

Fano threefolds with infinite automorphism groups

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Abstract. We classify smooth Fano threefolds with infinite automorphism groups.

Keywords: Fano threefolds, automorphism groups.

§ 1. Introduction

One of the most important mathematical results obtained by Iskovskikh is a classification of smooth Fano threefolds of Picard rank 1; see [1], [2]. In fact, it was he who introduced the notion of Fano variety. Using Iskovskikh's classification, Mori and Mukai classified all smooth Fano threefolds of higher Picard ranks; see [3] as well as [4] for a minor revision. Fano varieties currently play a central role in both algebraic and complex geometry and provide key examples for number theory and mathematical physics.

Let \mathbb{k} be an algebraically closed field of characteristic zero. Automorphism groups of (smooth) Fano varieties are important from the point of view of birational geometry, in particular, from the point of view of birational automorphism groups of rationally connected varieties. There is not much to study in dimension 1: the only one-dimensional smooth Fano variety is the projective line, whose automorphism group is $\mathrm{PGL}_2(\mathbb{k})$. The automorphism groups of two-dimensional Fano varieties (also known as del Pezzo surfaces) are already rather tricky. However, the structure of del Pezzo surfaces is classically known and their automorphism groups have been described in detail; see [5]. There are several non-trivial examples of smooth Fano threefolds of Picard rank 1 with infinite automorphism groups; see [6], Proposition 4.4, [7] and [8]. The following recent result was obtained in [9].

Theorem 1.1 ([9], Theorem 1.1.2). *Let X be a smooth Fano threefold of Picard rank 1. Then the group $\mathrm{Aut}(X)$ is finite unless one of the following cases occurs.*

- 1) *The threefold X is the projective space \mathbb{P}^3 , and $\mathrm{Aut}(X) \cong \mathrm{PGL}_4(\mathbb{k})$.*
- 2) *The threefold X is a smooth quadric Q in \mathbb{P}^4 , and $\mathrm{Aut}(X) \cong \mathrm{PSO}_5(\mathbb{k})$.*
- 3) *The threefold X is the smooth section V_5 of the Grassmannian $\mathrm{Gr}(2, 5) \subset \mathbb{P}^9$ by a linear subspace of dimension 6, and $\mathrm{Aut}(X) \cong \mathrm{PGL}_2(\mathbb{k})$.*

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- 4) *The threefold X has Fano index 1 and anticanonical degree 22. Here the following cases may occur:*
- (i) $X = X_{22}^{\text{MU}}$ is the Mukai–Umemura threefold and $\text{Aut}(X) \cong \text{PGL}_2(\mathbb{k})$;
 - (ii) $X = X_{22}^{\text{a}}$ is a unique threefold of this type such that the connected component of the identity in $\text{Aut}(X)$ is isomorphic to \mathbb{k}^+ ;
 - (iii) $X = X_{22}^{\text{m}}(u)$ is a threefold in a certain one-dimensional family such that the connected component of the identity in $\text{Aut}(X)$ is isomorphic to \mathbb{k}^\times .

Smooth Fano threefolds of Picard rank greater than 1 are more numerous than those of Picard rank 1. Some of them have large automorphism groups. They have been studied by various authors. For example, Batyrev classified all smooth toric Fano threefolds in [10]; see also [11]. Süß [12] classified all smooth Fano threefolds admitting a faithful action of a two-dimensional torus. Smooth Fano threefolds with a faithful action of $(\mathbb{k}^+)^3$ were classified in [13] (compare with [14], Theorem 6.1). Threefolds with an action of $\text{SL}_2(\mathbb{k})$ were studied in [7], [15]–[17]. Some results on higher-dimensional Fano varieties with infinite automorphism groups can be found in [18] and [19].

The goal of this paper is to provide a classification similar to that given by Theorem 1.1 in the case of higher Picard rank. Given a smooth Fano threefold X , we label it (or rather its deformation family) by a pair of numbers

$$\mathfrak{J}(X) = \rho.N,$$

where ρ is the Picard number of X and N is the number of X in the classification tables in [3], [20] and [4]. Note that the most complete list of smooth Fano threefolds is contained in [4].

The main result of this paper is the following theorem.

Theorem 1.2. *The following assertions hold.*

- (i) *The group $\text{Aut}(X)$ is infinite for every smooth Fano threefold X with*

$$\mathfrak{J}(X) \in \{1.15, 1.16, 1.17, 2.26, \dots, 2.36, 3.9, 3.13, \dots, 3.31, 4.2, \dots, 4.12, 5.1, 5.2, 5.3, 6.1, 7.1, 8.1, 9.1, 10.1\}.$$

- (ii) *There are smooth Fano threefolds with infinite automorphism group when*

$$\mathfrak{J}(X) \in \{1.10, 2.20, 2.21, 2.22, 3.5, 3.8, 3.10, 4.13\},$$

while the general threefold in each of these families has a finite automorphism group.

- (iii) *The group $\text{Aut}(X)$ is finite when X is contained in any of the remaining families of smooth Fano threefolds.*

In fact, we describe all connected components of the identity in the automorphism groups of all smooth Fano threefolds; see Table 1.

If X is a smooth Fano threefold and its automorphism group is infinite, then X is rational. However, unlike the case of Picard rank 1, such a threefold X may have a non-trivial Hodge number $h^{1,2}(X)$. The simplest example is given by the blow-up of \mathbb{P}^3 along a plane cubic, that is, a smooth Fano threefold X with $\mathfrak{J}(X) = 2.28$. In this case one has $h^{1,2}(X) = 1$. Using Theorem 1.2, we obtain the following result.

Corollary 1.3 (compare with Corollary 1.1.3 in [9]). *Let X be a smooth Fano threefold with $h^{1,2}(X) > 0$. Then $\text{Aut}(X)$ is infinite if and only if*

$$\mathfrak{I}(X) \in \{2.28, 3.9, 3.14, 4.2\}.$$

If, furthermore, X has no extremal contractions to threefolds with non-Gorenstein singularities, then $\mathfrak{I}(X) = 4.2$.

Remark 1.4. Let X be a smooth Fano threefold of Picard rank at least 2. Suppose that X cannot be obtained from a smooth Fano threefold by blowing up a smooth irreducible curve. In this case, X is said to be primitive; see Definition 1.3 in [21]. Moreover, by Theorem 1.6 in [21], there is a (standard) conic bundle $\pi: X \rightarrow S$ such that either $S \cong \mathbb{P}^2$ and the Picard rank of X is 2, or $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the Picard rank of X is 3. Write Δ for the discriminant curve of this conic bundle. We also suppose that the group $\text{Aut}(X)$ is infinite. Using Theorem 1.2 and the classification of primitive Fano threefolds in [21], we see that either the arithmetic genus of Δ is 1, or $\Delta = \emptyset$ and π is a \mathbb{P}^1 -bundle. Furthermore, in the former case it follows from the classification that X is a divisor of bidegree $(1, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ (and in particular X also has the structure of a \mathbb{P}^1 -bundle). When $\Delta = \emptyset$ and $S \cong \mathbb{P}^2$, the same classification (or [22], [23]) yields that X is either a divisor of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, or the projectivization of an indecomposable vector bundle of rank 2 on \mathbb{P}^2 (in this case X is toric). Finally, if $\Delta = \emptyset$ and $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, then either $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, or X is the blow-up of a quadric cone in \mathbb{P}^4 at its vertex.

Information about automorphism groups of smooth complex Fano threefolds can be used to study the problem of the existence of Kähler–Einstein metrics on them. For example, the Matsushima obstruction implies that a smooth Fano threefold admits no such metric if its automorphism group is not reductive; see [24]. Thus, by inspecting our Table 1 we obtain the following result.

Corollary 1.5. *If X is a smooth complex Fano threefold with*

$$\mathfrak{I}(X) \in \{2.28, 2.30, 2.31, 2.33, 2.35, 2.36, 3.16, 3.18, 3.21, \dots, 3.24, \\ 3.26, 3.28, \dots, 3.31, 4.8, \dots, 4.12\},$$

then X admits no Kähler–Einstein metric. Each family of smooth Fano threefolds with $\mathfrak{I}(X) \in \{1.10, 2.21, 2.26, 3.13\}$ contains a variety admitting no Kähler–Einstein metric.

If a smooth complex Fano threefold X has an infinite automorphism group, then the vanishing of its Futaki invariant (a character of the Lie algebra of holomorphic vector fields) is necessary for the existence of a Kähler–Einstein metric on X ; see [25]. This gives us a simple obstruction for the existence of Kähler–Einstein metrics. When X is toric, the vanishing of its Futaki invariant is also sufficient for X to be Kähler–Einstein; see [26]. In this case, the Futaki invariant vanishes if and only if the barycentre of the canonical weight polytope is at the origin. The Futaki invariants of smooth non-toric Fano threefolds admitting a faithful action of a two-dimensional torus were computed in [12], Theorem 1.1. We hope that the results in the present paper can be used to compute the Futaki invariants of other smooth Fano threefolds with infinite automorphism groups.

When a smooth complex Fano variety X is acted on by a reductive group G , one can use Tian's α -invariant

$$\alpha_G(X)$$

to prove the existence of a Kähler–Einstein metric on X . To be precise, if

$$\alpha_G(X) > \frac{\dim(X)}{\dim(X) + 1},$$

then X is Kähler–Einstein by [27]. The larger the group G , the larger the value taken by the α -invariant $\alpha_G(X)$. This simple criterion was used in [28]–[32] to prove the existence of Kähler–Einstein metrics on many smooth Fano threefolds.

Example 1.6. In the notation of Theorem 1.1, one has

$$\alpha_{\mathrm{PGL}_2(\mathbb{k})}(X_{22}^{\mathrm{MU}}) = \frac{5}{6};$$

see Theorem 3 in [29]. Moreover, $\alpha_{\mathrm{PGL}_2(\mathbb{k})}(V_5) = 5/6$ by [30]. Thus, the Fano threefolds X_{22}^{MU} and V_5 are both Kähler–Einstein (when $\mathbb{k} = \mathbb{C}$).

Thanks to the proof of the Yau–Tian–Donaldson conjecture in [33], there is an algebraic-geometric characterization of smooth complex Fano varieties admitting Kähler–Einstein metrics in terms of K -stability. However, this criterion appears to be ineffective in concrete cases since proving K -stability requires checking the positivity of the Donaldson–Futaki invariant for all possible degenerations of X . In the recent paper [34], Datar and Székelyhidi proved that given an action of a reductive group G on X , it suffices to consider only G -equivariant degenerations. For many smooth Fano threefolds, this equivariant version of K -stability was checked in [35]. We hope that our Theorem 1.2 can be used to check this in other cases.

In some applications, it is useful to know the full automorphism group of a Fano variety (compare with [36]). However, a complete classification of automorphism groups is available only in dimension two [5] and in some particular cases in dimension three; see [6], Proposition 4.4, [9], § 5, and [37]. For example, at the moment we have no information about the possible automorphism groups of smooth cubic threefolds.

There are several interesting examples of smooth Fano varieties with infinite automorphism groups in dimension four; see, for example, [38]. However, the situation here is very far from a classification similar to our Theorem 1.2.

The plan of the paper is as follows. We study automorphisms of smooth Fano threefolds by splitting them into several groups depending on their construction (some of them admit several natural constructions). In § 2 we list preliminary facts needed in the paper. In § 3 we study Fano varieties that are either direct products of lower-dimensional varieties or cones over them. In §§ 4, 5 and 6 we study Fano threefolds that are blow-ups of \mathbb{P}^3 , the smooth quadric, and V_5 respectively. In §§ 7 and 8 we study blow-ups and double covers of the flag variety $W = \mathrm{Fl}(1, 2; 3)$ and products of projective spaces respectively. The next three sections are devoted to three particularly remarkable families of varieties. In § 9 we study the blow-up of a smooth quadric along a twisted quartic. This variety is more complicated from our point of view than those in § 5, so we treat it separately. In § 10 we study divisors of bidegree $(1, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. In § 11 we study smooth Fano threefolds X

with $\mathfrak{I}(X) = 3.2$. Note that the varieties in this family are trigonal, but this family was omitted from Iskovskih’s list of smooth trigonal Fano threefolds in [2]. Finally, in §12 we study the remaining sporadic families of Fano threefolds.

Summarizing §§3–12, we give a table (Appendix A) containing an explicit description of the connected components of the identity in the infinite automorphism groups arising in Theorem 1.2.

Notation and conventions. All varieties are assumed to be projective and defined over an algebraically closed field \mathbb{k} of characteristic zero. Given a variety Y and a subvariety Z , we write $\text{Aut}(Y; Z)$ for the stabilizer of Z in $\text{Aut}(Y)$. The connected components of the identity in $\text{Aut}(Y)$ and $\text{Aut}(Y; Z)$ are denoted by $\text{Aut}^0(Y)$ and $\text{Aut}^0(Y; Z)$ respectively.

Throughout the paper we write \mathbb{F}_n for the Hirzebruch surface

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)).$$

In particular, \mathbb{F}_1 is the blow-up of \mathbb{P}^2 at a point. The blow-up of \mathbb{P}^3 at a point is denoted by V_7 . We write Q for the smooth three-dimensional quadric and V_5 for the smooth section of the Grassmannian $\text{Gr}(2, 5) \subset \mathbb{P}^9$ by a linear subspace of dimension 6. We denote the flag variety $\text{Fl}(1, 2; 3)$ of complete flags in a three-dimensional vector space by W . It can also be described as the projectivization of the tangent bundle of \mathbb{P}^2 or as a smooth divisor of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.

We write $\mathbb{P}(a_0, \dots, a_n)$ for the weighted projective space with weights a_0, \dots, a_n . Note that $\mathbb{P}(1, 1, 1, 2)$ is the cone in \mathbb{P}^6 over a Veronese surface in \mathbb{P}^5 . One has an isomorphism

$$\text{Aut}(\mathbb{P}(1, 1, 1, 2)) \cong (\mathbb{k}^+)^6 \rtimes ((\text{GL}_3(\mathbb{k}) \times \mathbb{k}^\times)/\mathbb{k}^\times), \tag{1.7}$$

where \mathbb{k}^\times is embedded in the product on the right-hand side by

$$t \mapsto (t \cdot \text{Id}_{\text{GL}_3(\mathbb{k})}, t^2);$$

compare with Proposition A.2.5 in [39].

Let $n > k_1 > \dots > k_r$ be positive integers. We write $\text{PGL}_{n;k_1, \dots, k_r}(\mathbb{k})$ for the parabolic subgroup in $\text{PGL}_n(\mathbb{k})$ consisting of the images of matrices in $\text{GL}_n(\mathbb{k})$ that preserve a (possibly incomplete) flag of subspaces of dimensions k_1, \dots, k_r . In particular, the group $\text{PGL}_{n;k}(\mathbb{k})$ is isomorphic to the group of $n \times n$ matrices with a zero lower-left rectangle of size $(n - k) \times k$. One has an isomorphism

$$\text{PGL}_{n;k}(\mathbb{k}) \cong (\mathbb{k}^+)^{k(n-k)} \rtimes ((\text{GL}_k(\mathbb{k}) \times \text{GL}_{n-k}(\mathbb{k}))/\mathbb{k}^\times). \tag{1.8}$$

Similarly, one has

$$\begin{aligned} &\text{PGL}_{n;k_1, k_2}(\mathbb{k}) \\ &\cong ((\mathbb{k}^+)^{k_1(n-k_1)} \rtimes (\mathbb{k}^+)^{k_2(k_1-k_2)}) \rtimes ((\text{GL}_{k_2}(\mathbb{k}) \times \text{GL}_{k_1-k_2}(\mathbb{k}) \times \text{GL}_{n-k_1}(\mathbb{k}))/\mathbb{k}^\times). \end{aligned} \tag{1.9}$$

In (1.8) and (1.9), the subgroup \mathbb{k}^\times is embedded in each factor as the group of scalar matrices. We denote the group $\text{PGL}_{2;1}(\mathbb{k}) \cong \mathbb{k}^+ \rtimes \mathbb{k}^\times$ by B for brevity. It is a Borel subgroup of $\text{PGL}_2(\mathbb{k})$.

When $n \geq 5$, we write $\text{PSO}_{n;k}(\mathbb{k})$ for the parabolic subgroup of $\text{PSO}_n(\mathbb{k})$ preserving an isotropic linear subspace of dimension k . In particular, $\text{PSO}_{n;1}(\mathbb{k})$ is the stabilizer of a point on a smooth $(n - 2)$ -dimensional quadric \mathcal{Q} in the group $\text{Aut}^0(\mathcal{Q})$. One can then check that $\text{PSO}_{n;1}(\mathbb{k})$ is isomorphic to the connected component of the identity in the automorphism group of the cone over a smooth $(n - 4)$ -dimensional quadric. Therefore, we have

$$\text{PSO}_{5;1}(\mathbb{k}) \cong (\mathbb{k}^+)^3 \rtimes (\text{SO}_3(\mathbb{k}) \times \mathbb{k}^\times) \cong (\mathbb{k}^+)^3 \rtimes (\text{PGL}_2(\mathbb{k}) \times \mathbb{k}^\times)$$

and

$$\text{PSO}_{6;1}(\mathbb{k}) \cong (\mathbb{k}^+)^4 \rtimes ((\text{SO}_4(\mathbb{k}) \times \mathbb{k}^\times) / \{\pm 1\}). \tag{1.10}$$

We write $\text{PGL}_{(2,2)}(\mathbb{k})$ for the image in $\text{PGL}_4(\mathbb{k})$ of the group of block-diagonal matrices in $\text{GL}_4(\mathbb{k})$ with two 2×2 blocks. One has an isomorphism

$$\text{PGL}_{(2,2)}(\mathbb{k}) \cong (\text{GL}_2(\mathbb{k}) \times \text{GL}_2(\mathbb{k})) / \mathbb{k}^\times,$$

where the group \mathbb{k}^\times is embedded in each factor $\text{GL}_2(\mathbb{k})$ as the group of scalar matrices. The group $\text{PGL}_{(2,2)}(\mathbb{k})$ acts on \mathbb{P}^3 preserving two skew lines. We write $\text{PGL}_{(2,2);1}(\mathbb{k})$ for the parabolic subgroup of $\text{PGL}_{(2,2)}(\mathbb{k})$ which is the stabilizer of a point on one of these lines. It is the image in $\text{PGL}_4(\mathbb{k})$ of the group of block-diagonal matrices in $\text{GL}_4(\mathbb{k})$ with two 2×2 blocks one of which is upper-triangular. Thus, one has

$$\text{PGL}_{(2,2);1}(\mathbb{k}) \cong (\text{GL}_2(\mathbb{k}) \times \tilde{\text{B}}) / \mathbb{k}^\times,$$

where $\tilde{\text{B}}$ is the subgroup of upper-triangular matrices in $\text{GL}_2(\mathbb{k})$, and \mathbb{k}^\times is embedded in each factor as the group of scalar matrices.

The explicit descriptions of Fano threefolds in [3], [20] and [4] will be used without special reference. We sometimes change these descriptions slightly for simplicity.

Occasionally, we need to compute the dimension of families of Fano varieties with certain properties considered *up to isomorphism*. Note that Fano varieties generally have no moduli spaces with nice properties (compare with Lemma 6.5 below). To appropriately describe a family parametrizing Fano varieties up to isomorphism, one has to deal with moduli stacks and coarse moduli spaces of these stacks. However, this is not our goal and we actually make only weaker claims in such cases. Namely, by saying that a family of Fano varieties is d -dimensional up to isomorphism, we mean that the corresponding parameter space \mathcal{P} (which is always obvious from the description of the family) contains an open subset where the natural automorphism group of \mathcal{P} acts with equidimensional orbits, and the corresponding quotient is d -dimensional. The question of the irreducibility of families will not be considered in such cases. We point out that the dimensions of families of Fano threefolds can often be computed in a straightforward manner. In several non-obvious cases we provide computations for the reader’s convenience.

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§ 2. Preliminaries

Any Fano variety X with at most Kawamata log terminal singularities admits only finitely many extremal contractions. In particular, this implies that every extremal contraction is $\text{Aut}^0(X)$ -equivariant and, for every birational extremal contraction $\pi: X \rightarrow Y$, the action of $\text{Aut}^0(X)$ on Y is faithful. In the latter case, $\text{Aut}^0(X)$ is naturally embedded in $\text{Aut}^0(Y; Z)$, where $Z \subset Y$ is the image of the exceptional set of π . We will use these facts many times throughout the paper without special reference.

The following assertion is well known to experts.

