Sing}(X)| \leq \frac{(2r-1)r}{3}$. Then $X$ is $\mathbb{Q}$-factorial.

**Theorem 4.** Suppose that $|\text{Sing}(V)| \leq \frac{(n-1)^2}{4}$. Then $V$ is $\mathbb{Q}$-factorial.

The bounds in Theorems 3 and 4 may not be sharp in general. For example, in the case $r = 3$ the 3-fold $X$ is $\mathbb{Q}$-factorial if $|\text{Sing}(X)| \leq 14$ due to [7], and in the case $n = 4$ the 3-fold $V$ is $\mathbb{Q}$-factorial if $|\text{Sing}(V)| \leq 8$ due to [5].

**Example 5.** Consider a hypersurface $X \subset \mathbb{P}(1^4, r)$ given by the equation

\[
u^2 = g_r^2(x, y, z, t) + h_1(x, y, z, t)f_{2r-1}(x, y, z, t)
\]

where $g_i$, $h_i$, and $f_i$ are sufficiently general polynomials of degree $i$. Let $\pi : X \to \mathbb{P}^3$ be a restriction of the natural projection $\mathbb{P}(1^4, r) \to \mathbb{P}^3$, induced by an embedding of the graded algebras $\mathbb{C}[x_0, \ldots, x_{2n}] \subset \mathbb{C}[x_0, \ldots, x_{2n}, y]$. Then $\pi : X \to \mathbb{P}^3$ is a double cover branched over a nodal hypersurface
\( g_r^2 + h_1 f_{2r-1} = 0 \) of degree \( 2r \) and \( |\text{Sing}(X)| = (2r - 1)r \); the 3-fold \( X \) is not \( \mathbb{Q} \)-factorial, i.e., the divisor \( h_1 = 0 \) splits into \( 2 \) non-\( \mathbb{Q} \)-Cartier divisors.

**Example 6.** Let \( V \subset \mathbb{P}^4 \) be a hypersurface,

\[
x g_{n-1}(x, y, z, t, w) + y f_{n-1}(x, y, z, t, w) \subset \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x, y, z, t, w]),
\]

where \( g_{n-1} \) and \( f_{n-1} \) are general polynomials of degree \( n-1 \). Then \( V \) is nodal and contains the plane \( x = y = 0 \). Hence, the 3-fold \( V \) is not \( \mathbb{Q} \)-factorial and \( |\text{Sing}(V)| = (n - 1)^2 \).

Therefore, asymptotically the bounds in Theorems 3 and 4 are not very far from being sharp. On the other hand, the following result is proved in [8].

**Proposition 7.** Every smooth surface on \( V \) is a Cartier divisor if \( |\text{Sing}(V)| < (n - 1)^2 \).

We expect the following to be true.

**Conjecture 8.** The 3-fold \( X \) is \( \mathbb{Q} \)-factorial whenever the inequality \( |\text{Sing}(X)| < (2r - 1)r \) holds; the 3-fold \( V \) is \( \mathbb{Q} \)-factorial whenever the inequality \( |\text{Sing}(V)| < (n - 1)^2 \) holds.

The claim of Conjecture 8 is proved only for \( r \leq 3 \) and \( n \leq 4 \) (see [16], [7], [5]), but for many \( r \) and \( n \) the bounds in Theorems 3 and 4 can be improved. For example, we prove the following result.

**Proposition 9.** Suppose that the equalities \( r = 4 \) and \( n = 5 \) hold. Then \( X \) is \( \mathbb{Q} \)-factorial whenever \( |\text{Sing}(X)| < 25 \), and the 3-fold \( V \) is \( \mathbb{Q} \)-factorial whenever \( |\text{Sing}(V)| < 14 \).

The following result is proved in [8].

**Theorem 10.** Suppose that the subset \( \text{Sing}(V) \subset \mathbb{P}^4 \) is a set-theoretic intersection of hypersurfaces of degree \( l < \frac{n}{2} \) and \( |\text{Sing}(V)| < \frac{(n-2)(n-1)^2}{n} \). Then \( V \) is \( \mathbb{Q} \)-factorial.

The saturated ideal of a set of \( k \) points in general position in \( \mathbb{P}^4 \) is generated by polynomials of degree at most \( \frac{n}{4} \) when \( k < (n - 1)^2 \) and \( n > 72 \) by [17]. Therefore, Theorem 10 implies the \( \mathbb{Q} \)-factoriality of \( V \) having less than \( \frac{1}{2}(n - 1)^2 \) nodes in assumption that the nodes of \( V \) are in general position. However, the latter condition implies the \( \mathbb{Q} \)-factoriality of \( V \) due to Proposition 2. We prove the following generalization of Theorem 10.

**Theorem 11.** Let \( \mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^3}(k)| \) and \( \mathcal{D} \subset |\mathcal{O}_{\mathbb{P}^4}(l)| \) be linear subsystems of hypersurfaces vanishing at \( \text{Sing}(S) \) and \( \text{Sing}(V) \) respectively. Put \( \mathcal{H} = \mathcal{H}|_S \) and \( \mathcal{D} = \mathcal{D}|_V \). Suppose that inequalities \( k < r \) and \( l < \frac{n}{2} \) hold. Then \( \dim(\text{Bs}(\mathcal{H})) = 0 \) implies the \( \mathbb{Q} \)-factoriality of the 3-fold \( X \), and \( \dim(\text{Bs}(D)) = 0 \) implies the \( \mathbb{Q} \)-factoriality of the 3-fold \( V \).

\(^3\)Namely, the 3-folds \( X \) and \( V \) are nodal Calabi-Yau 3-folds.
Corollary 12. Suppose \( \operatorname{Sing}(S) \subset \mathbb{P}^3 \) and \( \operatorname{Sing}(V) \subset \mathbb{P}^4 \) are set-theoretic intersections of hypersurfaces of degree \( k < r \) and \( l < \frac{n}{2} \) respectively. Then \( X \) and \( V \) are \( \mathbb{Q} \)-factorial.

From the point of view of birational geometry the most important application of Theorems 3 and 4 is the \( \mathbb{Q} \)-factoriality condition for a nodal quartic 3-fold and a sextic double solid, i.e., the cases \( r = 3 \) and \( n = 4 \) respectively, because in these cases the \( \mathbb{Q} \)-factoriality implies the non-rationality (see [24], [7]). However, it is possible to apply Theorems 3 and 4 to certain higher-dimensional problems in birational algebraic geometry.

Theorem 13. Let \( \tau : U \to \mathbb{P}^s \) be a double cover branched over a hypersurface \( F \) of degree \( 2r \) and \( D \) be a hyperplane in \( \mathbb{P}^s \) such that \( D_1 \cap \cdots \cap D_{s-3} \) is a \( \mathbb{Q} \)-factorial nodal 3-fold, where \( D_i \) is a general divisor in \( |\tau^*(D)| \). Then \( \operatorname{Cl}(U) \) and \( \operatorname{Pic}(U) \) are generated by \( \tau^*(D) \).

Theorem 14. Let \( W \subset \mathbb{P}^r \) be a hypersurface of degree \( n \) such that \( H_1 \cap \cdots \cap H_{r-4} \) is a \( \mathbb{Q} \)-factorial nodal 3-fold, where \( H_i \) is a general enough hyperplane section of \( W \). Then the groups \( \operatorname{Cl}(W) \) and \( \operatorname{Pic}(W) \) are generated by the class of a hyperplane section of \( W \subset \mathbb{P}^r \).

A priori Theorems 13 and 14 can be used to prove the non-rationality of certain singular hypersurfaces of degree \( r \) in \( \mathbb{P}^r \) and double covers of \( \mathbb{P}^s \) branched over singular hypersurfaces of degree \( 2s \) (see [6]). In the latter case the application of Theorem 13 can be explicit. For example, we prove the following result.

Proposition 15. Let \( \xi : Y \to \mathbb{P}^4 \) be a double cover branched over a hypersurface \( F \subset \mathbb{P}^4 \) of degree 8 such that \( F \) is smooth outside of a smooth curve \( C \subset F \), the singularity of the hypersurface \( F \) in a sufficiently general point of \( C \) is locally isomorphic to the singularity

\[
x_1^2 + x_2^2 + x_3^2 = 0 \subset \mathbb{C}^4 \cong \operatorname{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),
\]

the singularities of \( F \) in other points of \( C \) are locally isomorphic to the singularity

\[
x_1^2 + x_2^2 + x_3^2 x_4 = 0 \subset \mathbb{C}^4 \cong \operatorname{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),
\]

and a general 3-fold in the linear system \( |-K_Y| \) is \( \mathbb{Q} \)-factorial. Then \( Y \) is a birationally rigid \(^4\) terminal \( \mathbb{Q} \)-factorial Fano 4-fold with \( \operatorname{Pic}(Y) \cong \mathbb{Z} \) and \( \operatorname{Bir}(Y) \) is a finite group consisting of biregular automorphisms. In particular, the 4-fold \( Y \) is non-rational.

Example 16. Let \( Y \subset \mathbb{P}(1^5, 4) \) be a hypersurface

\[
u^2 = \sum_{i=1}^{3} f_i(x, y, z, t, w)g_i^2(x, y, z, t, w) \subset \mathbb{P}(1^5, 4) \cong \operatorname{Proj}(\mathbb{C}[x, y, z, t, w, u]),
\]

\(^4\)Namely, the 4-fold \( Y \) is a unique Mori fibration birational to \( Y \) (see [12]).
where \(f_i\) and \(g_i\) are sufficiently general non-constant homogeneous polynomials such that \(\text{deg}(f_i) + 2\text{deg}(g_i) = 8\). Then the natural projection \(\mathbb{P}(1^5, 4) \to \mathbb{P}^4\) induces a double cover \(\tau : Y \to \mathbb{P}^4\) branched over a hypersurface \(F \subset \mathbb{P}^4\), whose equation is \(\sum_{i=1}^3 f_i g_i^2 = 0\) and which is smooth outside of a curve \(g_1 = g_2 = g_3 = 0\). Therefore, the 4-fold \(X\) is not rational due to Proposition 15 and Theorems 3 and 11.

How many nodes can \(X\) and \(V\) have? The best known upper bounds (see [29]) are the following: \(|\text{Sing}(X)| \leq A_3(2r)\) and \(|\text{Sing}(V)| \leq A_4(n)\), where \(A_i(j)\) is the Arnold number, a number of points \((a_1, \ldots, a_i) \in \mathbb{Z}^i\) such that \((i - 2)\frac{j}{2} + 1 < \sum_{t=1}^i a_t \leq \frac{i j}{2}\) and \(a_t \in (0, j)\), which implies \(|\text{Sing}(X)| \leq 68\) and \(|\text{Sing}(X)| \leq 180\) when \(r = 3\) and \(4\), and \(|\text{Sing}(V)| \leq 45\) and \(|\text{Sing}(V)| \leq 135\) when \(n = 4\) and \(5\) respectively. This bound is sharp for \(n = 4\) (see [21]). There is a sharp bound \(|\text{Sing}(X)| \leq 65\) in the case \(r = 3\) (see [2], [19], [30]). However, there are no known example of a nodal quintic in \(\mathbb{P}^4\) having more than 130 nodes (see [28]).

