POINTS IN PROJECTIVE SPACES AND APPLICATIONS

IVAN CHELTSOV

Abstract

We prove the factoriality of a nodal hypersurface in $\mathbb{P}^4$ of degree $d$ that has at most $2(d - 1)^2/3$ singular points, and we prove the factoriality of a double cover of $\mathbb{P}^3$ branched over a nodal surface of degree $2r$ having less than $(2r - 1)r$ singular points.

1. Introduction

Let $\Sigma$ be a finite subset in $\mathbb{P}^n$ and $\xi \in \mathbb{N}$, where $n \geq 2$. Then the points of the set $\Sigma$ impose independent linear conditions on homogeneous forms of degree $\xi$ if and only if for every point $P \in \Sigma$ there is a homogeneous form of degree $\xi$ that vanishes at every point of the set $\Sigma \setminus P$, and does not vanish at the point $P$. The latter is equivalent to the equality

$$h^1 \left( I_{\Sigma} \otimes \mathcal{O}_{\mathbb{P}^n}(\xi) \right) = 0,$$

where $I_{\Sigma}$ is the ideal sheaf of the subset $\Sigma \subset \mathbb{P}^n$.

In this paper we prove the following result (see Section 2).

**Theorem 1.** Suppose that there is a natural number $\lambda \geq 2$ such that at most $\lambda k$ points of the set $\Sigma$ lie on a curve in $\mathbb{P}^n$ of degree $k$. Then

$$h^1 \left( I_{\Sigma} \otimes \mathcal{O}_{\mathbb{P}^n}(\xi) \right) = 0$$

in the case when one of the following conditions holds:

- $\xi = \lfloor 3\lambda/2 - 3 \rfloor$ and $|\Sigma| < \lambda \lfloor \lambda/2 \rfloor$;
- $\xi = \lfloor 3\mu - 3 \rfloor$, $|\Sigma| \leq \lambda \mu$ and $|3\mu| - \mu - 2 \geq \lambda \geq \mu$ for some $\mu \in \mathbb{Q}$;
- $\xi = \lfloor n\mu \rfloor$, $|\Sigma| \leq \lambda \mu$ and $(n - 1)\mu \geq \lambda$ for some $\mu \in \mathbb{Q}$.

Let us consider applications of Theorem 1.

**Definition 2.** An algebraic variety $X$ is factorial if $\text{Cl}(X) = \text{Pic}(X)$.
Let $\pi: X \to \mathbb{P}^3$ be a double cover branched over a surface $S \subset \mathbb{P}^3$ of degree $2r \geq 4$ such that the only singularities of the surface $S$ are isolated ordinary double points. Then $X$ is a hypersurface
\[ w^2 = f_{2r}(x, y, z, t) \subset \mathbb{P}(1, 1, 1, 1, r) \cong \operatorname{Proj}(\mathbb{C}[x, y, z, t, w]), \]
where $\text{wt}(x) = \cdots = \text{wt}(t) = 1$, $\text{wt}(w) = r$, and $f_{2r}(x, y, z, t)$ is a homogeneous polynomial of degree $2r$ such that $S \subset \mathbb{P}^3$ is given by
\[ f_{2r}(x, y, z, t) = 0 \subset \mathbb{P}^3 \cong \operatorname{Proj}(\mathbb{C}[x, y, z, t]). \]

The following conditions are equivalent (see \[10\] and \[8\]):
- the threefold $X$ is factorial;
- the singularities of the threefold $X$ are $\mathbb{Q}$-factorial\(^1\);
- the equality $\operatorname{rk} H_4(X, \mathbb{Z}) = 1$ holds;
- the ring
\[ \mathbb{C}[x, y, z, t, w]/\langle w^2 - f_{2r}(x, y, z, t) \rangle \]
is a unique factorization domain;
- the points of the set $\operatorname{Sing}(S)$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3r - 4$.

**Theorem 3.** Suppose that the inequality
\[ \left| \operatorname{Sing}(S) \right| < (2r - 1)r \]
holds. Then the threefold $X$ is factorial.

**Proof.** The subset $\operatorname{Sing}(S) \subset \mathbb{P}^3$ is a set-theoretic intersection of surfaces of degree $2r - 1$. Then $X$ is factorial by Theorem 1. q.e.d.

The assertion of Theorem 3 is proved in \[4\] in the case when $r = 3$.

**Example 4.** Suppose that the surface $S$ is given by an equation
\[ g^2(x, y, z, t) = g_1(x, y, z, t)g_{2r-1}(x, y, z, t) \subset \mathbb{P}^3, \]
where $g_i$ is a general homogeneous polynomial of degree $i$. Then
\[ \left| \operatorname{Sing}(S) \right| = (2r - 1)r, \]
and $S$ has at most ordinary double points. But $X$ is not factorial.

For $r = 3$, the threefold $X$ is non-rational if it is factorial (see \[4\]), but the threefold $X$ is rational if the surface $S$ is the Barth sextic (see \[1\]).

We prove the following generalization of Theorem 3 in Section 3.

\(^1\)A variety is $\mathbb{Q}$-factorial if some non-zero integral multiple of every Weil divisor on it is a Cartier divisor. This property is not local in the analytic topology, because ordinary double points of threefolds are not locally analytically $\mathbb{Q}$-factorial.
**Theorem 6.** Suppose that the inequality

$$ \left| \text{Sing}(S) \right| \leq (2r - 1)r + 1 $$

holds. Then $X$ is not factorial $\iff$ $S$ can be defined by equation 5.

The assertion of Theorem 6 is proved in [11] in the case when $r = 3$. Let $V$ be a hypersurface in $\mathbb{P}^4$ of degree $d$ such that $V$ has at most isolated ordinary double points. Then $V$ can be given by the equation

$$ f_d(x, y, z, t, u) = 0 \subset \mathbb{P}^4 \cong \text{Proj} \left( \mathbb{C}[x, y, z, t, u] \right), $$

where $f_d(x, y, z, t, u)$ is a homogeneous polynomial of degree $d$. The following conditions are equivalent (see [10] and [8]):

- the threefold $V$ is factorial;
- the threefold $V$ has $\mathbb{Q}$-factorial singularities;
- the equality $\text{rk} H_4(V, \mathbb{Z}) = 1$ holds;
- the ring

$$ \mathbb{C}[x, y, z, t, u]/\langle f_d(x, y, z, t, u) \rangle $$

is a unique factorization domain;
- the points of the set $\text{Sing}(V)$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^4$ of degree $2d - 5$.

The threefold $V$ is not rational if it is factorial and $d = 4$ (see [12]), but general determinantal quartic threefolds are known to be rational.

**Conjecture 7.** Suppose that the inequality

$$ \left| \text{Sing}(V) \right| < (d - 1)^2 $$

holds. Then the threefold $V$ is factorial.

The assertion of Conjecture 7 is proved in [3] and [5] for $d \leq 7$.

**Example 8.** Suppose that $V$ is given by the equation

$$ xg(x, y, z, t, u) + yf(x, y, z, t, u) = 0 \subset \mathbb{P}^4 \cong \text{Proj} \left( \mathbb{C}[x, y, z, t, u] \right), $$

where $g$ and $f$ are general homogeneous forms of degree $d - 1$. Then

$$ \left| \text{Sing}(V) \right| = (d - 1)^2 $$

and $V$ has at most ordinary double points. But $V$ is not factorial.

The threefold $V$ is factorial if $|\text{Sing}(V)| \leq (d - 1)^2/4$ by [2].

**Theorem 9.** Suppose that the inequality

$$ \left| \text{Sing}(V) \right| \leq \frac{2(d - 1)^2}{3} $$

holds. Then the threefold $V$ is factorial.
Proof. The set $\text{Sing}(V)$ is a set-theoretic intersection of hypersurfaces of degree $d-1$. Then $V$ is factorial for $d \geq 7$ by Theorem 1.

For $d \leq 6$, the threefold $V$ is factorial by Theorem 2 in [9]. q.e.d.

Let $Y$ be a complete intersection of hypersurfaces $F$ and $G$ in $\mathbb{P}^5$ of degree $m$ and $k$, respectively, such that $m \geq k$, and the complete intersection $Y$ has at most isolated ordinary double points.

**Example 10.** Let $F$ and $G$ be general hypersurfaces that contain a two-dimensional linear subspace in $\mathbb{P}^5$. Then

$$\left| \text{Sing}(Y) \right| = (m + k - 2)^2 - (m - 1)(k - 1)$$

and $Y$ has at most ordinary double points. But $Y$ is not factorial.

The threefold $Y$ is factorial if $G$ is smooth and singular points of $Y$ impose independent linear conditions on homogeneous forms of degree $2m + k - 6$ (see [8]).

**Theorem 11.** Suppose that $G$ is smooth, and the inequalities

$$\left| \text{Sing}(Y) \right| \leq (m + k - 2)(2m + k - 6)/5$$

and $m \geq 7$ hold. Then the threefold $Y$ is factorial.

**Proof.** The set $\text{Sing}(Y)$ is a set-theoretic intersection of hypersurfaces of degree $m + k - 2$. Then $Y$ is factorial by Theorem 1. q.e.d.

