Nonrational del Pezzo fibrations

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(Communicated by A. Sommese)

Abstract. Let $X$ be a general divisor in $|3M + nL|$ on the rational scroll $\text{Proj}(\bigoplus_{i=1}^{4} \mathcal{O}_{\mathbb{P}^1}(d_i))$, where $d_i$ and $n$ are integers, $M$ is the tautological line bundle, $L$ is a fibre of the natural projection to $\mathbb{P}^1$, and $d_1 \geq \cdots \geq d_4 = 0$. We prove that $X$ is rational $\iff d_1 = 0$ and $n = 1$.

1 Introduction

The rationality problem for threefolds\(^1\) splits in three cases: conic bundles, del Pezzo fibrations, and Fano threefolds. The cases of conic bundles and Fano threefolds are well studied.

Let $\psi: X \to \mathbb{P}^1$ be a fibration into del Pezzo surfaces of degree $k \geq 1$ such that $X$ is smooth and $\text{rk Pic}(X) = 2$. Then $X$ is rational if $k \geq 5$. The following result is due to [1] and [12].

Theorem 1.1. Suppose that fibres of $\psi$ are normal and $k = 4$. Then $X$ is rational if and only if

$$\chi(X) \in \{0, -8, -4\},$$

where $\chi(X)$ is the topological Euler characteristic.

The following result is due to [8].

Theorem 1.2. Suppose that $K_X^2 \not\in \text{Int NE}(X)$ and $k \leq 2$. Then $X$ is nonrational.

In the case when $k \leq 2$ and $K_X^2 \in \text{Int NE}(X)$, the threefold $X$ belongs to finitely many deformation families, whose general members are nonrational (see [13], [7], [5], Proposition 1.5).

*The author would like to thank A. Corti, M. Grinenko, V. Iskovskikh, V. Shokurov for fruitful conversations.

\(^1\)All varieties are assumed to be projective, normal, and defined over $\mathbb{C}$.
Suppose that $k = 3$. Then $X$ is a divisor in the linear system $|3M + nL|$ on the scroll

$$\text{Proj} \left( \bigoplus_{i=1}^{4} \mathcal{O}_{\mathbb{P}^1}(d_i) \right),$$

where $n$ and $d_i$ are integers, $M$ is the tautological line bundle, and $L$ is a fibre of the natural projection to $\mathbb{P}^1$. Suppose that $d_1 \geq d_2 \geq d_3 \geq d_4 = 0$.

Suppose that $X$ is a general\footnote{A complement to a countable union of Zariski closed subsets.} divisor in $|3M + nL|$. The following result is due to [8].

**Theorem 1.3.** Suppose that $K_X^2 \not\in \text{Int} \overline{\text{NE}}(X)$. Then $X$ is nonrational.

It follows from [4], [11], [2], [13], [3], [7] that $X$ is nonrational when $(d_1, d_2, d_3, n) \in \{(0, 0, 0, 2), (1, 0, 0, 0), (2, 1, 1, -2), (1, 1, 1, -1)\}$.

We prove the following result in Section 3.

**Theorem 1.4.** The threefold $X$ is rational $\iff d_1 = 0$ and $n = 1$.

Therefore, the threefold $X$ is nonrational if $\chi(X) \neq -14$. Indeed, we have

$$\chi(X) = -4K_X^3 - 54 = -4(18 - 6(d_1 + d_2 + d_3) - 8n) - 54 = 18 - 24(d_1 + d_2 + d_3) - 32n,$$

and $\chi(X) = -14$ implies $(d_1, d_2, d_3, n) = (0, 0, 0, 1)$ or $(d_1, d_2, d_3, n) = (2, 1, 1, -2)$.

The inequality $5n \geq 12 - 3(d_1 + d_2 + d_3)$ holds when $K_X^2 \not\in \text{Int} \overline{\text{NE}}(X)$. For $n < 0$, the inequality

$$5n \geq 12 - 3(d_1 + d_2 + d_3)$$

implies that $K_X^2 \not\in \text{Int} \overline{\text{NE}}(X)$ (see Lemma 36 in [3]). Hence, the threefold $X$ does not belong to finitely many deformation families in the case when $K_X^2 \in \text{Int} \overline{\text{NE}}(X)$ (see Section 2).