Lemma 2.1. *Let Y be a Fano variety with at most Kawamata log terminal singularities, and let $Z \subset Y$ be an irreducible subvariety. Suppose that there is a very ample divisor D on Y such that Z is not contained in any effective divisor linearly equivalent to D . Then the action of $\text{Aut}^0(Y; Z)$ on Z is faithful. Furthermore, if Z is non-ruled, then $\text{Aut}(Y; Z)$ is finite.*

Proof. Since the Picard group of Y is finitely generated, the linear system of D determines an $\text{Aut}^0(Y)$ -equivariant embedding $\varphi: Y \rightarrow \mathbb{P}^N$ such that the automorphisms in $\text{Aut}^0(Y)$ are induced by automorphisms of \mathbb{P}^N . Note that Y is not contained in a hyperplane in \mathbb{P}^N , and the same holds for Z by hypothesis. Thus, $\text{Aut}^0(Y)$ coincides with $\text{Aut}^0(\mathbb{P}^N; Y)$, and $\text{Aut}^0(Y; Z)$ acts faithfully on Z . We note that $\text{Aut}^0(Y)$ and $\text{Aut}^0(Y; Z)$ are linear algebraic groups. Thus, if the group $\text{Aut}^0(Y; Z)$ were non-trivial, it would contain a subgroup isomorphic to either \mathbb{k}^\times or \mathbb{k}^+ . In both cases this would imply that Z is covered by rational curves. We conclude that if Z is non-ruled, then the group $\text{Aut}^0(Y; Z)$ is trivial and, therefore, $\text{Aut}(Y; Z)$ is finite. \square

The following result is classical; see, for example, [40], §§ 8, 9.

Theorem 2.2. *Let X be a smooth del Pezzo surface of degree $d = K_X^2$. Then the following assertions hold.*

- If $d = 9$, then $\text{Aut}(X) \cong \text{PGL}_3(\mathbb{k})$.
- If $d = 8$, then either $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k}) \times \text{PGL}_2(\mathbb{k})$, or $X \cong \mathbb{F}_1$ and $\text{Aut}(X) \cong \text{PGL}_{3;1}(\mathbb{k})$.
- If $d = 7$, then $\text{Aut}^0(X) \cong \text{B} \times \text{B}$.
- If $d = 6$, then $\text{Aut}^0(X) \cong (\mathbb{k}^\times)^2$.
- If $d \leq 5$, then the group $\text{Aut}(X)$ is finite.

Let $\pi: X \rightarrow S$ be a proper flat morphism, where X is a threefold and S is a surface. If a general fibre of π is isomorphic to \mathbb{P}^1 , then we say that π is a *conic bundle*. We say that π is a *standard* conic bundle if X and S are smooth and

$$\text{Pic}(X) \cong \pi^* \text{Pic}(S) \oplus \mathbb{Z};$$

see, for example, [41], Definition 1.3, [42], Definition 1.12, or [43], § 3. In this case, the morphism $\pi: X \rightarrow S$ is a Mori fibre space. Let $\Delta \subset S$ be the discriminant locus of π , that is, the locus of points $P \in S$ such that the scheme fibre $\pi^{-1}(P)$ is not isomorphic to \mathbb{P}^1 .

Remark 2.3 (see [42], Corollary 1.11). If $\pi: X \rightarrow S$ is a standard conic bundle, then Δ is a reduced curve (possibly reducible) with at most nodes as singularities.

In this case, the fibre of π over P is isomorphic to a reducible reduced conic in \mathbb{P}^2 provided that $P \in \Delta$ and P is not a singular point of Δ . When P is a singular point of Δ , the fibre over P is isomorphic to a non-reduced conic in \mathbb{P}^2 .

The following assertion will be used in §§ 6 and 10.

Lemma 2.4. *Let $C \subset \mathbb{P}^2$ be an irreducible nodal cubic. Then the group $\text{Aut}(\mathbb{P}^2; C)$ is finite.*

Proof. The action of $\text{Aut}(\mathbb{P}^2; C)$ on C is faithful by Lemma 2.1. This action lifts to the normalization of C , so that $\text{Aut}(\mathbb{P}^2; C)$ acts on \mathbb{P}^1 preserving a pair of points. Therefore, we have $\text{Aut}^0(\mathbb{P}^2; C) \subset \mathbb{k}^\times$.

Suppose that $\text{Aut}^0(\mathbb{P}^2; C) \cong \mathbb{k}^\times$. Then the action of \mathbb{k}^\times lifts to the projective space

$$\mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))^\vee) \cong \mathbb{P}^3.$$

Moreover, it preserves a twisted cubic \tilde{C} (which is the image of \mathbb{P}^1 embedded by the latter linear system) and a point $P \in \mathbb{P}^3$ outside \tilde{C} (such that the projection of \tilde{C} from P provides the original embedding $C \subset \mathbb{P}^2$). Since the curve C is nodal, there is a unique line L in \mathbb{P}^3 that contains P and intersects \tilde{C} at two points, P_1 and P_2 . The line L is \mathbb{k}^\times -invariant. Furthermore, the points P, P_1 and P_2 are \mathbb{k}^\times -invariant, so that the action of \mathbb{k}^\times on L is trivial. This means that the group \mathbb{k}^\times (or an appropriate central extension of it) acts on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$ with certain weights w_1, w_2, w_3 , and w_4 , at least two of which coincide. This in turn means that \mathbb{P}^3 cannot contain a \mathbb{k}^\times -invariant twisted cubic. The resulting contradiction shows that the group $\text{Aut}^0(\mathbb{P}^2; C)$ is trivial. \square

The following assertion will be used in §§ 3, 5, 6 and 11.

Lemma 2.5. *Let H be a hyperplane section of a smooth n -dimensional quadric $Y \subset \mathbb{P}^{n+1}$, where $n \geq 2$, and let $\Gamma \subset \text{Aut}(Y)$ be the pointwise stabilizer of H . Then Γ is finite and every automorphism of H is induced by an automorphism of Y .*

Proof. Denote the homogeneous coordinates on \mathbb{P}^{n+1} by x_0, \dots, x_{n+1} . Since the group $\text{Aut}(Y)$ acts transitively on $\mathbb{P}^n \setminus Y$ and on Y , we can assume that H is given by $x_0 = 0$ and Y is given by

$$x_0^2 + \dots + x_{n+1}^2 = 0$$

when H is smooth, and by

$$x_0x_1 + x_2^2 + \dots + x_{n+1}^2 = 0$$

when H is singular (in this case the corresponding hyperplane is tangent to Y at the point $[0 : 1 : 0 : \dots : 0]$). In both cases, Γ acts trivially on the last n coordinates. Hence $\Gamma = \{\pm 1\}$ in the former case and trivial in the latter. The last assertion of the lemma is obvious. \square

We now prove several auxiliary assertions about two-dimensional quadrics.

Lemma 2.6. *Let C be a smooth curve of bidegree $(1, n)$, $n \geq 2$, on $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose that the projection of C to the first factor of $\mathbb{P}^1 \times \mathbb{P}^1$ is ramified at two points. Then C is given by $x_0y_0^n + x_1y_1^n = 0$ in some coordinates $[x_0 : x_1] \times [y_0 : y_1]$ on $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. It follows from the Riemann–Hurwitz formula that the ramification indices at both ramification points of the projection of C to the first factor of $\mathbb{P}^1 \times \mathbb{P}^1$ are equal to n .

Consider homogeneous coordinates on the factors of $\mathbb{P}^1 \times \mathbb{P}^1$ such that the branch points are $[0 : 1]$ and $[1 : 0]$ and the ramification points are $[0 : 1] \times [0 : 1]$ and $[1 : 0] \times [1 : 0]$. The equation of C in the local coordinates x, y at the point $[0 : 1] \times [0 : 1]$ is a polynomial in y of degree n with unique root $y = 0$. Hence this polynomial is proportional to y^n . The same applies to the other ramification point. \square

Corollary 2.7. *Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth curve of bidegree $(1, n)$, $n \geq 2$, such that $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1; C)$ is infinite. Then C is unique up to the action of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and one has $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1; C) \cong \mathbb{k}^\times$.*

Proof. The action of $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1; C)$ on C is faithful by Lemma 2.1. The action of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1; C)$ preserves the set of ramification points of the projection of C to the first factor of $\mathbb{P}^1 \times \mathbb{P}^1$. The cardinality of this set is at least 2 and, therefore, it is exactly 2. The rest follows from Lemma 2.6. \square

Lemma 2.8. *Up to the action of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$, there is a unique smooth curve of bidegree $(1, 1)$ or $(1, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, and a $(2n - 5)$ -dimensional family of smooth curves of bidegree $(1, n)$ when $n \geq 3$.*

Proof. The uniqueness in the cases of bidegree $(1, 1)$, $(1, 2)$ is obvious.

Suppose that $n \geq 3$. The dimension of the linear system of curves of bidegree $(1, n)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is

$$2 \cdot (n + 1) - 1 = 2n + 1.$$

Let C be a general smooth curve in this linear system, and let π_1 be the projection of C to the first factor of $\mathbb{P}^1 \times \mathbb{P}^1$. Then the ramification points of π_1 are $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1; C)$ -invariant. Since for a general C there are at least 4 such ramification points, we conclude that the group $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1; C)$ acts trivially on $C \cong \mathbb{P}^1$. On the other hand, its action on C is faithful by Lemma 2.1. Thus the group $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1; C)$ is finite. Since the group $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ has dimension 6, the assertion of the lemma follows. \square

Remark 2.9. Let C be a curve of bidegree $(1, n)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Then

$$\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1; C) \cong \mathbb{B} \times \text{PGL}_2(\mathbb{k})$$

when $n = 0$, and $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1; C) \cong \text{PGL}_2(\mathbb{k})$ when $n = 1$.

We conclude this section with an elementary (but useful) observation concerning the projection from $\text{GL}_n(\mathbb{k})$ to $\text{PGL}_n(\mathbb{k})$.

Remark 2.10. Let Γ be a subgroup of $\text{GL}_{n-1}(\mathbb{k})$ containing all scalar matrices. Consider a subgroup $\Gamma \times \mathbb{k}^\times \subset \text{GL}_n(\mathbb{k})$ embedded in the group of block-diagonal matrices with blocks of sizes $n - 1$ and 1. Then the image of $\Gamma \times \mathbb{k}^\times$ in $\text{PGL}_n(\mathbb{k})$ is isomorphic to Γ .

§ 3. Direct products and cones

In this section we consider smooth Fano threefolds X with

$$\mathfrak{I}(X) \in \{2.34, 2.36, 3.9, 3.27, 3.28, 3.31, 4.2, 4.10, 5.3, 6.1, 7.1, 8.1, 9.1, 10.1\}.$$

Lemma 3.1. *Let X_1 and X_2 be normal projective varieties. Then*

$$\text{Aut}^0(X_1 \times X_2) \cong \text{Aut}^0(X_1) \times \text{Aut}^0(X_2).$$

Furthermore, consider a subvariety $Z \subset X_1$ and a point $P \in X_2$. Identify Z with a subvariety of the fibre of the projection $X_1 \times X_2 \rightarrow X_2$ over the point P . Then

$$\text{Aut}^0(X_1 \times X_2; Z) \cong \text{Aut}^0(X_1; Z) \times \text{Aut}^0(X_2; P).$$

Proof. The group $\text{Aut}^0(X_1 \times X_2)$ acts trivially on the Néron–Severi group of $X_1 \times X_2$. In particular, it preserves the numerical class of the pullback of a very ample divisor on X_1 . It follows that the projection $X_1 \times X_2 \rightarrow X_1$ is $\text{Aut}^0(X_1 \times X_2)$ -equivariant. We similarly see that the projection $X_1 \times X_2 \rightarrow X_2$ is $\text{Aut}^0(X_1 \times X_2)$ -equivariant, and the first assertion of the lemma follows. The second follows easily from the first. \square

Corollary 3.2. *Let X be a smooth Fano threefold. Then the following assertions hold.*

- If $\mathfrak{I}(X) = 2.34$, then $\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k}) \times \text{PGL}_3(\mathbb{k})$.
- If $\mathfrak{I}(X) = 3.27$, then $\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k}) \times \text{PGL}_2(\mathbb{k}) \times \text{PGL}_2(\mathbb{k})$.
- If $\mathfrak{I}(X) = 3.28$, then $\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k}) \times \text{PGL}_{3;1}(\mathbb{k})$.
- If $\mathfrak{I}(X) = 4.10$, then $\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k}) \times \text{B} \times \text{B}$.
- If $\mathfrak{I}(X) = 5.3$, then $\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k}) \times (\mathbb{k}^\times)^2$.
- If $\mathfrak{I}(X) \in \{6.1, 7.1, 8.1, 9.1, 10.1\}$, then $\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k})$.

Proof. In all these cases, X is the product of \mathbb{P}^1 and a del Pezzo surface. Thus, the assertion follows from Lemma 3.1 and Theorem 2.2. \square

Lemma 3.3. *Let X be a smooth Fano threefold with $\mathfrak{I}(X) = 2.36$. Then*

$$\text{Aut}^0(X) \cong \text{Aut}(\mathbb{P}(1, 1, 1, 2)).$$

Proof. The threefold X is the blow-up of $\mathbb{P}(1, 1, 1, 2)$ at its (unique) singular point. \square

We refer the reader to (1.7) for a detailed description of $\text{Aut}(\mathbb{P}(1, 1, 1, 2))$.

Lemma 3.4. *Let Y be a smooth Fano variety embedded in \mathbb{P}^N by the complete linear system $|D|$, where D is a very ample divisor on Y such that $D \sim_{\mathbb{Q}} -\lambda K_Y$ for some positive rational number λ . Let $Z \subset Y$ be an irreducible non-ruled subvariety not lying in any effective divisor of the linear system $|D|$. Let \hat{Y} be the cone in \mathbb{P}^{N+1} with vertex P over Y . Then*

$$\text{Aut}^0(\hat{Y}; Z \cup P) \cong \mathbb{k}^\times.$$

Proof. Note that \hat{Y} is a Fano variety, and its singularity at P is Kawamata log terminal since D is proportional to the anticanonical class of Y . One has

$$\text{Aut}^0(\hat{Y}) \cong \text{Aut}^0(\mathbb{P}^{N+1}; \hat{Y}).$$

In particular, the group $\text{Aut}^0(\widehat{Y}; Z \cup P)$ is a subgroup of $\text{Aut}^0(\mathbb{P}^{N+1}; \widehat{Y})$. The group $\text{Aut}^0(\widehat{Y}; Z \cup P)$ preserves the linear span \mathbb{P}^N of Z and acts trivially on \mathbb{P}^N by Lemma 2.1. It follows that $\text{Aut}^0(\widehat{Y}; Z \cup P)$ is contained in the pointwise stabilizer of $\mathbb{P}^N \cup P$ in $\text{Aut}(\mathbb{P}^{N+1})$. This stabilizer is isomorphic to \mathbb{k}^\times . On the other hand, $\text{Aut}^0(\widehat{Y}; Z \cup P)$ contains an obvious subgroup isomorphic to \mathbb{k}^\times , and the assertion follows. \square

Corollary 3.5. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.9$ or $\mathfrak{J}(X) = 4.2$. Then $\text{Aut}^0(X) \cong \mathbb{k}^\times$.*

Lemma 3.6. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.31$. Then*

$$\text{Aut}^0(X) \cong \text{PSO}_{6;1}(\mathbb{k}).$$

Proof. The threefold X is the blow-up of the cone Y over a smooth quadric surface at its (unique) singular point. Therefore, we have $\text{Aut}^0(X) \cong \text{Aut}^0(Y)$. On the other hand, Y is isomorphic to the intersection of a smooth four-dimensional quadric $\mathcal{Q} \subset \mathbb{P}^5$ with the tangent space at some of its points. Using Lemma 2.5, we see that $\text{Aut}^0(Y)$ is isomorphic to the stabilizer of a point on \mathcal{Q} in the group $\text{Aut}^0(\mathcal{Q}) \cong \text{PSO}_6(\mathbb{k})$. \square

We refer the reader to (1.10) for a detailed description of the group $\text{PSO}_{6;1}(\mathbb{k})$.

§ 4. Blow-ups of the projective space

In this section we consider smooth Fano threefolds X with

$$\mathfrak{J}(X) \in \{2.4, 2.9, 2.12, 2.15, 2.25, 2.27, 2.28, 2.33, 2.35, 3.6, 3.11, 3.12, 3.14, 3.16, 3.23, 3.25, 3.26, 3.29, 3.30, 4.6, 4.9, 4.12, 5.2\}.$$

Lemma 2.1 immediately implies the following result.

Corollary 4.1. *Let X be a smooth Fano threefold with*

$$\mathfrak{J}(X) \in \{2.4, 2.9, 2.12, 2.15, 2.25\}.$$

Then the group $\text{Aut}(X)$ is finite.

Proof. These varieties are blow-ups of \mathbb{P}^3 along smooth curves of positive genus that are not contained in a plane. \square

Corollary 4.2. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) \in \{3.6, 3.11\}$. Then the group $\text{Aut}(X)$ is finite.*

Proof. The variety X is a blow-up of a smooth Fano variety Y with $\mathfrak{J}(Y) = 2.25$. Thus the assertion follows from Corollary 4.1. \square

Lemma 4.3. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 2.27$. Then*

$$\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k}).$$

Proof. The threefold X is the blow-up of \mathbb{P}^3 along a twisted cubic curve C . Applying Lemma 2.1, we see that the group $\text{Aut}^0(X) \cong \text{Aut}^0(\mathbb{P}^3; C)$ is a subgroup of $\text{Aut}(C) \cong \text{PGL}_2(\mathbb{k})$. On the other hand, since the curve $C \cong \mathbb{P}^1$ is embedded in \mathbb{P}^3 by a complete linear system, one has $\text{Aut}(C) \subset \text{Aut}^0(\mathbb{P}^3; C)$. \square

Lemma 4.4. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 2.28$. Then*

$$\text{Aut}^0(X) \cong (\mathbb{k}^+)^3 \rtimes \mathbb{k}^\times.$$

Proof. The threefold X is the blow-up of \mathbb{P}^3 along a plane cubic curve. Applying Lemma 2.1, we see that $\text{Aut}^0(X)$ is isomorphic to the pointwise stabilizer of a plane in $\text{Aut}(\mathbb{P}^3) \cong \text{PGL}_4(\mathbb{k})$. \square

Lemma 4.5. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 2.33$ or 2.35. Then $\text{Aut}^0(X)$ is isomorphic to $\text{PGL}_{4;2}(\mathbb{k})$ or $\text{PGL}_{4;1}(\mathbb{k})$ respectively.*

Proof. The threefold X with $\mathfrak{J}(X) = 2.33$ (resp. $\mathfrak{J}(X) = 2.35$) is the blow-up of \mathbb{P}^3 along a line (resp. at a point). Thus the assertion of the lemma follows from the definitions of the corresponding parabolic subgroups in $\text{Aut}(\mathbb{P}^3) \cong \text{PGL}_4(\mathbb{k})$. \square

Lemma 4.6. *There is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 3.12$ such that $\text{Aut}^0(X) \cong \mathbb{k}^\times$. For all other smooth Fano threefolds X with $\mathfrak{J}(X) = 3.12$, the group $\text{Aut}(X)$ is finite.*

Proof. The threefold X is the blow-up of \mathbb{P}^3 along the disjoint union of a line ℓ and a twisted cubic Z . There is an isomorphism $\text{Aut}^0(X) \cong \text{Aut}^0(\mathbb{P}^3; Z \cup \ell)$. Consider the pencil \mathcal{P} of planes through ℓ in \mathbb{P}^3 . This pencil is $\text{Aut}^0(\mathbb{P}^3; Z \cup \ell)$ -invariant. Thus there is an exact sequence of groups

$$1 \rightarrow \text{Aut}_{\mathcal{P}} \rightarrow \text{Aut}^0(\mathbb{P}^3; Z \cup \ell) \rightarrow \Gamma,$$

where $\text{Aut}_{\mathcal{P}}$ preserves every member of \mathcal{P} , and Γ is a subgroup of $\text{Aut}(\mathbb{P}^1)$. Since a general plane $\Pi \cong \mathbb{P}^2$ in \mathcal{P} intersects $Z \cup \ell$ along the union of ℓ and three non-collinear points outside ℓ , we see that the (connected) group $\text{Aut}_{\mathcal{P}}$ is trivial. On the other hand, Γ is also connected and all planes in \mathcal{P} tangent to Z are Γ -invariant. Since there are at least two such planes in \mathcal{P} , we conclude that Γ can be infinite only when there are exactly two of them. This condition means that ℓ is the line of intersection of two osculating planes of Z . Conversely, if ℓ is constructed in this way, then the group $\text{Aut}^0(X) \cong \text{Aut}^0(\mathbb{P}^3; Z \cup \ell)$ is isomorphic to the stabilizer of the two corresponding points of tangency on Z in $\text{Aut}(\mathbb{P}^3; Z) \cong \text{PGL}_2(\mathbb{k})$, that is, to \mathbb{k}^\times . It remains to notice that this configuration is unique up to the action of $\text{Aut}(\mathbb{P}^3; Z)$. \square

Lemma 4.7. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.14$. Then $\text{Aut}^0(X) \cong \mathbb{k}^\times$.*

Proof. The threefold X is the blow-up of \mathbb{P}^3 along the union of a point P and a smooth cubic curve Z contained in a plane Π disjoint from P . Thus, we have $\text{Aut}^0(X) \cong \text{Aut}^0(\mathbb{P}^3; Z \cup P)$. The plane Π is $\text{Aut}^0(X)$ -invariant. The action of $\text{Aut}^0(\mathbb{P}^3; Z \cup P)$ on Π is trivial by Lemma 2.1, and the desired assertion follows. \square

Lemma 4.8. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.16$. Then $\text{Aut}^0(X) \cong \text{B}$.*

Proof. The threefold X is the blow-up of V_7 along the proper transform of a twisted cubic Z passing through the centre P of the blow-up $V_7 \rightarrow \mathbb{P}^3$. Therefore $\text{Aut}^0(X)$ is isomorphic to the subgroup of $\text{Aut}(\mathbb{P}^3)$ preserving Z and P . It remains to note that the stabilizer of Z in $\text{Aut}(\mathbb{P}^3)$ is isomorphic to $\text{PGL}_2(\mathbb{k})$. \square

Lemma 4.9. *Let X be a smooth Fano threefold with $\mathfrak{I}(X) = 3.23$. Then*

$$\text{Aut}^0(X) \cong (\mathbb{k}^+)^3 \rtimes (\mathbb{B} \times \mathbb{k}^\times).$$

Proof. The threefold X is the blow-up of V_7 along the proper transform of a conic Z passing through the centre P of the blow-up $V_7 \rightarrow \mathbb{P}^3$. Therefore $\text{Aut}^0(X)$ is isomorphic to the subgroup Θ of $\text{Aut}(\mathbb{P}^3)$ preserving Z and P .