Arguing as in the proof of Theorem 11, we obtain the following result.

**Theorem 12.** Suppose that $G$ is smooth, and the inequalities

$$\left| \text{Sing}(Y) \right| \leq (2m + k - 3)(m + k - 2)/3$$

and $m \geq k + 6$ hold. Then the threefold $Y$ is factorial.

Let $H$ be a smooth hypersurface in $\mathbb{P}^4$ of degree $d \geq 2$, and let

$$\eta: U \longrightarrow H$$

be a double cover branched over a surface $R \subset H$ such that

$$R \sim \mathcal{O}_{\mathbb{P}^4}(2r)|_H$$

and $2r \geq d$. Suppose that $S$ has at most isolated ordinary double points.

**Theorem 13.** Suppose that the inequalities

$$\left| \text{Sing}(R) \right| \leq (2r + d - 2)r/2$$

and $r \geq d + 7$ hold. Then the threefold $U$ is factorial.

**Proof.** The subset $\text{Sing}(R) \subset \mathbb{P}^4$ is a set-theoretic intersection of hypersurfaces of degree $2r + d - 2$. Then $U$ is factorial by Theorem 1, because it is factorial if the points of $\text{Sing}(R)$ impose independent linear conditions on homogeneous forms of degree $3r + d - 5$ (see [8]). q.e.d.

2. Main result

Let $\Sigma$ be a finite subset in $\mathbb{P}^n$, where $n \geq 2$. Now we prove the following special case of Theorem 1, leaving the other cases to the reader.

**Proposition 14.** Let $r \geq 2$ be a natural number. Suppose that
\[ |\Sigma| < (2r - 1)r, \]
and at most $(2r - 1)k$ points in $\Sigma$ lie on a curve of degree $k$. Then
\[ h^1(I_\Sigma \otimes \mathcal{O}_{\mathbb{P}^n}(3r - 4)) = 0. \]

The following result is Corollary 4.3 in [7].

**Theorem 15.** Let $\pi: Y \to \mathbb{P}^2$ be a blow up of points $P_1, \ldots, P_\delta \in \mathbb{P}^2$, and let $E_i$ be the $\pi$-exceptional divisor such that $\pi(E_i) = P_i$. Then
\[ \left| \pi^*(\mathcal{O}_{\mathbb{P}^2}(\xi)) - \sum_{i=1}^\delta E_i \right| \]
does not have base points if at most $k(\xi + 3 - k) - 2$ points in $\{P_1, \ldots, P_\delta\}$ lie on a curve of degree $k$ for every $k \leq (\xi + 3)/2$, and the inequality
\[ \delta \leq \max \left\{ \left\lfloor \frac{\xi + 3}{2} \right\rfloor \left( \xi + 3 - \left\lfloor \frac{\xi + 3}{2} \right\rfloor \right) - 1, \left\lfloor \frac{\xi + 3}{2} \right\rfloor^2 \right\} \]
holds, where $\xi$ is a natural number such that $\xi \geq 3$.

Therefore, it follows from Theorem 15 that to prove Proposition 14, we may assume that $n = 3$ due to the following result.

**Lemma 16.** Let $\Pi \subset \mathbb{P}^n$ be an $m$-dimensional linear subspace, and let
\[ \psi: \mathbb{P}^n \dashrightarrow \Pi \cong \mathbb{P}^m \]
be a projection from a linear subspace $\Omega \subset \mathbb{P}^n$ such that
- the subspace $\Omega$ is sufficiently general and $\dim(\Omega) = n - m - 1$,
- there is a subset $\Lambda \subset \Sigma$ such that
\[ |\Lambda| \geq \lambda k + 1, \]
but the set $\psi(\Lambda)$ is contained in an irreducible curve of degree $k$, and $n > m \geq 2$. Let $\mathcal{M}$ be the linear system that contains all hypersurfaces in $\mathbb{P}^n$ of degree $k$ that pass through all points in $\Lambda$. Then
\[ \dim(\text{Bs}(\mathcal{M})) = 0, \]
and either $m = 2$, or $k > \lambda$. 


Proof. Suppose that there is an irreducible curve $Z$ such that

$$Z \subset \text{Bs}(\mathcal{M}),$$

and put $\Xi = Z \cap \Lambda$. We may assume that $\psi|_Z$ is a birational morphism, and

$$\psi(Z) \cap \psi(\Lambda \setminus \Xi) = \emptyset,$$

because $\Omega$ is general. Then $\deg(\psi(Z)) = \deg(Z)$.

Let $C$ be an irreducible curve in $\Pi$ of degree $k$ that contains $\psi(\Lambda)$, and let $W$ be the cone in $\mathbb{P}^n$ over the curve $C$ and with vertex $\Omega$. Then

$$W \in \mathcal{M},$$

which implies that $W$ contains the curve $Z$. Thus, we have

$$\psi(Z) = C,$$

which implies that $\Xi = \Lambda$ and $\deg(Z) = k$. But $|Z \cap \Sigma| \leq \lambda k$. We have

$$\dim \left( \text{Bs}(\mathcal{M}) \right) = 0.$$

Suppose that $m > 2$ and $k \leq \lambda$. Let us show that the latter assumption leads to a contradiction. We may assume that $m = 3$ and $n = 4$, because $\psi$ as a composition of $n - m$ projections from points.

Let $\mathcal{Y}$ be the set of all irreducible reduced surfaces in $\mathbb{P}^4$ of degree $k$ that contains all points of the set $\Lambda$, and let $\Upsilon$ be a subset of $\mathbb{P}^4$ consisting of points that are contained in every surface of $\mathcal{Y}$. Then

$$\Lambda \subseteq \Upsilon,$$

but the previous arguments imply that $\Upsilon$ is a finite set.

Let $\mathcal{S}$ be the set of all surfaces in $\mathbb{P}^3$ of degree $k$ such that

$$S \in \mathcal{S} \iff \exists Y \in \mathcal{Y} \mid \psi(Y) = S \text{ and } \psi|_Y \text{ is a birational morphism},$$

and let $\Psi$ be a subset of $\mathbb{P}^3$ consisting of points that are contained in every surface of the set $\mathcal{S}$. Then $\mathcal{S} \neq \emptyset$ and

$$\psi(\Lambda) \subseteq \psi(\Upsilon) \subseteq \Psi.$$

The generality of $\Omega$ implies that $\psi(\Upsilon) = \Psi$. Indeed, for every point

$$O \in \Pi \setminus \Psi$$

and for a general surface $Y \in \mathcal{Y}$, we may assume that the line passing through $O$ and $\Omega$ does not intersect $Y$, but $\psi|_Y$ is a birational morphism.

The set $\Psi$ is a set-theoretic intersection of surfaces in $\Pi$ of degree $k$, which implies that at most $\delta k$ points in $\Psi$ lie on a curve in $\Pi$ of degree $\delta$.

We see that at most $k^2$ points in $\Psi$ lie on a curve in $\Pi$ of degree $k$, but the set $\psi(\Lambda)$ contains at least $\lambda k + 1$ points that are contained in an irreducible curve in $\Pi$ of degree $k$, which is a contradiction. q.e.d.
We have a finite subset \( \Sigma \subset \mathbb{P}^3 \) and a natural number \( r \geq 2 \) such that
\[
|\Sigma| < (2r - 1) r,
\]
and at most \((2r - 1)k\) points in \( \Sigma \) lie on a curve of degree \( k \). Then
\[
|\Sigma| < (2r - 1)(r - \epsilon)
\]
for some integer \( \epsilon \geq 0 \). Let us prove the following result.

**Proposition 17.** The equality \( h^1(I_{\Sigma} \otimes \mathcal{O}_{\mathbb{P}^3}(3r - 4 - \epsilon)) = 0 \) holds.

Fix a point \( P \in \Sigma \). To prove Proposition 17, it is enough to construct a surface\(^2\) of degree \( 3r - 4 - \epsilon \) that contains \( \Sigma \setminus P \) and does not contain \( P \).

We assume that \( r \geq 3 \) and \( \epsilon \leq r - 3 \), because the assertion of Proposition 17 follows from Theorem 2 in [9] and Theorem 15 otherwise.

**Lemma 18.** Suppose that there is a hyperplane \( \Pi \subset \mathbb{P}^3 \) that contains the set \( \Sigma \). Then there is a surface of degree \( 3r - 4 - \epsilon \) that contains every point of the set \( \Sigma \setminus P \) and does not contain the point \( P \).

**Proof.** Suppose that \( |\Sigma \setminus P| > |(3r - 1 - \epsilon)/2|^2 \). Then
\[
(2r - 1)(r - \epsilon) - 2 \geq |\Sigma \setminus P| \geq \left[ \frac{3r - 1 - \epsilon}{2} \right]^2 + 1 \geq \frac{(3r - 2 - \epsilon)^4}{4} + 1,
\]
which implies that \((r - 4)^2 + 2r + \epsilon^2 \leq 0 \). We have \( r = 4 \) and \( \epsilon = 0 \). Then
\[
|\Sigma \setminus P| \leq \left[ \frac{3r - 1 - \epsilon}{2} \right] \left( 3r - 1 - \epsilon - \left[ \frac{3r - 1 - \epsilon}{2} \right] \right).
\]

Thus, in every possible case, the number \( |\Sigma \setminus P| \) does not exceed
\[
\max \left( \left[ \frac{3r - 1 - \epsilon}{2} \right], \left[ \frac{3r - 1 - \epsilon}{2} \right]^2 \right).
\]

At most \( 3r - 4 - \epsilon \) points of \( \Sigma \setminus P \) lie on a line, because \( 3r - 4 - \epsilon \geq 2r - 1 \).