Let us illustrate our methods by proving the following result.

**Proposition 1.5.** Let $X$ be double cover of the scroll

$$\text{Proj} \left( \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \right)$$

that is branched over a general\footnote{A complement to a countable union of Zariski closed subsets.} divisor $D \in |4M - 2L|$, where $M$ is the tautological line bundle, and $L$ is a fibre of the natural projection to $\mathbb{P}^1$. Then $X$ is nonrational.

**Proof.** Put $V = \text{Proj}((\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})$. The divisor $D$ is given by the equation

$$\alpha_6x_1^4 + \alpha_6x_1^3x_2 + \alpha_4x_1^3x_3 + \alpha_2x_1^2x_2 + \alpha_1x_1x_2x_3 + \alpha_2x_2^2x_3 + \alpha_0x_1x_2^3 + \alpha_2x_1^2x_2 + \alpha_1x_1x_2^3 + \alpha_0x_1x_2^3 = 0$$

in bihomogeneous coordinates on $V$ (see § 2.2 in [10]), where $\alpha_d = \alpha_d(t_1, t_2)$ is a sufficiently general homogeneous polynomial of degree $d \geq 0$. Let

$$\chi: Y \rightarrow \text{Proj} \left( \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \right)$$
be a double cover branched over a divisor $\Delta \subset V$ that is given by the same bihomogeneous equation as of divisor $D$ with the only exception that $\alpha_0 = \alpha_1 = 0$. Then $Y$ is singular, because the divisor $\Delta$ is singular along the curve $Y_3 \subset V$ that is given by the equations $x_1 = x_2 = 0$.

The Bertini theorem implies the smoothness of $\Delta$ outside of the curve $Y_3$. Let $C$ be a curve on the threefold $Y$ such that $\chi(C) = Y_3$. Then the threefold $Y$ has singularities of type $\mathbb{A}_1 \times \mathbb{C}$ at general point of the curve $C$. We may assume that the system 

$$\alpha_2(t_1, t_2) = \alpha_1^1(t_1, t_2) = \alpha_2^2(t_1, t_2) = 0$$

has no non-trivial solutions. Then $Y$ has singularities of type $\mathbb{A}_1 \times \mathbb{C}$ at every point of $C$.

Let $\alpha : \tilde{V} \to V$ be the blow up of $Y_3$, and $\beta : \tilde{Y} \to Y$ be the blow up of $C$. Then the diagram

\begin{equation}
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{\chi}} & \tilde{V} \\
\beta \downarrow & & \alpha \downarrow \\
Y & \xrightarrow{\chi} & V
\end{array}
\end{equation}

commutes, where $\tilde{\chi} : \tilde{Y} \to \tilde{V}$ is a double cover. The threefold $\tilde{Y}$ is smooth.

Let $E$ be the exceptional divisor of $\alpha$, and $\tilde{\Delta}$ be the proper transform of $\Delta$ via $\alpha$. Then

$$\tilde{\Delta} \sim \alpha^*(4M - 2L) - 2E,$$

hence $\tilde{\Delta}$ is nef and big, because the pencil $|\alpha^*(4M - 2L) - E|$ does not have base points. The morphism $\tilde{\chi}$ is branched over $\tilde{\Delta}$. Then $\text{rk Pic}(\tilde{Y}) = 3$ by Theorem 2 in [9].

The linear system $|g^*(M - L) - E|$ does not have base points and gives a $\mathbb{P}^1$-bundle

$$\tau : \tilde{V} \to \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \cong F_0,$$

which induces a conic bundle $\tilde{\tau} = \tau \circ \tilde{\chi} : \tilde{Y} \to F_0$. Let $Y_2 \subset V$ be the subscroll given by $x_1 = 0$, and $S$ be a proper transform of $Y_2$ via $\alpha$. Then

$$Y_2 \cong \text{Proj}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \cong F_2,$$

and $S \equiv Y_2$. But $\tau$ maps $S$ to the section of $F_2$ that has trivial self-intersection.