Choose a point P' not contained in the linear span of Z , and let Γ be the subgroup of Θ that fixes P' . Then $\Theta \cong (\mathbb{k}^+)^3 \rtimes \Gamma$. On the other hand, Γ is the image in $\text{PGL}_4(\mathbb{k})$ of a group Γ' such that the image of Γ' in $\text{PGL}_3(\mathbb{k}) \cong \text{Aut}(\mathbb{P}^2)$ is the group that preserves a conic in \mathbb{P}^2 and a point on this conic. The assertion now follows from Remark 2.10; compare with Lemma 5.3 below. \square

Lemma 4.10. *Let X be a smooth Fano threefold with $\mathfrak{I}(X) = 3.25$. Then*

$$\text{Aut}^0(X) \cong \text{PGL}_{(2,2)}(\mathbb{k}).$$

Proof. The threefold X is the blow-up of \mathbb{P}^3 along a disjoint union of two lines, ℓ_1 and ℓ_2 . Therefore $\text{Aut}^0(X)$ is isomorphic to the subgroup of $\text{Aut}(\mathbb{P}^3)$ that preserves ℓ_1 and ℓ_2 . \square

Lemma 4.11. *Let X be a smooth Fano threefold with $\mathfrak{I}(X) = 3.26$. Then*

$$\text{Aut}^0(X) \cong (\mathbb{k}^+)^3 \rtimes (\text{GL}_2(\mathbb{k}) \times \mathbb{k}^\times).$$

Proof. The threefold X is the blow-up of \mathbb{P}^3 along a disjoint union of a line ℓ and a point P . Therefore $\text{Aut}^0(X)$ is isomorphic to the subgroup of $\text{Aut}(\mathbb{P}^3)$ that preserves ℓ and P . The quotient of this group by its unipotent radical is isomorphic to the image in $\text{PGL}_4(\mathbb{k})$ of the subgroup of $\text{GL}_4(\mathbb{k})$ consisting of block-diagonal matrices with blocks of sizes 2, 1 and 1. The desired assertion now follows from Remark 2.10. \square

Lemma 4.12. *Let X be a smooth Fano threefolds with $\mathfrak{I}(X) = 3.29$. Then*

$$\text{Aut}^0(X) \cong \text{PGL}_{4;3,1}(\mathbb{k}).$$

Proof. The threefold X is the blow-up of V_7 along a line in the exceptional divisor $E \cong \mathbb{P}^2$ of the blow-up $V_7 \rightarrow \mathbb{P}^3$ of a point P on \mathbb{P}^3 . Therefore $\text{Aut}^0(X)$ is isomorphic to the subgroup of $\text{Aut}(\mathbb{P}^3)$ that preserves P and a plane Π through P . \square

Lemma 4.13. *Let X be a smooth Fano threefold with $\mathfrak{I}(X) = 3.30$. Then*

$$\text{Aut}^0(X) \cong \text{PGL}_{4;2,1}(\mathbb{k}).$$

Proof. The threefold X is the blow-up of V_7 along the proper transform of a line ℓ passing through the centre P of the blow-up $V_7 \rightarrow \mathbb{P}^3$. Therefore $\text{Aut}^0(X)$ is isomorphic to the subgroup of $\text{Aut}(\mathbb{P}^3)$ preserving ℓ and P . \square

Lemma 4.14. *There is a unique smooth Fano threefold X with $\mathfrak{I}(X) = 4.6$. Moreover, one has*

$$\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k}).$$

Proof. The variety X can be described as the blow-up of \mathbb{P}^3 along three disjoint lines, ℓ_1, ℓ_2 and ℓ_3 . Thus, $\text{Aut}^0(X) \cong \text{Aut}^0(\mathbb{P}^3; \ell_1 \cup \ell_2 \cup \ell_3)$. Note that there is a unique quadric Q' passing through ℓ_1, ℓ_2 and ℓ_3 ; see, for example, [44], Exercise 7.2. Hence Q' is preserved by $\text{Aut}^0(\mathbb{P}^3; \ell_1 \cup \ell_2 \cup \ell_3)$. Furthermore, the quadric Q' is smooth. Since the elements of $\text{Aut}(Q')$ are linear, one has

$$\text{Aut}^0(\mathbb{P}^3; \ell_1 \cup \ell_2 \cup \ell_3) \cong \text{Aut}^0(Q'; \ell_1 \cup \ell_2 \cup \ell_3).$$

Since the lines ℓ_i are disjoint, they are rulings of the same family of lines on $Q' \cong \mathbb{P}^1 \times \mathbb{P}^1$. The assertions of the lemma now follow since the subgroup of $\text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(\mathbb{k})$ preserving three points of \mathbb{P}^1 is finite and $\text{PGL}_2(\mathbb{k})$ acts transitively on triples of distinct points of \mathbb{P}^1 . \square

Lemma 4.15. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 4.9$. Then*

$$\text{Aut}^0(X) \cong \text{PGL}_{(2,2);1}(\mathbb{k}).$$

Proof. The threefold X is the blow-up of a one-dimensional fibre of the morphism $\pi: Y \rightarrow \mathbb{P}^3$, where π is the blow-up of \mathbb{P}^3 along two disjoint lines, ℓ_1 and ℓ_2 . Therefore, $\text{Aut}^0(X)$ is isomorphic to the subgroup of $\text{Aut}(\mathbb{P}^3)$ that preserves ℓ_1, ℓ_2 and a point on one of these lines. \square

Lemma 4.16. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 4.12$. Then*

$$\text{Aut}^0(X) \cong (\mathbb{k}^+)^4 \rtimes (\text{GL}_2(\mathbb{k}) \times \mathbb{k}^\times).$$

Proof. The threefold X is obtained by blowing up two one-dimensional fibres of the morphism $\pi: Y \rightarrow \mathbb{P}^3$, where π is the blow-up of \mathbb{P}^3 along a line ℓ . Therefore, $\text{Aut}^0(X)$ is isomorphic to the subgroup of $\text{Aut}(\mathbb{P}^3)$ that preserves ℓ and two points on it. The quotient of this group by its unipotent radical is isomorphic to the image in $\text{PGL}_4(\mathbb{k})$ of the subgroup of $\text{GL}_4(\mathbb{k})$ consisting of block-diagonal matrices with blocks of sizes 2, 1 and 1. The desired assertion now follows from Remark 2.10. \square

Lemma 4.17. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 5.2$. Then*

$$\text{Aut}^0(X) \cong \mathbb{k}^\times \times \text{GL}_2(\mathbb{k}).$$

Proof. The threefold X is obtained by blowing up two one-dimensional fibres contained in the same irreducible component of the exceptional divisor of the morphism $\pi: Y \rightarrow \mathbb{P}^3$, where π is the blow-up of \mathbb{P}^3 along two disjoint lines, ℓ_1 and ℓ_2 . Therefore, $\text{Aut}^0(X)$ is isomorphic to the subgroup of $\text{Aut}(\mathbb{P}^3)$ that preserves ℓ_1, ℓ_2 and two points on one of them. This group is in its turn isomorphic to the image in $\text{PGL}_4(\mathbb{k})$ of the subgroup of $\text{GL}_4(\mathbb{k})$ consisting of block-diagonal matrices with blocks of sizes 2, 1 and 1. The desired assertion now follows from Remark 2.10. \square

§ 5. Blow-ups of the quadric threefold

In this section we consider smooth Fano threefolds X with

$$\mathfrak{J}(X) \in \{2.7, 2.13, 2.17, 2.23, 2.29, 2.30, 2.31, 3.10, 3.15, 3.18, 3.19, 3.20, 4.4, 5.1\}.$$

Let $Q \subset \mathbb{P}^4 = \mathbb{P}(V)$ be a smooth quadric, and let $F: V \rightarrow \mathbb{k}$ be the corresponding quadratic form of rank 5. We say that a quadratic form (defined on a linear space U) has rank k on $\mathbb{P}(U)$ (or vanishes when $k = 0$) if it has rank k on U .

Since the ample generator of $\text{Pic}(Q)$ determines an embedding $Q \hookrightarrow \mathbb{P}^4$, Lemma 2.1 immediately implies the following assertion.

Corollary 5.1. *Let X be a smooth Fano threefold with*

$$\mathfrak{J}(X) \in \{2.7, 2.13, 2.17\}.$$

Then the group $\text{Aut}(X)$ is finite.

Proof. These varieties are blow-ups of Q along smooth curves of positive genus that are not contained in a hyperplane section. \square

Lemma 5.2. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 2.23$. Then the group $\text{Aut}(X)$ is finite.*

Proof. The threefold X is the blow-up of Q along a curve Z that is the complete intersection of a hyperplane section H of Q with another quadric. One has $\text{Aut}^0(X) \cong \text{Aut}^0(Q; Z)$. Thus, there is an exact sequence of groups

$$1 \rightarrow \Gamma_H \rightarrow \text{Aut}(Q; Z) \rightarrow \text{Aut}(H; Z),$$

where Γ_H is the pointwise stabilizer of H in $\text{Aut}(Q; Z)$. Since Z is an elliptic curve, the group $\text{Aut}(H; Z)$ is finite by Lemma 2.1. Furthermore, Γ_H is finite by Lemma 2.5. Thus, the group $\text{Aut}(X)$ is also finite. \square

Lemma 5.3. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 2.30$ or 2.31. Then the group $\text{Aut}^0(X)$ is isomorphic to $\text{PSO}_{5;1}(\mathbb{k})$ or $\text{PSO}_{5;2}(\mathbb{k})$ respectively.*

Proof. Both assertions are obvious since the threefolds X with $\mathfrak{J}(X) = 2.30$ (resp. $\mathfrak{J}(X) = 2.31$) are obtained by blowing up points (resp. lines) on a smooth quadric threefold. \square

To understand the automorphism groups of more complicated blow-ups of Q along conics and lines, we need some elementary auxiliary facts.

Lemma 5.4. *Let $C = \Pi \cap Q$ be the conic on Q cut out by a plane Π , and let ℓ_Π be the line orthogonal to Π with respect to F . Let F_Π and F_{ℓ_Π} be the restrictions of F to the cones over Π and ℓ_Π respectively. Then*

$$3 - \text{rk}(F_\Pi) = 2 - \text{rk}(F_{\ell_\Pi}).$$

In particular, $\ell_\Pi \subset Q$ if and only if C is a double line, ℓ_Π is tangent to Q if and only if C is reducible and reduced, and ℓ_Π intersects Q transversally if and only if C is smooth.

Proof. The numbers on both sides of the equality are the dimensions of the kernels of F_Π and F_{ℓ_Π} respectively. Both are equal to $\dim(\Pi \cap \ell_\Pi) + 1$. \square

Lemma 5.5. *Let $C = \Pi \cap Q$ be the conic on Q cut out by a plane Π , and let ℓ_Π be the line orthogonal to Π with respect to F . Let $\ell \subset Q$ be a line disjoint from C . Then the following assertions hold.*

(i) *The lines ℓ and ℓ_Π are disjoint.*

(ii) *If L is a linear three-dimensional subspace of \mathbb{P}^4 containing ℓ and ℓ_Π , then $L \cap Q$ is smooth.*

Proof. Suppose that ℓ and ℓ_Π are not disjoint. Put $\ell \cap \ell_\Pi = P$. Consider any point $P' \in C$ and the line $\ell_{P'}$ through P and P' . The quadratic form F vanishes at P and P' , and the corresponding vectors are orthogonal to each other with respect to F . It follows that F vanishes on $\ell_{P'}$, whence $\ell_{P'} \subset Q$. Thus, the cone T over C with vertex P is contained in Q . In particular, T is contained in the tangent space to Q at P and, since the intersection of this tangent space with Q is two-dimensional, T is exactly this intersection. This means that ℓ is contained in the cone and, therefore, intersects C . The resulting contradiction proves part (i).

Thus, the linear span L of ℓ and ℓ_Π is three-dimensional. Suppose that $L \cap Q$ is singular. Then the restriction of F to L is degenerate, so its kernel is non-trivial. Hence there is a point P lying in Q , in Π and, therefore, in C . It also lies on ℓ by Lemma 5.4 since ℓ lies on Q and the intersection of its orthogonal plane (containing P) with Q is ℓ . Thus, C intersects ℓ . The resulting contradiction proves part (ii). \square

Lemma 5.6. *Let C_1 and C_2 be disjoint smooth conics on Q , ℓ_1 and ℓ_2 their orthogonal lines, and L the linear span of ℓ_1 and ℓ_2 . Then $L \cong \mathbb{P}^3$ and $Q \cap L$ is smooth.*

Proof. Let Π_1 and Π_2 be the planes containing C_1 and C_2 . Then the linear space orthogonal to L is $\Pi_1 \cap \Pi_2$. However, Π_1 and Π_2 intersect each other at a point since otherwise the curves C_1 and C_2 intersect each other at the points of $\Pi_1 \cap \Pi_2 \cap Q$.

Suppose that $Q \cap L$ is singular. Then the rank of the restriction of F to L is not maximal. This means that there is a point P lying in the kernel of the restricted quadratic form. Then $P \in Q$, $P \in \Pi_1$ and $P \in \Pi_2$. Hence C_1 intersects C_2 . \square

Lemma 5.7. *Let $C \subset Q$ be a smooth conic. Then*

$$\text{Aut}(Q; C) \cong \text{PGL}_2(\mathbb{k}) \times \mathbb{k}^\times,$$

where the factor $\text{PGL}_2(\mathbb{k})$ acts faithfully on C while the factor \mathbb{k}^\times is the pointwise stabilizer of C in $\text{Aut}(Q)$.

Proof. Let $\Pi \cong \mathbb{P}^2$ be the linear span of C . By Lemma 5.4, the line ℓ orthogonal to C intersects Q at two points, P_1 and P_2 . Since the automorphisms of Q are linear, they preserve Π and ℓ . Choose coordinates x_0, \dots, x_4 in \mathbb{P}^4 such that the plane Π is given by $x_0 = x_1 = 0$, the line ℓ is given by

$$x_2 = x_3 = x_4 = 0,$$

the points P_1 and P_2 are $P_1 = [1 : 0 : 0 : 0 : 0]$ and $P_2 = [0 : 1 : 0 : 0 : 0]$, and the curve C is given by

$$x_0 = x_1 = x_2^2 + x_3^2 + x_4^2 = 0.$$

Then Q is given by

$$x_0x_1 + x_2^2 + x_3^2 + x_4^2 = 0.$$

One has

$$\text{Aut}^0(Q; C) = \text{Aut}^0(Q; C \cup P_1 \cup P_2).$$

The subgroup $\text{Aut}^0(Q; C \cup P_1 \cup P_2) \subset \text{PGL}_2(\mathbb{k})$ is the image of the subgroup $\Gamma \cong \text{O}_3(\mathbb{k})$ of $\text{GL}_5(\mathbb{k})$ that consists of block-diagonal matrices with blocks of sizes 3, 1 and 1, where the 3×3 -block is an orthogonal matrix and the entries of the 1×1 blocks

are inverse to each other. Since Γ contains no scalar matrices but $\pm \text{Id}_{\text{GL}_5(\mathbb{k})}$, we see that the subgroup $\text{SO}_3(\mathbb{k}) \times \mathbb{k}^\times \subset \Gamma$ of index 2 is mapped isomorphically onto the image of Γ in $\text{PGL}_5(\mathbb{k})$. Therefore, one has

$$\text{Aut}^0(Q; C) \cong \text{PSO}_3(\mathbb{k}) \times \mathbb{k}^\times \cong \text{PGL}_2(\mathbb{k}) \times \mathbb{k}^\times.$$

The other assertions of the lemma are obvious. \square

Lemma 5.8. *There is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 2.29$. Moreover,*

$$\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k}) \times \mathbb{k}^\times.$$

Proof. The variety X is the blow-up of Q along a smooth conic C . This means that $\text{Aut}^0(X) \cong \text{Aut}^0(Q; C)$. The desired assertion follows from Lemma 5.7. \square

Lemma 5.9. *There is a unique Fano threefold X with $\mathfrak{J}(X) = 3.10$ and $\text{Aut}^0(X) \cong (\mathbb{k}^\times)^2$. There is a one-dimensional family of Fano threefolds X with $\mathfrak{J}(X) = 3.10$ and $\text{Aut}^0(X) \cong \mathbb{k}^\times$. For any other smooth Fano threefold X with $\mathfrak{J}(X) = 3.10$, the group $\text{Aut}(X)$ is finite.*

Proof. The threefold X is the blow-up of Q along two disjoint conics, C_1 and C_2 . Thus, one has

$$\text{Aut}^0(X) \cong \text{Aut}^0(Q; C_1 \cup C_2).$$

By Lemma 5.4, the line orthogonal to C_i intersects Q at two points, $P_1^{(i)}$ and $P_2^{(i)}$. Thus,

$$\text{Aut}^0(Q; C_1 \cup C_2) = \text{Aut}^0(Q; \bigcup P_j^{(i)}).$$

By Lemma 5.6, the linear span L of the set $\{P_j^{(i)}\}$ is isomorphic to \mathbb{P}^3 and the corresponding quadric $Q' = Q \cap L$ is smooth. Moreover, the points $P_1^{(i)}$ and $P_2^{(i)}$ cannot lie on the same ruling of $Q' \cong \mathbb{P}^1 \times \mathbb{P}^1$ by Lemma 5.4.