Let us prove that at most \( k(3r - 1 - \epsilon - k) - 2 \) points in \( \Sigma \setminus P \) can lie on a curve of degree \( k \leq (3r - 1 - \epsilon)/2 \). It is enough to show that
\[
k(3r - 1 - \epsilon - k) - 2 \geq k(2r - 1)
\]
for all \( k \leq (3r - 1 - \epsilon)/2 \). We must prove this only for \( k > 1 \) such that
\[
k(3r - 1 - \epsilon - k) - 2 < |\Sigma \setminus P| \leq (2r - 1)(r - \epsilon) - 2,
\]
because otherwise the condition that at most \( k(3r - 1 - k) - 2 \) points in the set \( \Sigma \setminus P \) can lie on a curve of degree \( k \) is vacuous.

We may assume that \( k < r - \epsilon \). But
\[
k(3r - 1 - \epsilon - k) - 2 \geq k(2r - 1) \iff r > k - \epsilon,
\]
which immediately implies that at most \( k(3r - 1 - \epsilon - k) - 2 \) points in the set \( \Sigma \setminus P \) can lie on a curve of degree \( k \).

\(^2\)For simplicity we consider homogeneous forms on \( \mathbb{P}^n \) as hypersurfaces.
It follows from Theorem 15 that there is a curve
\[ C \subset \Pi \cong \mathbb{P}^2 \]
of degree \( 3r - 4 - \epsilon \) that contains \( \Sigma \setminus P \) and does not contain \( P \in \Sigma \).

A general cone in \( \mathbb{P}^3 \) over the curve \( C \) is the required surface. q.e.d.

Fix a general hyperplane \( \Pi \subset \mathbb{P}^3 \). Let \( \psi : \mathbb{P}^3 \dasharrow \Pi \) be a projection from a sufficiently general point \( O \in \mathbb{P}^3 \). Put \( \Sigma' = \psi(\Sigma) \) and \( P' = \psi(P) \).

**Lemma 19.** Suppose that at most \((2r-1)k\) points in \( \Sigma' \) lie on a curve of degree \( k \). Then there is a surface in \( \mathbb{P}^3 \) of degree \( 3r-4-\epsilon \) that contains all points of the set \( \Sigma \setminus P \) but does not contain the point \( P \in \Sigma \).

**Proof.** Arguing as in the proof of Lemma 18, we obtain a curve
\[ C \subset \Pi \cong \mathbb{P}^2 \]
of degree \( 3r - 4 - \epsilon \) that contains \( \Sigma' \setminus P' \) and does not pass through \( P' \).

Let \( Y \) be the cone in \( \mathbb{P}^3 \) over \( C \) whose vertex is \( O \). Then \( Y \) is a surface of degree \( 3r - 4 - \epsilon \) that contains all points of the set \( \Sigma \setminus P \) but does not contain the point \( P \in \Sigma \). q.e.d.

To conclude the proof of Proposition 14, we may assume that there is a natural number \( k \) such that at least \((2r-1)k+1\) points of \( \Sigma' \) lie on a curve of degree \( k \), where \( k \) is the smallest number of such property.

**Lemma 20.** The inequality \( k \geq 3 \) holds.

**Proof.** The inequality \( k \geq 2 \) holds by Lemma 16, which implies \( r \geq 3 \).

Suppose that there is a subset \( \Phi \subseteq \Sigma \) such that
\[ |\Phi| > 2(2r - 1), \]
but \( \psi(\Phi) \) is contained in a conic \( C \subset \Pi \). Then \( C \) is irreducible.

Let \( \mathcal{D} \) be a linear system of quadrics in \( \mathbb{P}^3 \) containing \( \Phi \). Then
\[ \dim(Bs(\mathcal{D})) = 0 \]
by Lemma 16. Let \( W \) be a cone in \( \mathbb{P}^3 \) over \( C \) with the vertex \( \Omega \). Then
\[ 8 = D_1 \cdot D_2 \cdot W \geq \sum_{\omega \in \Phi} \text{mult}_\omega(D_1)\text{mult}_\omega(D_2) \geq |\Phi| > 2(2r - 1) \geq 8, \]
where \( D_1 \) and \( D_2 \) are general divisors in \( \mathcal{D} \). q.e.d.

Therefore, there is a subset \( A_k^1 \subseteq \Sigma \) such that
\[ |A_k^1| > (2r - 1)k, \]
but the subset \( \psi(A_k^1) \subset \Pi \cong \mathbb{P}^2 \) is contained in an irreducible curve of degree \( k \geq 3 \). Similarly, we obtain a disjoint union
\[ \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} A_j^i, \]
where $\Lambda_j^i$ is a subset in $\Sigma$ such that
\[ |\Lambda_j^i| > (2r - 1)j, \]
the subset $\psi(\Lambda_j^i)$ is contained in an irreducible reduced curve of degree $j$, and at most $(2r - 1)\zeta$ points of the subset
\[ \psi\left(\Sigma \setminus \left( \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i \right) \right) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2 \]
lie on a curve in $\Pi$ of degree $\zeta$. Put $\Lambda = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i$.

Let $\Xi_j^i$ be the base locus of the linear system of surfaces of degree $j$ that pass through the set $\Lambda_j^i$. Then $\Xi_j^i$ is a finite set by Lemma 16, and
\[ (21) \quad |\Sigma \setminus \Lambda| < (2r - 1)(r - \epsilon) - 1 - \sum_{i=k}^{l} c_i(2r - 1)i. \]

**Corollary 22.** The inequality $\sum_{i=k}^{l} ic_i \leq r - \epsilon - 1$ holds.

We have $\Lambda_j^i \subseteq \Xi_j^i$. But the set $\Xi_j^i$ imposes independent linear conditions on homogeneous forms of degree $3(j - 1)$ by the following result.

**Lemma 23.** Let $\mathcal{M}$ be a linear subsystem in $|O_{\mathbb{P}^n}(\lambda)|$ such that
\[ \dim\left(\text{Bs}(\mathcal{M})\right) = 0, \]
where $\lambda \geq 2$. Then the points in $\text{Bs}(\mathcal{M})$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^n$ of degree $n(\lambda - 1)$.

**Proof.** See Lemma 22 in [2] or Theorem 3 in [6]. q.e.d.

Put $\Xi = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Xi_j^i$. Then $\Lambda \subseteq \Xi$.

**Lemma 24.** Suppose that $\Sigma$ is contained in $\Xi$. Then there is a surface of degree $3r - 4 - \epsilon$ that contains $\Sigma \setminus P$ and does not contain $P \in \Sigma$.

**Proof.** For every $\Xi_j^i$ containing $P$ there is a surface of degree $3(j - 1)$ that contains the set $\Xi_j^i \setminus \mathbb{P}^3$ and does not contain $P$ by Lemma 23.

For every $\Xi_j^i$ not containing $P$ there is a surface of degree $j$ that contains $\Xi_j^i$ and does not contain $P$ by the definition of the set $\Xi_j^i$.

We have $j < 3(j - 1)$, because $k \geq 2$. For every $\Xi_j^i$ there is a surface $F_j^i \subset \mathbb{P}^3$ of degree $3(j - 1)$ that contains the set $\Xi_j^i \setminus (\Xi_j^i \cap P)$ and does not contain the point $P$. The union $\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} F_j^i$ is a surface of degree
\[ \sum_{i=k}^{l} 3(i - 1)c_i \leq \sum_{i=k}^{l} 3ic_i - 3ck \leq 3r - 6 - 3\epsilon \leq 3r - 4 - \epsilon \]
that contains the set $\Sigma \setminus \mathbb{P}^3$ and does not contain the point $P$. q.e.d.
The proof of Lemma 24 implies that there is a surface of degree
\[
\sum_{i=k}^{l} 3(i - 1)c_i
\]
containing \((\Xi \cap \Sigma) \setminus (\Xi \cap P)\) and not containing \(P\), and a surface of degree
\[
\sum_{i=k}^{l} ic_i
\]
containing \(\Xi \cap \Sigma\) and not containing any point of the set \(\Sigma \setminus (\Xi \cap \Sigma)\).

**Lemma 25.** Let \(\Lambda\) and \(\Delta\) be disjoint finite subsets in \(\mathbb{P}^n\) such that
- there is a hypersurface in \(\mathbb{P}^n\) of degree \(\zeta\) that contains all points in the set \(\Lambda\) and does not contain any point in the set \(\Delta\),
- the points of the sets \(\Lambda\) and \(\Delta\) impose independent linear conditions on hypersurfaces in \(\mathbb{P}^n\) of degree \(\xi\) and \(\xi - \zeta\), respectively,
where \(\xi \geq \zeta\) are natural numbers. Then the points of the set \(\Lambda \cup \Delta\) impose independent linear conditions on hypersurfaces in \(\mathbb{P}^n\) of degree \(\xi\).