Let $\tilde{S}$ be a surface in $\tilde{Y}$ such that $\tilde{\chi}(\tilde{S}) = S$, and $Z \subset \tilde{Y}$ be a general fibre of the natural projection to $\mathbb{P}^1$. Then $-K_Z$ is nef and big and $K_Z^2 = 2$. But the morphism

$$\alpha \circ \tilde{\chi}|_S : \tilde{S} \to Y_2$$

is a double cover branched over a divisor that is cut out by the equation

$$\alpha_0^4(t_0, t_1)x_2^3 + \alpha_1^3(t_0, t_1)x_2x_3 + \alpha_2^2(t_0, t_1)x_3^2 = 0.$$  

Let $\Xi \subset F_0$ be a degeneration divisor of the conic bundle $\tilde{\tau}$. Then

$$\Xi \sim \lambda \tilde{\tau}(\tilde{S}) + \mu \tilde{\tau}(Z),$$
where \( \lambda \) and \( \mu \) are integers. But \( \lambda = 6 \), because \( K_S^2 = 2 \). We have \( \tilde{\tau}(S) \not\subseteq \Xi \). Then

\[
\mu = \tilde{\tau}(S) \cdot \Xi = 8 - K_S^2,
\]

because \( \mu \) is the number of reducible fibres of the conic bundle \( \tilde{\tau}|_S \). These fibers are given by

\[
(\alpha_3^2(t_0, t_1))^2 = 4\alpha_2^2(t_0, t_1)\alpha_0^4(t_0, t_1),
\]

which implies that \( \mu = \tilde{\tau}(S) \cdot \Xi = 8 \). Then \( \tilde{Y} \) is nonruled by Theorem 10.2 in [11], which implies the nonrationality of the threefold \( X \) by Theorem 1.8.3 in § IV of the book [6]. □

2 Preliminaries

All results of this section follow from [3]. Take a scroll

\[
V = \text{Proj} \left( \bigoplus_{i=1}^{4} \mathcal{O}_{\mathbb{P}^1}(d_i) \right),
\]

where \( d_i \) is an integer, and \( d_1 \geq d_2 \geq d_3 \geq d_4 = 0 \). Let \( M \) and \( L \) be the tautological line bundle and a fibre of the natural projection to \( \mathbb{P}^1 \), respectively. Then \( \text{Pic}(V) = \mathbb{Z}M \oplus \mathbb{Z}L \).

Let \( (t_1 : t_2 : x_1 : x_2 : x_3 : x_k) \) be bihomogeneous coordinates on \( V \) such that \( x_i = 0 \) defines a divisor in \( |M - d_i L| \), and \( L \) is given by \( t_1 = 0 \). Then \( |aM + bL| \) is spanned by divisors

\[
c_{i_1i_2i_3i_4}(t_1, t_2)x_1^{i_1}x_2^{i_2}x_3^{i_3}x_k^{i_4} = 0,
\]

where \( \sum_{j=1}^{4} i_j = a \) and \( c_{i_1i_2i_3i_4}(t_1, t_2) \) is a homogeneous polynomial of degree \( b + \sum_{j=1}^{4} i_j d_j \). Let \( Y_j \subseteq V \) be a subscroll \( x_1 = \cdots = x_{j-1} = 0 \). The following result holds (see § 2.8 in [10]).

Corollary 2.1. Take \( D \in |aM + bL| \) and \( q \in \mathbb{N} \), where \( a \) and \( b \) are integers. Then

\[
\text{mult}_{Y_j}(D) \geq q \iff ad_j + b + (d_1 - d_j)(q - 1) < 0.
\]

Let \( X \) be a general divisor \( |3M + nL| \), where \( n \) is an integer.

Lemma 2.2. Suppose \( X \) is smooth and \( \text{rk Pic}(X) = 2 \). Then \( d_1 \geq -n \) and \( 3d_3 \geq -n \).