Let π_1 and π_2 be the projections of $Q' \cong \mathbb{P}^1 \times \mathbb{P}^1$ to its factors. If $|\{\pi_1(P_j^{(i)})\}| \geq 3$ and $|\{\pi_2(P_j^{(i)})\}| \geq 3$, then $\text{Aut}(Q'; \bigcup P_j^{(i)})$ is finite since the stabilizer of three or more points on \mathbb{P}^1 is finite. Thus, the group $\text{Aut}^0(Q; C_1 \cup C_2) \cong \text{Aut}^0(X)$ is finite by Lemma 2.5. Hence we can assume that $|\{\pi_1(P_j^{(i)})\}| = 2$. If $|\{\pi_2(P_j^{(i)})\}| = 2$, then $\text{Aut}^0(Q'; \{P_j^{(i)}\}) \cong (\mathbb{k}^\times)^2$ and, since the automorphisms of Q' act on it by elements of $\text{PGL}_4(\mathbb{k})$, one has $\text{Aut}^0(X) \cong (\mathbb{k}^\times)^2$. Note that automorphisms of a non-singular quadric surface act transitively on quadruples of points of the type considered. Moreover, any two smooth hyperplane sections of a quadric threefold can be identified by means of an automorphism of the quadric threefold. It follows that all varieties X with $\text{Aut}^0(X) \cong (\mathbb{k}^\times)^2$ are isomorphic. In the case when $|\{\pi_2(P_j^{(i)})\}| \geq 3$, we similarly obtain a one-dimensional family of varieties X with $\text{Aut}^0(X) \cong \mathbb{k}^\times$. (One has $|\{\pi_2(P_j^{(i)})\}| = 4$ for a generic element of this family.) \square

Lemma 5.10. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.15$. Then X is unique up to isomorphism, and $\text{Aut}^0(X) \cong \mathbb{k}^\times$.*

Proof. The threefold X is the blow-up of Q along a disjoint union of a smooth conic C and a line ℓ . One has $\text{Aut}^0(X) \cong \text{Aut}^0(Q; C \cup \ell)$. By Lemma 5.4, the line orthogonal to C intersects Q at two points, P_1 and P_2 . Thus,

$$\text{Aut}^0(Q; C \cup \ell) \cong \text{Aut}^0(Q; \ell \cup P_1 \cup P_2).$$

By Lemma 5.5, the linear span L of the line ℓ and the points P_1 and P_2 is isomorphic to \mathbb{P}^3 and the quadric $Q' = Q \cap L$ is smooth. By Lemma 2.5, we have

$$\text{Aut}^0(Q; \ell \cup P_1 \cup P_2) \cong \text{Aut}^0(Q'; \ell \cup P_1 \cup P_2).$$

We notice that P_1 and P_2 lie on different rulings of $Q' \cong \mathbb{P}^1 \times \mathbb{P}^1$ because otherwise the line through P_1 and P_2 lies on Q . Hence the images of ℓ , P_1 and P_2 under the projection to the base of the family of lines on Q' containing ℓ are three points on \mathbb{P}^1 , so that their stabilizer is finite. On the other hand, projecting P_1 and P_2 to the base of the other family, we obtain two points on \mathbb{P}^1 , and their stabilizer in $\text{Aut}(\mathbb{P}^1)$ is \mathbb{k}^\times . Since the automorphisms of \mathbb{P}^1 preserving the two points are induced from automorphisms of Q' and Q , we see that $\text{Aut}^0(X) \cong \mathbb{k}^\times$. Moreover, any two smooth hyperplane sections of the quadric threefold Q can be mapped to each other by an automorphism of Q . Finally, for every line on a smooth quadric surface and any two points on this quadric lying outside the line and on different rulings, there is an automorphism of the quadric sending them to any other prescribed line and two points satisfying the same geometric conditions. This proves the remaining assertion of the lemma. \square

Lemma 5.11. *There is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 3.18$. Moreover, one has*

$$\text{Aut}^0(X) \cong \mathbb{B} \times \mathbb{k}^\times.$$

Proof. The variety X is the blow-up of the quadric Q at a point P and along the proper transform of some conic C passing through P . Thus, $\text{Aut}^0(X)$ is a group of automorphisms of Q preserving C and P . The desired assertion now follows from Lemma 5.7. \square

Remark 5.12. Another description of the smooth Fano threefold X with $\mathfrak{J}(X) = 3.18$ was given in [20], §12. Namely, X is described as the blow-up of \mathbb{P}^3 along the disjoint union of a line and a conic. These descriptions are equivalent. Indeed, after the blow-up of the conic on \mathbb{P}^3 , the normal bundle of the proper transform $\tilde{\Pi} \cong \mathbb{P}^2$ of the plane Π containing the conic is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-1)$. Hence one can contract $\tilde{\Pi}$ and obtain a smooth quadric threefold. The line on \mathbb{P}^3 becomes a conic passing through the point which is the image of $\tilde{\Pi}$.

Lemma 5.13. *There is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 3.19$. Moreover, one has*

$$\text{Aut}^0(X) \cong \mathbb{k}^\times \times \text{PGL}_2(\mathbb{k}).$$

Proof. The variety X is the blow-up of two distinct points P_1, P_2 on Q such that the line ℓ through P_1 and P_2 in \mathbb{P}^4 is not contained in Q . Thus, one has $\text{Aut}^0(X) \cong \text{Aut}^0(Q; P_1 \cup P_2)$, and all elements of this group preserve the line ℓ since they are linear. They also preserve its orthogonal plane Π and the corresponding conic $\Pi \cap Q$, which is smooth by Lemma 5.4. The desired assertion now follows from Lemma 5.7. \square

Lemma 5.14. *There is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 3.20$. One has*

$$\text{Aut}^0(X) \cong \mathbb{k}^\times \times \text{PGL}_2(\mathbb{k}).$$

Proof. The threefold X is the blow-up of Q along two disjoint lines, ℓ_1 and ℓ_2 . One has $\text{Aut}^0(X) \cong \text{Aut}^0(Q; \ell_1 \cup \ell_2)$. Let L be the linear span of $\ell_1 \cup \ell_2$. Then $L \cong \mathbb{P}^3$ and the quadric surface $Q' = L \cap Q$ is smooth. The lines ℓ_1 and ℓ_2 are contained in the same family of lines on Q' because they are disjoint by assumption. Since the stabilizer in $\text{Aut}(\mathbb{P}^1)$ of two points on \mathbb{P}^1 is the multiplicative group \mathbb{k}^\times and since the natural restriction homomorphism $\text{PGL}_4(\mathbb{k}) \rightarrow \text{Aut}(Q')$ is surjective, we see that $\text{Aut}^0(X) \cong \mathbb{k}^\times \times \text{PGL}_2(\mathbb{k})$ by Lemma 2.5. The uniqueness of X follows easily since any two smooth hyperplane sections of the quadric threefold Q can be mapped to each other by an automorphism of this quadric. \square

Lemma 5.15. *There is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 4.4$. Moreover, one has*

$$\text{Aut}^0(X) \cong (\mathbb{k}^\times)^2.$$

Proof. The variety X is the blow-up of two points P_1, P_2 on the quadric Q followed by the blow-up of the proper transform of a smooth conic C passing through them. Thus, $\text{Aut}^0(X)$ is a group of automorphisms of Q preserving C, P_1 and P_2 . Hence the assertion of the lemma follows from Lemma 5.7. \square

Lemma 5.16. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 5.1$. Then*

$$\text{Aut}^0(X) \cong \mathbb{k}^\times.$$

Proof. Let $\pi: Y \rightarrow Q$ be the blow-up of Q along a smooth conic C . Then X is the blow-up of Y at three distinct (one-dimensional) fibres of π over three points of C . Therefore, $\text{Aut}^0(X)$ is isomorphic to the connected component of the identity in the pointwise stabilizer of C in the group $\text{Aut}(Q)$. The rest is straightforward; see the proof of Lemma 5.7. \square

§ 6. Blow-ups of the quintic del Pezzo threefold

In this section we consider smooth Fano threefolds X with

$$\mathfrak{J}(X) \in \{2.14, 2.20, 2.22, 2.26\}.$$

Let V_5 be a smooth section of the Grassmannian $\text{Gr}(2, 5) \subset \mathbb{P}^9$ by a linear subspace of dimension 6. Then V_5 is a smooth Fano threefold of Picard rank 1 and anticanonical degree 40. It is known that the group $\text{Pic}(V_5)$ is generated by an ample divisor H such that $-K_{V_5} \sim 2H$ and $H^3 = 5$. The linear system $|H|$ is base-point-free and gives an embedding $V_5 \hookrightarrow \mathbb{P}^6$.

The group $\text{Aut}(V_5)$ is known to be isomorphic to $\text{PGL}_2(\mathbb{k})$; see, for example, [6], Proposition 4.4, or [45], Proposition 7.1.10. Moreover, the threefold V_5 consists of three $\text{PGL}_2(\mathbb{k})$ -orbits that can be described as follows (see [7], Lemma 1.5, [20], Remark 3.4.9, [46], Proposition 2.13). The unique one-dimensional orbit is a rational normal curve $\mathcal{C} \subset V_5$ of degree 6. The unique two-dimensional orbit is $\mathcal{S} \setminus \mathcal{C}$, where \mathcal{S} is an irreducible surface in the linear system $|2H|$ whose singular locus consists of the curve \mathcal{C} .

Corollary 6.1. *Let P be a point of $V_5 \setminus \mathcal{S}$. Then the stabilizer of P in $\text{Aut}(V_5)$ is finite.*

One can show that the stabilizer of a point in $V_5 \setminus \mathcal{S}$ is isomorphic to the octahedral group, but we will not use this fact.

Lemma 6.2. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 2.14$. Then $\text{Aut}(X)$ is finite.*

Proof. The threefold X is the blow-up of V_5 along a smooth complete intersection C of two surfaces in the linear system $|H|$. Then

$$\text{Aut}^0(X) \cong \text{Aut}^0(V_5; C).$$

Since C is a smooth elliptic curve, the group $\text{Aut}^0(V_5; C)$ acts trivially on it. On the other hand, it follows from Lemma 7.2.3 in [45] that C is not contained in the surface \mathcal{S} because $\text{deg}(C) = 5$. Hence there is a point $P \in C$ such that $P \notin \mathcal{S}$ and P is fixed by $\text{Aut}^0(X)$. Now, applying Corollary 6.1, we see that the group $\text{Aut}^0(V_5; C)$ is trivial, whence $\text{Aut}(X)$ is finite. \square

Remark 6.3. There is a $\text{PGL}_2(\mathbb{k})$ -equivariant identification of the Hilbert scheme \mathfrak{H}_ℓ of lines on V_5 with the plane \mathbb{P}^2 ; see [46], Proposition 2.20 (compare with Theorem I in [47]). This plane contains a unique $\text{PGL}_2(\mathbb{k})$ -invariant conic, which we denote by \mathfrak{C} . By [48], § 1.2.1, and [20], Remark 3.4.9, the lines on V_5 contained in \mathcal{S} are parametrized by the points of the conic \mathfrak{C} , and they are exactly the tangent lines to \mathfrak{C} . Moreover, if C is a line on V_5 contained in \mathcal{S} , then its normal bundle is

$$\mathcal{N}_{C/V_5} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

by Proposition 2.27 in [46]. Likewise, if $C \not\subset \mathcal{S}$, then $\mathcal{N}_{C/V_5} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$.

Remark 6.4. Let C be either a line or an irreducible conic in V_5 , $\pi: X \rightarrow V_5$ the blow-up of C , and E the exceptional surface of π . Then the linear system $|\pi^*(H) - E|$ is base-point-free because V_5 is a scheme-theoretic intersection of quadrics in \mathbb{P}^6 , and V_5 contains no planes. Thus, the divisor $-K_X \sim \pi^*(2H) - E$ is ample.

Lemma 6.5. *Up to isomorphism, there are exactly two smooth Fano threefolds X with $\mathfrak{J}(X) = 2.26$. For one of them, we have*

$$\text{Aut}^0(X) \cong \mathbb{k}^\times$$

and for the other, $\text{Aut}^0(X) \cong \text{B}$.

Proof. In this case, the threefold X is the blow-up of V_5 along a line C . Moreover, by Remark 6.4, the blow-up of any line on V_5 is a smooth Fano variety. By Remark 6.3, the Hilbert scheme \mathfrak{H}_ℓ of lines on V_5 is isomorphic to \mathbb{P}^2 and the action of $\text{Aut}(V_5) \cong \text{PGL}_2(\mathbb{k})$ on \mathfrak{H}_ℓ is faithful by Lemma 4.2.1 in [9]. Therefore we have

$$\text{Aut}^0(X) \cong \text{Aut}^0(V_5; C) \cong \Gamma,$$

where Γ is the stabilizer in $\text{PGL}_2(\mathbb{k})$ of the point $[C] \in \mathfrak{H}_\ell$. Furthermore, the Hilbert scheme \mathfrak{H}_ℓ consists of two $\text{PGL}_2(\mathbb{k})$ -orbits, one of which is the conic \mathfrak{C} and the other is $\mathfrak{H}_\ell \setminus \mathfrak{C}$. If $[C] \in \mathfrak{C}$, then $\Gamma \cong \text{B}$. But if $[C] \in \mathfrak{H}_\ell \setminus \mathfrak{C}$, then $\Gamma \cong \mathbb{k}^\times$. Thus, up to isomorphism, we have exactly two smooth Fano threefolds X with $\mathfrak{J}(X) = 2.26$, and the corresponding group $\text{Aut}^0(X)$ is isomorphic to B , \mathbb{k}^\times respectively. \square

Remark 6.6. Let C be a line on the threefold V_5 , and let $\pi: X \rightarrow V_5$ be the blow-up of this line. By Remark 6.4, the threefold X is smooth and $\mathfrak{J}(X) = 2.26$. By Lemma 6.5, either $\text{Aut}^0(X) \cong \mathbb{k}^\times$ or $\text{Aut}^0(X) \cong B$. This could be proved without using the Hilbert scheme of lines on V_5 . Indeed, it follows from [21], p. 117, or [20], Proposition 3.4.1, that there is a commutative diagram

$$\begin{array}{ccc} & X & \\ \pi \swarrow & & \searrow \eta \\ V_5 & \overset{\phi}{\dashrightarrow} & Q, \end{array}$$

where Q is a smooth quadric threefold in \mathbb{P}^4 , the rational map ϕ is the projection from the line C , and the morphism η is the blow-up of a twisted cubic in Q (we denote this cubic by C_3). If C is not contained in the surface \mathcal{S} , then

$$E \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Likewise, if C is contained in \mathcal{S} , then $E \cong \mathbb{F}_2$. This follows from Remark 6.3. In both cases, $\eta(E)$ is a hyperplane section of Q that passes through the curve C_3 ; see [20], Proposition 3.4.1(iii). If C is not contained in \mathcal{S} , then this hyperplane section is smooth. Otherwise the surface $\eta(E)$ is a quadric cone, so that the induced morphism $E \rightarrow \eta(E)$ contracts the (-2) -curve of the surface $E \cong \mathbb{F}_2$. Thus, one has

$$\text{Aut}(X) \cong \text{Aut}(V_5; C) \cong \text{Aut}(Q; C_3),$$

where the group $\text{Aut}(Q; C_3)$ can easily be described explicitly since the pair (Q, C_3) is unique up to projective equivalence (in each of the two cases). Indeed, fix homogeneous coordinates $[x : y : z : t : w]$ on \mathbb{P}^4 . We may assume that $\eta(E)$ is cut out on Q by the hyperplane $\{w = 0\}$. Then we can identify C_3 with the image of the map

$$[\lambda : \mu] \mapsto [\lambda^3 : \lambda^2\mu : \lambda\mu^2 : \mu^3 : 0].$$

If $\eta(E)$ is smooth, then we may assume that Q is given by

$$xt - yz + w^2 = 0. \tag{6.7}$$

In this case, it follows from Lemma 2.5 and Corollary 2.7 that $\text{Aut}^0(Q; C_3) \cong \mathbb{k}^\times$. Here the action of the group \mathbb{k}^\times is given by

$$\zeta : [x : y : z : t : w] \mapsto [x : \zeta^2 y : \zeta^4 z : \zeta^6 t : \zeta^3 w].$$

Similarly, if $\eta(E)$ is singular, one can show that $\text{Aut}^0(Q; C_3) \cong B$.

For every line on V_5 there is a unique surface in $|H|$ that is singular along this line. This surface is spanned by the lines in V_5 that intersect the given line. More precisely, we have the following result.

Lemma 6.8. *Let S be a surface in $|H|$ that has non-isolated singularities. Then S is singular along a line C and smooth outside C . If $C \subset S$, then S contains no irreducible curves of degree 3. Likewise, if $C \not\subset S$, then S contains no irreducible curves of degree 3 that intersect C . In this case, the surface S contains a unique $\text{Aut}^0(V_5; C)$ -invariant irreducible cubic curve disjoint from C . Moreover, this curve is a twisted cubic curve.*

Proof. One can assume that H is a general surface in $|H|$. Then H is a smooth del Pezzo surface of degree 5, and

$$S|_H \in |-K_H|,$$

so that $S|_H$ is an irreducible singular curve of arithmetic genus 1. It follows that $S|_H$ has a unique singular point (an ordinary isolated double point or an ordinary cusp). This shows that S is singular along a certain line C and S has only isolated singularities outside C .

We use the notation of Remark 6.6. Let \tilde{S} be the proper transform of the surface S on the threefold X . Then \tilde{S} is the exceptional surface of the birational morphism η . In particular, the surface S is smooth outside the line C .

Note that $\tilde{S} \cong \mathbb{F}_n$ for some integer $n \geq 0$. If C is not contained in S , then $n = 1$. This follows from the proof of Lemma 13.2.1 in [45]. Indeed, let \mathbf{s} be an irreducible curve in \mathbb{F}_n such that $\mathbf{s}^2 = -n$, and let \mathbf{f} be a general fibre of the natural projection $\xi: \mathbb{F}_n \rightarrow \mathbb{P}^1$. Then

$$-\tilde{S}|_{\tilde{S}} \sim \mathbf{s} + k\mathbf{f}$$

for some integer k . We have

$$-7 = 2 + K_Q \cdot C_3 = \tilde{S}^3 = (\mathbf{s} + k\mathbf{f})^2 = -n + 2k,$$

whence $k = (n - 7)/2$. Thus, if C is not contained in S , then $\eta(E)$ is a smooth surface. It follows that

$$E|_{\tilde{S}} \sim \mathbf{s} + \frac{n-1}{2}\mathbf{f}$$

is a section of the natural projection $\tilde{S} \rightarrow \mathbb{P}^1$ and, therefore, $n = 1$ because otherwise $0 \leq E|_{\tilde{S}} \cdot \mathbf{s} = -(n + 1)/2$. In a similar vein, if C is contained in S , then $\eta(E)$ is a quadric cone, whence

$$E|_{\tilde{S}} = Z + F,$$

where Z and F are irreducible curves in \tilde{S} such that Z is a section of the projection ξ and F is the fibre of ξ over the singular point of the cone $\eta(E)$. In this case we have

$$Z \sim \mathbf{s} + \frac{n-3}{2}\mathbf{f},$$

so that $0 \leq Z \cdot \mathbf{s} = -(n + 3)/2$ when $n \neq 3$. Hence, if $C \subset S$, then $n = 3$.

Let M be an irreducible curve in S such that $M \neq C$, and let \tilde{M} be the proper transform of M on the threefold X . Then

$$\tilde{M} \sim a\mathbf{s} + b\mathbf{f}$$

for some non-negative integers a and b . Moreover, $M \neq \mathbf{s}$ since $\mathbf{s} \subset E \cap \tilde{S}$. In particular, we have

$$0 \leq \mathbf{s} \cdot \tilde{M} = \mathbf{s} \cdot (a\mathbf{s} + b\mathbf{f}) = b - na,$$

whence $b \geq na$. Thus, if C is contained in S , then, since $n = 3$, we have

$$\deg(M) = \pi^*(H) \cdot \tilde{M} = (\mathbf{s} + 4\mathbf{f}) \cdot \tilde{M} = b + a \geq 4a.$$

It follows that $\deg(M) \neq 3$.

To complete the proof of the lemma, we may assume that C is not contained in \mathcal{S} and M is an irreducible cubic curve. We need to show that such a curve M indeed exists and is unique. As above, we see that

$$3 = \deg(M) = \pi^*(H) \cdot \widetilde{M} = (\mathbf{s} + 3\mathbf{f}) \cdot \widetilde{M} = b + 2a \geq 3a,$$

whence $\widetilde{M} \sim \mathbf{s} + \mathbf{f}$. In particular, the curve M is disjoint from C since $E \cap \widetilde{\mathcal{S}} = \mathbf{s}$.

It follows from Remark 6.6 that $\text{Aut}^0(X) \cong \text{Aut}^0(V_5; C) \cong \text{Aut}^0(Q; C_3) \cong \mathbb{k}^\times$ and

$$\text{Aut}(X) \cong \text{Aut}(V_5; C) \cong \text{Aut}(Q; C_3).$$

Thus, to complete the proof of the lemma, it is enough to show that the linear system $|\mathbf{s} + \mathbf{f}|$ contains a unique irreducible $\text{Aut}^0(X)$ -invariant curve. To do this, observe that the group $\text{Aut}(Q; C_3)$ contains a slightly larger subgroup $\Gamma \cong \mathbb{k}^\times \times \mathbb{Z}/2\mathbb{Z}$. Indeed, let Q be the quadric given by the equation (6.7). Then the additional involution acts on Q as

$$[x : y : z : t : w] \mapsto [t : z : y : x : w].$$

This group Γ acts faithfully on the surface $\widetilde{\mathcal{S}} \cong \mathbb{F}_1$.