**Proof.** Let \(Q\) be a point in \(\Lambda \cup \Delta\). To conclude the proof we must find a hypersurface of degree \(\xi\) that passes through the set \((\Lambda \cup \Delta) \setminus Q\) and does not contain the point \(Q\). We may assume that \(Q \in \Lambda\).

Let \(F\) be the homogeneous form of degree \(\xi\) that vanishes at every point of the set \(\Lambda \setminus Q\) and does not vanish at the point \(Q\). Put
\[
\Delta = \{Q_1, \ldots, Q_6\},
\]
where \(Q_i\) is a point. There is a homogeneous form \(G_i\) of degree \(\xi\) that vanishes at every point in \((\Lambda \cup \Delta) \setminus Q_i\) and does not vanish at \(Q_i\). Then
\[
F(Q_i) + \mu_i G_i(Q_i) = 0
\]
for some \(\mu_i \in \mathbb{C}\), because \(g_i(Q_i) \neq 0\). Then the homogeneous form
\[
F + \sum_{i=1}^{6} \mu_i G_i
\]
vanishes on set \((\Lambda \cup \Delta) \setminus Q\) and does not vanish at the point \(Q\). q.e.d.

Put \(d = 3r - 4 - \epsilon - \sum_{i=k}^{l} ic_i\) and
\[
\bar{\Sigma} = \psi\left(\Sigma \setminus (\Xi \cap \Sigma)\right).
\]
To prove Proposition 17, we may assume that \(\emptyset \neq \bar{\Sigma} \subseteq \Sigma'\).

It follows from Lemma 25 that to prove Proposition 17 it is enough to show that \(\bar{\Sigma} \subset \Pi\) and \(d\) satisfy the hypotheses of Theorem 15.

**Lemma 26.** The inequality \(|\bar{\Sigma}| \leq \lfloor (d + 3)/2 \rfloor^2\) holds.
Proof. Suppose that the inequality $|\bar{\Sigma}| \geq \lfloor (d+3)/2 \rfloor^2 + 1$ holds. Then
\[
(2r-1) \left( r - \epsilon - \sum_{i=k}^{l} c_i i \right) - 2 \geq \left( \frac{3r - 2 - \epsilon - \sum_{i=k}^{l} c_i}{4} \right) + 1
\]
by Corollary 22. Put $\Delta = \epsilon + \sum_{i=k}^{l} c_i i$. Then $\Delta \geq k \geq 3$ and
\[
4(2r-1)(r-\Delta) - 12 \geq (3r - 2 - \Delta)^2,
\]
which implies that $0 < r^2 - 8r + 16 + 2r\Delta + \Delta^2 \leq 0$. q.e.d.

The inequality $d \geq 3$ holds by Corollary 22, because $r \geq 3$.

**Lemma 27.** Suppose that at least $d + 1$ points in the set $\bar{\Sigma}$ are contained in a line. Then there is a surface in $\mathbb{P}^3$ of degree $3r - 4 - \epsilon$ that contains all points of the set $\bar{\Sigma} \setminus P$ and does not contain the point $P \in \Sigma$.

Proof. We have $|\bar{\Sigma}| \geq d + 1$. It follows from inequality 21 that
\[
3r - 3 - \epsilon - \sum_{i=k}^{l} ic_i < (2r-1)(r-\epsilon) - 1 - \sum_{i=k}^{l} c_i (2r-1)i,
\]
which gives $\sum_{i=k}^{l} ic_i \neq r - \epsilon - 1$. Now it follows from Corollary 22 that
\[
\sum_{i=k}^{l} ic_i \leq r - \epsilon - 2,
\]
but $2r-1 \geq 3r-3-\epsilon - \sum_{i=k}^{l} ic_i$. Then $\sum_{i=k}^{l} ic_i = r - \epsilon - 2$ and $d = 2r-2$.

We have a surface of degree $\sum_{i=k}^{l} 3(i-1)c_i \leq 3r - 4 - \epsilon$ that contains
\[
\left( \Xi \cap \Sigma \right) \setminus \left( \Xi \cap P \right)
\]
and does not contain $P$. But we have a surface of degree $r - \epsilon - 2$ that contains $\Xi \cap \Sigma$ and does not contain any point of the set $\Sigma \setminus (\Xi \cap \Sigma)$.

The set $\Sigma \setminus (\Xi \cap \Sigma)$ contains at most $4r - 4$ points, at most $2r-1$ points of the set $\Sigma$ lie on a line. It follows from Theorem 2 in [9] that the set $\Sigma \setminus (\Xi \cap \Sigma)$ imposes independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $2r - 2$. Applying Lemma 25, we complete the proof. q.e.d.

So, we may assume that at most $d$ points in $\bar{\Sigma}$ lie on a line.

**Lemma 28.** For every $t \leq (d + 3)/2$, at most
\[
t(d + 3 - t) - 2
\]
points in $\bar{\Sigma}$ lie on a curve of degree $t$ in $\Pi \cong \mathbb{P}^2$. 


Proof. At most \((2r - 1)t\) of the points in \(\Sigma\) lie on a curve of degree \(t\), which implies that to conclude the proof it is enough to show that
\[ t(d + 3 - t) - 2 \geq (2r - 1)t \]
for every \(t \leq (d + 3)/2\) such that \(t > 1\) and \(t(d + 3 - t) - 2 < |\Sigma|\). But
\[ t(d + 3 - t) - 2 \geq t(2r - 1) \iff r - \sum_{i=k}^{l} ic_i > t, \]
because \(t > 1\). Thus, we may assume that \(t(d + 3 - t) - 2 < |\Sigma|\) and
\[ r - \sum_{i=k}^{l} ic_i < t \leq \frac{d + 3}{2}. \]

Let \(g(x) = x(d + 3 - x) - 2\). Then
\[ g(t) \geq g\left(r - \sum_{i=k}^{l} ic_i\right), \]
because \(g(x)\) is increasing for \(x < (d + 3)/2\). Therefore, we have
\[ (2r - 1)\left(r - \sum_{i=k}^{l} ic_i\right) - 2 \geq |\Sigma| > g(t) \geq \left(r - \sum_{i=k}^{l} ic_i\right)(2r - 1) - 2, \]
because inequality 21 holds.

We can apply Theorem 15 to the blow up of the plane \(\Pi\) at the points of the set \(\Sigma\) and to the integer \(d\). Then applying Lemma 25, we obtain a surface in \(\mathbb{P}^3\) of degree \(3r - 4 - \epsilon\) containing \(\Sigma \setminus P\) and not containing \(P\).

The assertion of Proposition 17 is completely proved, which implies the assertion of Proposition 14. The proof of Theorem 1 is similar.

3. Auxiliary result

Now we prove Theorem 6. Let \(\pi: X \to \mathbb{P}^3\) be a double cover branched over a surface \(S\) of degree \(2r \geq 4\) with isolated ordinary double points.

**Lemma 29.** Let \(F\) be a hypersurface in \(\mathbb{P}^n\) of degree \(d\) that has isolated singularities, and let \(C\) be a curve in \(\mathbb{P}^n\) of degree \(k\). Then
- the inequality \(|\text{Supp}(C) \cap \text{Sing}(F)| \leq k(d - 1)|\) holds,
- the equality \(|\text{Supp}(C) \cap \text{Sing}(F)| = k(d - 1)|\) implies that
  \[ \text{Sing}(C) \cap \text{Sing}(F) = \emptyset. \]

**Proof.** Let \(f(x_0, \ldots, x_n)\) be the homogeneous form of degree \(d\) such that \(f(x_0, \ldots, x_n) = 0\) defines \(F \subset \mathbb{P}^n\), where \((x_0 : \ldots : x_n)\) are homogeneous coordinates on \(\mathbb{P}^n\). Put
\[ \mathcal{D} = \sum_{i=0}^{n} \lambda_i \frac{\partial f}{\partial x_i} = 0 \quad \subset |\mathcal{O}_{\mathbb{P}^n}(d - 1)|, \]
where \( \lambda_0, \ldots, \lambda_n \) are complex numbers. Then
\[
Bs(D) = \text{Sing}(F),
\]
which implies that the curve \( C \) intersects a generic member of the linear system \( D \) at most \((d-1)k\) times, which implies the assertion. q.e.d.

**Lemma 30.** Let \( \Pi \subset \mathbb{P}^3 \) be a hyperplane, and let \( C \subset \Pi \) be a reduced curve of degree \( r \). Suppose that the equality
\[
\text{Supp}(C) \cap \text{Sing}(S) = (2r-1)r
\]
holds. Then \( S \) can be defined by equation 5.

**Proof.** Put
\[
S \big|_{\Pi} = \sum_{i=1}^{\alpha} m_i C_i,
\]
where \( C_i \) is an irreducible reduced curve, and \( m_i \) is a natural number.

