# ON K-STABILITY OF $\mathbb{P}^{3}$ BLOWN UP ALONG A QUINTIC ELLIPTIC CURVE 

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#### Abstract

In this note, we study K-stability of smooth Fano threefolds that can be obtained by blowing up the three-dimensional projective space along a smooth elliptic curve of degree five.


Let $C_{5}$ be be a smooth quintic elliptic curve in $\mathbb{P}^{3}$, and let $\pi: X \rightarrow \mathbb{P}^{3}$ be the blow up of this curve. Then $X$ is a smooth Fano threefold in the family №2.17, and every smooth Fano 3-fold in this family can be obtained by blowing up $\mathbb{P}^{3}$ along a suitable smooth quintic elliptic curve.

It is well known that there exists the following Sarkisov link:

where $Q$ is a smooth quadric threefold in $\mathbb{P}^{4}$, and $q$ is a blow up of a smooth quintic elliptic curve $C_{5}^{\prime}$. Let $E_{\mathbb{P}^{3}}$ and $E_{Q}$ be the exceptional divisors of $\pi$ and $q$, respectively. If $\ell$ is a general fiber of the natural projection $E_{\mathbb{P}^{3}} \rightarrow C_{5}$, then $\pi(\ell)$ is a trisecant of the quintic elliptic curve $C_{5}$. Similarly, if $\ell^{\prime}$ is a general fiber of the projection $E_{Q} \rightarrow C_{5}^{\prime}$, then $q\left(\ell^{\prime}\right)$ is a secant of the curve $C_{5}^{\prime}$ contained in $Q$.

Example 1. Let $\mathcal{E}$ be the harmonic elliptic curve, and let $\theta$ be an element in $\operatorname{Aut}(\mathcal{E})$ of order 4 that fixes a point $P \in C_{5}$. Then it follows from [9] that

$$
\operatorname{Aut}(\mathcal{E},[5 P]) \cong \boldsymbol{\mu}_{5}^{2} \rtimes \boldsymbol{\mu}_{4},
$$

and there exists an $\operatorname{Aut}(\mathcal{E},[5 P])$-equivariant embedding $\phi: \mathcal{E} \hookrightarrow \mathbb{P}^{4}$ such that $\phi(\mathcal{E})$ is a smooth quintic elliptic curve. Let $G$ be a subgroup in $\operatorname{Aut}(\mathcal{E},[5 P])$ such that $G \cong \boldsymbol{\mu}_{5} \rtimes \boldsymbol{\mu}_{4}$. Then $G$ fixes a unique point in $\mathbb{P}^{4}$ that is not contains in the hypersurface spanned by the secants of the quintic curve $\phi(\mathcal{E})$. Let $\psi: \mathbb{P}^{4} \rightarrow \mathbb{P}^{3}$ be the projection from this point. Then $\psi \circ \phi(\mathcal{E})$ is a smooth quintic elliptic curve. Let $C_{5}=\psi \circ \phi(\mathcal{E})$. Then $\operatorname{Aut}(X) \cong \boldsymbol{\mu}_{5} \rtimes \boldsymbol{\mu}_{4}$, and $X$ is K-stable [2, Section 5.7]

Since being K-stable is an open condition, a general member of the family № 2.17 is K-stable. In fact, all smooth Fano threefolds in the deformation family №2.17 are expected to be K-stable [2]. To show this it is enough to prove that $\beta(\mathbf{F})=A_{X}(\mathbf{F})-S_{X}(\mathbf{F})>0$ for every prime divisor $\mathbf{F}$ over the Fano threefold $X$ [7, 10], where $A_{X}(\mathbf{F})$ is the $\log$ discrepancy of the divisor $\mathbf{F}$, and

$$
S_{X}(\mathbf{F})=\frac{1}{\left(-K_{X}\right)^{3}} \int_{0}^{\infty} \operatorname{vol}\left(-K_{X}-u \mathbf{F}\right) d u .
$$

Unfortunately, we are unable to prove this. Instead, we prove the following weaker result:
Main Theorem. Let $\mathbf{F}$ be a prime divisor over $X$ such that $\beta(\mathbf{F}) \leqslant 0$, let $Z$ be its center on $X$. Then $Z$ is a point in $E_{\mathbb{P} 3} \cap E_{Q}$.