We claim that the linear system $|\mathbf{s} + \mathbf{f}|$ contains a unique Γ -invariant curve (compare with the proof of Lemma 13.2.1 in [45], where a similar claim was proved for a subgroup of Γ isomorphic to D_{10}). Indeed, let $\theta: \widetilde{\mathcal{S}} \rightarrow \mathbb{P}^2$ be the contraction of the curve \mathbf{s} . Then θ determines a faithful action of Γ on \mathbb{P}^2 . It is easy to check that \mathbb{P}^2 contains a unique Γ -invariant line. We denote this line by ℓ and write $\widetilde{\ell}$ for its proper transform on $\widetilde{\mathcal{S}}$. Then ℓ does not contain the point $\theta(\mathbf{s})$, whence

$$\widetilde{\ell} \sim \mathbf{s} + \mathbf{f}.$$

Thus, the curve $\widetilde{\ell}$ is a unique Γ -invariant curve in the linear system $|\mathbf{s} + \mathbf{f}|$.

Note that $\widetilde{\ell}$ is an $\text{Aut}^0(X)$ -invariant curve in $|\mathbf{s} + \mathbf{f}|$. Moreover, it is a unique irreducible $\text{Aut}^0(X)$ -invariant curve in $|\mathbf{s} + \mathbf{f}|$. This completes the proof of the lemma. \square

Corollary 6.9. *Let C be a twisted cubic curve in V_5 , $\pi: X \rightarrow V_5$ the blow-up of C , and E the exceptional surface of π . Then the linear system $|\pi^*(H) - E|$ is base-point-free and the divisor $-K_X$ is ample.*

Proof. It is enough to show that $|\pi^*(H) - E|$ is base-point-free. Suppose that this is not the case. Then V_5 contains a line L which is either a secant of C or tangent to C . This follows since V_5 is a scheme-theoretic intersection of quadrics and V_5 contains no quadric surfaces. Let S be the surface in $|H|$ that is singular along L . Then C is contained in S . This contradicts Lemma 6.8. \square

Lemma 6.10. *Up to isomorphism, there is a unique smooth Fano threefold X with $\mathfrak{I}(X) = 2.20$ such that the group $\text{Aut}(X)$ is infinite. Moreover, in this case one has $\text{Aut}^0(X) \cong \mathbb{k}^\times$.*

Proof. In this case, the threefold X is the blow-up of V_5 along a twisted cubic curve. Denote this cubic curve by C . Then

$$\text{Aut}^0(X) \cong \text{Aut}^0(V_5; C).$$

Note that C is not contained in the surface \mathcal{S} by Lemma 7.2.3 in [45].

By Corollary 6.9, there is a commutative diagram

$$\begin{array}{ccc}
 & X & \\
 \pi \swarrow & & \searrow \eta \\
 V_5 & \xrightarrow{\phi} & \mathbb{P}^2,
 \end{array} \tag{6.11}$$

where ϕ is a linear projection from the twisted cubic C and the morphism η is the standard conic bundle given by the linear system $|\pi^*(H) - E|$. Simple computations show that the discriminant of the conic bundle η is a curve of degree 3. Denote it by Δ . Then Δ has at most isolated ordinary double points by Corollary 2.3.

Note that the diagram (6.11) is $\text{Aut}(X)$ -equivariant. Moreover, if the group $\text{Aut}^0(X)$ is not trivial, then it acts non-trivially on the plane \mathbb{P}^2 in (6.11) because $\text{Aut}^0(X)$ acts non-trivially on the π -exceptional surface E (this follows from Corollary 6.1 since C is not contained in S).

Using Lemmas 2.1 and 2.4, we see that the curve Δ must be reducible. Thus, we can write $\Delta = \ell + M$, where ℓ is a line and M is a conic (possibly reducible).

Let \tilde{S} be a surface in $|\pi^*(H) - E|$ such that $\eta(\tilde{S}) = \ell$. We put $S = \pi(\tilde{S})$. Then \tilde{S} and S are non-normal by construction. Thus, it follows from Lemma 6.8 that S is singular along a line in V_5 . Denote this line by L . Then L is not contained in S by Lemma 6.8. Hence $\text{Aut}^0(V_5; L) \cong \mathbb{k}^\times$ by Remark 6.6.

Since L must be $\text{Aut}^0(V_5; C)$ -invariant, we see that either $\text{Aut}^0(V_5; C)$ is trivial or

$$\text{Aut}^0(V_5; C) \cong \mathbb{k}^\times.$$

In the latter case, Lemma 6.8 yields that C is a unique $\text{Aut}^0(V_5; C)$ -invariant twisted cubic curve contained in the surface S . Thus, up to the action of $\text{Aut}(V_5)$, there is a unique choice for C such that the group $\text{Aut}^0(V_5; C)$ is non-trivial, and in this case one has $\text{Aut}^0(V_5; C) \cong \mathbb{k}^\times$. In fact, Lemma 6.8 also implies that this case does indeed occur. \square

We need the following fact about rational quartic curves in \mathbb{P}^3 .

Lemma 6.12. *Let C_4 be a smooth rational quartic curve in \mathbb{P}^3 . Then C_4 is contained in a unique quadric surface. Moreover, this quadric surface is smooth.*

Proof. A dimension count shows that C_4 is contained in a quadric surface, which we denote by S . This surface is unique since otherwise C_4 would be a complete intersection of two quadric surfaces in \mathbb{P}^3 , which is not the case since C_4 is smooth and rational.

Suppose that S is singular. Then S is an irreducible quadric cone. Let $\alpha: \mathbb{F}_2 \rightarrow S$ be the blow-up of the vertex of this cone, and let \tilde{C}_4 be the proper transform of the curve C_4 on the surface \mathbb{F}_2 . We write \mathbf{s} for the unique (-2) -curve on \mathbb{F}_2 and \mathbf{f} for the general fibre of the natural projection $\mathbb{F}_2 \rightarrow \mathbb{P}^1$. Then

$$\tilde{C}_4 \sim a\mathbf{s} + b\mathbf{f}$$

for some non-negative integers a and b . Moreover,

$$b = (\mathbf{s} + 2\mathbf{f}) \cdot (a\mathbf{s} + b\mathbf{f}) = \text{deg}(C_4) = 4$$

because α is given by the linear system $|\mathbf{s} + 2\mathbf{f}|$. Since \tilde{C}_4 is smooth and rational, we see that $a = 1$. Then

$$\mathbf{s} \cdot \tilde{C}_4 = \mathbf{s} \cdot (\mathbf{s} + 4\mathbf{f}) = 2.$$

It follows that the curve $C_4 = \alpha(\tilde{C}_4)$ is singular. The resulting contradiction shows that S is a smooth quadric. \square

We conclude this section by proving the following result.

Lemma 6.13. *Up to isomorphism, there is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 2.22$ such that the group $\text{Aut}(X)$ is infinite. Moreover, in this case one has $\text{Aut}^0(X) \cong \mathbb{k}^\times$.*

Proof. The threefold X is the blow-up of V_5 along a smooth conic. Denote this conic by C . It follows easily from Remark 6.4 that there is a commutative diagram

$$\begin{array}{ccc}
 & X & \\
 \pi \swarrow & & \searrow \eta \\
 V_5 & \overset{\phi}{\dashrightarrow} & \mathbb{P}^3,
 \end{array} \tag{6.14}$$

where π is the blow-up of C , the morphism ϕ is a linear projection from the conic C , and the morphism η is the blow-up of a smooth rational quartic curve, which we denote by C_4 . The diagram (6.14) is $\text{Aut}(X)$ -equivariant. Thus, we see that

$$\text{Aut}(X) \cong \text{Aut}(V_5; C) \cong \text{Aut}(\mathbb{P}^3; C_4).$$

By Lemma 6.12, the curve C_4 is contained in a unique quadric surface, which we denote by S . This surface is smooth, again by Lemma 6.12. Thus, we have

$$S \cong \mathbb{P}^1 \times \mathbb{P}^1$$

and C_4 is a curve of bidegree $(1, 3)$. Note that $\text{Aut}^0(\mathbb{P}^3; C_4) \cong \text{Aut}^0(S; C_4)$ by Lemma 2.1. Thus, by Corollary 2.7, there is a unique (up to projective equivalence) choice for C_4 such that the group $\text{Aut}^0(\mathbb{P}^3; C_4)$ is non-trivial. In this case, Corollary 2.7 also yields that $\text{Aut}^0(\mathbb{P}^3; C_4) \cong \mathbb{k}^\times$. This case does indeed occur. For example, let C_4 be given by the parametrization

$$[u^4 : u^3v : uv^3 : v^4],$$

where $[u : v] \in \mathbb{P}^1$. In this case, the quadric S is given by $xt = yz$, where $[x : y : z : t]$ are the homogeneous coordinates on \mathbb{P}^3 . Since C_4 is a scheme-theoretic intersection of cubic surfaces in \mathbb{P}^3 , we see that the blow-up of \mathbb{P}^3 along C_4 is indeed a Fano threefold, which can also be obtained by blowing up V_5 along a smooth conic. \square

§ 7. Blow-ups of the flag variety

In this section we consider smooth Fano threefolds X with

$$\mathfrak{J}(X) \in \{2.32, 3.7, 3.13, 3.24, 4.7\}.$$

Recall that we are denoting the (unique) smooth Fano threefold with $\mathfrak{J}(X) = 2.32$ by W . This threefold is isomorphic to the flag variety $\text{Fl}(1, 2; 3)$ of complete flags in a three-dimensional vector space as well as to the projectivization of the tangent bundle of \mathbb{P}^2 and to a smooth divisor of bidegree $(1, 1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$.

We start with the following well-known result.

Lemma 7.1. *One has $\text{Aut}^0(W) \cong \text{PGL}_3(\mathbb{k})$, and each of the two natural projections $W \rightarrow \mathbb{P}^2$ induces an isomorphism*

$$\text{Aut}^0(W) \cong \text{Aut}(\mathbb{P}^2) \cong \text{PGL}_3(\mathbb{k}).$$

Proof. The proof is left to the reader. \square

Lemma 7.2. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.7$. Then the group $\text{Aut}(X)$ is finite.*

Proof. The threefold X is the blow-up of the flag variety W along a smooth curve C which is a complete intersection of two divisors in the linear system $|-\frac{1}{2}K_W|$. By the adjunction formula, C is an elliptic curve. We have $\text{Aut}^0(X) \subset \text{Aut}^0(W; C)$.

Let $\pi_1: W \rightarrow \mathbb{P}^2$ and $\pi_2: W \rightarrow \mathbb{P}^2$ be the natural projections. They are $\text{Aut}^0(W)$ -equivariant. We put $C_1 = \pi_1(C)$ and $C_2 = \pi_2(C)$. Since the intersection numbers of the fibres of each projection π_i with divisors in the linear system $|-\frac{1}{2}K_W|$ are equal to 1, we see that both curves C_1 and C_2 are isomorphic to C . Hence we have

$$\text{Aut}^0(W; C) \subset \text{Aut}(\mathbb{P}^2; C_1) \times \text{Aut}(\mathbb{P}^2; C_2).$$

On the other hand, the groups $\text{Aut}(\mathbb{P}^2; C_1)$ and $\text{Aut}(\mathbb{P}^2; C_2)$ are finite by Lemma 2.1. \square

We will need the following simple auxiliary result.

Lemma 7.3 ([38], Lemma 6.2(a)). *Let C_1 and C_2 be irreducible conics in \mathbb{P}^2 . Then the following assertions hold.*

- (i) *If $|C_1 \cap C_2| = 1$, then $\text{Aut}^0(\mathbb{P}^2; C_1 \cup C_2) \cong \mathbb{k}^+$.*
- (ii) *If C_1 and C_2 are tangent to each other at two distinct points, then $\text{Aut}^0(\mathbb{P}^2; C_1 \cup C_2) \cong \mathbb{k}^\times$.*

We now proceed to Fano varieties X with $\mathfrak{J}(X) = 3.13$.

Lemma 7.4. *The following assertions hold.*

- *There is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 3.13$ and*

$$\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k}).$$

- *There is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 3.13$ and $\text{Aut}^0(X) \cong \mathbb{k}^+$.*

- *For all other smooth Fano threefolds X with $\mathfrak{J}(X) = 3.13$, one has $\text{Aut}^0(X) \cong \mathbb{k}^\times$.*

Proof. A smooth Fano threefold X with $\mathfrak{J}(X) = 3.13$ is the blow-up of the flag variety W along a curve C such that the natural projections π_1 and π_2 map C isomorphically onto smooth conics in \mathbb{P}^2 , which we denote by C_1 and C_2 respectively. Put $S_1 = \pi_1^{-1}(C_1)$ and $S_2 = \pi_2^{-1}(C_2)$. Then $S_i \cong \mathbb{P}^1 \times \mathbb{P}^1$, the intersection $S_1 \cap S_2$

is a curve of bidegree $(2, 2)$ on $S_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and C is its irreducible component of bidegree $(1, 1)$. One has

$$\text{Aut}^0(X) \cong \text{Aut}^0(W; C).$$

The curve C and the surfaces S_i are $\text{Aut}^0(W; C)$ -invariant. Moreover, the projections $\pi_i: W \rightarrow \mathbb{P}^2$ are $\text{Aut}^0(W; C)$ -equivariant and the conics C_i are invariant under the corresponding action of $\text{Aut}^0(W; C)$ on \mathbb{P}^2 .

Note that the threefold W can be used to identify one copy of \mathbb{P}^2 with the dual of the other. On the other hand, C_1 can be used to identify the plane $\mathbb{P}^2 \supset C_1$ with its dual. Using these identifications, we may regard C_2 as a conic contained in the same projective plane \mathbb{P}^2 as the conic C_1 . Then C_1 and C_2 are $\text{Aut}^0(W; C)$ -invariant under the natural action of $\text{Aut}^0(W; C)$ on \mathbb{P}^2 . We conclude that

$$\text{Aut}^0(W; C) \cong \text{Aut}(\mathbb{P}^2; C_1 \cup C_2).$$

Since W is the flag variety $\text{Fl}(1, 2; 3)$, we can describe the curve $S_1 \cap S_2 \subset W$ as

$$S_1 \cap S_2 = \{(P, \ell) \in W \mid P \in C_1 \text{ and } \ell \text{ is tangent to } C_2\}.$$

The induced double covers $\pi_i: S_1 \cap S_2 \rightarrow C_i$ are branched exactly over the points of $C_1 \cap C_2$. If $S_1 \cap S_2$ is a reduced curve, then its arithmetic genus is equal to 1. In this case, since $S_1 \cap S_2$ contains an irreducible component of bidegree $(1, 1)$ on $S_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, we see that one of the following cases must occur: either $|C_1 \cap C_2| = 2$ and C_1 is tangent to C_2 at both points of their intersection $C_1 \cap C_2$, or $|C_1 \cap C_2| = 1$. If the intersection $S_1 \cap S_2$ is not reduced, then it is just the curve C taken with multiplicity 2, so that the conics C_1 and C_2 coincide. Recall that the conics C_1 and C_2 are irreducible in all of these cases.

Suppose that the conics C_1 and C_2 are tangent at two distinct points. (Note that there is a one-parameter family of such pairs of conics, up to isomorphism.) Then one has $\text{Aut}^0(\mathbb{P}^2; C_1 \cup C_2) \cong \mathbb{k}^\times$ by Lemma 7.3(ii).

Suppose that the conics C_1 and C_2 are tangent with multiplicity 4 at a single point. (Note that there is exactly one such pair of conics, up to isomorphism.) Then one has $\text{Aut}^0(\mathbb{P}^2; C_1 \cup C_2) \cong \mathbb{k}^+$ by Lemma 7.3(i).

Finally suppose that the conics C_1 and C_2 coincide. Then

$$\text{Aut}^0(\mathbb{P}^2; C_1 \cup C_2) \cong \text{PGL}_2(\mathbb{k}). \quad \square$$

Remark 7.5. A smooth Fano threefold X with $\mathfrak{J}(X) = 3.13$ and $\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k})$ was considered in [36], Example 2.4; see also [16].

Lemma 7.6. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.24$. Then $\text{Aut}^0(X) \cong \text{PGL}_{3,1}(\mathbb{k})$.*

Proof. The threefold X is the blow-up of the flag variety W along a fibre of one of the natural projections $W \rightarrow \mathbb{P}^2$. The required assertion follows since the morphisms $X \rightarrow W$ and $X \rightarrow \mathbb{P}^2$ are $\text{Aut}^0(X)$ -equivariant. \square

Lemma 7.7. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 4.7$. Then*

$$\text{Aut}^0(X) \cong \text{GL}_2(\mathbb{k}).$$

Proof. The threefold X is the blow-up of the flag variety W along two disjoint fibres C_1, C_2 of the projections $\pi_1, \pi_2: W \rightarrow \mathbb{P}^2$ respectively. Therefore, we have $\text{Aut}^0(X) \cong \text{Aut}^0(W; C_1 \cup C_2)$. Put $P = \pi_1(C_1)$ and $\ell = \pi_1(C_2)$. Then ℓ is a line on \mathbb{P}^2 . Note that $P \not\subset \ell$ since the curves C_1 and C_2 are disjoint. Recall that the $\text{Aut}^0(X)$ -equivariant projection π_1 induces an isomorphism $\text{Aut}^0(W) \cong \text{Aut}(\mathbb{P}^2)$. Under this isomorphism, the subgroup $\text{Aut}^0(W; C_1 \cup C_2)$ is mapped to a subgroup of $\text{Aut}(\mathbb{P}^2; \ell \cup P)$.

We claim that the group $\text{Aut}^0(X)$ is actually isomorphic to $\text{Aut}^0(\mathbb{P}^2; \ell \cup P)$. Indeed, let σ be an arbitrary element of $\text{Aut}^0(\mathbb{P}^2; \ell \cup P)$ and let $\hat{\sigma}$ be its (unique) preimage in $\text{Aut}^0(W)$. Then $\hat{\sigma}$ preserves the curve $C_1 = \pi_1^{-1}(P)$ and the surface $S = \pi_1^{-1}(\ell)$. Moreover, C_2 is the unique fibre of π_2 contained in S . It follows that $\hat{\sigma}$ also preserves C_2 .

We conclude that $\text{Aut}^0(X)$ is isomorphic to the stabilizer of the disjoint union of a line and a point in \mathbb{P}^2 . The desired assertion now follows from Remark 2.10. \square

§ 8. Blow-ups and double covers of direct products

In this section we consider smooth Fano threefolds X with

$$\mathfrak{J}(X) \in \{2.2, 2.18, 3.1, 3.3, 3.4, 3.5, 3.8, 3.17, 3.21, 3.22, 4.1, 4.3, 4.5, 4.8, 4.11, 4.13\}.$$

Lemma 8.1. *Suppose that $Y = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ and let $\theta: X \rightarrow Y$ be a smooth double cover of Y branched over a smooth divisor of multidegree (d_1, \dots, d_k) . Suppose that $n_i + 1 \leq d_i \leq 2n_i$ for every i . Then the group $\text{Aut}(X)$ is finite.*

Proof. Let π_i be the projection of Y to the i th factor and let H_i be a hyperplane in this factor. Then the divisor class

$$H = \sum_{i=1}^k \pi_i^*(H_i)$$

determines the Segre embedding of X . On the other hand, the branch divisor Z is divisible by 2 in the group $\text{Pic}(Y)$ and, therefore, is not contained in any effective divisor in the linear system $|H|$. Moreover, the canonical class of Z is numerically effective by the adjunction formula, whence Z is not uniruled. Thus it follows from Lemma 2.1 that the group $\text{Aut}(Y; Z)$ is finite. On the other hand, X is a Fano variety and the projections $\pi_i \circ \theta: X \rightarrow \mathbb{P}^{n_i}$ are extremal contractions. It follows that they are $\text{Aut}^0(X)$ -equivariant. Hence the map θ is $\text{Aut}^0(X)$ -equivariant, so that $\text{Aut}^0(X)$ is a subgroup of $\text{Aut}(Y; Z)$. \square

Corollary 8.2. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) \in \{2.2, 3.1\}$. Then the group $\text{Aut}(X)$ is finite.*

Proof. A smooth Fano threefold with $\mathfrak{J}(X) = 2.2$ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ with branch divisor of bidegree $(2, 4)$. A smooth Fano threefold with $\mathfrak{J}(X) = 3.1$ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with branch divisor of tridegree $(2, 2, 2)$. Hence the desired assertion follows from Lemma 8.1. \square

Lemma 8.3. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 2.18$. Then the group $\text{Aut}(X)$ is finite.*

Proof. The variety X is a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ branched over a divisor Z of bidegree $(2, 2)$. The natural morphisms from X to \mathbb{P}^1 and \mathbb{P}^2 are extremal contractions, whence the double cover $\theta: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is $\text{Aut}^0(X)$ -equivariant. Thus, $\text{Aut}^0(X)$ is a subgroup of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2; Z)$. By Lemma 2.1, the action of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2; Z)$ on Z is faithful. Considering the projection $Z \rightarrow \mathbb{P}^2$, we see that Z is a double cover of \mathbb{P}^2 branched over a quartic. Hence Z is a smooth del Pezzo surface of degree 2. Therefore the automorphism group of Z is finite by Theorem 2.2, and the assertion follows. \square

Corollary 8.4. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.4$. Then the group $\text{Aut}(X)$ is finite.*

Proof. The variety X is the blow-up of a smooth Fano threefold Y with $\mathfrak{J}(Y) = 2.18$. Thus, the assertion follows from Lemma 8.3. \square

Lemma 8.5. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.3$. Then the group $\text{Aut}(X)$ is finite.*

Proof. The variety X is a divisor of tridegree $(1, 1, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ and the natural projection $\pi: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is $\text{Aut}^0(X)$ -equivariant. One can check that π is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a smooth curve C which is the complete intersection of two divisors of bidegree $(1, 2)$. It follows that $\text{Aut}^0(X)$ is a subgroup of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2; C)$. Since C is not contained in any effective divisor of bidegree $(1, 1)$, the action of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2; C)$ on C is faithful by Lemma 2.1. On the other hand, we see from the adjunction formula that C has genus 3. Therefore the automorphism group of C is finite, and the assertion follows. \square

The following fact was explained to us by A. Kuznetsov.