Proof. We see that \( Y_2 \not\subseteq X \). Then \( Y_3 \not\subseteq X \), because \( \text{rk Pic}(X) = 2 \). But \( \text{mult}_{Y_1}(X) \leq 1 \), because the threefold \( X \) is smooth. The assertion of Corollary 2.1 concludes the proof. □

Lemma 2.3. Suppose \( X \) is smooth and \( \text{rk Pic}(X) = 2 \). Then we have either \( d_1 = -n \) or \( d_2 \geq -n \).

\(^4\)A complement to a Zariski closed subset in moduli.
Suppose that $r = d_1 + n > 0$ and $d_2 < -n$. Then $X$ can be given by the equation
\[
\sum_{i+j+k=2} \gamma_{ijk}(t_0, t_2)x_1^i x_2^j x_3^k x_4 = \alpha_r(t_1, t_2)x_1 x_2^2 + \sum_{i+j+k=3} \beta_{ijk}(t_0, t_2)x_1^i x_2^j x_3^k,
\]
where $\alpha_r(t_1, t_2)$ is a homogeneous polynomial of degree $r$, $\beta_{ijk}$ and $\gamma_{ijk}$ are homogeneous polynomials of degree $n + id_1 + jd_2 + kd_3$. Then every point of the intersection
\[
x_1 = x_2 = x_3 = 0
\]
must be singular on the threefold $X$, which is a contradiction. \hfill \Box

**Lemma 2.4.** Suppose $X$ is smooth, $d_2 = d_3$, $n < 0$ and $\text{rk} \, \text{Pic}(X) = 2$. Then $3d_3 \neq -n$.

**Proof.** Suppose that $3d_3 = -n$. Then $X$ can be given by the following bihomogeneous equation
\[
\sum_{i+j+k=2} \gamma_{ijk}(t_0, t_2)x_1^i x_2^j x_3^k x_4 = f_3(x_2, x_3) + \alpha_r(t_0, t_2)x_1 x_2^3 + \sum_{i+j+k=3} \beta_{ijk}(t_0, t_2)x_1^i x_2^j x_3^k x_4,
\]
where $f_3(x_2, x_3)$ is a homogeneous polynomial of degree $3$, $\beta_{ijk}$ and $\gamma_{ijk}$ are homogeneous polynomials of degree $n + 2d_1 + jd_2 + kd_3$ and $n + 1 + jd_2 + kd_3$ respectively, $\alpha_r$ is a homogeneous polynomial of degree $r = 3d_1 + n$. The threefold $X$ contains 3 subscrolls given by the equations $x_1 = f_3(x_2, x_3) = 0$, which is impossible, because $\text{rk} \, \text{Pic}(X) = 2$. \hfill \Box

The following result follows from Lemmas 2.2, 2.3 and 2.4.

**Lemma 2.5.** The threefold $X$ is smooth and $\text{rk} \, \text{Pic}(X) = 2$ whenever

1. in the case when $d_1 = 0$, the inequality $n > 0$ holds,
2. either $d_1 = -n$ and $3d_3 \geq -n$, or $d_1 > -n$, $d_2 \geq -n$ and $3d_3 \geq -n$,
3. in the case when $d_2 = d_3$ and $n < 0$, the inequality $3d_3 > -n$ holds.

**Proof.** Suppose that all these conditions are satisfied. We must show that $X$ is smooth, because the equality $\text{rk} \, \text{Pic}(X) = 2$ holds by Proposition 32 in [3].

The linear system $|3M + nL|$ does not have base points if $n \geq 0$. So, the threefold $X$ is smooth by the Bertini theorem in the case $n \geq 0$. Therefore, we may assume that $n < 0$.

The base locus of $|3M + nL|$ consists of the curve $Y_4$, which implies that $X$ is smooth outside of the curve $Y_4$ and in a general point of $Y_4$ by the Bertini theorem and Corollary 2.1, respectively.