By [13, Corollary 4.14], the Main Theorem implies the following corollary.
Corollary 2. Suppose that $\operatorname{Aut}\left(\mathbb{P}^{3}, C_{5}\right)$ does not fix a point in $C_{5}$. Then $X$ is $K$-stable.

[^0]Observe that $\operatorname{Aut}(X) \cong \operatorname{Aut}\left(\mathbb{P}^{3}, C_{5}\right)$, and all possibilities for the group $\operatorname{Aut}\left(\mathbb{P}^{3}, C_{5}\right)$ can be easily derived from [9]. Namely, if $C_{5}$ is general, then $\operatorname{Aut}\left(\mathbb{P}^{3}, C_{5}\right)$ is trivial, so Corollary 2 is not applicable. If $\operatorname{Aut}\left(\mathbb{P}^{3}, C_{5}\right)$ is not trivial, then it must be isomorphic to one of the following finite groups:

$$
\boldsymbol{\mu}_{5} \rtimes \boldsymbol{\mu}_{4}, \boldsymbol{\mu}_{5} \rtimes \boldsymbol{\mu}_{2}, \boldsymbol{\mu}_{6}, \boldsymbol{\mu}_{5}, \boldsymbol{\mu}_{4}, \boldsymbol{\mu}_{2} .
$$

Furthermore, if $\operatorname{Aut}\left(\mathbb{P}^{3}, C_{5}\right)$ contains a subgroup isomorphic to $\boldsymbol{\mu}_{5}$, it acts on $C_{5}$ by translations. This implies that $\operatorname{Aut}\left(\mathbb{P}^{3}, C_{5}\right)$ does not fix a point in $C_{5} \Longleftrightarrow \operatorname{Aut}\left(\mathbb{P}^{3}, C_{5}\right)$ contains a subgroup isomorphic to $\boldsymbol{\mu}_{5}$. Therefore, Corollary 2 gives the following generalization of Example 1.

Corollary 3. Suppose that $\operatorname{Aut}(X)$ contains a subgroup isomorphic to $\boldsymbol{\mu}_{5}$. Then $X$ is $K$-stable.
Example 4 ( 9 ). Fix $a \in \mathbb{C}$ such that $a \neq 0$ and $a^{10}+11 a^{5}-1 \neq 0$. Let $C_{5}^{\prime}$ be the quintic elliptic curve in $\mathbb{P}^{4}$ given by the following system of equations:

$$
\left\{\begin{array}{l}
x_{0}^{2}+a x_{2} x_{3}-\frac{x_{1} x_{4}}{a}=0, \\
x_{1}^{2}+a x_{3} x_{4}-\frac{x_{2} x_{0}}{a}=0, \\
x_{2}^{2}+a x_{4} x_{0}-\frac{x_{3} x_{1}}{a}=0, \\
x_{3}^{2}+a x_{0} x_{1}-\frac{x_{4} x_{2}}{a}=0, \\
x_{4}^{2}+a x_{1} x_{2}-\frac{x_{0} x_{3}}{a}=0,
\end{array}\right.
$$

where $\left[x_{0}: x_{1}: x_{2}: x_{3}: x_{4}\right]$ are coordinates on $\mathbb{P}^{4}$. Let $\sigma, \tau, \iota$ be the automorphisms of $\mathbb{P}^{4}$ given by

$$
\begin{aligned}
\sigma\left(\left[x_{0}: x_{1}: x_{2}: x_{3}: x_{4}\right]\right) & =\left[x_{1}: x_{2}: x_{3}: x_{4}: x_{0}\right], \\
\tau\left(\left[x_{0}: x_{1}: x_{2}: x_{3}: x_{4}\right]\right) & =\left[x_{0}: \omega_{5} x_{1}: \omega_{5}^{2} x_{2}: \omega_{5}^{3} x_{3}: \omega_{5}^{4} x_{4}\right], \\
\iota\left(\left[x_{0}: x_{1}: x_{2}: x_{3}: x_{4}\right]\right) & =\left[x_{0}: x_{4}: x_{3}: x_{2}: x_{1}\right],
\end{aligned}
$$