Lemma 8.6. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.8$. Then X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a complete intersection of divisors of bidegree $(0, 2)$ and $(1, 2)$.*

Proof. The variety X can be described as follows. Let $g: \mathbb{F}_1 \rightarrow \mathbb{P}^2$ be the blow-up of a point, and let p_1, p_2 be the natural projections of $\mathbb{F}_1 \times \mathbb{P}^2$ to \mathbb{F}_1 and \mathbb{P}^2 respectively. Then X is a divisor in the linear system $|p_1^*g^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(2)|$. We now restate this description. Put $Y = \mathbb{P}^1 \times \mathbb{P}^2$ and let a and b be the pullbacks to Y of the divisors $\mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{\mathbb{P}^2}(1)$ respectively. One has

$$\mathbb{P} = \mathbb{F}_1 \times \mathbb{P}^2 \cong \mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(-a)).$$

Suppose that $h \in |\mathcal{O}_{\mathbb{P}}(1)|$. Then X is a divisor in the linear system $|h + 2b|$.

Suppose that E is a vector bundle of rank 2 over a smooth variety M , $\phi: \mathbb{P}_M(E) \rightarrow M$ is the natural projection, and $V \in |\mathcal{O}_{\mathbb{P}_M(E)}(1)|$. Then the induced birational map $V \rightarrow M$ is the blow-up of $Z = \{s = 0\}$, where

$$s \in H^0(M, E^*) \cong H^0(\mathbb{P}_M(E), \mathcal{O}_{\mathbb{P}_M(E)}(1)).$$

This means that if $\pi: \mathbb{P} \rightarrow Y$ is the natural projection, then $\pi_*\mathcal{O}_{\mathbb{P}}(h)$ is dual to $\mathcal{O}_Y \oplus \mathcal{O}_Y(-a)$. Hence X is the blow-up of a section of

$$\pi_*\mathcal{O}_{\mathbb{P}}(h + 2b) \cong (\mathcal{O}_Y \oplus \mathcal{O}_Y(a)) \otimes \mathcal{O}_Y(2b) \cong \mathcal{O}_Y(2b) \oplus \mathcal{O}_Y(a + 2b).$$

In other words, the threefold X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a complete intersection of divisors of bidegree $(0, 2)$ and $(1, 2)$. \square

Lemma 8.7. *Let n, m be integers with $1 \leq n$ and $1 \leq m \leq 2$ and let \mathcal{C} be the family of smooth curves C of bidegree (n, m) on $\mathbb{P}^1 \times \mathbb{P}^2$ which project isomorphically to \mathbb{P}^2 (so that the projection of C to \mathbb{P}^1 is an n -sheeted covering and the image of C under the projection to \mathbb{P}^2 is a curve of degree m). Then, up to the action of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2)$, the family of curves in \mathcal{C} has dimension 0 when $1 \leq n \leq 2$ and dimension $2n - 5$ when $n \geq 3$. Furthermore, up to the action of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2)$, there is a unique curve C_0 in this family such that the group $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2; C_0)$ is infinite. One has*

- $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2; C_0) \cong (\mathbb{k}^+)^2 \rtimes (\mathbb{k}^\times)^2$ if $m = 1$ and $n \geq 2$;
- $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2; C_0) \cong \text{PGL}_2(\mathbb{k})$ if $m = 2$ and $n = 1$;
- $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2; C_0) \cong \mathbb{k}^\times$ if $m = 2$ and $n \geq 2$.

Proof. Choose a curve C in \mathcal{C} . Let $\pi: \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be the natural projection, so that $\pi(C)$ is a line when $m = 1$ and a smooth conic when $m = 2$. Put $S = \pi^{-1}(\pi(C))$. Then $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and C is a curve of bidegree $(n, 1)$ on S . Moreover, the surface S is $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2; C)$ -invariant. Note that the action of $\text{Aut}^0(S; C)$ on S comes from the restriction of the action of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2; C)$. Therefore, the assertions concerning the number of parameters follow from Corollary 2.7 and Lemma 2.8.

If $m = 2$, then the action of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2; C)$ on S is faithful by Lemma 2.1, whence $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2; C) \cong \text{Aut}(S; C)$. In this case, the assertions of the lemma follow from Corollary 2.7 and Remark 2.9.

We now assume that $m = 1$ and $n \geq 2$. Then

$$\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2; C) \cong \Gamma \rtimes \text{Aut}(S; C),$$

where Γ is the pointwise stabilizer of S in $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^2)$. On the other hand, the group Γ is isomorphic to the pointwise stabilizer of the line $\pi(S)$ on \mathbb{P}^2 , whence

$$\Gamma \cong (\mathbb{k}^+)^2 \rtimes \mathbb{k}^\times.$$

Therefore, the assertion of the lemma follows from Corollary 2.7 and Remark 2.9. \square

Corollary 8.8. *Up to isomorphism, smooth Fano threefolds X with $\mathfrak{J}(X) = 3.5, 3.8, 3.17$ and 3.21 form families of dimensions 5, 3, 0 and 0 respectively. Each of these families contains a unique variety X_0 with infinite automorphism group. If $\mathfrak{J}(X_0) = 3.17$, then $\text{Aut}^0(X_0) \cong \text{PGL}_2(\mathbb{k})$. If $\mathfrak{J}(X_0) = 3.5$ or 3.8 , then $\text{Aut}^0(X_0) \cong \mathbb{k}^\times$. If $\mathfrak{J}(X_0) = 3.21$, then*

$$\text{Aut}^0(X_0) \cong (\mathbb{k}^+)^2 \rtimes (\mathbb{k}^\times)^2.$$

Proof. A variety X with $\mathfrak{J}(X) = 3.5$ is the blow-up of a curve C of bidegree $(5, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^2$. By Lemma 8.6, a variety X with $\mathfrak{J}(X) = 3.8$ is the blow-up of a curve C of bidegree $(4, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^2$. A variety X with $\mathfrak{J}(X) = 3.17$ is the blow-up of a curve C of bidegree $(1, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^2$. A variety X with $\mathfrak{J}(X) = 3.21$ is the blow-up of a curve C of bidegree $(2, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^2$. We conclude that $\text{Aut}^0(X) \cong \text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2; C)$. Hence everything follows from Lemma 8.7. \square

Remark 8.9 (compare with Lemma 7.6). One can use an argument similar to the proof of Lemma 8.7 in order to show that there is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 3.24$ and that $\text{Aut}^0(X) \cong \text{PGL}_{3;1}(\mathbb{k})$. Indeed, such a variety can be obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^2$ along a curve of bidegree $(1, 1)$.

Remark 8.10. Theorem 1.1 in [12] asserts that there is a smooth Fano threefold X with $\mathfrak{J}(X) = 3.8$ admitting a faithful action of $(\mathbb{k}^\times)^2$. Actually, this is not the case. Corollary 8.8 above shows that the two-dimensional torus cannot act faithfully on this variety. This also follows since every smooth Fano threefold in this family admits a fibration into del Pezzo surfaces of degree 5 by means of the natural projection $\mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^1$ in Lemma 8.6. These Fano threefolds can be obtained by blowing up a divisor of bidegree $(1, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ along a smooth conic. By Lemma 8.6, this conic is mapped isomorphically onto a smooth conic in \mathbb{P}^2 under the projection to the second factor. In [12], the description of smooth Fano varieties X with $\mathfrak{J}(X) = 3.8$ uses another conic which is mapped to a point in \mathbb{P}^2 by this projection. Blowing up this *wrong* conic yields a weak Fano threefold that is not actually a Fano threefold. As a consequence of this mistake, we still do not know whether there exists a smooth Fano threefold X with $\mathfrak{J}(X) = 3.8$ admitting a non-trivial Kähler–Ricci soliton as stated in [35], Theorem 6.2.

The following assertion can be proved in the same way as Lemma 8.7.

Lemma 8.11. *Given any positive integer n , we write \mathcal{C} for the family of smooth curves of tridegree $(1, 1, n)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then, up to the action of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$, the family \mathcal{C} has dimension 0 when $1 \leq n \leq 2$ and dimension $2n - 5$ when $n \geq 3$. Furthermore, up to the action of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$, this family contains a unique curve C_0 such that the group $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; C_0)$ is infinite. One has*

- $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; C_0) \cong \text{PGL}_2(\mathbb{k})$ when $n = 1$;
- $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; C_0) \cong \mathbb{k}^\times$ when $n \geq 2$.

Proof. The proof is left to the reader. \square

Corollary 8.12. *Up to isomorphism, the smooth Fano threefolds X with $\mathfrak{J}(X) = 4.3$ and $\mathfrak{J}(X) = 4.13$ form families of dimensions 0 and 1 respectively. In both cases, there is a unique variety X_0 with infinite automorphism group. In both cases one has $\text{Aut}^0(X_0) \cong \mathbb{k}^\times$.*

Proof. A variety X with $\mathfrak{J}(X) = 4.3$ or 4.13 is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve C of tridegree $(1, 1, 2)$ or $(1, 1, 3)$ respectively. We conclude that

$$\text{Aut}^0(X) \cong \text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; C).$$

Thus, the assertion follows from Lemma 8.11. \square

Remark 8.13 (compare with Lemma 4.14). One can use Lemma 8.11 to prove that there is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 4.6$ and that one has $\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k})$. Indeed, such a variety can be obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of tridegree $(1, 1, 1)$.

Lemma 8.14. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.22$. Then*

$$\text{Aut}^0(X) \cong \text{B} \times \text{PGL}_2(\mathbb{k}).$$

Proof. The threefold X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a conic Z lying in a fibre of the projection $\mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^1$. The morphisms $X \rightarrow \mathbb{P}^1$ and $X \rightarrow \mathbb{P}^2$ are $\text{Aut}^0(X)$ -equivariant. Thus the assertion follows from Lemma 3.1. \square

Lemma 8.15. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 4.1$. Then the group $\text{Aut}(X)$ is finite.*

Proof. The threefold X is a divisor of multidegree $(1, 1, 1, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Considering a projection $X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we see that X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a smooth curve C which is the intersection of two divisors of tridegree $(1, 1, 1)$. Thus, one has

$$\text{Aut}^0(X) \cong \text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; C).$$

By the adjunction formula, C is an elliptic curve. Consider the projections

$$\pi_i: \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1, \quad i = 1, 2, 3.$$

They are $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; C)$ -equivariant and the restriction of each π_i to C is a double cover $C \rightarrow \mathbb{P}^1$. The group $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; C)$ is non-trivial if and only if its action on one of the factors \mathbb{P}^1 is non-trivial. However, this action must preserve the set of the four branch points of the double cover. It follows that the group $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; C)$ is trivial. \square

Lemma 8.16. *Let X be a smooth Fano threefold with $\mathfrak{I}(X) = 4.5$. Then*

$$\text{Aut}^0(X) \cong (\mathbb{k}^\times)^2.$$

Proof. The threefold X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a disjoint union of smooth curves Z_1 and Z_2 of bidegrees $(2, 1)$ and $(1, 0)$ respectively. One has

$$\text{Aut}^0(X) \cong \text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2; Z_1 \cup Z_2).$$

By Lemma 8.7 one has

$$\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2; Z_1) \cong (\mathbb{k}^+)^2 \rtimes (\mathbb{k}^\times)^2,$$

where the subgroup $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2; Z_1) \subset \text{Aut}(\mathbb{P}^2)$ acts as the stabilizer of two points in \mathbb{P}^2 , namely, the points P_1 and P_2 that are the images under the projection $\pi_2: \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ of the ramification points of the double cover $Z_1 \rightarrow \mathbb{P}^1$ given by the projection $\pi_1: \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^1$.

Consider the action of $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2)$ on \mathbb{P}^2 induced by the $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2)$ -equivariant projection π_2 . It is easy to see that $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2; Z_1 \cup Z_2)$ is the subgroup of $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2; Z_1)$ consisting of all elements that preserve the point $\pi_2(Z_2)$ on \mathbb{P}^2 . Note that $\pi_2(Z_1)$ is the line through P_1 and P_2 . The point $\pi_2(Z_2)$ is not on this line since otherwise $Z_1 \cap Z_2 \neq \emptyset$. Therefore $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2; Z_1 \cup Z_2)$ acts on \mathbb{P}^2 preserving three points P_1, P_2 and $\pi_2(Z_2)$ in general position. Hence,

$$\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^2; Z_1 \cup Z_2) \cong (\mathbb{k}^\times)^2. \quad \square$$

Lemma 8.17. *Let X be a smooth Fano threefold with $\mathfrak{I}(X) = 4.8$. Then*

$$\text{Aut}^0(X) \cong \text{B} \times \text{PGL}_2(\mathbb{k}).$$

Proof. The threefold X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve Z of bidegree $(1, 1)$ contained in a fibre of the projection $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. The morphisms $X \rightarrow \mathbb{P}^1$ and $X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ are $\text{Aut}^0(X)$ -equivariant. Thus, the assertion follows from Remark 2.9 and Lemma 3.1. \square

Lemma 8.18. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 4.11$. Then*

$$\text{Aut}^0(X) \cong \mathbb{B} \times \text{PGL}_{3;1}(\mathbb{k}).$$

Proof. The threefold X is the blow-up of $\mathbb{P}^1 \times \mathbb{F}_1$ along a (-1) -curve Z contained in a fibre of the projection $\mathbb{P}^1 \times \mathbb{F}_1 \rightarrow \mathbb{P}^1$. The morphisms $X \rightarrow \mathbb{P}^1$ and $X \rightarrow \mathbb{F}_1$ are $\text{Aut}^0(X)$ -equivariant. Moreover, the (-1) -curve on \mathbb{F}_1 is unique and, therefore, it is invariant under the whole group $\text{Aut}(\mathbb{F}_1)$. Thus, the assertion of the lemma follows from Theorem 2.2 and Lemma 3.1. \square

§ 9. Blow-up of a quadric along a twisted quartic

To deal with the case $\mathfrak{J}(X) = 2.21$, we need some auxiliary information about representations of the group $\text{SL}_2(\mathbb{k})$.

Lemma 9.1. *Let U_4 be the (unique) irreducible five-dimensional representation of the group $\text{SL}_2(\mathbb{k})$ (or $\text{PGL}_2(\mathbb{k})$), and let U_0 be its trivial (one-dimensional) representation. Consider the projective space $\mathbb{P} = \mathbb{P}(U_0 \oplus U_4) \cong \mathbb{P}^5$ and, for every point $R \in \mathbb{P}$, write Γ_R^0 for the connected component of the identity in the stabilizer of R in $\text{PGL}_2(\mathbb{k})$. Then the following assertions hold.*

- There is a unique point $Q_0 \in \mathbb{P}$ such that $\Gamma_{Q_0}^0 = \text{PGL}_2(\mathbb{k})$.
- Up to the action of $\text{PGL}_2(\mathbb{k})$, there is a unique $Q_a \in \mathbb{P}$ such that $\Gamma_{Q_a}^0 \cong \mathbb{k}^+$.
- Up to the action of $\text{PGL}_2(\mathbb{k})$, there is a unique $Q_B \in \mathbb{P}$ such that $\Gamma_{Q_B}^0 \cong \mathbb{B}$.
- Up to the action of $\text{PGL}_2(\mathbb{k})$, there are a one-dimensional family of points $Q_m^\xi \in \mathbb{P}$ parametrized by an open subset of \mathbb{k} and an isolated point $Q_m^{3,1} \in \mathbb{P}$ such that

$$\Gamma_{Q_m^\xi}^0 \cong \Gamma_{Q_m^{3,1}}^0 \cong \mathbb{k}^\times.$$

- The point Q_0 is contained in the closure of the $\text{PGL}_2(\mathbb{k})$ -orbit of Q_a and in the closure of the family Q_m^ξ .

Proof. There is a $\text{PGL}_2(\mathbb{k})$ -equivariant (set-theoretical) identification of \mathbb{P} with the disjoint union $U_4 \sqcup \mathbb{P}(U_4)$. Thus, we need to find points with infinite stabilizers in U_4 and $\mathbb{P}(U_4)$.

The representation U_4 can be identified with the space of homogeneous polynomials of degree 4 in two variables u and v , where the action of $\text{PGL}_2(\mathbb{k})$ comes from the natural action of $\text{SL}_2(\mathbb{k})$. The point $Q_0 = 0$ is clearly the only one with stabilizer $\text{PGL}_2(\mathbb{k})$. The point Q_a can be chosen in the $\text{PGL}_2(\mathbb{k})$ -orbit of the polynomial u^4 , and the points Q_m^ξ can be chosen as $\xi^{-1}u^2v^2$.

Consider the projectivization $\mathbb{P}(U_4)$. The point Q_B can be chosen as the equivalence class of the polynomial u^4 . Furthermore, up to the action of $\text{PGL}_2(\mathbb{k})$, there are exactly two points $Q_m^{3,1}$ and $Q_m^{2,2}$ in $\mathbb{P}(U_4)$ such that the connected component of the identity in their stabilizers is isomorphic to \mathbb{k}^\times . These points may be chosen as the classes of the polynomials u^3v and u^2v^2 respectively. Clearly, the point $Q_m^{2,2}$ is the limit of the points Q_m^ξ as $\xi \rightarrow 0$ (while Q_0 is their limit as $\xi \rightarrow \infty$).

Finally, we note that $Q_m^{3,1}$ is not contained in the closure of the family Q_m^ξ (nor in the union of the $\text{PGL}_2(\mathbb{k})$ -orbits of the corresponding points) because a polynomial of u and v with a simple root cannot be the limit of polynomials having only multiple roots. \square

Lemma 9.2. *The following assertions hold.*

– *There is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 2.21$ and*

$$\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k}).$$

– *There is a unique smooth Fano threefold X with $\mathfrak{J}(X) = 2.21$ and $\text{Aut}^0(X) \cong \mathbb{k}^+$.*

– *There is a one-parameter family of smooth Fano threefolds X with $\mathfrak{J}(X) = 2.21$ and $\text{Aut}^0(X) \cong \mathbb{k}^\times$.*

– *For all other smooth Fano threefolds X with $\mathfrak{J}(X) = 2.21$, the group $\text{Aut}(X)$ is finite.*

Proof. The threefold X is the blow-up of the quadric Q along a twisted quartic Z . As in Lemma 4.3, we conclude that $\text{Aut}(X)$ is the stabilizer of the quadric Q in the subgroup $\Gamma \cong \text{PGL}_2(\mathbb{k})$ of $\text{Aut}(\mathbb{P}^4)$ that acts naturally on Z .