We assume that \( C_i \neq C_j \) for \( i \neq j \), and \( C = \sum_{i=1}^{\beta} C_i \), where \( \beta \leq \alpha \).

It follows from Lemma 29 and from the equalities
\[
(31) \quad \sum_{i=1}^{\beta} \deg(C_i) = r = \frac{\sum_{i=1}^{\alpha} m_i \deg(C_i)}{2}
\]
that \( C_i \cap \text{Sing}(S) = (2r-1)\deg(C_i) \) if \( i \leq \beta \), and
\[
\text{Sing}(C) \cap \text{Sing}(S) = \emptyset.
\]

Suppose that \( m_\gamma = 1 \) for some \( \gamma \leq \beta \). Then
\[
C_\gamma \cap \text{Sing}(S) = (2r-1)\deg(C_\gamma),
\]
but the curve \( S|_{\Pi} = \sum_{i=1}^{\alpha} m_i C_i \) must be singular at every singular point of the surface \( S \) that is contained in \( C_\gamma \). Thus, we have
\[
\text{Sing}(S) \cap \text{Supp}(C_\gamma) \subseteq \bigcup_{i \neq \gamma} C_i \cap C_\gamma,
\]
but \(|C_i \cap C_\gamma| \leq (C_i \cdot C_\gamma)_{\Pi} = \deg(C_i)\deg(C_\gamma)\) for \( i \neq \gamma \). Hence, we have
\[
\sum_{i \neq \gamma} \deg(C_i)\deg(C_\gamma) \geq (2r-1)\deg(C_\gamma),
\]
but on the plane \( \Pi \) we have the equalities
\[
(2r - \deg(C_\gamma))\deg(C_\gamma) = \left(S \big|_{\Pi} - C_\gamma\right) \cdot C_\gamma = \sum_{i \neq \gamma} m_i \deg(C_i)\deg(C_\gamma),
\]
which implies that \( \deg(C_\gamma) = 1 \) and \( m_i = 1 \) for every \( i \).

Now, equalities 31 imply that \( \beta < \alpha \), but every singular point of the surface \( S \) that is contained in the curve \( C \) must lie in the set
\[
C \cap \bigcup_{i=\beta+1}^{\alpha} C_i
\]
that consists of at most \( r^2 \) points, which is a contradiction.

Thus, we see that \( m_i \geq 2 \) for every \( i \leq \beta \). Therefore, it follows from the equalities \( 31 \) that \( \alpha = \beta \) and \( m_i = 2 \) for every \( i \).

Let \( f(x, y, z, w) \) be the homogeneous form of degree \( 2r \) such that
\[
f(x, y, z, w) = 0
\]
defines the surface \( S \subset \mathbb{P}^3 \), where \((x : y : z : w)\) are homogeneous coordinates on \( \mathbb{P}^3 \). We may assume that \( \Pi \) is given by \( x = 0 \). Then
\[
f(0, y, z, w) = g_r^2(y, z, w),
\]
where \( g_r(y, z, w) \) is a form of degree \( r \) such that \( C \) is given by
\[
x = g_r(y, z, w) = 0,
\]
which implies that \( S \) can be defined by equation \( 5 \).

It follows from Lemma 29 that at most \((2r - 1)k\) singular points of the surface \( S \) can lie on a curve in \( \mathbb{P}^3 \) of degree \( k \).

**Lemma 32.** Let \( C \) be an irreducible reduced curve in \( \mathbb{P}^3 \) of degree \( k \) that is not contained in a hyperplane. Then
\[
\left| C \cap \operatorname{Sing}(S) \right| \leq (2r - 1)k - 2.
\]

**Proof.** Suppose that the curve \( C \) contains at least \((2r - 1)k - 1\) singular points of the surface \( S \). Then \( C \subset S \), because otherwise we have
\[
2rk = \deg(C) \deg(S) \leq 2(2r - 1)k - 2 = 4rk - 2k - 2,
\]
which leads to \( 2k(r - 1) \leq 2 \). But \( r \geq 2 \) and \( k \geq 3 \).

Let \( O \) be a sufficiently general point of the curve \( C \), and let \( \psi : \mathbb{P}^3 \dashrightarrow \Pi \)
be a projection from \( O \), where \( \Pi \) is a general plane in \( \mathbb{P}^3 \). Then
\[
\psi|_C : C \dashrightarrow \psi(C)
\]
is a birational morphism, because \( C \) is not a plane curve.

Put \( Z = \psi(C) \). Then \( Z \) has degree \( k - 1 \).

Let \( Y \) be a cone in \( \mathbb{P}^3 \) over \( Z \) with the vertex \( O \). Then \( C \subset Y \).

The point \( O \) is not contained in a hyperplane in \( \mathbb{P}^3 \) that is tangent to the surface \( S \) at some point of the curve \( C \), because \( C \) is not contained in a hyperplane. Then \( Y \) does not tangent \( S \) along the curve \( C \). Put
\[
S|_Y = C + R,
\]
where \( R \) is a curve of degree \( 2rk - k - 2r \). The generality in the choice of the point \( O \) implies that \( R \) does not contain rulings of the cone \( Y \).
Let $\alpha : \bar{Z} \to Z$ be the normalization of $Z$. Then the diagram

$$
\begin{array}{ccc}
\bar{Y} & \xrightarrow{\beta} & Y \\
\downarrow \pi & & \downarrow \psi_Y \\
\bar{Z} & \xrightarrow{\alpha} & Z
\end{array}
$$

commutes, where $\beta$ is a birational morphism, the surface $\bar{Y}$ is smooth, and $\pi$ is a $\mathbb{P}^1$-bundle. Let $L$ be a general fiber of $\pi$, and $E$ be a section of the $\mathbb{P}^1$-bundle $\pi$ such that $\beta(E) = O$. Then $E^2 = -k + 1$ on $\bar{Y}$.

Let $Q$ be an arbitrary point of the set $\text{Sing}(S) \cap C$, and let $\bar{C}$ and $\bar{R}$ be proper transforms of the curves $C$ and $R$ on the surface $\bar{Y}$, respectively. Then there is a point $\bar{Q} \in \bar{Y}$ such that $\bar{Q} \in \text{Supp}(\bar{C} \cdot \bar{R})$ and $\beta(\bar{Q}) = Q$. But we have

$$
\bar{R} \equiv (2r - 2)E + (2rk - k - 2r)L
$$

and $\bar{C} \equiv E + kL$. Therefore, we have

$$
(2r - 1)k - 2 = \bar{C} \cdot \bar{R} \geq (2r - 1)k - 1,
$$

which is a contradiction. q.e.d.

Now we prove Theorem 6 by reductio ad absurdum, where we assume that $r \geq 4$, because the case $r = 3$ is done in [11].

Put $\Sigma = \text{Sing}(S)$, and suppose that the following conditions hold:

- the inequalities $|\Sigma| \leq (2r - 1)r + 1$ and $r \geq 3$ hold;
- the surface $S$ can not be defined by equation 5;
- the threefold $X$ is not factorial.

There is a point $P \in \Sigma$ such that every surface in $\mathbb{P}^3$ of degree $3r - 4$ that pass through the set $\Sigma \setminus P$ contains the point $P$ as well.

**Lemma 33.** Let $\Pi$ be a hyperplane in $\mathbb{P}^3$. Then $|\Pi \cap \Sigma| \leq 2r$.

**Proof.** Suppose that the inequality $|\Pi \cap \Sigma| > 2r$ holds. Let us show that this assumption leads to a contradiction.

Let $\Gamma$ be the subset of the set $\Sigma$ that consists of all points that are not contained in the plane $\Pi$. Then $\Gamma$ contains at most

$$
(2r - 1)(r - 1) - 1
$$

points, which impose independent linear conditions on homogeneous forms of degree $3r - 5$ by Proposition 17.
Suppose that \( P \notin \Pi \). There is a surface \( F \subset \mathbb{P}^3 \) of degree \( 3r - 5 \) that contains the set \( \Gamma \setminus P \) and does not contain the point \( P \). Then
\[
F \cup \Pi \subset \mathbb{P}^3
\]
is the surface of degree \( 3r - 4 \) that contains the set \( \Sigma \setminus P \) and does not contain the point \( P \), which is impossible. Therefore, we have \( P \in \Pi \).

Arguing as in the proof of Lemma 29, we see that
\[
\left| \Pi \cap \Sigma \right| \leq (2r - 1)r,
\]
because \( S|_{\Pi} \) is singular in every point of the set \( \Pi \cap \Sigma \).

It follows from Lemma 30 that \( \Pi \cap \Sigma \) is not contained in a curve of degree \( r \) if \( |\Pi \cap \Sigma| = (2r - 1)r \). Arguing as in the proof of Lemma 18, we see that there is a surface of degree \( 3r - 4 \) that contains the set
\[
\left( \Pi \cap \Sigma \right) \setminus P
\]
and does not contain \( P \), which concludes the proof by Lemma 25. q.e.d.

The inequality \( |\Sigma| \geq (2r - 1)r \) holds by Proposition 14.