where $\omega_{5}$ is a primitive fifth root of unity. Set $G=\langle\sigma, \tau, \iota\rangle$. Then $G \cong \boldsymbol{\mu}_{5}^{2} \rtimes \boldsymbol{\mu}_{2}$, and $C_{5}^{\prime}$ is $G$-invariant. Consider the following quadric hypersurface:

$$
Q=\left\{x_{0}^{2}+a x_{2} x_{3}-\frac{x_{1} x_{4}}{a}=0\right\} \subset \mathbb{P}^{4} .
$$

Observe that $Q$ is smooth, and $Q$ is $\langle\tau, \iota\rangle$-invariant. Let $q: X \rightarrow Q$ be the blow up of the curve $C_{5}^{\prime}$. Then we have $\langle\tau, \iota\rangle$-equivariant Sarkisov link ( $\star$ for an appropriate non-singular quintic elliptic curve $C_{5} \subset \mathbb{P}^{3}$. Since $\langle\tau, \iota\rangle \cong \boldsymbol{\mu}_{5} \rtimes \boldsymbol{\mu}_{2}, X$ is K-stable by Corollary 3,

Let $\mathbb{k}$ be a subfield in $\mathbb{C}$ such that $C_{5}$ is defined over $\mathbb{k}$. Then the Sarkisov link $\star$ is also defined over the field $\mathbb{k}$. Moreover, the Main Theorem and [13, Corollary 4.14] imply the following result.

Corollary 5. If the intersection $E_{\mathbb{P}^{3}} \cap E_{Q}$ does not have $\mathbb{k}$-points, then $X$ is $K$-stable.
Corollary 6. If $C_{5}(\mathbb{k})=\varnothing$ or $C_{5}^{\prime}(\mathbb{k})=\varnothing$, then $X$ is $K$-stable.
In fact, one can show that $C_{5}(\mathbb{k})=\varnothing$ if and only if $C_{5}^{\prime}(\mathbb{k})=\varnothing$.
Corollary 6 has many applications. For instance, if $\mathbb{k}$ is a number field, there are infinitely many smooth quintic genus one curves in $\mathbb{P}^{3}$ defined over $\mathbb{k}$ that do not have $\mathbb{k}$-rational points [3, 4, 11]. Thus, using Corollary 6 and Pfaffian representations of quintic elliptic curves [5], one can construct infinitely many explicit examples of K-stable smooth Fano threefolds in the family №2.17.

Example 7 (T. Fisher). Fix a prime $p \geqslant 2$. Let $C_{5}^{\prime}$ be the quintic elliptic curve in $\mathbb{P}^{4}$ given by

$$
\left\{\begin{array}{l}
x_{0}^{2}+p x_{1} x_{4}-p x_{2} x_{3}=0 \\
x_{1}^{2}+x_{0} x_{2}-p x_{3} x_{4}=0 \\
x_{2}^{2}+x_{1} x_{3}-x_{0} x_{4}=0 \\
p x_{3}^{2}+p x_{2} x_{4}-x_{0} x_{1}=0 \\
p x_{4}^{2}+x_{0} x_{3}-x_{1} x_{2}=0
\end{array}\right.
$$

let $Q$ be the quadric $\left\{x_{0}^{2}+p x_{1} x_{4}-p x_{2} x_{3}=0\right\}$, and let $q: X \rightarrow Q$ be the blow up along the curve $C_{5}^{\prime}$, where $\left[x_{0}: x_{1}: x_{2}: x_{3}: x_{4}\right]$ are the coordinates on $\mathbb{P}^{4}$. Then $\star$ exists for an appropriate quintic elliptic curve $C_{5} \subset \mathbb{P}^{3}$. We can set $\mathbb{k}=\mathbb{Q}$. Then $C_{5}^{\prime}(\mathbb{k})=\varnothing$, so $X$ is K-stable by Corollary 6 .