In the case when $d_1 = -n$ and $d_2 < -n$, the bihomogeneous equation of the threefold $X$ is
\[
\sum_{i+j+k=2} \gamma_{ijk}(t_0, t_2)x_1^i x_2^j x_3^k x_4 = \alpha_0 x_1 x_4^2 + \sum_{i+j+k=3} \beta_{ijk}(t_0, t_2)x_1^i x_2^j x_3^k,
\]
where $\beta_{ijk}$ and $\gamma_{ijk}$ are homogeneous polynomials of degree $n + id_1 + jd_2 + kd_3$ and $\alpha_0$ is a nonzero constant. The curve $Y_4$ is given by $x_1 = x_2 = x_3 = 0$, which implies that $X$ is smooth.
In the case when \( d_1 > -n \) and \( d_2 \geq -n \), the bihomogeneous equation of \( X \) is

\[
\sum_{i,j,k \geq 0 \atop i+j+k=2} \gamma_{ijk}(t_0, t_2)x_1^ix_2^jx_3^kx_4 = \sum_{i=1}^{3} \alpha_i(t_0, t_2)x_i^2 + \sum_{i,j,k \geq 0 \atop i+j+k=3} \beta_{ijk}(t_0, t_2)x_1^ix_2^jx_3^k,
\]

where \( \alpha_i \) is a homogeneous polynomial of degree \( d_i + n \), and \( \beta_{ijk} \) and \( \gamma_{ijk} \) are homogeneous polynomials of degree \( n + id_1 + jd_2 + kd_3 \). Therefore, either \( \alpha_1x_1^2 \) or \( \alpha_2x_2^2 \) does not vanish at any given point of the curve \( Y_4 \), which implies that \( X \) is smooth. \( \square \)

Thus, there is an infinite series of quadruples \((d_1, d_2, d_3, n)\) such that the threefold \( X \) is smooth, the equality \( \text{rk Pic}(X) = 2 \) holds, the inequality \( 5n < 12 - 3(d_1 + d_2 + d_3) \) holds and \( n < 0 \).

### 3 Nonrationality

We use the notation of Section 2. Let \( X \) be a general\(^5\) divisor in \(|3M + nL|\), and suppose that the threefold \( X \) is smooth, \( \text{rk Pic}(X) = 2 \), and \( X \) is rational. Let us show that \( d_1 = 0 \) and \( n = 1 \).

The threefold \( X \) is given by a bihomogeneous equation

\[
\sum_{i=0}^{3} \alpha_i(t_0, t_2)x_3^ix_4^{3-i} + x_1F(t_0, t_1, x_1, x_2, x_3, x_4) + x_2G(t_0, t_1, x_1, x_2, x_3, x_4) = 0,
\]

where \( \alpha_i \) is a general homogeneous polynomial of degree \( n + id_3 \), and \( F \) and \( G \) stand for

\[
\sum_{i,j,k \geq 0 \atop i+j+k=2} \beta_{ijkl}(t_0, t_2)x_1^ix_2^jx_3^kx_4^l \quad \text{and} \quad \sum_{i,j,k,l \geq 0 \atop i+j+k+l=2} \gamma_{ijkl}(t_0, t_2)x_1^ix_2^jx_3^kx_4^l
\]

respectively, where \( \beta_{ijkl} \) is a general homogeneous polynomial of degree \( n + (i+1)d_1 + jd_2 + kd_3 \), and \( \gamma_{ijkl} \) is a general homogeneous polynomial of degree \( n + id_1 + (j + 1)d_2 + kd_3 \).

Let \( Y \) be a threefold given by \( x_1F + x_2G = 0 \). Then \( Y_3 \subset Y \), where \( Y_3 \) is given by \( x_1 = x_2 = 0 \).

**Lemma 3.1.** The threefold \( Y \) has \( 2d_1 + 2d_2 + 4d_3 + 4n > 0 \) isolated ordinary double points.

**Proof.** The threefold \( Y \) is singular exactly at the points of \( V \) where

\[
x_1 = x_2 = F(t_0, t_1, x_1, x_2, x_3, x_4) = G(t_0, t_1, x_1, x_2, x_3, x_4) = 0
\]

by the Bertini theorem. But \( Y_3 \cong \text{Proj}(\mathcal{O}_{d_1}(d_3) \oplus \mathcal{O}_{d_1}) \cong F_{d_3} \), where \((t_0 : t_1 : t_3 : t_4)\) can be considered as natural bihomogeneous coordinates on the surface \( Y_3 \).