2. Preliminaries

Let \(X\) be a variety and \(B_X\) be a boundary on \(X\), i.e., \(B_X = \sum_{i=1}^k a_i B_i\), where \(B_i\) is a prime divisor on \(X\) and \(a_i \in \mathbb{Q}\) (see [22]). The log pair \((X, B_X)\) is called movable when every component \(B_i\) is a linear system on \(X\) such that the base locus of \(B_i\) has codimension at least 2 (see [12], [4]). We assume that \(K_X\) and \(B_X\) are \(\mathbb{Q}\)-Cartier divisors.

**Definition 17.** A log pair \((V, B^V)\) is a log pull-back of the log pair \((X, B_X)\) with respect to a birational morphism \(f : V \to X\) if \(B^V = f^{-1}(B_X) - \sum_{i=1}^n a(X, B_X, E_i) E_i\) such that the equivalence \(K_V + B^V \sim_\mathbb{Q} f^*(K_X + B_X)\) holds, where \(E_i\) is an \(f\)-exceptional divisor and \(a(X, B_X, E_i) \in \mathbb{Q}\). The number \(a(X, B_X, E_i)\) is called a discrepancy of \((X, B_X)\) in the \(f\)-exceptional divisor \(E_i\).

**Definition 18.** A birational morphism \(f : V \to X\) is called a log resolution of the log pair \((X, B_X)\) if the variety \(V\) is smooth and the union of all proper transforms of the divisors \(B_i\) and all \(f\)-exceptional divisors forms a divisor with simple normal crossing.

**Definition 19.** A proper irreducible subvariety \(Y \subset X\) is called a center of log canonical singularities of the log pair \((X, B_X)\) if there are a birational morphism \(f : V \to X\) together with a not necessarily \(f\)-exceptional divisor \(E \subset V\) such that \(E\) is contained in the support of the effective part of the

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5 Usually boundaries are assumed to be effective (see [22]), but we do not assume this.
divisor \([B^V]\) and \(f(E) = Y\). The set of all the centers of log canonical singularities of the log pair \((X, B_X)\) is denoted by \(\text{LCS}(X, B_X)\).

**Definition 20.** For a log resolution \(f : V \rightarrow X\) of \((X, B_X)\) the subscheme \(\mathcal{L}(X, B_X)\) associated to the ideal sheaf \(\mathcal{I}(X, B_X) = f_*(\mathcal{O}_V([−B^V]))\) is called a log canonical singularity subscheme of the log pair \((X, B_X)\).

The support of the log canonical singularity subscheme \(\mathcal{L}(X, B_X)\) is a union of all elements in the set \(\text{LCS}(X, B_X)\). The following result is due to [27] (see [23], [1], [4]).

**Theorem 21.** Suppose that \(B_X\) is effective and for some nef and big divisor \(H\) on \(X\) the divisor \(D = K_X + B_X + H\) is Cartier. Then

\[H^i(X, \mathcal{I}(X, B_X) \otimes \mathcal{O}_X(D)) = 0\] for \(i > 0\).

Consider the following application of Theorem 21.

**Lemma 22.** Let \(Σ \subset \mathbb{P}^n\) be a finite subset, and \(\mathcal{M}\) be a linear system of hypersurfaces of degree \(k\) passing through all points of the set \(Σ\). Suppose that the base locus of the linear system \(\mathcal{M}\) is zero-dimensional. Then the points of the set \(Σ\) impose independent linear conditions on the homogeneous forms on \(\mathbb{P}^n\) of degree \(n(k−1)\).

**Proof.** Let \(Λ \subset \mathbb{P}^n\) be a base locus of the linear system \(\mathcal{M}\). Then \(Σ \subseteq Λ\) and \(Λ\) is a finite subset in \(\mathbb{P}^n\). Now consider sufficiently general different divisors \(H_1, \ldots, H_s\) in the linear system \(\mathcal{M}\) for \(s \gg 0\). Let \(X = \mathbb{P}^n\) and \(B_X = \frac{n}{s} \sum_{i=1}^s H_i\). Then \(\text{Supp}(\mathcal{L}(X, B_X)) = Λ\).

To prove the claim it is enough to prove that for every point \(P \in Σ\) there is a hypersurface in \(\mathbb{P}^n\) of degree \(n(k−1)\) that passes through all the points in the set \(Σ \setminus P\) and does not pass through the point \(P\). Let \(Σ \setminus P = \{P_1, \ldots, P_k\}\), where \(P_i\) is a point of \(X = \mathbb{P}^n\), and let \(f : V \rightarrow X\) be a blowup at the points of \(Σ \setminus P\). Then

\[K_V + (B_V + \sum_{i=1}^k (\text{mult}_{P_i}(B_X) − n)E_i) + f^*(H) = f^*(n(k−1)H) − \sum_{i=1}^k E_i,\]

where \(E_i = f^{-1}(P_i)\), \(B_V = f^{-1}(B_X)\) and \(H\) is a hyperplane in \(\mathbb{P}^n\). By construction we have \(\text{mult}_{P_i}(B_X) = n\text{mult}_{P_i}(\mathcal{M}) \geq n\) and \(B_V = B_X + \sum_{i=1}^k (\text{mult}_{P_i}(B_X) − n)E_i\) is effective.

Let \(\bar{P} = f^{-1}(P)\). Then \(\bar{P} \in \text{LCS}(V, \bar{B}_V)\) and \(\bar{P}\) is an isolated center of log canonical singularities of the log pair \((V, \bar{B}_V)\), because in the neighborhood of the point \(P\) the birational morphism \(f : V \rightarrow X\) is an isomorphism. On the other hand, the map

\[H^0(\mathcal{O}_V(f^*(n(k−1)H)−\sum_{i=1}^k E_i)) \rightarrow H^0(\mathcal{O}_{\mathcal{L}(V, \bar{B}_V)} \otimes \mathcal{O}_V(f^*(n(k−1)H)−\sum_{i=1}^k E_i))\]
is surjective by Theorem [24]. However, in the neighborhood of the point \( \bar{P} \) the support of the subscheme \( \mathcal{L}(V, B_Y) \) consists just of the point \( \bar{P} \). The latter implies the existence of a divisor \( D \in |f^*(n(k-1)H) - \sum_{i=1}^{k} E_i| \) that does not pass through \( \bar{P} \). Thus, \( f(D) \) is a hypersurface in \( \mathbb{P}^n \) of degree \( n(k-1) \) that passes through the points of \( \Sigma \setminus P \) and does not pass through the point \( P \in \Sigma \). \( \square \)

Actually, arguing as in the proof of Lemma [22] we can prove Theorem [11].

Proof of Theorem [11]. We have a double cover \( \pi : X \to \mathbb{P}^3 \) branched over a nodal hypersurface \( S \subset \mathbb{P}^3 \) of degree \( 2r \), a linear subsystem \( \mathcal{H} \subset |O_{\mathbb{P}^3}(k)| \) of hypersurfaces vanishing at \( \text{Sing}(S) \) for \( k < r \) such that \( \dim(B_s(\mathcal{H})) = 0 \), where \( \mathcal{H} = \mathcal{H}|_S \). We must show that the nodes of \( S \) impose independent linear conditions on homogeneous forms of degree \( 3r - 4 \) due to Proposition [2]. Suppose that \( \dim(B_s(\mathcal{H})) = 0 \). Then Lemma [22] implies that the nodes of \( S \) impose independent linear conditions on homogeneous forms of degree \( 3r - 4 \), which proves Corollary [12]. In the general case we can repeat the proof of Lemma [22] replacing \( \frac{1}{r} \sum_{i=1}^{s} H_i \) by \( S + \frac{1}{r} \sum_{i=1}^{s} H_i \). The proof of the \( \mathbb{Q} \)-factoriality of the nodal hypersurface \( V \subset \mathbb{P}^4 \) is similar. \( \square \)

Definition 23. A proper irreducible subvariety \( Y \subset X \) is called a center of canonical singularities of \((X, B_X)\) if there is a birational morphism \( f : W \to X \) and an \( f \)-exceptional divisor \( E \subset W \) such that the discrepancy \( a(X, B_X, E) \leq 0 \) and \( f(E) = Y \). The set of all centers of canonical singularities of the log pair \((X, B_X)\) is denoted by \( \mathcal{CS}(X, B_X) \).

The following result is a corollary of Theorem 17.6 in [23].

Proposition 24. Let \( H \) be an effective Cartier divisor on \( X \) and \( Z \in \mathcal{CS}(X, B_X) \). Suppose that \( X \) and \( H \) are smooth in the generic point of \( Z \), \( Z \subset H \), \( H \not\subseteq \text{Supp}(B_X) \) and \( B_X \) is an effective boundary. Then \( \text{LCS}(H, B_X|_H) \neq \emptyset \).

The following result is Corollary 7.3 in [26] (see [20], [12]).

Theorem 25. Suppose that \( X \) is smooth, \( \dim(X) \geq 3 \), the boundary \( B_X \) is effective and movable, and the set \( \mathcal{CS}(X, B_X) \) contains a closed point \( O \in X \). Then \( \text{mult}_O(B_X^2) \geq 4 \) and the equality implies \( \text{mult}_O(B_X) = 2 \) and \( \dim(X) = 3 \).

The following result is implied by Theorem 3.10 in [12] and Proposition 24.

Theorem 26. Suppose that \( \dim(X) \geq 3 \), \( B_X \) is effective, and the set \( \mathcal{CS}(X, B_X) \) contains an ordinary double point \( O \) of \( X \). Then the equality \( \text{mult}_O(B_X) \geq 1 \) holds\(^6\), moreover, the equality \( \text{mult}_O(B_X) = 1 \) implies that \( \dim(X) = 3 \).

The following result is an easy modification of Theorem 26.

\(^6\)The rational number \( \text{mult}_O(B_X) \) is defined by the equivalence \( f^*(B_X) \sim_\mathbb{Q} f^{-1}(B_X) + \text{mult}_O(B_X)E \), where \( f : W \to X \) is a blowup of \( O \) and \( E \) is an \( f \)-exceptional divisor.
Proposition 27. Suppose that $\dim(X) = 3$, $B_X$ is effective, and the set $\mathcal{CS}(X,B_X)$ contains an isolated singular point $O$ of the variety $X$, which is locally isomorphic to the singularity $y^3 = \sum_{i=1}^3 x_i^2$. Then the inequality $\operatorname{mult}_O(B_X) \geq \frac{1}{2}$ holds.

Proof. The 3-fold $W$ is smooth, $E$ is isomorphic to a cone in $\mathbb{P}^3$ over a smooth conic, the restriction $-E|_E$ is rationally equivalent to a hyperplane section of $E \subset \mathbb{P}^3$, and

$$K_W + B_W \sim \mathbb{Q} f^*(K_X + B_X) + (1 - \operatorname{mult}_O(B_X))E,$$

where $B_W = f^{-1}(B_X)$. Suppose that $\operatorname{mult}_O(B_X) < \frac{1}{2}$. Then

$$\mathcal{CS}(W,B_W) \subset \mathcal{CS}(W,B_W + (\operatorname{mult}_O(B_X) - 1)E),$$

because $\operatorname{mult}_O(B_X) - 1 < 0$. However, the log pair $(W,B_W + (\operatorname{mult}_O(B_X) - 1)E)$ is a log pull-back of $(X,B_X)$ and $O \in \mathcal{CS}(X,B_X)$. Therefore, there is a proper irreducible subvariety $Z \subset E$ such that $Z \in \mathcal{CS}(W,B_W)$. Hence, $\mathcal{LCS}(E,B_W|_E) \neq \emptyset$ by Proposition 24.