Let us prove the Main Theorem. Let $\mathbf{F}$ be a prime divisor over $X$, and let $Z$ be its center on $X$. Suppose that $Z$ is not a point in $E_{\mathbb{P}^{3}} \cap E_{Q}$. Let us show that $\beta(\mathbf{F})>0$.

If $Z$ is a surface, then it follows from [6] that $\beta(\mathbf{F})>0$. Thus, we may assume that

- either $Z$ is a point,
- or $Z$ is an irreducible curve.

Let $P$ be any point in $Z$. Choose an irreducible smooth surface $S \subset X$ such that $P \in S$. Set

$$
\tau=\sup \left\{u \in \mathbb{Q}_{\geqslant 0} \mid \text { the divisor }-K_{X}-u S \text { is pseudo-effective }\right\} .
$$

For $u \in[0, \tau]$, let $P(u)$ be the positive part of the Zariski decomposition of the divisor $-K_{X}-u S$, and let $N(u)$ be its negative part. Then $\beta(S)=1-S_{X}(S)$, where

$$
S_{X}(S)=\frac{1}{-K_{X}^{3}} \int_{0}^{\infty} \operatorname{vol}\left(-K_{X}-u S\right) d u=\frac{1}{24} \int_{0}^{\tau} P(u)^{3} d u
$$

Let us show how to compute $P(u)$ and $N(u)$. Set $H_{\mathbb{P}^{3}}=\pi^{*}\left(\mathcal{O}_{\mathbb{P}^{3}}(1)\right)$ and $H_{Q}=q^{*}\left(\mathcal{O}_{Q}(1)\right)$. Then

$$
\begin{aligned}
H_{\mathbb{P}^{3}} & \sim 2 H_{Q}-E_{Q}, & H_{Q} & \sim 3 H_{\mathbb{P}^{3}}-E_{\mathbb{P}^{3}}, \\
E_{\mathbb{P}^{3}} & \sim 5 H_{Q}-3 E_{Q}, & E_{Q} & \sim 5 H_{\mathbb{P}^{3}}-2 E_{\mathbb{P}^{3}} .
\end{aligned}
$$

Let us compute $P(u)$ and $N(u)$ in the following cases: $S \in\left|H_{\mathbb{P}^{3}}\right|, S \in\left|H_{Q}\right|$, and $S=E_{\mathbb{P}^{3}}$.
Example 8. Suppose that $S \in\left|H_{\mathbb{P}^{3}}\right|$. Then $\tau=\frac{3}{2}$, since $-K_{X}-u S \sim_{\mathbb{R}} \frac{3-2 u}{2} S+\frac{1}{2} E_{Q}$. Based on that, the positive part of the Zariski decomposition has the following form

$$
P(u) \sim_{\mathbb{R}}\left\{\begin{array}{l}
(4-u) H_{\mathbb{P}^{3}}-E_{\mathbb{P}^{3}} \text { for } 0 \leqslant u \leqslant 1, \\
(3-2 u) H_{Q} \text { for } 1 \leqslant u \leqslant \frac{3}{2}
\end{array}\right.
$$

and the negative part

$$
N(u)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant u \leqslant 1, \\
(u-1) E_{Q} \text { for } 1 \leqslant u \leqslant \frac{3}{2},
\end{array}\right.
$$

which gives

$$
S_{X}(S)=\frac{1}{24} \int_{0}^{\frac{3}{2}}(P(u))^{3} d u=\frac{1}{24} \int_{0}^{1} 24-u^{3}+12 u^{2}-33 u d u+\frac{1}{24} \int_{1}^{\frac{3}{2}} 2(3-2 u)^{3} d u=\frac{23}{48}
$$