**Lemma 34.** Let \( L_1 \neq L_2 \) be lines in \( \mathbb{P}^3 \). Then
\[
\left| (L_1 \cup L_2) \cap \Sigma \right| < 4r - 2.
\]

**Proof.** Suppose that \( |(L_1 \cup L_2) \cap \Sigma| \geq 4r - 2 \). Then
\[
|L_1 \cap \Sigma| = |L_1 \cap \Sigma| = 2r - 1
\]
by Lemma 29. Then \( L_1 \cap L_2 = \emptyset \) by Lemma 33.

Fix two points \( Q_1 \) and \( Q_2 \) in the set
\[
\Sigma \setminus \left( (L_1 \cup L_2) \cap \Sigma \right)
\]
different from \( P \) such that \( Q_1 \neq Q_2 \). Let \( \Pi_i \) be a hyperplane in \( \mathbb{P}^3 \) that contains \( L_i \) and \( Q_i \). Then \( |\Pi_i \cap \Sigma| = 2r \) by Lemma 33.

Suppose that \( P \notin \Pi_1 \cup \Pi_2 \). There is a surface \( F \subset \mathbb{P}^3 \) of degree \( 3r - 6 \) that does not contain the point \( P \) and contains all points of the set
\[
\left( \Sigma \setminus \left( \Sigma \cap (\Pi_1 \cup \Pi_2) \right) \right) \setminus P
\]
by Proposition 17. Hence, the union
\[
F \cup \Pi_1 \cup \Pi_2
\]
is a surface in \( \mathbb{P}^3 \) of degree \( 3r - 4 \) that contains \( \Sigma \setminus P \) and does not contain \( P \), which is impossible. Therefore, we have \( P \in \Pi_1 \cup \Pi_2 \).

The set \( \Sigma \cap (\Pi_1 \cup \Pi_2) \) consists of \( 4r \) points by Lemma 33. The points in
\[
\Sigma \cap \left( \Pi_1 \cup \Pi_2 \right)
\]
impose independent linear conditions on homogeneous forms $\mathbb{P}^3$ of degree $3r - 4$ by Theorem 2 in [9]. On the other hand, the inequality
\[
\left| \Sigma \setminus \left( \Sigma \cap \left( \Pi_1 \cup \Pi_2 \right) \right) \right| < (2r - 1)(r - 2)
\]
holds. Then the points in $\Sigma \setminus \left( \Sigma \cap \left( \Pi_1 \cup \Pi_2 \right) \right)$ impose independent linear conditions homogeneous forms of degree $3r - 6$ by Proposition 17, which leads to a contradiction by applying Lemma 25.

Lemma 35. Let $C$ be a curve in $\mathbb{P}^3$ of degree $k \geq 2$. Then
\[
\left| C \cap \Sigma \right| < (2r - 1)k.
\]

Proof. Suppose that $\left| C \cap \Sigma \right| \geq (2r - 1)k$. Then
\[
\left| C \cap \Sigma \right| = (2r - 1)k
\]
by Lemma 29, and $C$ is not contained in a hyperplane by Lemma 33. The curve $C$ must be reducible by Lemma 32. Put
\[
C = \sum_{i=1}^{\alpha} C_i,
\]
where $\alpha \geq 2$ and $C_i$ is an irreducible curve. Then
\[
k = \sum_{i=1}^{\alpha} d_i,
\]
where $d_i = \deg(C_i)$. Then $\left| C_i \cap \Sigma \right| = (2r - 1)d_i$ by Lemma 29.

The curve $C_i$ is contained in a hyperplane in $\mathbb{P}^3$ by Lemma 32. Then
\[
d_1 = d_2 = \cdots = d_\alpha = 1
\]
and $\alpha = k \neq 1$ by Lemma 33, which contradicts Lemma 34. \quad q.e.d.

Lemma 36. Let $L$ be a line in $\mathbb{P}^3$. Then $\left| L \cap \Sigma \right| \leq 2r - 2$.

Proof. Suppose that the inequality $\left| L \cap \Sigma \right| \geq 2r - 1$ holds. Then
\[
\left| L \cap \Sigma \right| = 2r - 1
\]
by Lemma 29. Let $\Phi$ be a hyperplane in $\mathbb{P}^3$ such that $\Phi$ passes through the line $L$, and $\Phi$ contains a point of the set $\Sigma \setminus (L \cap \Sigma)$. Then
\[
\left| \Phi \cap \Sigma \right| = 2r
\]
by Lemma 33. Put $\Delta = \Sigma \setminus (\Phi \cap \Sigma)$. Then $\left| \Delta \right| \leq (2r - 1)(r - 1)$.

The points in $\Delta$ impose dependent linear conditions on homogeneous forms of degree $3r - 5$, because otherwise the points in $\Sigma$ impose independent linear conditions on forms of degree $3r - 4$ by Lemma 25.

Therefore, we see that there is a point $Q \in \Delta$ such that every surface of degree $3r - 5$ containing $\Delta \setminus Q$ must pass through $Q$. Then
\[
\left| \Delta \right| = (2r - 1)(r - 1)
\]
and $\left| \Sigma \right| = (2r - 1)r + 1$ by Proposition 17.
Fix sufficiently general hyperplane $\Pi \subset \mathbb{P}^3$ and a point $O \in \mathbb{P}^3$. Let $\psi: \mathbb{P}^3 \dasharrow \Pi$ be a projection from $O$. Put $\Delta' = \psi(\Delta)$ and $Q' = \psi(Q)$.

At most $2r - 2$ points in $\Delta'$ lie on a line by Lemmas 16 and 34.

Suppose that at most $(2r - 1)k$ points in the set $\Delta'$ lie on any curve of degree $k$ for every $k$, and there is a curve $Z \subset \Pi$ of degree $r - 1$ that contains the whole set $\Delta'$. Then

$$h^1(I_\Delta \otimes \mathcal{O}_{\mathbb{P}^3}(3r - 5)) = 0$$

by Lemmas 16, 23 and 35 in the case when $Z$ is irreducible. So, we have

$$Z = \sum_{i=1}^{\alpha} Z_i,$$

where $\alpha \geq 2$, and $Z_i$ is an irreducible curve of degree $d_i$. Then

$$|Z_i \cap \Delta'| = (2r - 1)d_i,$$

because $r = \sum_{i=1}^{\alpha} d_i$. Then every point of the set $\Delta'$ is contained in one irreducible component of the curve $Z$. We have $d_i \neq 1$ for every $i$.

Let $Z_\beta$ be the unique component of the curve $Z$ such that $Q' \in Z_\beta$, and let $\Gamma \subset \Delta$ be a subset such that

$$\psi(\Gamma) = \Delta' \cap Z_\beta \subset \Pi \cong \mathbb{P}^2,$$

which implies that $Q \in \Gamma$. There is a surface $F_\beta \subset \mathbb{P}^3$ of degree $3(d_\beta - 1)$ that contains $\Gamma \setminus Q$ and does not contain $Q$ by Lemmas 16, 23 and 35.

Let $Y_i$ be a cone over $Z_i$ whose vertex is the point $O$. Then $F_\beta \cup \bigcup_{i \neq \beta} Y_i$ is a surface of degree $3d_\beta - 3 + \sum_{i \neq \beta} d_i = 2d_i + r - 4$ containing $\Delta \setminus Q$ and not containing $Q$, which is impossible, because $2d_i + r - 4 \leq 3r - 5$.

Hence, we proved that

- either at least $(2r - 1)k + 1$ points in $\Delta'$ lie on a curve of degree $k$;
- or there is no curve of degree $r - 1$ that contains the set $\Delta'$.

Suppose that at most $(2r - 1)k$ points of the set $\Delta'$ lie on every curve of degree $k$ for every natural $k$. Then it follows from Theorem 15 that there is a curve in $\Pi$ of degree $3r - 5$ that contains $\Delta' \setminus Q'$ and does not contain the point $Q'$, which is a contradiction.

So, at least $(2r - 1)k + 1$ points in $\Delta'$ lie on some curve in $\Pi$ of degree $k$, where $k \geq 3$ by Lemma 20. Thus, the proof of Proposition 17 implies the existence of a subset $\Xi \subseteq \Delta$ such that

- at most $(2r - 1)k$ points in $\psi(\Delta \setminus \Xi)$ lie on a curve of degree $k$;
- there is a surface in $\mathbb{P}^3$ of degree $\mu \leq r - 2$ that contains all points of the set $\Xi$ and does not contain any point of the set $\Delta \setminus \Xi$. 


• the inequality $|\Delta \setminus \Xi| \leq (2r-1)(r-1-\mu) - 1$ holds and
  
  $h^1(I_{\Xi} \otimes \mathcal{O}_{P^3}(3r-5)) = 0$.

Put $\bar{\Delta} = \psi(\Delta \setminus \Xi)$ and $d = 3r-5-\mu$. The points of $\bar{\Delta}$ impose dependent linear conditions on homogeneous forms of degree $d$ by Lemma 25, which implies that there is a point $Q \in \Delta$ such that $\Delta \setminus Q$ and $d$ do not satisfy one of the hypotheses of Theorem 15.