Let $B_E = B_W|_E$. Then $\mathcal{LCS}(E,B_E)$ does not contain curves on $E$, because otherwise the intersection of $B_E$ with the ruling of $E$ is greater than $\frac{1}{2}$, which is impossible due to our assumption $\operatorname{mult}_O(B_X) < \frac{1}{2}$. Therefore, $\dim(\operatorname{Supp}(\mathcal{L}(E,B_E))) = 0$.

Let $H$ be a hyperplane section of $E \subset \mathbb{P}^3$. Then

$$K_E + B_E + (1 - \operatorname{mult}_O(B_X))H \sim \mathbb{Q} - H$$

and $H^0(\mathcal{O}_E(-H)) = 0$. On the other hand, the sequence of groups

$$H^0(\mathcal{O}_E(-H)) \to H^0(\mathcal{O}_{\mathcal{L}(E,B_E)}) \to H^1(E, \mathcal{I}(E,B_E) \otimes \mathcal{O}_E(-H))$$

is exact and $H^1(E, \mathcal{I}(E,B_E) \otimes \mathcal{O}_E(-H)) = 0$ by Theorem 21. Therefore, the latter implies the vanishing of $H^0(\mathcal{O}_{\mathcal{L}(E,B_E)})$, which contradicts $\mathcal{LCS}(E,B_E) \neq \emptyset$. 

The following result is due to [11] (see [26], [4]).

Theorem 28. Let $X$ be a Fano variety with $\operatorname{Pic}(X) \cong \mathbb{Z}$ with terminal $\mathbb{Q}$-factorial singularities such that either $X$ is not birationally rigid or $\operatorname{Bir}(X) \neq \operatorname{Aut}(X)$. Then there is a linear system $\mathcal{M}$ on $X$ whose base locus has codimension at least 2 such that the singularities of the log pair $(X,\mu \mathcal{M})$ are not canonical, where $\mu \in \mathbb{Q}_{>0}$ such that $\mu \mathcal{M} \sim \mathbb{Q} - K_X$.

The following result is due to [3].

Theorem 29. Let $\pi : Y \to \mathbb{P}^2$ be the blowup at points $P_1, \ldots, P_s$ on $\mathbb{P}^2$, $s \leq \frac{d^2 + 3d + 10}{2}$, such that at most $k(d + 3 - k) - 2$ of the points $P_i$ lie on a curve of degree $k \leq \frac{d+3}{2}$, where $d \geq 3$ is a natural number. Then $|\pi^*(\mathcal{O}_{\mathbb{P}^2}(d)) - \sum_{i=1}^s E_i|$ is free, where $E_i = \pi^{-1}(P_i)$.
Corollary 30. Let $\Sigma \subset \mathbb{P}^2$ be a finite subset such that the inequality $|\Sigma| \leq \frac{d^2 + 9d + 16}{2}$ holds and at most $k(d + 3 - k) - 2$ points of $\Sigma$ lie on a curve of degree $k \leq \frac{d + 3}{2}$, where $d \geq 3$ is a natural number. Then for every point $P \in \Sigma$ there is a curve $C \subset \mathbb{P}^2$ of degree $d$ that passes through all the points in $\Sigma \setminus P$ and does not pass through the point $P$.

In the case $d = 3$ the claim of Theorem 29 is nothing but the freeness of the anticanonical linear system of a weak del Pezzo surface of degree $9 - s \geq 2$ (see [14]).

3. Double solids

In this section we prove Theorem 3. Let $\pi : X \to \mathbb{P}^3$ be a double cover branched over a nodal hypersurface $S \subset \mathbb{P}^3$ of degree $2r$ such that $|\text{Sing}(S)| \leq (2r - 1)r$. We must show that the nodes of $S \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms of degree $3r - 4$ on $\mathbb{P}^3$ due to Proposition 2. Moreover, we may assume $r \geq 3$, because in the case $r \leq 2$ the required claim is trivial.

Definition 31. The points of a subset $\Gamma \subset \mathbb{P}^3$ satisfy the property $\nabla$ if at most $t(2r - 1)$ points of the set $\Gamma$ can lie on a curve in $\mathbb{P}^3$ of degree $t \in \mathbb{N}$.

Proposition 32. The points of the subset $\Sigma \subset \mathbb{P}^3$ satisfy the property $\nabla$.

Proof. Let $F(x_0, x_1, x_2, x_3) = 0$ be a homogeneous equation of degree $2r$ that defines $S \subset \mathbb{P}^3$, where $(x_0 : x_1 : x_2 : x_3)$ are homogeneous coordinates on $\mathbb{P}^3$. Consider the linear system

$$L = \left\{ \sum_{i=0}^{3} \lambda_i \frac{\partial F}{\partial x_i} = 0 \right\} \subset |\mathcal{O}_{\mathbb{P}^3}(2r - 1)|,$$

where $\lambda_i \in \mathbb{C}$. The base locus of $L$ consists of singular points of $S$. A curve in $\mathbb{P}^3$ of degree $t$ intersects a generic member of $L$ at most $(2r - 1)t$ times, which implies the claim. $\Box$

Fix a point $P \in \Sigma$. To prove that the points of $\Sigma \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms of degree $3r - 4$ it is enough to construct a hypersurface in $\mathbb{P}^3$ of degree $3r - 4$ that passes through $\Sigma \setminus P$ and does not pass through $P \in \Sigma$.

Lemma 33. Suppose $\Sigma \subset \Pi$ for some hyperplane $\Pi \subset \mathbb{P}^3$. Then there is a hypersurface in $\mathbb{P}^3$ of degree $3r - 4$ that passes through $\Sigma \setminus P$ and does not pass through $P \in \Sigma$.

Proof. Let us apply Corollary 30 to $\Sigma \subset \Pi$ and $d = 3r - 4 \geq 5$. We must check that all the conditions of Corollary 30 are satisfied, which is easy but
not obvious. First of all,

$$|\Sigma| \leq \frac{(2r - 1)r}{3} \Rightarrow |\Sigma| \leq \frac{d^2 + 9d + 16}{6}$$

and at most $d = 3r - 4$ points of $\Sigma$ can lie on a line in $\Pi$ because $r \geq 3$ and the points of the subset $\Sigma \subset \Pi$ satisfy the property $\nabla$ due to Proposition 32.

Now we must prove that at most $t(3r - 1 - t) - 2$ points of $\Sigma$ can lie on a curve of degree $t \leq \frac{3r - 1}{2}$. The case $t = 1$ is already done. Moreover, at most $t(2r - 1)$ points of the set $\Sigma$ can lie on a curve of degree $t$ by Proposition 32. Thus, we must show that

$$t(3r - 1 - t) - 2 \geq t(2r - 1)$$

for all $t \leq \frac{3r - 1}{2}$. Moreover, we must prove the latter inequality only for such $t > 1$ that the inequality $t(3r - 1 - t) - 2 < |\Sigma|$ holds, because otherwise the corresponding condition on the points of the set $\Sigma$ is vacuous. Moreover, we have

$$t(3r - 1 - t) - 2 \geq t(2r - 1) \iff r > t,$$

because $t > 1$. Suppose that the inequality $r \leq t$ holds for some natural number $t$ such that $t \leq \frac{3r - 1}{2}$ and $t(3r - 1 - t) - 2 < |\Sigma|$. Let $g(x) = x(3r - 1 - x) - 2$. Then $g(x)$ is increasing for $x < \frac{3r - 1}{2}$. Thus, we have $g(t) \geq g(r)$, because $\frac{3r - 1}{2} \geq t \geq r$. Hence,

$$\frac{(2r - 1)r}{3} \geq |\Sigma| > g(t) \geq g(r) = r(2r - 1) - 2,$$

which is impossible when $r \geq 3$.

Therefore, there is a curve $C \subset \Pi$ of degree $3r - 4$ that passes through $\Sigma \setminus P$ and does not pass through $P$ by Corollary 33. Let $Y \subset \mathbb{P}^3$ be a sufficiently general cone over the curve $C \subset \Pi \cong \mathbb{P}^2$. Then $Y \subset \mathbb{P}^3$ is a hypersurface of degree $3r - 4$ that passes through all the points of the set $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$.

Take a sufficiently general hyperplane $\Pi \subset \mathbb{P}^3$. Let $\psi : \mathbb{P}^3 \dashrightarrow \Pi$ be a projection from a sufficiently general point $O \in \mathbb{P}^3$, $\Sigma' = \psi(\Sigma) \subset \Pi \cong \mathbb{P}^2$ and $\hat{P} = \psi(P) \in \Sigma'$.

**Lemma 34.** Suppose that the points of $\Sigma' \subset \Pi$ satisfy the property $\nabla$. Then there is a hypersurface in $\mathbb{P}^3$ of degree $3r - 4$ containing $\Sigma \setminus P$ and not passing through $P$.

**Proof.** Arguing as in the proof of Lemma 33 we obtain a curve $C \subset \Pi$ of degree $3r - 4$ that passes through $\Sigma' \setminus P$ and does not pass through $\hat{P}$. Let $Y \subset \mathbb{P}^3$ be a cone over the curve $C$ with the vertex $O$. Then $Y \subset \mathbb{P}^3$ is a hypersurface of degree $3r - 4$ that passes through $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$. □
Perhaps the points of the set \( \Sigma' \subset \Pi \) always satisfy the property \( \nabla \), but we are unable to prove it. We may assume that the points of \( \Sigma' \subset \Pi \) do not satisfy the property \( \nabla \).

**Definition 35.** The points of a subset \( \Gamma \subset \mathbb{P}^s \) satisfy the property \( \nabla_k \) if at most \( i(2r-1) \) points of the set \( \Gamma \) can lie on a curve in \( \mathbb{P}^s \) of degree \( i \in \mathbb{N} \) for all \( i \leq k \).

Therefore, there is a smallest \( k \in \mathbb{N} \) such that the points of \( \Sigma' \subset \Pi \) do not satisfy the property \( \nabla_k \), i.e., there is a subset \( \Lambda_k^1 \subset \Sigma \) such that \( |\Lambda_k^1| > k(2r-1) \) and all points of 
\[
\Lambda_k^1 = \psi(\Lambda_k^1) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2
\]
lie on a curve \( C \subset \Pi \) of degree \( k \). Moreover, the curve \( C \) is irreducible and reduced due to the minimality of \( k \). In the case when the points of the subset \( \Sigma' \setminus \Lambda_k^1 \subset \Pi \) do not satisfy the property \( \nabla_k \) we can find a subset \( \Lambda_k^2 \subset \Sigma \setminus \Lambda_k^1 \) such that \( |\Lambda_k^2| > k(2r-1) \) and all the points of the set \( \Lambda_k^2 = \psi(\Lambda_k^2) \) lie on an irreducible curve of degree \( k \). Thus, we can iterate this construction \( c_k \) times and get \( c_k > 0 \) disjoint subsets 
\[
\Lambda_k^i \subset \Sigma \setminus \bigcup_{j=1}^{i-1} \Lambda_k^j \subset \Sigma
\]
such that \( |\Lambda_k^i| > k(2r-1) \), all the points of the subset \( \Lambda_k^i = \psi(\Lambda_k^i) \subset \Sigma' \) lie on an irreducible reduced curve on \( \Pi \) of degree \( k \), and all the points of the subset 
\[
\Sigma' \setminus \bigcup_{i=1}^{c_k} \Lambda_k^i \subset \Pi \cong \mathbb{P}^2
\]
satisfy the property \( \nabla_k \). Now we can repeat this construction for the property \( \nabla_{k+1} \) and find \( c_{k+1} \geq 0 \) disjoint subsets 
\[
\Lambda_{k+1}^i \subset (\Sigma \setminus \bigcup_{i=1}^{c_k} \Lambda_k^i) \setminus \bigcup_{j=1}^{i-1} \Lambda_{k+1}^j \subset \Sigma \setminus \bigcup_{i=1}^{c_k} \Lambda_k^i \subset \Sigma
\]
such that \( |\Lambda_{k+1}^i| > (k+1)(2r-1) \), the points of \( \Lambda_{k+1}^i = \psi(\Lambda_{k+1}^i) \subset \Sigma' \) lie on an irreducible reduced curve on \( \Pi \) of degree \( k+1 \), and the points of the subset 
\[
\Sigma' \setminus \bigcup_{i=1}^{c_k} \bigcup_{j=k}^{k+1} \Lambda_j^i \subset \Sigma' \subset \Pi \cong \mathbb{P}^2
\]
satisfy the property \( \nabla_{k+1} \). Now we can iterate this construction for \( \nabla_{k+2} \), \( \ldots \), \( \nabla_l \) and get disjoint subsets \( \Lambda_j^i \subset \Sigma \) for \( j = k, \ldots, l \geq k \) such that
|\Lambda_j| > j(2r - 1), all the points of the subset \( \tilde{\Lambda}_j \subset \Sigma' \) lie on an irreducible reduced curve of degree \( j \) in \( \Pi \), and all the points of the subset