\(^5\)A complement to a countable union of Zariski closed subsets.
Let $C$ and $Z$ be the curves on $Y_3$ that are cut out by the equations $F = 0$ and $G = 0$, respectively. Then $C$ and $Z$ are given by the equations

$$
\sum_{k,l \geq 0 \atop k+l \leq 2} \beta_{kl}(t_0, t_2)x_k^l = 0 \quad \text{and} \quad \sum_{k,l \geq 0 \atop k+l \leq 2} \gamma_{kl}(t_0, t_2)x_k^l = 0
$$

respectively, where $\beta_{kl} = \beta_{00kl}$ and $\gamma_{kl} = \gamma_{00kl}$. The degrees of $\beta_{kl}$ and $\gamma_{kl}$ are $n + d_1 + kd_3$ and $n + d_2 + kd_3$, respectively.

Let $O$ be a point of the scroll $V$ such that the set

$$x_1 = x_2 = F(t_0, t_1, x_1, x_2, x_3, x_4) = G(t_0, t_1, x_1, x_2, x_3, x_4) = 0$$

contains the point $O$. Then $O \in C \cap Z$ and $O \in \text{Sing}(Y)$. It is easy to see that $O$ is an isolated ordinary double point of the threefold $Y$ in the case when the curves $C$ and $Z$ are smooth and intersect each other transversally at the point $O$.

Put $\tilde{M} = M|_{Y_3}$ and $\tilde{L} = L|_{Y_3}$. Then $C \in |2\tilde{M} + (n + d_1)\tilde{L}|$ and $Z \in |2\tilde{M} + (n + d_2)\tilde{L}|$. But

$$|2\tilde{M} + (n + d_1)\tilde{L}|$$

does not have base points, because $d_1 + n \geq 0$ by Lemma 2.2. So, the curve $C$ is smooth.

The linear system $|2\tilde{M} + (n + d_2)\tilde{L}|$ may have base components, and $Z$ may not be reduced or irreducible. We have to show that $C$ intersects $Z$ transversally at smooth points of $Z$, because

$$|C \cap Z| = C \cdot Z = 2d_1 + 2d_2 + 4d_3 + 4n,$$

where $2d_1 + 2d_2 + 4d_3 + 4n > 0$ by Lemmas 2.2, 2.3 and 2.4.

Suppose that $d_1 > -n$. Then $d_2 \geq -n$ by Lemma 2.3. We see that $|2\tilde{M} + (n + d_2)\tilde{L}|$ does not have base points. Then $Z$ is smooth and $C$ intersects $Z$ transversally at every point of $C \cap Z$.

We may assume that $d_1 = -n$. Let $Y_4 \subset Y_3$ be a curve given by $x_3 = 0$. Then

$$C \cap Y_4 = \emptyset,$$

and either the linear system $|2\tilde{M} + (n + d_2)\tilde{L}|$ does not have base points, or the base locus of the linear system $|2\tilde{M} + (n + d_2)\tilde{L}|$ consist of the curve $Y_4$. However, we have

$$C \cap Z \subset Y_3 \setminus Y_4,$$

which implies that $C$ intersects the curve $Z$ transversally at smooth points of $Z$. \hfill \Box

Let $\pi: \hat{V} \to V$ be the blow up of $Y_3$, and $\hat{Y}$ be and the proper transforms of $Y$ via $\pi$. Then

$$\hat{Y} \sim \pi^*(3M + nL) - E,$$

where $E$ is and exceptional divisor of $\pi$. The threefold $\hat{Y}$ is smooth.

**Lemma 3.2.** The equality $\text{rk Pic}(\hat{Y}) = 3$ holds.
Proof. The linear system $|\pi^*(M - d_2 L) - E|$ does not have base points. Thus, the divisor
$$
\tilde{Y} \sim \pi^*(3M + nL) - E
$$
is nef and big when $n \geq 0$ by Lemmas 2.2, 2.3 and 2.4. Hence, the equality $\text{rk Pic}(\tilde{Y}) = 3$ holds in the case when $n > 0$ by Theorem 2 in [9]. So, we may assume that $n < 0$.