We have $d \geq 3$, because $r \geq 4$. The proof of Lemma 26 gives

$$|\Delta \setminus Q| \leq \left\lfloor \frac{d+3}{2} \right\rfloor^2,$$

which implies that at least $t(d+3-t) - 1$ points of the finite set $\bar{\Delta} \setminus Q$ lie on a curve of degree $t$ for some natural number $t$ such that $t \leq \frac{d+3}{2}$. Suppose that $t = 1$. Then at least $d+1$ points of $\Delta$ lie on a line, but at most $2r-2$ points of $\Delta'$ lie on a line by Lemmas 16 and 34, which implies that $d = 2r-3$ and $|\Delta| = 2r-2$. Then the points in $\bar{\Delta}$ impose dependent linear conditions on homogeneous forms of degree $d$, which is impossible. Therefore, we see that $t \geq 2$.

At least $t(d+3-t) - 1$ points in $\Delta \setminus Q$ lie on a curve of degree $t$. Then

$$t(d+3-t) - 1 \leq |\Delta \setminus Q| \leq (2r-1)(r-1) - 2 - \mu(2r-1),$$

but $t(d+3-t) - 1 \leq (2r-1)t$, because at most $(2r-1)t$ points in $\bar{\Delta}$ lie on a curve of degree $t$. Hence, we have $t \geq r-1-\mu$, which gives

$$(2r-1)(r-1-\mu) - 2 \geq |\Delta \setminus Q| \geq t(d+3-t) - 1 \geq \left( r-1-\mu \right)(2r-1)-1,$$

which is a contradiction. q.e.d.

**Corollary 37.** Let $C$ be any curve in $P^3$ of degree $k$. Then

$$|C \cap \Sigma| < (2r-1)k.$$

Fix a hyperplane $\Pi \subset P^3$ and a general point $O \in P^3$. Let

$$\psi: P^3 \to \Pi \subset P^3$$

be a projection from $O$. Put $\Sigma' = \psi(\Sigma)$ and $P' = \psi(P)$.

**Lemma 38.** Let $C$ be an irreducible curve in $\Pi$ of degree $r$. Then

$$|C \cap \Sigma'| < (2r-1)r.$$

**Proof.** Suppose that $|C \cap \Sigma'| \geq (2r-1)r$. Let $\Psi$ be a subset in $\Sigma$ that contains all points mapped to the curve $C$ by the projection $\psi$. Then

$$|\Psi| \geq (2r-1)r,$$

but less than $(2r-1)r$ points in $\Sigma$ lie on a curve of degree $r$.

Let $\mathcal{H}$ be a linear system of surfaces in $P^3$ of degree $r$ that pass through the set $\Psi$, and let $\Phi$ be the base locus of $\mathcal{H}$. Then

$$\dim(\Phi) = 0.$$
is finite by Lemma 16. Put $\Upsilon = \Sigma \cap \Phi$. The points in $\Upsilon$ impose independent linear conditions on homogeneous forms of degree $3r - 3$ by Lemma 23.

Let $\Gamma$ be a subset in $\Upsilon$ such that $\Upsilon \setminus \Gamma$ consists of $4r - 6$ points. Then

$$|\Gamma| \leq 2r^2 - 5r - 5 \leq \frac{(r + 2)(r + 1)r}{6} - 1,$$

because $r \geq 4$. Therefore, there is a surface $F \subset P^3$ of degree $r - 1$ that contains all points of the set $\Gamma$.

Let $\Theta$ be a subset of the set $\Upsilon$ such that $\Theta$ consists of all points that are contained in the surface $F$. Then $\Theta$ imposes independent linear conditions on homogeneous forms of degree $3r - 4$ by Theorem 3 in [9].

Put $\Delta = \Upsilon \setminus \Theta$. Using Theorem 2 in [9], we easily see that the points of the set $\Delta$ impose independent linear conditions on homogeneous forms of degree $2r - 3$ by Lemmas 33 and 36. Then

$$h^1(I_{\Upsilon} \otimes O_{P^3}(3r - 4)) = 0$$

by Lemma 25, which also follows from Theorem 3 in [6].

We have $|\Sigma \setminus \Upsilon| \leq 1$. Thus, the points in $\Sigma$ impose independent linear conditions on homogeneous forms of degree $3r - 4$ by Lemma 25. q.e.d.

**Lemma 39.** There is a curve $Z \subset \Pi$ of degree $k$ such that

$$|Z \cap \Sigma'| \geq (2r - 1)k + 1.$$

**Proof.** Suppose that at most $(2r - 1)k$ points of the set $\Sigma'$ lie on a curve of degree $k$ for every integer $k \geq 1$. Let us derive a contradiction.

The finite subset $\Sigma' \setminus P' \subset \Pi$ and the natural number $3r - 4$ do not satisfy at least one of the hypotheses of Theorem 15. But

$$|\Sigma' \setminus P'| \leq \max \left( \left\lfloor \frac{3r - 1}{2} \right\rfloor \left( 3r - 1 - \left\lfloor \frac{3r - 1}{2} \right\rfloor \right), \left\lfloor \frac{3r - 1}{2} \right\rfloor^2 \right),$$

and at most $2r - 1 \leq 3r - 4$ points in $\Sigma' \setminus P'$ lie on a line by Lemma 16.

We see that at least

$$k(3r - 1 - k) - 1$$

points in $\Sigma' \setminus P'$ lie on a curve of degree $k$ such that $2 \leq k \leq (3r - 1)/2$, which implies that $k = r$, because at most $k(2r - 1)$ points in $\Sigma'$ lie on a curve of degree $k$, and $|\Sigma' \setminus P'| \leq (2r - 1)r$.

Thus, there is a curve $C \subset \Pi$ of degree $r$ such that

$$\left| \text{Supp}(C) \cap (\Sigma' \setminus P') \right| \geq (2r - 1)r - 1,$$

which implies that $P' \in C$, because otherwise there is a curve in $\Pi$ of degree $3r - 4$ that contains $\Sigma' \setminus P'$ and does not contain $P'$. Then

$$\left| \text{Supp}(C) \cap \Sigma' \right| \geq (2r - 1)r,$$
which implies that \( C \) is reducible by Lemma 38. Put

\[
C = \sum_{i=1}^{\alpha} C_i,
\]

where \( C_i \) is an irreducible curve of degree \( d_i \geq 1 \) and \( \alpha \geq 2 \). Then

\[
(2r - 1)r \leq |C \cap \Sigma'| \leq \sum_{i=1}^{\alpha} |C_i \cap \Sigma'| \leq \sum_{i=1}^{\alpha} (2r - 1)\deg(C_i) = (2r - 1)r,
\]

which implies that \( C_i \) contains \( (2r - 1)d_i \) points of the set \( \Sigma \), and every point of the set \( \Sigma \) is contained in at most one curve \( C_i \).

Let \( C_v \) be the component of \( C \) that contains \( P' \), and let \( \Upsilon \) be a subset of the set \( \Sigma \) that contains all points of the set \( \Sigma \) that are mapped to the curve \( C_v \) by the projection \( \psi \). Then

\[
|\Upsilon| = (2r - 1)d_v,
\]

but less than \( (2r - 1)d_v \) points of the set \( \Sigma \) lie on a curve of degree \( d_v \).

The points in \( \Upsilon \) impose independent linear conditions on the homogeneous forms of degree \( 3(d_v - 1) \) by Lemmas 16 and 23.

There is a surface \( F \subset \mathbb{P}^3 \) of degree such that

\[
\Upsilon \setminus P \subset F \nsubseteq P
\]

and \( \deg(F) = 3(d_v - 1) \). Let \( Y_i \) be a cone in \( \mathbb{P}^3 \) over the curve \( C_i \) whose vertex is the point \( O \). Then the surface

\[
F \cup \bigcup_{i \neq v} Y_i \notin \mathcal{O}_{\mathbb{P}^3}(2d_v - 3 + r)
\]

contains the set \( \Sigma \setminus P \) and does not contain the point \( P \). But

\[
2d_v - 3 + r \leq 3r - 4,
\]

which is a contradiction.

Arguing as in the proof of Theorem 1, we construct a disjoint union

\[
\bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda^j_i \subseteq \Sigma
\]

such that \( |\Lambda^j_i| > (2r - 1)j \), the subset \( \psi(\Lambda^j_i) \) is contained in an irreducible curve of degree \( j \), and at most \( (2r - 1)t \) points of the subset

\[
\psi \left( \Sigma \setminus \left( \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda^j_i \right) \right) \subseteq \Sigma' \subset \Pi \cong \mathbb{P}^2
\]

lie on a curve in \( \Pi \) of degree \( t \). Then \( r > k \geq 3 \) by Lemmas 20 and 38.
 Put \( \Lambda = \bigcup_{j=k}^{l} \bigcup_{i=1}^{j} \Lambda^i_j \). Let \( \Xi^j_i \) be the base locus of the linear system of surfaces in \( \mathbb{P}^3 \) of degree \( j \) that pass through \( \Lambda^i_j \). Then

\[
(40) \quad |\Sigma \setminus \Lambda| \leq (2r-1)r + 1 - \sum_{i=k}^{l} c_i(2r-1)i + 1 \leq (2r-1)^2 - \sum_{i=k}^{l} ic_i,
\]

which implies that \( \sum_{i=k}^{l} ic_i \leq r \). The set \( \Xi^j_i \) is finite by Lemma 16.