\[
\tilde{\Sigma} = \Sigma' \setminus \bigcup_{j=k}^{l} \tilde{\Lambda}_j \subseteq \Sigma' \subset \Pi \cong \mathbb{P}^2
\]

satisfy the property \( \nabla \), where \( c_j \geq 0 \) is the number of subsets \( \tilde{\Lambda}_j \). The subset \( \Lambda_k \subset \Sigma \) is non-empty, i.e., \( c_k > 0 \), but every subset \( \Lambda_j \subset \Sigma \) can be empty when \( j \neq k \) or \( i \neq 1 \), and the subset \( \tilde{\Sigma} \subset \Sigma' \) can be empty as well. Nevertheless, we always have the inequality

\[
(36) \quad |\Sigma| < \frac{(2r-1)r}{3} - \sum_{i=k}^{l} c_i(2r-1)i = \frac{(2r-1)}{3}(r-3)\sum_{i=k}^{l} i c_i.
\]

**Corollary 37.** The inequality \( \sum_{i=k}^{l} i c_i < \frac{r}{3} \) holds.

In particular, \( \Lambda_j \neq \emptyset \) implies \( j < \frac{r}{3} \).

**Lemma 38.** Suppose that \( \Lambda_j \neq \emptyset \). Let \( \mathcal{M} \) be a linear system of hypersurfaces of degree \( j \) in \( \mathbb{P}^3 \) passing through all the points in \( \Lambda_j \). Then the base locus of \( \mathcal{M} \) is zero-dimensional.

**Proof.** By the construction of the set \( \Lambda_j \) all the points of the subset

\( \tilde{\Lambda}_j = \psi(\Lambda_j) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2 \)

lie on an irreducible reduced curve \( C \subset \Pi \) of degree \( j \). Let \( Y \subset \mathbb{P}^3 \) be a cone over \( C \) with the vertex \( O \). Then \( Y \) is a hypersurface in \( \mathbb{P}^3 \) of degree \( j \) that contains all the points of the set \( \Lambda_j \). Therefore, \( Y \in \mathcal{M} \).

Suppose that the base locus of the linear system \( \mathcal{M} \) contains an irreducible reduced curve \( Z \subset \mathbb{P}^3 \). Then \( Z \subset Y \) and \( \psi(Z) = C \). Moreover, \( \Lambda_j \subset Z \), because \( \Lambda_j \not\subset \emptyset \) implies that \( \Lambda_j \not\subset \emptyset \) due to the generality of \( \psi \). Finally, the restriction \( \psi|_Z : Z \to C \) is a birational morphism, because the projection \( \psi \) is general. Hence, \( \deg(Z) = j \) and \( Z \) contains at least \( |\Lambda_j| > j(2r-1) \) points of \( \Sigma \). The latter contradicts Proposition 32.

**Corollary 39.** The inequality \( k \geq 2 \) holds.

For every \( \Lambda_j \neq \emptyset \) let \( \Xi_j \subset \mathbb{P}^3 \) be a base locus of the linear system of hypersurfaces of degree \( j \) in \( \mathbb{P}^3 \) passing through all the points in \( \Lambda_j \). For \( \Lambda_j = \emptyset \) put \( \Xi_j = \emptyset \). Then \( \Xi_j \) is a finite set by Lemma 38 and \( \Lambda_j \subset \Xi_j \) by construction.

**Lemma 40.** Suppose that \( \Xi_j \neq \emptyset \). Then the points of the subset \( \Xi_j \subset \mathbb{P}^3 \) impose independent linear conditions on homogeneous forms on \( \mathbb{P}^3 \) of degree \( 3(j-1) \).

**Proof.** The claim follows from Lemma 22. \( \square \)
Corollary 41. Suppose that $\Lambda_i^j \neq \emptyset$. Then the points of the subset $\Lambda_i^j \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3(j - 1)$.

Lemma 42. Suppose that $\bar{\Sigma} = \emptyset$. Then there is a hypersurface in $\mathbb{P}^3$ of degree $3r - 4$ containing $\Sigma \setminus P$ and not passing through the point $P$.

Proof. The set $\Sigma$ is a disjoint union $\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_i^j$, and there is a unique set $\Lambda_i^j$ containing the point $P$. In particular, $P \in \Xi_i^j$. On the other hand, the union $\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Xi_i^j$ is not necessarily disjoint. Thus, a priori the point $P$ can be contained in many sets $\Xi_i^j$.

For every $\Xi_i^j \neq \emptyset$ containing $P$ there is a hypersurface of degree $3(j - 1)$ that passes through $\Xi_i^j \setminus P$ and does not pass through the point $P$ by Lemma 40. For every $\Xi_i^j \neq \emptyset$ not containing the point $P$ there is a hypersurface $F_i^j \subset \mathbb{P}^3$ of degree $3(j - 1)$ that passes through $\Xi_i^j \setminus P$ and does not pass through the point $P$. Let

$$F = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} F_i^j \subset \mathbb{P}^3$$

be a possibly reducible hypersurface of degree $\sum_{i=k}^{l} 3(i - 1)c_i$. Then $F$ passes through all the points of the set $\Sigma \setminus P$ and does not pass through the point $P$. Moreover, we have

$$\text{deg}(F) = \sum_{i=k}^{l} 3(i - 1)c_i < \sum_{i=k}^{l} 3ic_i < r < 3r - 4$$

by Corollary 37 which implies the claim. □

Let $\hat{\Sigma} = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_i^j$ and $\bar{\Sigma} = \Sigma \setminus \hat{\Sigma}$. Then $\Sigma = \hat{\Sigma} \cup \bar{\Sigma}$ and $\psi(\hat{\Sigma}) = \hat{\Sigma} \subset \Pi$. Therefore, we proved Theorem 3 in the extreme cases: $\hat{\Sigma} = \emptyset$ and $\bar{\Sigma} = \emptyset$. Now we must combine the proofs of the Lemmas 34 and 42 to prove Theorem 3 in the case when $\hat{\Sigma} \neq \emptyset$ and $\bar{\Sigma} \neq \emptyset$.

Remark 43. Arguing as in the proof of Lemma 42 we obtain a hypersurface $F \subset \mathbb{P}^3$ of degree $\sum_{i=k}^{l} 3(i - 1)c_i$ that passes through all the points of the subset $\hat{\Sigma} \setminus P \subset \Sigma$ and does not pass through the point $P \in \Sigma$.

Put $d = 3r - 4 - \sum_{i=k}^{l} 3(i - 1)c_i$. Let us check that the subset $\hat{\Sigma} \subset \Pi \cong \mathbb{P}^2$ and the number $d$ satisfy all the hypotheses of Theorem 29. We may assume that $\emptyset \neq \hat{\Sigma} \subset \Sigma$.

Lemma 44. The inequality $d \geq 6$ holds.

Proof. The claim is implied by Corollary 37 and $c_k \geq 1$. □
Lemma 45. The inequality $|\Sigma| \leq \frac{d^2 + 9d + 10}{6}$ holds.

Proof. To prove the claim it is enough to show that

$$2(2r-1)(r-3) \sum_{i=k}^{l} ic_i \leq (3r-4-\sum_{i=k}^{l} 3(i-1)c_i)^2 + 9(3r-4-\sum_{i=k}^{l} 3(i-1)c_i) + 10,$$

because $|\Sigma| < \frac{(2r-1)}{3}(r - 3 \sum_{i=k}^{l} ic_i)$ by the inequality \[36\] However, we have

$$(3r-4-\sum_{i=k}^{l} 3(i-1)c_i)^2 + 9(3r-4-\sum_{i=k}^{l} 3(i-1)c_i) + 10$$

$$> (2r-4+3c_k)^2 + 9(2r-4+3c_k) + 10,$$

because $c_k \geq 1$ and $\sum_{i=k}^{l} 3ic_i < r$ by Corollary \[37\] Thus, we have

$$(2r-4+3c_k)^2 + 9(2r-4+3c_k) + 10 \geq (2r-1)^2 + 9(2r-1) + 10 = 4r^2 + 14r + 2,$$

which implies $4r^2 + 14r + 2 > 4r^2 - 2r > 2(2r-1)(r - 3 \sum_{i=k}^{l} ic_i).$ \[□\]

Lemma 46. At most $t(d + 3 - t) - 2$ points of $\Sigma$ lie on a curve in $\mathbb{P}^2$ of degree $t \leq \frac{d + 3}{2}$.

Proof. In the case $t = 1$ the claim is implied by Proposition \[32\] Corollary \[37\] and the inequality $c_k \geq 1$. Hence, we may assume that $t > 1$.

The points of the subset $\bar{\Sigma} \subset \mathbb{P}^2$ satisfy the property $\nabla$. Thus, at most $(2r-1)t$ of the points of $\bar{\Sigma}$ lie on a curve in $\mathbb{P}^2$ of degree $t$. Therefore, to conclude the proof it is enough to show that $t(d + 3 - t) - 2 \geq (2r-1)t$ for all $t \leq \frac{d + 3}{2}$. Moreover, it is enough to prove the latter inequality only for $t > 1$ such that $t(d + 3 - t) - 2 < |\bar{\Sigma}|$, because otherwise the corresponding condition on the points of the set $\bar{\Sigma}$ is vacuous.

Now we have

$$t(d + 3 - t) - 2 \geq t(2r-1) \iff t(r - \sum_{i=k}^{l} 3(i-1)c_i - t) \geq 2$$

$$\iff r - \sum_{i=k}^{l} 3(i-1)c_i > t,$$

because $t > 1$. We may assume that the inequalities $t(d + 3 - t) - 2 < |\bar{\Sigma}|$ and

$$r - \sum_{i=k}^{l} 3(i-1)c_i \leq t \leq \frac{d + 3}{2}$$
hold. Let \( g(x) = x(d + 3 - x) - 2 \). Then \( g(x) \) is increasing for \( x < \frac{d+3}{2} \). Therefore, the inequality \( g(t) \geq g(r - \sum_{i=k}^l 3(i - 1)c_i) \) holds. Hence, we have

\[
\frac{(2r - 1)}{3}(r - 3 \sum_{i=k}^l ic_i) > |\Sigma| > g(t) \geq (r - \sum_{i=k}^l 3(i - 1)c_i)(2r - 1) - 2
\]

and \((2r - 1)(6 \sum_{i=k}^l ic_i - 2r) + 6 - 9 \sum_{i=k}^l c_i(2r - 1) > 0\). Now we have

\[
(2r - 1)(6 \sum_{i=k}^l ic_i - 2r) + 6 - 9 \sum_{i=k}^l c_i(2r - 1) < 6 - 9 \sum_{i=k}^l c_i(2r - 1)
\]

because \( \sum_{i=k}^l 3ic_i < r \) by Corollary 37. The obtained contradiction implies the claim.