Let $\omega: \tilde{Y} \to \mathbb{P}^1$ be the natural projection and $S$ be the generic fibre of $\omega$, which is considered as a surface defined over the function field $\mathbb{C}(t)$. Then $S$ is a smooth cubic surface in $\mathbb{P}^3$, which contains a line in $\mathbb{P}^3$ defined over the field $\mathbb{C}(t)$, because $Y_3 \subset Y$. Then $\text{rk Pic}(S) \geq 2$.

To conclude the proof we must prove that $\text{rk Pic}(S) = 2$, because there is an exact sequence
$$
0 \to \mathbb{Z}[\pi^*(L)] \to \text{Pic}(\tilde{Y}) \to \text{Pic}(S) \to 0,
$$
because every fibre of $\tau$ is reduced and irreducible (see the proof of Proposition 32 in [3]).

Let $\tilde{S}$ be an example of the surface $S$ that is given by the equation

$$
x(q(t)x^2 + p(t)u^2) + y(r(t)y^2 + s(t)z^2) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t]),
$$
where $q(t), p(t), r(t), s(t)$ are polynomials such that the inequalities

$$
\deg(q(t)) > 0, \quad \deg(p(t)) \geq 0, \quad \deg(r(t)) > 0, \quad \deg(q(t)) \geq 0
$$

hold. The existence of the surface $\tilde{S}$ follows from the equation of the threefold $Y$.

Let $\mathbb{K}$ be an algebraic closure of the field $\mathbb{C}(t)$, let $L$ be a line $x = y = 0$, and let

$$
\gamma: \tilde{S} \to \mathbb{P}^1
$$
be a projection from $L$. Then $\gamma$ is a conic bundle defined over $\mathbb{C}(t)$. But $\gamma$ has five geometrically reducible fibres $F_1, F_2, F_3, F_4, F_5$ defined over $\mathbb{P}^3$ such that

- $F_i = \bar{F}_i \cup \bar{F}_i$, where $\bar{F}_i$ and $\bar{F}_i$ are geometrically irreducible curves,
- the curve $L \cup F_1$ is cut out on the surface $\tilde{S}$ by the equation

$$
y = e_x \sqrt[3]{q(t)/r(t)}x,
$$
where $e_x = -(1 + \sqrt{-3})/2$ and $i \in \{1, 2, 3\}$,
- the curve $F_4 \cup L$ is cut out on the surface $\tilde{S}$ by the equation $x = 0$, 
- the curve $F_5 \cup L$ is cut out on the surface $\tilde{S}$ by the equation $y = 0$.

The group $\text{Gal}(\mathbb{K}/\mathbb{C}(t))$ acts naturally on the set

$$
\Sigma = \{\bar{F}_1, \bar{F}_2, \bar{F}_3, \bar{F}_4, \bar{F}_5, \bar{F}_1, \bar{F}_2, \bar{F}_3, \bar{F}_4, \bar{F}_5\},
$$
because the conic bundle $\gamma$ is defined over $\mathbb{C}(t)$. The inequality $\text{rk Pic}(\tilde{S}) > 2$ implies the existence of a subset $\Gamma \subseteq \Sigma$ consisting of disjoint curves such that $\Gamma \subseteq \Sigma$ is $\text{Gal}(\mathbb{K}/\mathbb{C}(t))$-invariant.

The action of $\text{Gal}(\mathbb{K}/\mathbb{C}(t))$ on the set $\Sigma$ is easy to calculate explicitly. Putting

$$
\Delta = \{\bar{F}_1, \bar{F}_2, \bar{F}_3, \bar{F}_4, \bar{F}_5\}, \quad \Lambda = \{\bar{F}_4, \bar{F}_5\}, \quad \Xi = \{\bar{F}_5, \bar{F}_5\},
$$
we see that the group $\text{Gal}(K/C(t))$ acts transitively on each subset $\Lambda, \Xi, \Delta$, because we may assume that $q(t), p(t), r(t), s(t)$ are sufficiently general. But each subset $\Lambda, \Xi, \Delta$ does not consist of disjoint curves. Hence, the equality $\text{rk} \text{Pic}(\tilde{S}) = 2$ holds, which implies that $\text{rk} \text{Pic}(S) = 2$.