Let U_1 be a two-dimensional vector space such that the twisted quartic $Z \cong \mathbb{P}^1$ is identified with $\mathbb{P}(U_1)$. Then U_1 has the natural structure of an $\text{SL}_2(\mathbb{k})$ -representation which induces the action of Γ on Z . The projective space \mathbb{P}^4 is identified with the projectivization of the $\text{SL}_2(\mathbb{k})$ -representation $\text{Sym}^4(U_1)$, and the linear system \mathcal{Q} of quadrics in \mathbb{P}^4 passing through Z is identified with the projectivization of an $\text{SL}_2(\mathbb{k})$ -invariant six-dimensional vector subspace U in $\text{Sym}^2(\text{Sym}^4(U_1))$. By Exercise 11.31 in [49], the latter $\text{SL}_2(\mathbb{k})$ -representation splits into irreducible summands:

$$\text{Sym}^2(\text{Sym}^4(U_1)) \cong U_0 \oplus U_4 \oplus U_8,$$

where U_i is the (unique) irreducible $\text{SL}_2(\mathbb{k})$ -representation of dimension $i+1$. Therefore, one has an isomorphism $U \cong U_0 \oplus U_4$.

Let Q_0 be the quadric corresponding to the trivial $\text{SL}_2(\mathbb{k})$ -representation $U_0 \subset U$. Then Q_0 is $\text{PGL}_2(\mathbb{k})$ -invariant and $\text{Aut}(Q_0; Z) \cong \text{PGL}_2(\mathbb{k})$. We observe that Q_0 is smooth. Indeed, suppose that it is singular. If it is a cone (whose vertex is either a point or a line), then its vertex determines an $\text{SL}_2(\mathbb{k})$ -subrepresentation in $\text{Sym}^4(U_1) \cong U_4$. This contradicts the irreducibility of the $\text{SL}_2(\mathbb{k})$ -representation U_4 . We similarly see that Q_0 cannot be reducible or non-reduced. Thus we conclude that there is a unique smooth Fano threefold X_0 with $\mathfrak{J}(X_0) = 2.21$ and $\text{Aut}^0(X_0) \cong \text{PGL}_2(\mathbb{k})$.

We now use the results in Lemma 9.1. They yield all the desired assertions provided that we can check the smoothness (or otherwise) of the corresponding varieties. For the threefold X with $\text{Aut}^0(X) \cong \mathbb{k}^+$ and for a general threefold X with $\text{Aut}^0(X) \cong \mathbb{k}^\times$, the smoothness follows from the existence of a smooth variety X_0 in the closure of the corresponding family.

It remains to notice that the quadrics Q_B and $Q_m^{3,1}$ are singular. Indeed, one can choose homogeneous coordinates $[x : y : z : t : w]$ on \mathbb{P}^4 in such a way that the group \mathbb{k}^\times acts on \mathbb{P}^4 as

$$\zeta : [x : y : z : t : w] \mapsto [x : \zeta y : \zeta^2 z : \zeta^3 t : \zeta^4 w], \tag{9.3}$$

whence the quadrics Q_B and $Q_m^{3,1}$ are given by the equations $y^2 = xz$ and $xt = yz$ respectively. \square

Remark 9.4. There is a simple geometric way to construct the singular B-invariant quadric Q_B containing the twisted quartic Z . Indeed, the group B has a fixed

point P on Z . The projection from P maps \mathbb{P}^4 (resp. Z) to a projective space \mathbb{P}^3 acted on by B (resp. to a B -invariant twisted cubic Z' in \mathbb{P}^3). Furthermore, there is a B -fixed point P' on Z' . The projection from P' maps \mathbb{P}^3 (resp. Z') to a projective plane \mathbb{P}^2 acted on by B (resp. to a B -invariant conic in \mathbb{P}^2). Taking the cone with vertex P' over this conic, we obtain a B -invariant quadric in \mathbb{P}^3 passing through Z' . Taking the cone with vertex P over this quadric, we obtain a B -invariant quadric Q_B in \mathbb{P}^4 passing through Z . Note that this quadric is singular and, therefore, different from Q_0 . By Lemma 9.1, every B -invariant quadric passing through Z is either Q_0 or Q_B . This means that there is no smooth Fano threefold X with $\mathfrak{J}(X) = 2.21$ and $\text{Aut}^0(X) \cong B$.

Remark 9.5. The Fano threefold X with $\mathfrak{J}(X) = 2.21$ and $\text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k})$ was considered in [36], Example 2.3.

Smooth Fano threefolds with $\mathfrak{J}(X) = 2.21$ and $\text{Aut}^0(X) \cong \mathbb{k}^\times$ can be described very explicitly. Namely, every such threefold X is the blow-up of the quadric Q_λ in \mathbb{P}^4 given by

$$z^2 = \lambda xw + (1 - \lambda)yt \tag{9.6}$$

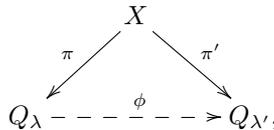
along a twisted quartic Z given by the parametrization

$$[u^4 : u^3v : u^2v^2 : uv^3 : v^4],$$

where $[u : v] \in \mathbb{P}^1$. Here $[x : y : z : t : w]$ are homogeneous coordinates in \mathbb{P}^4 , the group \mathbb{k}^\times acts on \mathbb{P}^4 as in (9.3), and $\lambda \in \mathbb{k}$ is such that $\lambda \neq 0$ and $\lambda \neq 1$. Note that $\text{Aut}(Q_\lambda; Z)$ also contains an additional involution

$$\iota : [x : y : z : t : w] \mapsto [w : t : z : y : x].$$

Along with the group \mathbb{k}^\times , it generates the subgroup $\mathbb{k}^\times \rtimes \mathbb{Z}/2\mathbb{Z}$. The action of this subgroup lifts to X . Observe that there is an $\text{Aut}(Q_\lambda; Z)$ -commutative diagram



where π is the blow-up of Q_λ along Z , the morphism π' is the blow-up of some smooth quadric $Q_{\lambda'}$ along Z , and ϕ is the birational map given by a linear system of quadrics passing through Z . In fact, it follows from Remark 2.13 in [32] that $\lambda = \lambda'$, and ϕ can be chosen to be an involution. In the case when $\text{Aut}^0(Q_\lambda; Z) \cong \text{Aut}^0(X) \cong \text{PGL}_2(\mathbb{k})$, this follows from Example 2.3 in [36].

Remark 9.7. A. Kuznetsov pointed out to us that $\text{Aut}^0(X_{-1/3}) \cong \text{PGL}_2(\mathbb{k})$ for the variety $X_{-1/3}$ corresponding to $\lambda = -1/3$ in the notation above. To check this, it suffices to write down the condition for the quadric (9.6) to be invariant under the generators of the Lie algebra of the group $\text{SL}_2(\mathbb{k})$.

§ 10. Divisors of bidegree (1, 2) on $\mathbb{P}^2 \times \mathbb{P}^2$

In this section we consider smooth Fano threefolds X with $\mathfrak{J}(X) = 2.24$. All of them are divisors of bidegree (1, 2) on $\mathbb{P}^2 \times \mathbb{P}^2$.

Lemma 10.1. *Let C and ℓ be a conic and a line in \mathbb{P}^2 respectively. Suppose that C and ℓ intersect each other transversally (at two distinct points). Then $\text{Aut}^0(\mathbb{P}^2; C \cup \ell) \cong \mathbb{k}^\times$.*

Proof. Denote the intersection points in $C \cap \ell$ by P_1 and P_2 . Let ℓ' be the tangent line to C at P_1 . Choose coordinates $[x : y : z]$ on \mathbb{P}^2 in such a way that the lines ℓ and ℓ' are given by $x = 0$ and $y = 0$. In particular, $P_1 = [0 : 0 : 1]$. We can also assume that $P_2 = [0 : 1 : 0]$. In these coordinates, the conic C is given by $x^2 = yz$. An automorphism of \mathbb{P}^2 preserving ℓ and C acts on the tangent space $T_{P_1}(C \cup \ell) \cong \mathbb{k}^2$ by dilation of the coordinates x and y (regarded as coordinates on $T_{P_1}(C \cup \ell)$). Therefore it acts in the same way on the original plane \mathbb{P}^2 . Keeping in mind that this automorphism preserves $C = \{x^2 = yz\}$, we get the assertion of the lemma. \square

Lemma 10.2 (compare with Theorem 1.1 in [12]). *Any smooth divisor of bidegree $(1, 2)$ on $\mathbb{P}^2 \times \mathbb{P}^2$ has a finite automorphism group with two exceptions. The connected component of the identity of the automorphism group of the first (resp. second) exception is isomorphic to \mathbb{k}^\times (resp. $(\mathbb{k}^\times)^2$).*

Proof. Let X be a smooth divisor of bidegree $(1, 2)$ on $\mathbb{P}^2 \times \mathbb{P}^2$. The projection ϕ to the first factor provides X with the structure of a conic bundle. Its discriminant curve Δ is a curve of degree 3 given by the vanishing of the discriminant of the quadratic form (whose coordinates are linear functions on the base of the conic bundle). The curve Δ is at worst nodal by Remark 2.3.

We denote coordinates on $\mathbb{P}^2 \times \mathbb{P}^2 = \mathbb{P}_x^2 \times \mathbb{P}_y^2$ by $[x_0 : x_1 : x_2] \times [y_0 : y_1 : y_2]$. Define Θ as the maximal subgroup of $\text{Aut}^0(X)$ acting by fibrewise transformations with respect to ϕ . There is an exact sequence of groups

$$1 \rightarrow \Theta \rightarrow \text{Aut}^0(X) \rightarrow \Gamma,$$

where Γ acts faithfully on \mathbb{P}_x^2 .

We claim that the group Θ is finite. Indeed, suppose that it is not. Let ℓ be a general line on \mathbb{P}_x^2 , and let S be the surface $\phi^{-1}(\ell)$. Then S is Θ -invariant and the image of Θ in $\text{Aut}(S)$ is infinite. On the other hand, S is a smooth del Pezzo surface of degree 5, so that $\text{Aut}(S)$ is finite by Theorem 2.2. The resulting contradiction shows that the kernel of the action of $\text{Aut}^0(X)$ on \mathbb{P}_x^2 is finite.

The variety X is given by

$$x_0Q_0 + x_1Q_1 + x_2Q_2 = 0, \tag{10.3}$$

where Q_i are quadratic forms in y_j . Note that they are linearly independent since X is smooth.

The curve Δ is Γ -invariant. If Δ is a smooth cubic, then the group Γ is finite by Lemma 2.1. If Δ is singular but irreducible, then Γ is finite by Remark 2.3 and Lemma 2.4.

Suppose that Δ is the union of a line and a smooth conic. Then the line intersects the conic transversally since Δ is nodal. In particular, Γ (and hence $\text{Aut}^0(X)$) is a subgroup of \mathbb{k}^\times . We shall find an equation for X in appropriate coordinates.

First, we can assume that the line is given by $x_0 = 0$, and its intersection points with the conic are $[0 : 1 : 0]$ and $[0 : 0 : 1]$. This means that by putting $x_0 = x_1 = 0$

or $x_0 = x_2 = 0$ in (10.3), we obtain squares of linear forms because the fibres over the nodes of Δ are double lines by Remark 2.3. Taking these (linearly independent!) linear forms for coordinates in \mathbb{P}_y^2 , we obtain $Q_1 = y_1^2$ and $Q_2 = y_2^2$.

We now put

$$Q_0 = a_0y_0^2 + a_1y_0y_1 + a_2y_0y_2 + a_3y_1^2 + a_4y_1y_2 + a_5y_2^2.$$

Notice that $a_0 \neq 0$ since otherwise the point of X given by $x_0 = y_1 = y_2 = 0$ is singular. Hence we can assume that $a_0 = 1$. Making the linear change of coordinates

$$y_0 = y'_0 - \frac{a_1}{2}y'_1 - \frac{a_2}{2}y'_2, \quad y_1 = y'_1, \quad y_2 = y'_2$$

and dropping the primes, we may assume that $a_1 = a_2 = 0$. Making another linear change of coordinates,

$$x_0 = x'_0 - a_3x'_1 - a_5x'_2, \quad x_1 = x'_1, \quad x_2 = x'_2$$

and dropping the primes, we may assume that $a_3 = a_5 = 0$. Finally, scaling the coordinates, we may assume that $a_4 = -1$ since $a_4 \neq 0$ (otherwise Δ would be the union of three lines). To summarize, there are coordinates in which X is given by

$$x_0(y_0^2 - y_1y_2) + x_1y_1^2 + x_2y_2^2 = 0.$$

The action of \mathbb{k}^\times in Lemma 10.1 is given by the weights

$$\begin{aligned} \text{wt}(x_0) &= 0, & \text{wt}(x_1) &= 2, & \text{wt}(x_2) &= -2, \\ \text{wt}(y_0) &= 0, & \text{wt}(y_1) &= -1, & \text{wt}(y_2) &= 1, \end{aligned}$$

whence we have $\text{Aut}^0(X) \cong \mathbb{k}^\times$ in this case.

Similarly, if Δ is the union of three lines in general position, then Γ (and hence $\text{Aut}^0(X)$) is a subgroup of $(\mathbb{k}^\times)^2$. Letting the intersection points of the lines be $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$, we easily see that X can be given by

$$x_0y_0^2 + x_1y_1^2 + x_2y_2^2 = 0.$$

The toric structure on \mathbb{P}_x^2 defined by the three lines induces the action of $(\mathbb{k}^\times)^2$ on X , whence we have $\text{Aut}^0(X) \cong (\mathbb{k}^\times)^2$ in this case. \square

§ 11. The manifold missing from Iskovskikh’s trigonal list

Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.2$. It may be described as follows. Put

$$U = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)).$$

Let $\pi : U \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the natural projection and let L be the tautological line bundle on U . Then X is a smooth threefold in the linear system $|2L + \pi^*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 3))|$.

By [1], the threefold X is not hyperelliptic; see also [50]. Thus the linear system $|-K_X|$ determines an embedding $X \hookrightarrow \mathbb{P}^9$. Note that X is not an intersection of quadrics in \mathbb{P}^9 . Indeed, let $\omega : X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the restriction of the projection π to X , and let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the projections to the first and second factors respectively. Put $\phi_1 = \pi_1 \circ \omega$ and $\phi_2 = \pi_2 \circ \omega$. Then the general fibre of ϕ_1 is a smooth cubic surface. It follows immediately that X is not an intersection of quadrics in \mathbb{P}^9 .

Remark 11.1. In the terminology of [2], § 2, the threefold X is trigonal. However, it is missing from the classification of smooth trigonal Fano threefolds obtained in [2], Theorem 2.5. More precisely, in the proof of this theorem, Iskovskikh showed that X can be obtained as follows. The scheme intersection of all quadrics in \mathbb{P}^9 containing X is a scroll

$$R = \mathbb{F}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)).$$

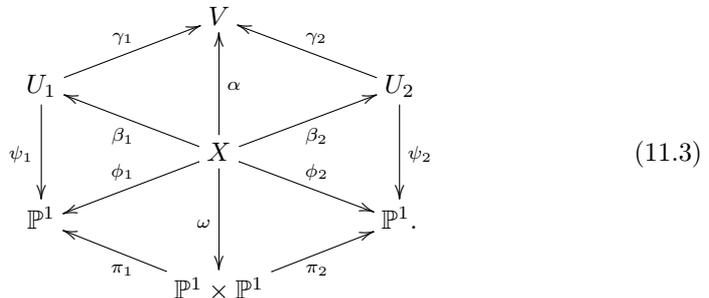
It is embedded in \mathbb{P}^9 by the tautological linear system, which we denote by M . Write F for the fibre of a general projection $R \rightarrow \mathbb{P}^1$. Then X is contained in the linear system $|3M - 4F|$. In the notation of [51], § 2, we have $R = \mathbb{F}(2, 2, 1, 1)$, and X is given by

$$\begin{aligned} &\alpha_2^1(t_1, t_2)x_1^3 + \alpha_2^2(t_1, t_2)x_1^2x_2 + \alpha_1^1(t_1, t_2)x_1^2x_3 + \alpha_1^2(t_1, t_2)x_1^2x_4 \\ &+ \alpha_2^3(t_1, t_2)x_1x_2^2 + \alpha_1^3(t_1, t_2)x_1x_2x_3 + \alpha_1^4(t_1, t_2)x_1x_2x_4 + \alpha_0^1(t_1, t_2)x_1x_3^2 \\ &+ \alpha_0^2(t_1, t_2)x_1x_3x_4 + \alpha_0^3(t_1, t_2)x_1x_4^2 + \alpha_2^4(t_1, t_2)x_2^3 + \alpha_1^5(t_1, t_2)x_2^2x_3 \\ &+ \alpha_1^6(t_1, t_2)x_2^2x_4 + \alpha_0^4(t_1, t_2)x_2x_3^2 + \alpha_0^5(t_1, t_2)x_2x_3x_4 + \alpha_0^6(t_1, t_2)x_2x_4^2 = 0, \end{aligned}$$

where each $\alpha_d^i(t_1, t_2)$ is a homogeneous polynomial of degree d . Thus, X is the threefold T_{11} in [50]. Note that the restriction to X of the natural projection $R \rightarrow \mathbb{P}^1$ determines the morphism ϕ_1 . In the proof of Theorem 2.5 in [2], Iskovskikh applied the Lefschetz theorem for X to deduce that its Picard group is cut out by divisors on the scroll R and exclude this case (this is case 4 in his proof). However, the threefold X is not an ample divisor on R since its restriction to the subscroll $x_3 = x_4 = 0$ is negative. Hence the Lefschetz theorem is not applicable here.

Lemma 11.2. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 3.2$. Then the group $\text{Aut}(X)$ is finite.*

Proof. In the notation of Remark 11.1, let S be the subscroll given by $x_3 = x_4 = 0$. Then $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and S is contained in X . Moreover, the normal bundle of S in X is $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$. This yields the existence of a commutative diagram,



Here U_1 and U_2 are smooth threefolds, the morphisms β_1 and β_2 are contractions of the surface S to curves in these threefolds, the morphism α is a contraction of S to an isolated ordinary double point of the threefold V , the morphism ϕ_2 is a fibration into del Pezzo surfaces of degree 6, the morphism ψ_1 is a fibration into del Pezzo surfaces of degree 4, and ψ_2 is a fibration into quadric surfaces. By construction,

V is a Fano threefold with a single isolated ordinary double point and the morphisms γ_1 and γ_2 are small resolutions of this singular point. We also note that $-K_V^3 = 16$.

Observe that the diagram (11.3) is $\text{Aut}(X)$ -equivariant. In particular, there is an exact sequence of groups

$$1 \longrightarrow G_{\phi_1} \longrightarrow \text{Aut}(X) \longrightarrow G_{\mathbb{P}^1} \longrightarrow 1,$$

where G_{ϕ_1} is a subgroup of $\text{Aut}(X)$ that leaves the general fibre of ϕ_1 invariant and $G_{\mathbb{P}^1}$ is a subgroup of $\text{Aut}(\mathbb{P}^1)$. Since the general fibre of ϕ_1 is a cubic surface, we see from Theorem 2.2 that the group G_{ϕ_1} is finite. We claim that $G_{\mathbb{P}^1}$ is also finite.

Indeed, consider an exact sequence of groups,

$$1 \longrightarrow G_\omega \longrightarrow \text{Aut}(X) \longrightarrow G_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow 1,$$

where G_ω is a subgroup of $\text{Aut}(X)$ that leaves the general fibre of ω invariant and $G_{\mathbb{P}^1 \times \mathbb{P}^1}$ is a subgroup of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$. If the group $G_{\mathbb{P}^1 \times \mathbb{P}^1}$ is finite, then so is $G_{\mathbb{P}^1}$ since there is a natural surjective homomorphism $G_{\mathbb{P}^1 \times \mathbb{P}^1} \rightarrow G_{\mathbb{P}^1}$.

To prove the lemma, it suffices to show that $G_{\mathbb{P}^1 \times \mathbb{P}^1}$ is finite. Note that this group preserves the projections π_1 and π_2 because ϕ_1 is a fibration into cubic surfaces while ϕ_2 is a fibration into del Pezzo surfaces of degree 6. Thus, the group $G_{\mathbb{P}^1 \times \mathbb{P}^1}$ is contained in $\text{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1) \cong \text{PGL}_2(\mathbb{k}) \times \text{PGL}_2(\mathbb{k})$.

The morphism ω in (11.3) is a standard conic bundle and its discriminant curve Δ is a curve of bidegree $(5, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The curve Δ is $G_{\mathbb{P}^1 \times \mathbb{P}^1}$ -invariant. Moreover, it is reduced and has at most isolated ordinary double points as singularities; see Remark 2.3. If C is an irreducible component of Δ , then the intersection number

$$C \cdot (\Delta - C)$$

must be even ([52], Corollary 2.1). In particular, this means that Δ has no irreducible components of bidegree $(0, 1)$.