Therefore, we can apply Theorem 29 to the blowup of the hyperplane \( \Pi \) at the points of the set \( \Sigma \setminus \hat{P} \subset \Pi \) due to Lemmas 44, 45 and 46. The application of Theorem 29 gives a curve \( C \subset \Pi \cong \mathbb{P}^2 \) of degree \( 3r - 4 - \sum_{i=k}^l 3(i - 1)c_i \) that passes through all the points of the set \( \Sigma \setminus \hat{P} \) and does not pass through the point \( \hat{P} = \psi(P) \). It should be pointed out that the subset \( \Sigma \subset \Sigma' \) may not contain \( \hat{P} \in \Sigma' \). Namely, \( \hat{P} \in \Sigma \) if and only if \( P \in \Sigma \).

Let \( G \subset \mathbb{P}^3 \) be a cone over the curve \( C \) with the vertex \( O \), where \( O \in \mathbb{P}^3 \) is the center of the projection \( \psi : \mathbb{P}^3 \to \Pi \). Then \( G \) is a hypersurface of degree \( 3r - 4 - \sum_{i=k}^l 3(i - 1)c_i \) that passes through the points of \( \Sigma \setminus P \) and does not pass through \( P \). On the other hand, we already have the hypersurface \( F \subset \mathbb{P}^3 \) of degree \( \sum_{i=k}^l 3(i - 1)c_i \) that passes through the points of \( \Sigma \setminus P \) and does not pass through \( P \). Therefore, \( F \cup G \subset \mathbb{P}^3 \) is a hypersurface of degree \( 3r - 4 \) that passes through all the points of the set \( \Sigma \setminus P \) and does not pass through the point \( P \in \Sigma \). Hence, we proved Theorem 4.

4. Hypersurfaces in \( \mathbb{P}^4 \)

In this section we prove Theorem 4. Let \( V \subset \mathbb{P}^4 \) be a nodal hypersurface of degree \( n \) such that \( |\text{Sing}(V)| \leq \frac{(n-1)^2}{4} \). In order to prove Theorem 4 it is enough to show that the nodes of the hypersurface \( V \) impose independent linear conditions on homogeneous forms of degree \( 2n - 5 \) on \( \mathbb{P}^4 \) due to Proposition 2. Moreover, we may always assume that \( n \geq 4 \), because in the case \( n \leq 3 \) the required claim is trivial.

Definition 47. The points of a subset \( \Gamma \subset \mathbb{P}^r \) satisfy the property \( \star \) if at most \( k(n - 1) \) points of the set \( \Gamma \) can lie on a curve in \( \mathbb{P}^r \) of degree \( k \in \mathbb{N} \).
Let $\Sigma = \text{Sing}(V) \subset P^4$. Then arguing as in the proof of Proposition 32 we obtain the following result.

**Proposition 48.** The points of the subset $\Sigma \subset P^4$ satisfy the property $\star$.

Fix a point $P \in \Sigma$. To prove that the points of $\Sigma \subset P^4$ impose independent linear conditions on homogeneous forms on $P^4$ of degree $2n - 5$ it is enough to construct a hypersurface in $P^4$ of degree $2n - 5$ that passes through the points of the set $\Sigma \setminus P$ and does not pass through $P \in \Sigma$. Arguing as in the proof of Lemma 33 we obtain the following result.

**Lemma 49.** Suppose that the subset $\Sigma \subset P^4$ is contained in some two-dimensional linear subspace $\Pi \subset P^4$. Then there is a hypersurface in $P^4$ of degree $2n - 5$ that passes through the points of the set $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$.

Fix a general two-dimensional linear subspace $\Pi \subset P^4$. Let $\psi : P^4 \to \Pi$ be a projection from a general line $L \subset P^4$, $\Sigma' = \psi(\Sigma)$ and $\hat{P} = \psi(P)$. Then $\psi|_\Sigma : \Sigma \to \Sigma'$ is a bijection.

**Lemma 50.** Suppose that the points in $\Sigma' \subset \Pi$ satisfy the property $\star$. Then there is a hypersurface in $P^4$ of degree $2n - 5$ containing $\Sigma \setminus P$ and not passing through $P \in \Sigma$.

**Proof.** Arguing as in the proof of Lemma 33 we prove the existence of a curve $C \subset \Pi$ of degree $2n - 5$ that passes through $\Sigma' \setminus \hat{P}$ and does not pass through $\hat{P}$. Let $Y \subset P^4$ be a three-dimensional cone over the curve $C$ with the vertex $L \subset P^4$. Then $Y \subset P^4$ is the required hypersurface. □

We may assume that the points of $\Sigma' \subset \Pi$ do not satisfy the property $\star$. Arguing as in the proof of Theorem 3 we can construct disjoint subsets $\Lambda^j_i \subset \Sigma$ for $j = r, \ldots, l \geq r$ such that the inequality $|\Lambda^j_i| > j(n - 1)$ holds, all the points of the subset $\Lambda^j_i = \psi(\Lambda^j_i) \subset \Sigma'$ lie on an irreducible reduced curve in $\Pi \cong P^2$ of degree $j$, and all the points in the subset

$$\Sigma = \Sigma' \setminus \bigcup_{j=r}^l \bigcup_{i=1}^{c_j} \Lambda^j_i \subset \Pi \cong P^2$$

satisfy the property $\star$, where $c_j \geq 0$ is a number of subsets $\Lambda^j_i$ and $c_r > 0$.

In particular,

$$0 \leq |\Sigma| < \frac{(n-1)^2}{4} - \sum_{i=r}^l c_i(n-1)i = \frac{n-1}{4}(n-1 - 4 \sum_{i=r}^l ic_i). \quad (51)$$

**Corollary 52.** The inequality $\sum_{i=r}^l ic_i < \frac{n-1}{4}$ holds.

For every $\Lambda^j_i \neq \emptyset$ let $\Xi^j_i \subset P^4$ be a base locus of the linear system of hypersurfaces of degree $j$ in $P^4$ passing through all the points in $\Lambda^j_i$; otherwise put $\Xi^j_i = \emptyset$. Then $\Xi^j_i$ is a finite set (see the proof of Lemma 33) and, in
particular, \( r \geq 2 \). Moreover, \( \Lambda^1_j \subseteq \Xi^1_j \) by definition of \( \Xi^1_j \subseteq \mathbb{P}^4 \). Therefore, the points of the set \( \Xi^1_j \subseteq \mathbb{P}^4 \) impose independent linear conditions on the homogeneous forms on \( \mathbb{P}^4 \) of degree \( 4(j - 1) \) by Lemma 22. In particular, the points of the set \( \Lambda^1_j \) impose independent linear conditions on the homogeneous forms on \( \mathbb{P}^4 \) of degree \( 4(j - 1) \).

Let \( \Sigma = \bigcup_{j=r}^l \Lambda^1_j \) and \( \hat{\Sigma} = \Sigma \setminus \Sigma \). Then \( \Sigma = \hat{\Sigma} \cup \Sigma \) and \( \psi(\Sigma) = \Sigma \subset \Pi \). Then arguing as in the proof of Lemma 42 we obtain a hypersurface in \( \mathbb{P}^4 \) of degree \( 2n - 5 \) containing all points in \( \Sigma \setminus P \) and not passing through \( P \) in the case when \( \hat{\Sigma} = \emptyset \). Actually, arguing as in the proof of Lemma 42 we prove the existence of a hypersurface \( F \subset \mathbb{P}^4 \) of degree \( \sum_{i=r}^l 4(i - 1)c_i \) that passes through all the points of the subset \( \hat{\Sigma} \setminus P \subset \Sigma \) and does not pass through the point \( P \in \Sigma \). Put \( d = 2n - 5 - \sum_{i=r}^l 4(i - 1)c_i \). Let us check that the subset \( \hat{\Sigma} \subset \Pi \) and the number \( d \) satisfy all hypotheses of Theorem 29. We may assume \( \hat{\Sigma} \neq \emptyset \) and \( \Sigma \neq \emptyset \).

**Lemma 53.** The inequality \( d \geq 5 \) holds.

**Proof.** We have \( \sum_{i=r}^l 4ic_i < n - 1 \) by Corollary 52. Thus, \( d > n - 4 + 4c_r \geq n \geq 4 \).

**Lemma 54.** The inequality \( |\hat{\Sigma}| \leq \frac{d^2 + 9d + 10}{6} \) holds.

**Proof.** Suppose that \( |\hat{\Sigma}| > \frac{d^2 + 9d + 10}{6} \). Then

\[
3(n - 1)(n - 1 - 4 \sum_{i=r}^l ic_i) \\
> 2(2n - 5 - \sum_{i=r}^l 4(i - 1)c_i)^2 + 18(2n - 5 - \sum_{i=r}^l 4(i - 1)c_i) + 20,
\]

because \( |\hat{\Sigma}| < \frac{\pi - 1}{4}(n - 1 - 4 \sum_{i=r}^l ic_i) \). Let \( A = \sum_{i=r}^l ic_i \) and \( B = \sum_{i=r}^l c_i \). Then

\[
3(n - 1)^2 - 12(n - 1)A > 2(2n - 1)^2 - 16A(2n - 1) + 32A^2 + 18(2n - 1) - 72A + 20,
\]

because \( B \geq c_r \geq 1 \). Thus, for \( n \geq 4 \) we have

\[
3(n - 1)^2 > 8n^2 + 28n + 4 + 32A^2 - A(20n + 68) > 5n^2 + 12n + 23 > 3(n - 1)^2,
\]

because \( A < \frac{\pi - 1}{4} \) by Corollary 52.

**Lemma 55.** At most \( k(d + 3 - k) - 2 \) points of \( \hat{\Sigma} \) lie on a curve in \( \mathbb{P}^3 \) of degree \( k \leq \frac{d + 3}{4} \).

**Proof.** The case \( k = 1 \) follows from Corollary 52 and \( c_r \geq 1 \). Therefore, we may assume that \( k > 1 \). The points of \( \hat{\Sigma} \subset \mathbb{P}^2 \) satisfy the property \( \star \). So, at most \( k(n - 1) \) of the points of \( \hat{\Sigma} \) lie on a curve of degree \( k \). To conclude the proof is enough to prove that

\[
k(d + 3 - k) - 2 \geq k(n - 1)
\]
for all \( k \leq \frac{d+3}{2} \). Moreover, it is enough to prove the latter inequality only for such natural numbers \( k > 1 \) that the inequality \( k(d + 3 - k) - 2 < |\Sigma| \) holds, because otherwise the corresponding condition on the points of the set \( \Sigma \) is vacuous.