The linear system $|\pi^*(M - d_2L) - E|$ does not have base points and induces a $\mathbb{P}^2$-bundle

$$\tau: \tilde{V} \longrightarrow \text{Proj} \left(O_{\mathbb{P}^1}(d_1) \oplus O_{\mathbb{P}^1}(d_2)\right) \cong \mathbb{F}_r,$$

where $r = d_1 - d_2$. Let $l$ be a fibre of the natural projection $\mathbb{F}_r \to \mathbb{P}^1$, and $s_0$ be an irreducible curve on the surface $\mathbb{F}_r$ such that $s_0^2 = r$, and $s_0$ is a section of the projection $\mathbb{F}_r \to \mathbb{P}^1$. Then

$$\pi^*(M - d_2L) - E \sim \pi^*(s_0)$$

and $\pi^*(L) \sim \pi^*(l)$. The morphism $\tau$ induces a conic bundle $\tilde{\tau} = \tau|_{\tilde{V}}: \tilde{V} \to \mathbb{F}_r$.

Let $\Delta$ be the degeneration divisor of the conic bundle $\tilde{\tau}$. Then

$$\Delta \sim 5s_{\infty} + \mu l,$$

where $\mu$ is a natural number, and $s_{\infty}$ is the exceptional section of the surface $\mathbb{F}_r$.

Let $S$ be a surface in $\tilde{V}$ and $B$ be a threefold in $\tilde{V}$ dominating the curve $s_0$. Then

$$B \cong \text{Proj} \left(O_{\mathbb{P}^1}(d_1) \oplus O_{\mathbb{P}^1}(d_3) \oplus O_{\mathbb{P}^1}\right)$$

and $\pi(B) \cong B$. But $\pi(B) \cap Y = \pi(S) \cup Y_3$.

The surface $Y_3$ is cut out on $\pi(B)$ by the equation $x_1 = 0$, where $\pi(B) \in |M - d_2L|$. We have

$$S \sim 2T + (d_1 + n)F,$$

where $T$ is a tautological line bundle on $B$, and $F$ is a fibre of the projection $B \to \mathbb{P}^1$. Then

$$K_S^2 = -5d_1 + 2d_3 - 4d_2 - 3n + 8$$

and $\mu = s_0 \cdot \Delta = 5d_1 - 2d_3 + 4d_2 + 3n$.

It follows from the equivalence $2K_{\tilde{X}} + \Delta \sim s_{\infty} + (3d_1 - 2d_3 + 6d_2 + 3n - 4)l$ that

$$|2K_{\tilde{X}} + \Delta| \not\equiv \emptyset \iff 3d_1 - 2d_3 + 6d_2 + 3n \geq 4,$$

which implies that $Y$ is nonrational by Theorem 10.2 in [11] if $3d_1 - 2d_3 + 6d_2 + 3n \geq 4$.

The threefold $Y$ is nonruled if and only if it is nonrational, because the threefold $Y$ is rationally connected. So, the threefold $X$ is nonrational by Theorem 1.8.3 in § IV of the book [6] whenever

$$3d_1 - 2d_3 + 6d_2 + 3n \geq 4,$$

which implies that $3d_1 - 2d_3 + 6d_2 + 3n < 4$, because we assume that $X$ is rational.

We see that either $d_1 = 0$ and $n = 1$ or $d_1 = 1$ and $d_2 = n = 0$ by Lemmas 2.2, 2.3 and 2.4, but the threefold $X$ is birational to a smooth cubic threefold in the case when $d_1 = 1$ and $d_2 = n = 0$, which is nonrational by [4]. Then $d_1 = 0$ and $n = 1$. The assertion of Theorem 1.4 is proved.
References


Received 23 January, 2007; revised 25 June, 2007
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