**Remark 41.** We have \( \sum_{i=k}^{l} ic_i \leq r - 1 \), because the equality

\[
\sum_{i=k}^{l} ic_i = r
\]

and inequalities 40 imply that \( k = l = r \), but \( k < r \) by Lemma 38.

It follows from Lemma 23 that the points of \( \Xi^j_i \) impose independent linear conditions on homogeneous forms on \( \mathbb{P}^3 \) of degree \( 3(j - 1) \).

Put \( \Xi = \bigcup_{j=k}^{l} \bigcup_{i=1}^{j} \Xi^j_i \). Then

\[
(42) \quad |\Sigma \setminus (\Xi \cap \Sigma)| \leq (2r-1)r - \sum_{i=k}^{l} c_i(2r-1)i.
\]

Therefore, we can find surfaces \( F \) and \( G \) in \( \mathbb{P}^3 \) of degree \( \sum_{i=k}^{l} 3(i-1)c_i \) and \( \sum_{i=k}^{l} ic_i \), respectively, such that

\[
(\Xi \cap \Sigma) \setminus P \subset F \not\subset P,
\]

the surface \( G \) contains the set \( \Xi \cap \Sigma \), and the surface \( G \) does not contain any point in \( \Sigma \setminus (\Xi \cap \Sigma) \). In particular, we have \( \Sigma \not\subset \Xi \), because

\[
\sum_{i=k}^{l} 3(i-1)c_i \leq \sum_{i=k}^{l} 3ic_i - 3c_k \leq 3r - 6 < 3r - 4.
\]

Put \( \Sigma = \psi(\Sigma \setminus (\Xi \cap \Sigma)) \) and \( d = 3r - 4 - \sum_{i=k}^{l} ic_i \).

It follows from Lemma 25 that there is a point \( \bar{Q} \in \Sigma \) such that every curve in \( \Pi \) of degree \( d \) that contains the set \( \Sigma \setminus \bar{Q} \) must pass through the point \( \bar{Q} \) as well. Therefore, we can not apply Theorem 15 to the points of the subset \( \Sigma \setminus \bar{Q} \subset \Pi \) and the natural number \( d \).

The proof of Lemma 26 implies that the inequality

\[
|\Sigma \setminus \bar{Q}| \leq (2r-1)^2 - \left( r - \sum_{i=k}^{l} c_i \right) - 1 \leq \left[ \frac{d + 3}{2} \right]^2
\]

holds, but \( d = 3r - 4 - \sum_{i=k}^{l} ic_i \geq 2r - 3 \geq 3 \), because \( \sum_{i=k}^{l} ic_i \leq r - 1 \), which implies that at least \( t(d + 3 - t) - 1 \) points of the set \( \Sigma \setminus \bar{Q} \) lie on a curve in \( \Pi \) of degree \( t \leq (d + 3)/2 \).

**Lemma 43.** The inequality \( t \neq 1 \) holds.
Proof. Suppose that \( t = 1 \). Then at least \( d + 1 \) points in \( \Sigma \setminus \bar{Q} \) lie on a line, which implies that \( d + 1 \leq 2r - 2 \) by Lemmas 16 and 36.

The inequality \( d + 1 \leq 2r - 2 \) gives \( \sum_{i=k}^{l} ic_i = r - 1 \) and \( d = 2r - 3 \). It follows from inequality 42 that

\[
|\Sigma \setminus (\Xi \cap \Sigma)| \leq 2r - 1,
\]

which implies that the set \( \Sigma \setminus (\Xi \cap \Sigma) \) imposes independent linear conditions on the homogeneous forms of degree \( 2r - 3 \) by Theorem 2 in [9], which is impossible by Lemma 25. q.e.d.

There is a curve \( C \subset \Pi \) of degree \( t \geq 2 \) that contains at least

\[
t(d + 3 - t) - 1
\]

points of the set \( \Sigma \setminus \bar{Q} \), which implies that

\[
t(d + 3 - t) - 1 \leq |\Sigma \setminus \bar{Q}|
\]

and \( t(d + 3 - t) - 1 \leq (2r - 1)t \). Therefore, we see that

\[
t \geq r - \sum_{i=k}^{l} ic_i,
\]

because \( t \geq 2 \). It follows from inequalities 40 that

\[
(2r - 1)\left(r - \sum_{i=k}^{l} ic_i\right) - 1 \geq |\Sigma \setminus \bar{Q}| \geq t(d + 3 - t) - 1
\]

\[
\geq \left(r - \sum_{i=k}^{l} ic_i\right)(2r - 1) - 1,
\]

which implies that \( t = r - \sum_{i=k}^{l} ic_i \), the curve \( C \) contains \( \Sigma \setminus \bar{Q} \), and inequalities 40 are actually equalities. We have \( \Sigma \cap \Xi = \Lambda \) and

\[
|\Sigma \setminus \Lambda| = (2r - 1)r + 1 - \sum_{i=k}^{l} c_i (2r - 1)i + 1
\]

\[
= (2r - 1)\left(r - \sum_{i=k}^{l} ic_i\right),
\]

which implies that \( l = k, c_k = 1, d = 3r - 4 - k \) and \( \sum_{i=k}^{l} ic_i = k \).

Lemma 44. The curve \( C \) contains the set \( \Sigma \).

Proof. Suppose that \( \Sigma \not\subseteq C \). Then \( \bar{Q} \not\subseteq C \), which implies that there is a curve in \( \Pi \) of degree \( r - k \) that contains the set \( \Sigma \setminus \bar{Q} \) but does not contain the point \( \bar{Q} \). The latter is impossible, because \( d \geq r - k \). q.e.d.
We have \( \text{deg}(C) = r - k \) and \( \psi(\Sigma \setminus \Lambda) \subset C \). The equality
\[
|\psi(\Sigma \setminus \Lambda)| = (r - k)(2r - 1)
\]
holds. But there is an irreducible curve \( Z \subset \Pi \) of degree \( k \) that contains all points of the set \( \psi(\Lambda) \), which consists of \( k(2r - 1) + 1 \). Then
\[
|\Sigma| = |\Sigma \setminus \Lambda| + |\Lambda| = (r - k)(2r - 1) + k(2r - 1) + 1 = (2r - 1)r + 1.
\]

**Lemma 45.** The curve \( C \) is reducible.

*Proof.* Suppose that \( C \) is irreducible. Then \( \Sigma \setminus \Lambda \) imposes independent linear conditions on forms of degree \( 3(r - k - 1) \) by Lemmas 16, 23, and 35, but the points in \( \Lambda \) impose independent linear conditions on forms of degree \( 3(k - 1) \) by Lemmas 16 and 23. Then \( \Sigma \) imposes independent linear conditions on forms of degree \( 3r - 4 \) by Lemma 25. q.e.d.

Put \( C = \sum_{i=1}^{\alpha} C_i \), where \( C_i \) is an irreducible curve of degree \( d_i \). Then
\[
r - k = \sum_{i=1}^{\alpha} d_i,
\]
the curve \( C_i \) contains \( (2r - 1)d_i \) points of the set \( \bar{\Sigma} \), and every point of the set \( \bar{\Sigma} \) is contained in a single irreducible component of the curve \( C \).

**Lemma 46.** The curve \( Z \) contains the point \( P' \).

*Proof.* Suppose that \( P' \notin Z \). Let \( C_v \) be a component of \( C \) such that
\[
P' \in C_v,
\]
and let \( \Upsilon \) be a subset of the set \( \Sigma \) that contains all points that are mapped to the curve \( C_v \) by the projection \( \psi \). Then \( |\Upsilon| = (2r - 1)d_v \).

The set \( \Upsilon \) imposes independent linear conditions on the homogeneous forms of degree \( 3(d_v - 1) \) by Lemmas 16, 23 and 35. There is a surface
\[
F \subset \mathbb{P}^3
\]
of degree \( 3(d_v - 1) \) that contains \( \Upsilon \setminus P \) and does not contain \( P \).

Let \( Y_i \) and \( Y \) be the cones in \( \mathbb{P}^3 \) over the curves \( C_i \) and \( Z \), respectively, whose vertex is the point \( O \). Then the union
\[
F \cup Y \cup \bigcup_{i \neq v} Y_i
\]
is a surface of degree \( 2d_v - 3 + r \leq 3r - 4 \) that contains the set \( \Sigma \setminus P \) and does not contain the point \( P \), which is a contradiction. q.e.d.

The proof of Lemma 46 implies that the set \( \Sigma \setminus \Lambda \) imposes independent linear conditions on homogeneous forms on \( \mathbb{P}^3 \) of degree \( 3r - 4 - k \), but we already know that the set \( \Lambda \) imposes independent linear conditions on homogeneous forms of degree \( 3(k - 1) \) by Lemmas 16 and 23.

Applying Lemma 25, we obtain a contradiction.
References


School of Mathematics
University of Edinburgh
Edinburgh EH9 3JZ, UK
E-mail address: I.Cheltsov@ed.ac.uk