The inequality \( k(d + 3 - k) - 2 \geq k(n - 1) \) holds if and only if \( n - 1 - \sum_{i=r}^l 4(i-1)c_i > k \), because \( k > 1 \). Thus, we may assume that the inequalities \( k(d + 3 - k) - 2 < |\Sigma| \) and

\[
n - 1 - \sum_{i=r}^l 4(i-1)c_i \leq k \leq \frac{d+3}{2}
\]

hold. Let \( g(x) = x(d + 3 - x) - 2 \). Then \( g(x) \) is increasing for \( x < \frac{d+3}{2} \). Thus, we have

\[
\frac{(n-1)}{4}(n-1 - 4 \sum_{i=r}^l ic_i) > |\Sigma| > g(k) \geq g(n-1 - \sum_{i=r}^l 4(i-1)c_i).
\]

Let \( A = \sum_{i=r}^l ic_i \) and \( B = \sum_{i=r}^l c_i \). Then the inequality

\[
\frac{(n-1)}{4}(n-1 - 4A) > 4(n-1 - 4A + 4B)(n-1) - 2
\]

holds. Therefore, we have

\[
n - 1 - 4A > 4(n-1) - 16A + 16B - 1 > 4(n-1) - 16A,
\]

because \( B \geq c_r \geq 1 \). Thus, \( 4A > n - 1 \), but \( A < \frac{n-1}{4} \) by Corollary 52.

Now we can apply Corollary 30 to get a curve \( C \subset \Pi \) of degree \( 2n - 5 - \sum_{i=r}^l 4(i-1)c_i \) that passes through the points of the subset \( \Sigma \setminus \hat{P} \subset \Pi \cong \mathbb{P}^2 \) and does not pass through the point \( \hat{P} \subset \Sigma' \). Let \( G \subset \mathbb{P}^4 \) be a cone over \( C \) with the vertex in the center \( L \) of the projection \( \psi : \mathbb{P}^4 \dashrightarrow \Pi \). Then \( G \subset \mathbb{P}^4 \) is a hypersurface of degree \( 2n - 5 - \sum_{i=r}^l 4(i-1)c_i \) that passes through \( \Sigma \setminus P \) and does not pass through \( P \). However, we already have the hypersurface \( F \subset \mathbb{P}^4 \) of degree \( \sum_{i=r}^l 4(i-1)c_i \) that passes through \( \Sigma \setminus P \) and does not pass through \( P \). Therefore, \( F \cup G \subset \mathbb{P}^4 \) is a hypersurface of degree \( 2n - 5 \) that passes through \( \Sigma \setminus P \) and does not pass through \( P \in \Sigma \). Thus, Theorem 4 is proved.

5. Calabi-Yau 3-folds

In this section we prove Proposition 9. Let \( \pi : X \to \mathbb{P}^3 \) be a double cover branched over a nodal hypersurface \( S \subset \mathbb{P}^3 \) of degree 8 such that \( |\text{Sing}(S)| \leq 25 \), and let \( V \subset \mathbb{P}^4 \) be a nodal hypersurface of degree 5 such that \( |\text{Sing}(V)| \leq 14 \). Due to Proposition 2 it is enough to prove that the nodes of the surface
S \subset \mathbb{P}^3 \) impose independent linear conditions on homogeneous forms of degree 8 on \( \mathbb{P}^3 \) and the nodes of the hypersurface \( V \subset \mathbb{P}^4 \) impose independent linear conditions on homogeneous forms of degree 5 on \( \mathbb{P}^4 \).

Let \( \Sigma = \text{Sing}(S) \subset \mathbb{P}^3 \) and \( \Lambda = \text{Sing}(V) \subset \mathbb{P}^4 \). Arguing as in the proof of Proposition 32 we see that no more than 7 points of \( \Sigma \) and no more than 4 points of \( \Lambda \) can lie on a curve of degree \( k = 1, 2, 3 \). Let us fix a point \( P \in \Sigma \) and a point \( Q \in \Lambda \). To prove Proposition 9 we must construct a hypersurface in \( \mathbb{P}^3 \) of degree 8 that passes through \( \Sigma \setminus P \) and does not pass through \( P \) and a hypersurface in \( \mathbb{P}^4 \) of degree 5 that passes through \( \Lambda \setminus Q \) and does not pass through the point \( Q \).

Take general two-dimensional linear subspaces \( \Pi \subset \mathbb{P}^3 \) and \( \Omega \subset \mathbb{P}^4 \). Let \( \psi : \mathbb{P}^3 \to \Pi \) be a projection from a general point \( P \in \mathbb{P}^3 \), and \( \xi : \mathbb{P}^4 \to \Omega \) be a projection from a general line \( L \subset \mathbb{P}^4 \). Put \( \Sigma' = \psi(\Sigma) \), \( \hat{P} = \psi(P) \), \( \Lambda' = \xi(\Lambda) \) and \( \hat{Q} = \xi(Q) \). Then no more than 7 points of the subset \( \Sigma' \subset \Pi \) and no more than 5 points of the subset \( \Lambda' \subset \Omega \) can lie on a line (see the proof of Lemma 38).

**Lemma 56.** No more than 14 points of the subset \( \Sigma' \subset \Pi \) and no more than 10 points of the subset \( \Lambda' \subset \Omega \) can lie on a conic.

**Proof.** Let \( \Phi \subset \Lambda \) be a subset with \( |\Phi| > 10 \). Consider the projection \( \xi \) as a composition of a projection \( \alpha : \mathbb{P}^4 \to \mathbb{P}^3 \) from some point \( A \in L \) and a projection \( \beta : \mathbb{P}^3 \to \Omega \) from the point \( B = \alpha(L) \). The generality in the choice of the line \( L \) implies the generality of the projections \( \alpha \) and \( \beta \). We claim that the points of the sets \( \alpha(\Phi) \) and \( \xi(\Phi) \) do not lie on a conic in \( \mathbb{P}^3 \) and \( \Omega \cong \mathbb{P}^2 \) respectively.

Suppose that the points of \( \alpha(\Phi) \) lie on a conic \( C \subset \mathbb{P}^3 \). Then conic \( C \) is irreducible. Let \( D \) be a linear system of quadric hypersurfaces in \( \mathbb{P}^4 \) passing through the points of \( \Phi \). As in the proof of Lemma 38 we see that the base locus of \( D \) is zero-dimensional, because the points of \( \Phi \subset \mathbb{P}^4 \) do not lie on a conic in \( \mathbb{P}^4 \). Take a cone \( W \subset \mathbb{P}^4 \) over the conic \( C \) with the vertex \( A \). Then \( \Phi \subset W \). Moreover, we have \( \Phi \subset \text{Bs}(D|_W) \) and \( D|_W \) has no base components. Let \( D_1 \) and \( D_2 \) be general curves in \( D|_W \). Then

\[
8 = D_1 \cdot D_2 \geq \sum_{\omega \in \Phi} \text{mult}_\omega(D_1) \text{mult}_\omega(D_2) \geq |\Phi| > 10,
\]

which is a contradiction. Therefore, the points of \( \alpha(\Phi) \) do not lie on a conic in \( \mathbb{P}^3 \).

Suppose that the points of \( \xi(\Phi) \) lie on a conic \( C \subset \Pi \). Then we can repeat the previous arguments to get a contradiction. The rest of the claim can be proved in a similar way. \( \square \)
Now we can apply Corollary 30 to the subset $\Lambda' \setminus \hat{Q} \subset \mathbb{F}^2$ and point $\hat{Q}$ to prove the existence of a hypersurface in $\mathbb{P}^4$ of degree 5 that passes through $\Lambda \setminus Q$ and does not pass through the point $Q \in \Lambda$ (see the proof of Theorem 4).

Similarly, in the case when at most 22 points of the subset $\Sigma' \subset \Pi$ can lie on a cubic curve in $\Pi \cong \mathbb{P}^2$ we can construct a hypersurface in $\mathbb{P}^3$ of degree 8 that passes through the points of the set $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$.

**Lemma 57.** Suppose that there is a subset $\Upsilon \subset \Sigma$ such that $|\Upsilon| > 22$ and all the points of the set $\psi(\Upsilon)$ lie on a cubic curve in $\Pi \cong \mathbb{P}^2$. Then there is a hypersurface in $\mathbb{P}^3$ of degree 8 that passes through the points of $\Sigma \setminus P$ and does not pass through the point $P$.

**Proof.** Let $\mathcal{H}$ be a linear system of cubic hypersurfaces in $\mathbb{P}^3$ passing through the points of the set $\Upsilon$. Then the base locus of $\mathcal{H}$ is zero-dimensional by Lemma 38.

Suppose $P \in \Upsilon$. Then there is a hypersurface $F \subset \mathbb{P}^3$ of degree 6 that passes through the points of $\Upsilon \setminus P$ and does not pass through the point $P$ by Lemma 22. On the other hand, the subset $\Sigma \setminus \Upsilon \subset \mathbb{P}^3$ contains at most 2 points. Hence, there is a quadric $G \subset \mathbb{P}^3$ that passes through the points of $\Sigma \setminus \Upsilon$ and does not pass through $P$. Thus, $F \cup G$ is the required hypersurface.

In the case when $P \notin \Upsilon$ and $P \in \operatorname{Bs}(\mathcal{H})$ we can repeat every step of the proof of the previous case. In the case when $P \notin \Upsilon$ and $P \notin \operatorname{Bs}(\mathcal{H})$ there is a cubic hypersurface in $\mathbb{P}^3$ that passes through the points of $\Upsilon$ and does not pass through the point $P$, which easily implies the existence of the required hypersurface.

Hence, Proposition 9 is proved.

### 6. Non-isolated singularities

In this section we prove Theorem 13 but we omit the proof of Theorem 14 because it is similar. Let $\tau : U \rightarrow \mathbb{P}^s$ be a double cover branched over a hypersurface $F \subset \mathbb{P}^s$ of degree $2r$ such that $D_1 \cap \cdots \cap D_{s-3}$ is a $\mathbb{Q}$-factorial nodal 3-fold, where $D_i$ is a general divisor in $|\tau^*(\mathcal{O}_{\mathbb{P}^s}(1))|$ and $s \geq 4$. Let $D$ be a general divisor in $|\tau^*(\mathcal{O}_{\mathbb{P}^s}(1))|$. We must show that the group $\operatorname{Cl}(U)$ is generated by $D$. Note that $U$ is normal.

**Lemma 58.** The group $H^1(\mathcal{O}_U(-nD))$ for $n > 0$ vanishes.

**Proof.** In the case when the singularities of the variety $U$ are mild enough the claim is implied by the Kawamata-Viehweg vanishing (see [22]). In general let us prove the claim by induction on $s$. Suppose that $s = 4$. Then we have
an exact sequence of sheaves
\[ 0 \to \mathcal{O}_U(-(n+1)D) \to \mathcal{O}_U(-nD) \to \mathcal{O}_D(-nD) \to 0 \]
for any \( n \in \mathbb{Z} \). Therefore, we have an exact sequence of the cohomology groups
\[ 0 \to H^1(\mathcal{O}_U(-(n+1)D)) \to H^1(\mathcal{O}_U(-nD)) \to H^1(\mathcal{O}_D(-nD)) \to \cdots \]
for \( n > 0 \). However, the 3-fold \( D \) is nodal by assumption. Thus, the group \( H^1(\mathcal{O}_D(-nD)) \) vanishes by the Kawamata-Viehweg vanishing. Hence, we have
\[ H^1(\mathcal{O}_U(-D)) \cong H^1(\mathcal{O}_U(-2D)) \cong \cdots \cong H^1(\mathcal{O}_U(-nD)) \]
for every \( n > 0 \). On the other hand, the group \( H^1(\mathcal{O}_U(-nD)) \) vanishes for \( n \gg 0 \) by the lemma of Enriques-Severi-Zariski (see [32]).