Let C be an irreducible component of Δ of bidegree (a, b) , where $b \geq 1$. Then C is $G_{\mathbb{P}^1 \times \mathbb{P}^1}^0$ -invariant, where $G_{\mathbb{P}^1 \times \mathbb{P}^1}^0$ is the connected component of the identity in the group $G_{\mathbb{P}^1 \times \mathbb{P}^1}$. Moreover, the action of $G_{\mathbb{P}^1 \times \mathbb{P}^1}^0$ on C is faithful. This follows from Lemma 2.1 when $b \geq 2$ and from Lemma 2.5 when $b = 1$.

Assume that the curve Δ has an irreducible component C of bidegree $(a, 1)$. Then the intersection of C with $\Delta - C$ consists of $a + 5 - a = 5$ points. Thus, the group $G_{\mathbb{P}^1 \times \mathbb{P}^1}^0$ is trivial in this case.

This means that we may assume that Δ has an irreducible component C of bidegree $(a, 2)$. Suppose that $a \geq 4$. If the normalization of C has positive genus, then the group $G_{\mathbb{P}^1 \times \mathbb{P}^1}^0$ is trivial by Lemma 2.1. Thus we may suppose that C has at least $p_a(C) \geq 3$ singular points. This again means that $G_{\mathbb{P}^1 \times \mathbb{P}^1}^0$ is trivial because the action of $G_{\mathbb{P}^1 \times \mathbb{P}^1}^0$ lifts to the normalization of C and preserves the preimage of the singular locus of C .

We are left with the case when $1 \leq a \leq 3$. Then the intersection of C with $\Delta - C$ consists of $2(5 - a) \geq 6$ points. Thus, the group $G_{\mathbb{P}^1 \times \mathbb{P}^1}^0$ is trivial in this case. \square

Remark 11.4. The commutative diagram (11.3) is well known to experts. For example, it occurs in the proofs of Theorem 2.3 in [53], Proposition 3.8 in [54], and Lemma 8.2 in [55].

§ 12. Remaining cases

In this section we consider smooth Fano threefolds X with

$$\mathfrak{J}(X) \in \{2.1, 2.3, 2.5, 2.6, 2.8, 2.10, 2.11, 2.16, 2.19\}.$$

Theorem 1.1 immediately implies the following assertion.

Corollary 12.1. *Let X be a smooth Fano threefold with*

$$\mathfrak{J}(X) \in \{2.1, 2.3, 2.5, 2.10, 2.11, 2.16, 2.19\}.$$

Then the group $\text{Aut}(X)$ is finite.

Proof. These varieties can be described as blow-ups of smooth Fano threefolds Y with $\mathfrak{J}(Y) \in \{1.11, 1.12, 1.13, 1.14\}$. \square

We need the following auxiliary fact.

Lemma 12.2. *Let $\Delta \subset \mathbb{P}^2$ be a nodal curve of degree at least 4. Then the group $\text{Aut}(\mathbb{P}^2; \Delta)$ is finite.*

Proof. The proof is left to the reader. \square

Lemma 12.3. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 2.6$. Then the group $\text{Aut}(X)$ is finite.*

Proof. The threefold X is either a divisor of bidegree $(2, 2)$ on $\mathbb{P}^2 \times \mathbb{P}^2$ or a double cover of the flag variety W branched over a divisor $Z \sim -K_W$.

Let X be a divisor of bidegree $(2, 2)$ on $\mathbb{P}^2 \times \mathbb{P}^2$. We write $\phi_i: X \rightarrow \mathbb{P}^2$, $i = 1, 2$, for the natural projections. They determine $\text{Aut}^0(X)$ -equivariant standard conic bundles whose discriminant curves Δ_i are sextics. By Remark 2.3, the curves Δ_i are at worst nodal. Hence the group $\text{Aut}(\mathbb{P}^2; \Delta_i)$ is finite by Lemma 12.2.

There are exact sequences of groups

$$1 \rightarrow \Theta_i \rightarrow \text{Aut}^0(X) \rightarrow \Gamma_i,$$

where the action of Θ_i is fibrewise with respect to ϕ_i and Γ_i acts faithfully on \mathbb{P}^2 and preserves Δ_i . In particular, the groups $\Gamma_i \subset \text{Aut}(\mathbb{P}^2; \Delta_i)$ are finite. Since $\text{Aut}^0(X)$ is connected, we conclude from these sequences that

$$\Theta_1 = \text{Aut}^0(X) = \Theta_2.$$

On the other hand, the intersection $\Theta_1 \cap \Theta_2$ acts trivially on $\mathbb{P}^2 \times \mathbb{P}^2$ and, therefore, is a trivial group. This means that the group $\text{Aut}^0(X)$ is trivial.

We now suppose that X is a double cover of the flag variety W branched over a divisor $Z \sim -K_W$. The divisor class $-\frac{1}{2}K_W$ is very ample. Hence the group $\text{Aut}(W; Z)$ is finite by Lemma 2.1. On the other hand, both conic bundles $X \rightarrow \mathbb{P}^2$ are $\text{Aut}^0(X)$ -equivariant, whence the double cover $\theta: X \rightarrow W$ is also $\text{Aut}^0(X)$ -equivariant. Thus, $\text{Aut}^0(X)$ is a subgroup of $\text{Aut}(W; Z)$, and the assertion follows. \square

Lemma 12.4. *Let X be a smooth Fano threefold with $\mathfrak{J}(X) = 2.8$. Then the group $\text{Aut}(X)$ is finite.*

Proof. There is a commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\phi} & V \\
 \alpha \downarrow & & \downarrow \beta \\
 V_7 & \xrightarrow{\pi} & \mathbb{P}^3,
 \end{array}$$

where π is the blow-up of a point $O \in \mathbb{P}^3$, the morphism β is a double cover branched over an irreducible quartic surface S that has a single isolated double point at O , the morphism ϕ is the blow-up of the (singular) threefold V at the preimage of O under β , and α is a double cover branched over the proper transform of S under π . The surface S has a singularity of type \mathbb{A}_1 or \mathbb{A}_2 at the point O . (The exceptional divisor of ϕ is a smooth quadric in the former case and a quadric cone in the latter.) In both cases, the morphism ϕ is a contraction of an extremal ray, so that ϕ is $\text{Aut}^0(X)$ -equivariant. Furthermore, the morphism β is given by the linear system $|-\frac{1}{2}K_V|$ and, therefore, it is also $\text{Aut}^0(X)$ -equivariant. This means that the group $\text{Aut}^0(X)$ is isomorphic to a subgroup of $\text{Aut}(\mathbb{P}^3; S)$. Since S is not uniruled, the group $\text{Aut}(\mathbb{P}^3; S)$ is finite by Lemma 2.1. Thus, we see that the group $\text{Aut}^0(X)$ is trivial. \square

Appendix A. The big table

In this appendix we provide an explicit description of the infinite automorphism groups arising in Theorem 1.2 and give more details about the corresponding Fano varieties. The notation for some frequently appearing groups is given at the end of § 1. We write S_d for a smooth del Pezzo surface of degree d , except for the quadric surface.

The first column of Table 1 contains the identifier $\mathfrak{J}(X)$ of a smooth Fano threefold X . The second column gives the anticanonical degree $-K_X^3$. In the third column we briefly describe the variety, following mainly [3], [20] and [4]. The fourth column contains the dimension of the family of Fano threefolds of the given type. In columns 5 and 6 we put the group $\text{Aut}^0(X)$ (if it is non-trivial) and the dimension of the family of varieties with the given group $\text{Aut}^0(X)$. The last column contains a reference to the statement in the text of the present paper where the variety is discussed.

Table 1. Automorphisms of smooth Fano threefolds

\mathfrak{J}	$-K^3$	Brief description	δ	Aut^0	δ^0	ref.
1.10	22	the zero locus of three sections of the rank 3 vector bundle $\wedge^2 \mathcal{Q}$, where \mathcal{Q} is the universal quotient bundle on $\text{Gr}(3, 7)$	6	\mathbb{k}^\times	1	1.1
				\mathbb{k}^+	0	
				$\text{PGL}_2(\mathbb{k})$	0	
1.15	40	the section V_5 of $\text{Gr}(2, 5) \subset \mathbb{P}^9$ by a linear subspace of codimension 3	0	$\text{PGL}_2(\mathbb{k})$	0	1.1
1.16	54	the hypersurface Q of degree 2 in \mathbb{P}^4	0	$\text{PSO}_5(\mathbb{k})$	0	1.1

1.17	64	\mathbb{P}^3	0	$\mathrm{PGL}_4(\mathbb{k})$	0	1.1
2.20	26	the blow-up of $V_5 \subset \mathbb{P}^6$ along a twisted cubic	3	\mathbb{k}^\times	0	6.10
2.21	28	the blow-up of $Q \subset \mathbb{P}^4$ along a twisted quartic	2	\mathbb{k}^\times	1	§ 9
				$\mathrm{PGL}_2(\mathbb{k})$	0	
				\mathbb{k}^+	0	
2.22	30	the blow-up of $V_5 \subset \mathbb{P}^6$ along a conic	1	\mathbb{k}^\times	0	6.13
2.24	30	a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree (1, 2)	1	$(\mathbb{k}^\times)^2$	0	§ 10
				\mathbb{k}^\times	0	
2.26	34	the blow-up of $V_5 \subset \mathbb{P}^6$ along a line	0	\mathbb{k}^\times	0	6.5
				B	0	
2.27	38	the blow-up of \mathbb{P}^3 along a twisted cubic	0	$\mathrm{PGL}_2(\mathbb{k})$	0	4.3
2.28	40	the blow-up of \mathbb{P}^3 along a plane cubic	1	$(\mathbb{k}^+)^3 \rtimes \mathbb{k}^\times$	1	4.4
2.29	40	the blow-up of $Q \subset \mathbb{P}^4$ along a conic	0	$\mathbb{k}^\times \times \mathrm{PGL}_2(\mathbb{k})$	0	5.8
2.30	46	the blow-up of \mathbb{P}^3 along a conic	0	$\mathrm{PSO}_{5;1}(\mathbb{k})$	0	5.3
2.31	46	the blow-up of $Q \subset \mathbb{P}^4$ along a line	0	$\mathrm{PSO}_{5;2}(\mathbb{k})$	0	5.3
2.32	48	a divisor W on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree (1, 1)	0	$\mathrm{PGL}_3(\mathbb{k})$	0	7.1
2.33	54	the blow-up of \mathbb{P}^3 along a line	0	$\mathrm{PGL}_{4;2}(\mathbb{k})$	0	4.5
2.34	54	$\mathbb{P}^1 \times \mathbb{P}^2$	0	$\mathrm{PGL}_2(\mathbb{k}) \times \mathrm{PGL}_3(\mathbb{k})$	0	3.2
2.35	56	the blow-up V_7 of a point on \mathbb{P}^3	0	$\mathrm{PGL}_{4;1}(\mathbb{k})$	0	4.5
2.36	62	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$	0	$\mathrm{Aut}(\mathbb{P}(1, 1, 1, 2))$	0	3.3
3.5	20	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a curve C of bidegree (5, 2) such that the composite $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is an embedding	5	\mathbb{k}^\times	0	8.8
3.8	24	a divisor in the linear system $ (\alpha \circ \pi_1)^*(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{P}^2}(2)) $, where $\pi_1: \mathbb{F}_1 \times \mathbb{P}^2 \rightarrow \mathbb{F}_1$ and $\pi_2: \mathbb{F}_1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ are the projections and $\alpha: \mathbb{F}_1 \rightarrow \mathbb{P}^2$ is the blow-up of a point	3	\mathbb{k}^\times	0	8.8
3.9	26	the blow-up of a cone over the Veronese surface $R_4 \subset \mathbb{P}^5$ with centre at the disjoint union of the vertex and a quartic on $R_4 \cong \mathbb{P}^2$	6	\mathbb{k}^\times	6	3.5

3.10	26	the blow-up of $Q \subset \mathbb{P}^4$ along the disjoint union of two conics	2	\mathbb{k}^\times	1	5.9
				$(\mathbb{k}^\times)^2$	0	
3.12	28	the blow-up of \mathbb{P}^3 along the disjoint union of a line and a twisted cubic	1	\mathbb{k}^\times	0	4.6
3.13	30	the blow-up of $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ along a curve of bidegree $(2, 2)$ that is mapped by the natural projections $\pi_2: W \rightarrow \mathbb{P}^2$ and $\pi_1: W \rightarrow \mathbb{P}^2$ to irreducible conics	1	\mathbb{k}^\times	1	7.4
				\mathbb{k}^+	0	
				$\mathrm{PGL}_2(\mathbb{k})$	0	
3.14	32	the blow-up of \mathbb{P}^3 along the disjoint union of a plane cubic curve contained in a plane $\Pi \subset \mathbb{P}^3$ and a point not belonging to Π	1	\mathbb{k}^\times	1	4.7
3.15	32	the blow-up of $Q \subset \mathbb{P}^4$ along the disjoint union of a line and a conic	0	\mathbb{k}^\times	0	5.10
3.16	34	the blow-up of V_7 along the proper transform under the blow-up $\alpha: V_7 \rightarrow \mathbb{P}^3$ of a twisted cubic passing through the centre of the blow-up α	0	B	0	4.8
3.17	36	a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1, 1, 1)$	0	$\mathrm{PGL}_2(\mathbb{k})$	0	8.8
3.18	36	the blow-up of \mathbb{P}^3 along the disjoint union of a line and a conic	0	$B \times \mathbb{k}^\times$	0	5.11
3.19	38	the blow-up of $Q \subset \mathbb{P}^4$ at two non-collinear points	0	$\mathbb{k}^\times \times \mathrm{PGL}_2(\mathbb{k})$	0	5.13
3.20	38	the blow-up of $Q \subset \mathbb{P}^4$ along the disjoint union of two lines	0	$\mathbb{k}^\times \times \mathrm{PGL}_2(\mathbb{k})$	0	5.14
3.21	38	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a curve of bidegree $(2, 1)$	0	$(\mathbb{k}^+)^2 \times (\mathbb{k}^\times)^2$	0	8.8
3.22	40	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a conic lying in a fibre of the projection $\mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^1$	0	$B \times \mathrm{PGL}_2(\mathbb{k})$	0	8.14
3.23	42	the blow-up of V_7 along the proper transform under the blow-up $\alpha: V_7 \rightarrow \mathbb{P}^3$ of an irreducible conic passing through the centre of the blow-up α	0	$(\mathbb{k}^+)^3 \times (B \times \mathbb{k}^\times)$	0	4.9
3.24	42	the blow-up of W along a fibre of the projection $W \rightarrow \mathbb{P}^2$	0	$\mathrm{PGL}_{3;1}(\mathbb{k})$	0	7.6
3.25	44	the blow-up of \mathbb{P}^3 along the disjoint union of two lines	0	$\mathrm{PGL}_{(2,2)}(\mathbb{k})$	0	4.10
3.26	46	the blow-up of \mathbb{P}^3 along the disjoint union of a point and a line	0	$(\mathbb{k}^+)^3 \times (\mathrm{GL}_2(\mathbb{k}) \times \mathbb{k}^\times)$	0	4.11

3.27	48	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	0	$(\mathrm{PGL}_2(\mathbb{k}))^3$	0	3.2
3.28	48	$\mathbb{P}^1 \times \mathbb{F}_1$	0	$\mathrm{PGL}_2(\mathbb{k}) \times \mathrm{PGL}_{3;1}(\mathbb{k})$	0	3.2
3.29	50	the blow-up of V_7 along a line lying in $E \cong \mathbb{P}^2$, where E is the exceptional divisor of the blow-up $V_7 \rightarrow \mathbb{P}^3$	0	$\mathrm{PGL}_{4;3,1}(\mathbb{k})$	0	4.12
3.30	50	the blow-up of V_7 along the proper transform under the blow-up $\alpha: V_7 \rightarrow \mathbb{P}^3$ of the line passing through the centre of the blow-up α	0	$\mathrm{PGL}_{4;2,1}(\mathbb{k})$	0	4.13
3.31	52	the blow-up of a cone over a smooth quadric in \mathbb{P}^3 at the vertex	0	$\mathrm{PSO}_{6;1}(\mathbb{k})$	0	3.6
4.2	28	the blow-up of the cone over a smooth quadric $S \subset \mathbb{P}^3$ along the disjoint union of the vertex and an elliptic curve on S	1	\mathbb{k}^\times	1	3.5
4.3	30	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of tridegree $(1, 1, 2)$	0	\mathbb{k}^\times	0	8.12
4.4	32	the blow-up of the smooth Fano threefold Y with $\mathfrak{J}(Y) = 3.19$ along the proper transform of a conic on a quadric $Q \subset \mathbb{P}^4$ that passes through both centres of the blow-up $Y \rightarrow Q$	0	$(\mathbb{k}^\times)^2$	0	5.15
4.5	32	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along the disjoint union of two irreducible curves of bidegrees $(2, 1)$ and $(1, 0)$	0	$(\mathbb{k}^\times)^2$	0	8.16
4.6	34	the blow-up of \mathbb{P}^3 along the disjoint union of three lines	0	$\mathrm{PGL}_2(\mathbb{k})$	0	4.14
4.7	36	the blow-up of $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ along the disjoint union of two curves of bidegrees $(0, 1)$ and $(1, 0)$	0	$\mathrm{GL}_2(\mathbb{k})$	0	7.7
4.8	38	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of tridegree $(0, 1, 1)$	0	$\mathrm{B} \times \mathrm{PGL}_2(\mathbb{k})$	0	8.17
4.9	40	the blow-up of the smooth Fano threefold Y with $\mathfrak{J}(Y) = 3.25$ along a curve contracted by the blow-up $Y \rightarrow \mathbb{P}^3$	0	$\mathrm{PGL}_{(2,2);1}(\mathbb{k})$	0	4.15
4.10	42	$\mathbb{P}^1 \times S_7$	0	$\mathrm{PGL}_2(\mathbb{k}) \times \mathrm{B} \times \mathrm{B}$	0	3.2
4.11	44	the blow-up of $\mathbb{P}^1 \times \mathbb{F}_1$ along a curve $C \cong \mathbb{P}^1$ contained in a fibre $F \cong \mathbb{F}_1$ of the projection $\mathbb{P}^1 \times \mathbb{F}_1 \rightarrow \mathbb{P}^1$ and satisfying $C \cdot C = -1$ on F	0	$\mathrm{B} \times \mathrm{PGL}_{3;1}(\mathbb{k})$	0	8.18
4.12	46	the blow-up of the smooth Fano threefold Y with $\mathfrak{J}(Y) = 2.33$ along two curves contracted by the blow-up $Y \rightarrow \mathbb{P}^3$	0	$(\mathbb{k}^+)^4 \rtimes (\mathrm{GL}_2(\mathbb{k}) \times \mathbb{k}^\times)$	0	4.16

4.13	26	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of tridegree $(1, 1, 3)$	1	\mathbb{k}^\times	0	8.12
5.1	28	the blow-up of the smooth Fano threefold Y with $\mathfrak{J}(Y) = 2.29$ along three curves contracted by the blow-up $Y \rightarrow Q$	0	\mathbb{k}^\times	0	5.16
5.2	36	the blow-up of the smooth Fano threefold Y with $\mathfrak{J}(Y) = 3.25$ along two curves $C_1 \neq C_2$ contracted by the blow-up $\phi: Y \rightarrow \mathbb{P}^3$ and contained in the same fibre of ϕ	0	$\mathbb{k}^\times \times \mathrm{GL}_2(\mathbb{k})$	0	4.17
5.3	36	$\mathbb{P}^1 \times S_6$	0	$\mathrm{PGL}_2(\mathbb{k}) \times (\mathbb{k}^\times)^2$	0	3.2
6.1	30	$\mathbb{P}^1 \times S_5$	0	$\mathrm{PGL}_2(\mathbb{k})$	0	3.2
7.1	24	$\mathbb{P}^1 \times S_4$	2	$\mathrm{PGL}_2(\mathbb{k})$	2	3.2
8.1	18	$\mathbb{P}^1 \times S_3$	4	$\mathrm{PGL}_2(\mathbb{k})$	4	3.2
9.1	12	$\mathbb{P}^1 \times S_2$	6	$\mathrm{PGL}_2(\mathbb{k})$	6	3.2
10.1	6	$\mathbb{P}^1 \times S_1$	8	$\mathrm{PGL}_2(\mathbb{k})$	8	3.2

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