Suppose that \( s > 4 \). Then we have an exact sequence of sheaves
\[ 0 \to \mathcal{O}_U(-(n+1)D) \to \mathcal{O}_U(-nD) \to \mathcal{O}_D(-nD) \to 0 \]
for any \( n \in \mathbb{N} \). Therefore, we have an exact sequence of the cohomology groups
\[ 0 \to H^1(\mathcal{O}_U(-(n+1)D)) \to H^1(\mathcal{O}_U(-nD)) \to H^1(\mathcal{O}_D(-nD)) \to \cdots \]
for \( n > 0 \). However, the group \( H^1(\mathcal{O}_D(-nD)) \) vanishes by the induction. Hence,
\[ H^1(\mathcal{O}_U(-D)) \cong H^1(\mathcal{O}_U(-2D)) \cong \cdots \cong H^1(\mathcal{O}_U(-nD)) \]
for \( n > 0 \), but \( H^1(\mathcal{O}_U(-nD)) = 0 \) for \( n \gg 0 \) by the lemma of Enriques-Severi-Zariski.

Consider a Weil divisor \( G \) on \( U \). Let us prove by induction on \( s \) that \( G \sim kD \) for some \( k \in \mathbb{Z} \). Suppose that \( s = 4 \). Then the 3-fold \( D \) is nodal and \( \mathbb{Q} \)-factorial by assumption. Moreover, the group \( \text{Cl}(D) \) is generated by the class of the divisor \( R|_D \), where \( R \) is a general divisor in \( |D| \). Thus, there is an integer \( k \) such that we have the equivalence \( G|_D \sim kR|_D \). Let \( \Delta = G - kR \). We may assume that \( \Delta \not\sim 0 \).

The sequence of sheaves
\[ 0 \to \mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(D) \to \mathcal{O}_U(\Delta) \to \mathcal{O}_D \to 0 \]
is exact, because \( \mathcal{O}_U(\Delta) \) is locally free in the neighborhood of \( D \).

Every section \( \eta \in H^0(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D)) \) gives an effective Weil divisor \( S \) different from the divisor \( D \), because the divisor \( D \) is the pull-back of a sufficiently general hyperplane on \( \mathbb{P}^n \). Thus, the divisor \( S \cap D \) is effective and
$S \cap D \sim -D|_D$, which is impossible. Hence, we have $H^0(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D)) = 0$. Therefore, the sequence

$$0 \to H^0(\mathcal{O}_U(\Delta)) \to H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D))$$

is exact.

**Lemma 59.** The group $H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-nD))$ vanishes for every $n > 0$.

**Proof.** The sheaf $\mathcal{O}_U(\Delta)$ is reflexive (see [18]). Thus, there is an exact sequence of sheaves

$$0 \to \mathcal{O}_U(\Delta) \to E \to F \to 0$$

where $E$ is a locally free sheaf and $F$ is a torsion free sheaf. Hence, the sequence of groups

$$H^0(F \otimes \mathcal{O}_U(-nD)) \to H^1(\mathcal{O}_D(\Delta) \otimes \mathcal{O}_D(-nD)) \to H^1(E \otimes \mathcal{O}_U(-nD))$$

is exact. However, for $n \gg 0$ the cohomology group $H^0(F \otimes \mathcal{O}_U(-nD))$ vanishes because the sheaf $F$ is torsion free, and the cohomology group $H^1(E \otimes \mathcal{O}_U(-nD))$ vanishes by the lemma of Enriques-Severi-Zariski. Therefore, $H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-nD)) = 0$ for $n \gg 0$.

Now consider an exact sequence of sheaves

$$0 \to \mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-(n+1)D) \to \mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-nD) \to \mathcal{O}_D(-nD) \to 0$$

and the induced sequence of cohomology groups

$$0 \to H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-(n+1)D)) \to H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-nD)) \to H^1(\mathcal{O}_D(-nD)) \to \cdots$$

for $n > 0$. Then the group $H^1(\mathcal{O}_D(-nD))$ vanishes by Lemma [18] Hence, we have

$$H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D)) \cong H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-2D)) \cong \cdots$$

$$\cong H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-nD))$$

for $n > 0$, but we already proved that $H^1(\mathcal{O}_U(-nD))$ vanishes for $n \gg 0$. □

Therefore, $H^0(\mathcal{O}_U(\Delta)) \cong \mathbb{C}$. Similarly $H^0(\mathcal{O}_U(-\Delta)) \cong \mathbb{C}$. Thus, the Weil divisor $\Delta$ is rationally equivalent to zero and $G \sim kD$ in the case $s = 4$, which contradicts our assumption $\Delta \not\sim 0$. Thus, the case $s = 4$ is done.

Suppose that $s > 4$. By the induction we may assume that the group $\text{Cl}(D)$ is generated by the class of the divisor $R|_D$, where $R$ is a general divisor in $|D|$. Thus, there is an integer $k$ such that $G|_D \sim kR|_D$. Put $\Delta = G - kR$. Then the sequence of sheaves

$$0 \to \mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D) \to \mathcal{O}_U(\Delta) \to \mathcal{O}_D \to 0$$
is exact, because $O_U(\Delta)$ is locally free in the neighborhood of $D$. Therefore, the sequence

$$0 \rightarrow H^0(O_U(\Delta)) \rightarrow H^0(O_D) \rightarrow H^1(O_U(\Delta) \otimes O_U(-D))$$

is exact. However, the proof of Lemma 59 holds for $s > 4$. Thus, the cohomology group $H^1(O_U(\Delta) \otimes O_U(-D))$ vanishes. Hence, $H^0(O_U(-\Delta)) \cong \mathbb{C}$. The same arguments prove that $H^0(O_U(-\Delta)) \cong \mathbb{C}$. Therefore, the Weil divisor $\Delta$ is rationally equivalent to zero and $G \sim kD$. Thus, we proved Theorem 13.

7. Birational rigidity

In this section we prove Proposition 15. Let $\xi : Y \rightarrow \mathbb{P}^4$ be a double cover branched over a hypersurface $F \subset \mathbb{P}^4$ of degree 8 such that the hypersurface $F$ is smooth outside of a smooth curve $C \subset F$, the singularity of the hypersurface $F$ in a sufficiently general point of the curve $C$ is locally isomorphic to the singularity

$$x_1^2 + x_2^2 + x_3^2 = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x_1, x_2, x_3]),$$

the singularities of $F$ in other points of $C$ are locally isomorphic to the singularity

$$x_1^2 + x_2^2 + x_3^2 x_4 = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),$$

and a general 3-fold in $|-K_Y|$ is $\mathbb{Q}$-factorial. Then $Y$ is a Fano 4-fold with terminal singularities and $-K_Y \sim \xi^*(O_{\mathbb{P}^4}(1))$. Moreover, $\text{Cl}(Y)$ and $\text{Pic}(Y)$ are generated by the divisor $-K_Y$ by Theorem 13. Hence, $Y$ is a Mori fibration (see [22]). We must prove that the 4-fold $Y$ is a unique Mori fibration birational to $Y$ and $\text{Bir}(Y) = \text{Aut}(Y)$. It is well known that the latter implies the finiteness of the group $\text{Bir}(Y)$.

Suppose that either $Y$ is not birationally rigid or $\text{Bir}(Y) \neq \text{Aut}(Y)$. Then Theorem 28 implies the existence of a linear system $\mathcal{M}$ on $Y$ such that $\mathcal{M}$ has no fixed components and the singularities of $(X, \frac{1}{n}\mathcal{M})$ are not canonical, where $\mathcal{M} \sim -nK_Y$. Thus, there is a rational number $\mu < \frac{1}{n}$ such that $(X, \mu\mathcal{M})$ is not canonical, i.e., $\text{CS}(Y, \mu\mathcal{M}) \neq \emptyset$.

Let $Z$ be an element of the set $\text{CS}(Y, \mu\mathcal{M})$. Then $\text{mult}_Z(\mathcal{M}) > n$.

**Lemma 60.** The subvariety $Z \subset Y$ is not a smooth point of $Y$.

**Proof.** Suppose $Z$ is a smooth point of $Y$. Then $\text{mult}_Z(\mathcal{M}) > 4n^2$ by Theorem 25 and

$$2n^2 = \mathcal{M}^2 \cdot H_1 \cdot H_2 \geq \text{mult}_Z(\mathcal{M})\text{mult}_Z(H_1)\text{mult}_Z(H_2) > 4n^2$$

for general divisors $H_1$ and $H_2$ in $|-K_Y|$ containing $Z$, which is a contradiction. \hfill \Box
Lemma 6.1. The subvariety $Z \subset Y$ is not a singular point of $Y$.

Proof. Let $\xi(Z) = O$. Then $O$ is a singular point of the hypersurface $F \subset \mathbb{P}^4$. Therefore, the point $O$ is contained in the curve $C \subset F$ by assumption. There are two possible cases, i.e., either the singularity of $F$ in the point $O$ is locally isomorphic to the singularity

$$x_1^2 + x_2^2 + x_3^3 = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),$$

or the singularity of $F$ in the point $O$ is locally isomorphic to the singularity

$$x_1^2 + x_2^2 + x_3^3 x_4 = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),$$

where $x_1 = x_2 = x_3$ are local equations of the curve $C \subset F$. Let us call the former case ordinary and the latter case non-ordinary.

Let $X$ be a sufficiently general divisor in the linear system $| - K_Y |$ passing through the point $Z$. Then the double cover $\xi$ induces the double cover $\tau : X \rightarrow \mathbb{P}^3$ ramified along an octic surface. The singularities of $X \setminus Z$ are ordinary double points. Moreover, $Z$ is an ordinary double point of $X$ in the ordinary case. In the non-ordinary case the singularity of the 3-fold $X$ at the point $Z$ is locally isomorphic to

$$x_1^2 + x_2^2 + x_3^3 + x_4^3 = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]).$$

Let $\mathcal{D} = \mathcal{M}|_X$ and $H = -K_Y|_X$. Then $\mathcal{D}$ has no fixed components, $\mathcal{D} \sim nH$ and we have $Z \in \text{LCS}(X, \mu \mathcal{D})$ by Proposition 24. In particular, $Z \in \text{CS}(X, \mu \mathcal{D})$.

Let $f : V \rightarrow X$ be a blowup of $Z$, $E = f^{-1}(Z)$ and $\mathcal{H}$ be a proper transform of the linear system $\mathcal{D}$ on $V$. Then $V$ is smooth in the neighborhood of $E$ and $E$ is isomorphic to a quadric surface in $\mathbb{P}^3$. In the ordinary case $E$ is smooth. In the non-ordinary case the quadric surface $E$ has one singular point $P \in E$, i.e., the surface $E$ is isomorphic to a quadric cone in $\mathbb{P}^3$. Note that $K_V \sim E$.

Let $\text{mult}_Z(\mathcal{D}) \in \mathbb{N}$ such that $\mathcal{H} \sim f^*(nH) - \text{mult}_Z(\mathcal{D})E$. Then $\text{mult}_Z(\mathcal{D}) > n$ in the ordinary case by Theorem 24. On the other hand, in the non-ordinary case we have the inequality $\text{mult}_Z(\mathcal{D}) > \frac{2}{3}$ due to Proposition 27.

By construction the linear system $| f^*(H) - E |$ is free and gives a morphism $\psi : V \rightarrow \mathbb{P}^2$ such that $\psi = \phi \circ \tau \circ f$, where $\phi : \mathbb{P}^3 \rightarrow \mathbb{P}^2$ is a projection from the point $O$. Moreover, the restriction $\psi|_E : E \rightarrow \mathbb{P}^2$ is a double cover. Let $L$ be a sufficiently general fiber of the morphism $\psi$. Then $L$ is a smooth curve of genus 2 and $L \cdot E = L \cdot f^*(H) = 2$. Thus,

$$L \cdot \mathcal{H} = L \cdot f^*(nH) - \text{mult}_Z(\mathcal{D})L \cdot E = 2n - 2\text{mult}_Z(\mathcal{D}) \geq 0,$$

because $\mathcal{H}$ has no base components. Hence, $\text{mult}_Z(\mathcal{D}) \leq n$. In particular, the ordinary case is impossible and it remains to eliminate the non-ordinary case.
The inequalities $\mathrm{mult}_Z(D) \leq n$ and $\mu < \frac{1}{n}$, the equivalence

$$K_V + \mu \mathcal{H} \sim f^*(K_X + \mu D) + (1 - \mu \mathrm{mult}_Z(D))E$$

and $Z \in \mathcal{CS}(X, \mu D)$ imply the existence of a proper irreducible subvariety $S \subset E$ such that $S \in \mathcal{CS}(V, \mu \mathcal{H} + (\mu \mathrm{mult}_Z(D) - 1)E)$. In particular, $S \in \mathcal{CS}(V, \mu \mathcal{H})$.

Suppose that $S$ is a curve. Then $\mathrm{mult}_S(\mathcal{H}) > n$. Let $L_\omega$ be a fiber of $\psi$ passing through a general point $\omega \in S$. Then $L_\omega$ describes a divisor in $V$ when we vary $\omega$ on $S$. Hence,

$$L_\omega \cdot \mathcal{H} = L_\omega \cdot f^*(nH) - \mathrm{mult}_Z(D)L_\omega \cdot E = 2n - 2\mathrm{mult}_Z(D) \geq \mathrm{mult}_S(L_\omega)\mathrm{mult}_S(\mathcal{H}) > n,$$

which contradicts the inequality $\mathrm{mult}_Z(D) > \frac{n}{2}$.

Therefore, $S$ is a point on $E$. Then $\mathrm{mult}_S(\mathcal{H}) > n$ and $\mathrm{mult}_S(\mathcal{H}^2) > 4n^2$ by Theorem 25, because $S$ is smooth on $V$. It is easy to see that the point $S$ is not a vertex $P$ of the quadric cone $E$, because the numerical intersection of a general ruling of $E$ with a general divisor in $\mathcal{H}$ is equal to $\mathrm{mult}_Z(D) \leq n$.

Let $\Gamma$ be a fiber of the morphism $\psi$ that passes through the point $S$, and let $D$ be a general divisor in the linear system $|f^*(H) - E|$ that passes through the point $S$. Then $\Gamma \subset D$. Note that $\Gamma$ may be reducible and singular, but we always have the inequality $\mathrm{mult}_S(\Gamma) \leq 2$, because $\tau \circ f(\Gamma)$ is a line passing through the point $O$ and $\tau |_{f(\Gamma)}$ is a double cover.

Suppose that $\Gamma$ is irreducible. Let $\mathcal{H}^2 = \lambda \Gamma + T$, where $\lambda \in \mathbb{N}$ and $T$ is a one-cycle such that $\Gamma \not\subset \mathrm{Supp}(T)$. Then the inequalities

$$\mathrm{mult}_S(T) > 4n^2 - \lambda \mathrm{mult}_S(\Gamma) \geq 4n^2 - 2\lambda$$

hold. On the other hand, the inequalities

$$\mathrm{mult}_S(T) \leq \mathrm{mult}_S(T)\mathrm{mult}_S(D) \leq T \cdot D = \mathcal{H}^2 \cdot D = 2n^2 - \mathrm{mult}_Z(2D) < \frac{7}{4}n^2$$

hold. Thus, we have $\lambda > \frac{9}{8}n^2$. Let $\tilde{D}$ be a general divisor in $|f^*(H)|$. Then

$$2n^2 = \tilde{D} \cdot \mathcal{H}^2 \geq \lambda \Gamma \cdot \tilde{D} = 2\lambda > \frac{9}{4}n^2,$$

which is a contradiction.

Therefore, the fiber $\Gamma$ is reducible. Then $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_i$ is a smooth rational curve such that $\tau \circ f(\Gamma_1) = \tau \circ f(\Gamma_2)$ is a line in $\mathbb{P}^3$ containing point $O$. Let

$$\mathcal{H}^2 = \lambda_1 \Gamma_1 + \lambda_2 \Gamma_2 + T,$$
where $\lambda_1 \in \mathbb{N}$ and $T$ is a one-cycle such that $\Gamma_i \not\subset \text{Supp}(T)$. Then the inequalities
\[
\frac{7}{4} n^2 > 2n^2 - \text{mult}_f^2(D) \geq T \cdot D \geq \text{mult}_S(T) > 4n^2 - \lambda_1 - \lambda_2
\]
hold. Thus, $\lambda_1 + \lambda_2 > \frac{9}{4} n^2$. Hence, we have
\[
2n^2 = \hat{D} \cdot H^2 \geq \lambda_1 \Gamma_1 \cdot \hat{D} + \lambda_2 \Gamma_2 \cdot \hat{D} = \lambda_1 + \lambda_2 > \frac{9}{4} n^2
\]
for a general divisor $\hat{D} \in |f^*(H)|$, which is a contradiction. \hfill \Box

**Lemma 62.** The subvariety $Z \subset Y$ is not a curve.

*Proof.* Suppose $Z$ is a curve. Let $X$ be a general divisor in $|−K_Y|$ and $P$ be a point in the intersection $Z \cap X$. Then $X$ is a nodal Calabi-Yau 3-fold. The point $P$ is smooth on the 3-fold $X$ if and only if $Z \not\subset \text{Sing}(X)$. In the case $Z \subset \text{Sing}(X)$ the point $P$ is an ordinary double point on $X$. Moreover, $P \subset \mathbb{C}S(X, \mu D)$, where $D = M|_X$. In the case when the point $P$ is smooth on $X$ we can proceed as in the proof of Lemma 60 to get a contradiction. In the case when the point $P$ is an ordinary double point on $X$ we can proceed as in the proof of Lemma 61 to get a contradiction. \hfill \Box

**Lemma 63.** The subvariety $Z \subset Y$ is not a surface.

*Proof.* Suppose $Z$ is a surface. Then $\text{mult}_Z(M) > n$. Let $V$ be a general divisor in the linear system $|−K_Y|$, $S = Z \cap V$ and $D = M|_V$. Then $V$ is a nodal Calabi-Yau 3-fold, the linear system $D$ has no base components, $S \subset V$ is an irreducible reduced curve and $\text{mult}_S(D) > n$. The double cover $\xi$ induces a double cover $\tau : V \to \mathbb{P}^3$ ramified along a nodal hypersurface $G \subset \mathbb{P}^3$ of degree 8.

Take a sufficiently general divisor $H$ in $|\tau^*(O_{\mathbb{P}^3}(1))|$. Then
\[
2n^2 = D^2 \cdot H \geq \text{mult}_Z^2(D)S \cdot H > n^2 S \cdot H,
\]
which implies $S \cdot H = 1$. Hence, $\tau(S)$ is a line in $\mathbb{P}^3$ and $\tau|_S$ is an isomorphism.

Suppose that $\tau(S) \not\subset G$. Then there is a smooth rational curve $\hat{S} \subset V$ such that $S \neq \hat{S}$ and $\tau(S) = \tau(\hat{S})$. Take a sufficiently general surface $D \subset |\tau^*(O_{\mathbb{P}^3}(1))|$ passing through the curve $S$. Then $D$ is smooth outside of $S \cap \hat{S}$. Moreover, the surface $D$ is smooth in every point of $S \cap \hat{S}$ that is smooth on $V$, and $D$ has an ordinary double point in every point of $S \cap \hat{S}$ that is an ordinary double point on $V$. On the other hand, at most 4 nodes of the hypersurface $G \subset \mathbb{P}^3$ can lie on the line $\tau(S)$, i.e., $|\text{Sing}(D)| \leq 4$. The sub-adjunction formula (see [22], [23]) implies
\[
(K_D + \hat{S})|_\hat{S} = K_\hat{S} + \text{Diff}_\hat{S}(0)
\]
and $\deg(\text{Diff}_\hat{S}(0)) = \frac{k}{2}$, where $k = |\text{Sing}(D)|$. Thus, the self-intersection $\hat{S}^2$ is negative on the surface $D$, because $K_D \cdot \hat{S} = 1$. Put $H = D|_D$. A priori the
linear system $\mathcal{H}$ can have a base component. However, the generality in the choice of $D$ implies

$$\mathcal{H} = \text{mult}_S(D)S + \text{mult}_{\tilde{S}}(D)\tilde{S} + \mathcal{B}$$

where $\mathcal{B}$ is a linear system on $D$ having no base components. Moreover, the equivalence

$$(n - \text{mult}_{\tilde{S}}(D))\tilde{S} \sim_Q (\text{mult}_S(D) - n)S + \mathcal{B}$$

holds, because $\tilde{S} + S \sim D|_D$ and $\mathcal{H} \sim nD|_D$. Therefore, the inequality $\tilde{S}^2 < 0$ implies the inequality $\text{mult}_{\tilde{S}}(D) > n$. Take a general divisor $H$ in $|\tau^*(O_{\mathbb{P}^3}(1))|$. Then

$$2n^2 = D^2 \cdot H \geq \text{mult}_{\tilde{S}}(D) \cdot H + \text{mult}_{\tilde{S}}(D) \cdot \tilde{S} \cdot H > n^2S \cdot H + n^2\tilde{S} \cdot H = 2n^2,$$

which is a contradiction.

Therefore, we have $\tau(S) \subset G$. Let $O$ be a general point on $\tau(S)$ and $\Pi$ be a hyperplane in $\mathbb{P}^3$ that is tangent to $G$ at the point $O$. Consider a sufficiently general line $L \subset \Pi$ passing through $O$. Let $\tilde{L} = \tau^{-1}(L)$ and $\tilde{O} = \tau^{-1}(O)$. Then $\tilde{L}$ is singular at $\tilde{O}$. Therefore, the curve $\tilde{L}$ is contained in the base locus of the linear system $D$, because otherwise

$$2n = \tilde{L} \cdot D \geq \text{mult}_{\tilde{O}}(\tilde{L}) \cdot \text{mult}_{\tilde{O}}(D) \geq 2\text{mult}_S(D) > 2n,$$

which is impossible. On the other hand, the curve $\tilde{L}$ describes a divisor in $V$ when we vary the line $L$ in $\Pi$. The latter is impossible, because $D$ has no base components.

Therefore, Proposition 15 is proved. □

References


**Steklov Institute of Mathematics, 8 Gubkin street, Moscow 117966, Russia**

**E-mail address:** cheltsov@yahoo.com

**Current address:** School of Mathematics, The University of Edinburgh, Kings Buildings, Mayfield Road, Edinburgh EH9 3JZ, UK

**E-mail address:** I.Cheltsov@ed.ac.uk