Example 9. Suppose that $S \in\left|H_{Q}\right|$. Then $-K_{X}-u S \sim_{\mathbb{R}} \frac{4-3 u}{3} S+\frac{1}{3} E_{\mathbb{P}^{3}}$. Then $\tau=\frac{4}{3}$,

$$
P(u) \sim_{\mathbb{R}}\left\{\begin{array}{l}
(3-u) H_{Q}-E_{Q} \text { for } 0 \leqslant u \leqslant 1 \\
(4-3 u) H_{\mathbb{P}^{3}} \text { for } 1 \leqslant u \leqslant \frac{4}{3}
\end{array}\right.
$$

and

$$
N(u)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant u \leqslant 1 \\
(u-1) E_{\mathbb{P}^{3}} \text { for } 1 \leqslant u \leqslant \frac{4}{3},
\end{array}\right.
$$

which gives

$$
S_{X}(S)=\frac{1}{24} \int_{0}^{1} 24-2 u^{3}+18 u^{2}-39 u d u+\frac{1}{24} \int_{1}^{\frac{4}{3}}(4-3 u)^{3} d u=\frac{121}{288}
$$

Example 10. Suppose that $S=E$. Then $-K_{X}-u S \sim_{\mathbb{R}} \frac{3-5 u}{5} E_{\mathbb{P}^{3}}+\frac{4}{5} E_{Q}$. Then $\tau=\frac{3}{5}$,

$$
P(u) \sim_{\mathbb{R}}\left\{\begin{array}{l}
4 H_{\mathbb{P}^{3}}-(1+u) E_{\mathbb{P}^{3}} \text { for } 0 \leqslant u \leqslant \frac{1}{3}, \\
(3-5 u) H_{Q} \text { for } \frac{1}{3} \leqslant u \leqslant \frac{3}{5}
\end{array}\right.
$$

and

$$
N(u)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant u \leqslant \frac{1}{3} \\
(3 u-1) E_{Q} \text { for } \frac{1}{3} \leqslant u \leqslant \frac{3}{5}
\end{array}\right.
$$

which gives

$$
S_{X}(S)=\frac{1}{24} \int_{0}^{\frac{1}{3}} 20 u^{3}-60 u+24 d u+\frac{1}{24} \int_{\frac{1}{3}}^{\frac{3}{5}} 2(3-5 u)^{3} d u=\frac{227}{1080}
$$

Now, we choose an irreducible curve $C \subset S$ that contains the point $P$. For instance, if $Z$ is a curve, and $S$ contains $Z$, then we can choose $C=Z$. Since $S \not \subset \operatorname{Supp}(N(u))$, we can write

$$
\left.N(u)\right|_{S}=d(u) C+N^{\prime}(u)
$$

where $N^{\prime}(u)$ is an effective $\mathbb{R}$-divisor on $S$ such that $C \not \subset \operatorname{Supp}\left(N^{\prime}(u)\right)$, and $d(u)=\operatorname{ord}_{C}\left(\left.N(u)\right|_{S}\right)$. Now, for every $u \in[0, \tau]$, we set

$$
t(u)=\sup \left\{v \in \mathbb{R}_{\geqslant 0} \mid \text { the divisor }\left.P(u)\right|_{S}-v C \text { is pseudo-effective }\right\} .
$$

For $v \in[0, t(u)]$, we let $P(u, v)$ be the positive part of the Zariski decomposition of $\left.P(u)\right|_{S}-v C$, and we let $N(u, v)$ be its negative part. Following [1, 2], we let

$$
S\left(W_{\bullet, \bullet}^{S} ; C\right)=\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{\tau} d(u)\left(\left.P(u)\right|_{S}\right)^{2} d u+\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{\tau} \int_{0}^{\infty} \operatorname{vol}\left(\left.P(u)\right|_{S}-v C\right) d v d u
$$

which we can rewrite as

$$
S\left(W_{\bullet, \bullet}^{S} ; C\right)=\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{\tau} d(u)(P(u, 0))^{2} d u+\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{\tau} \int_{0}^{t(u)}(P(u, v))^{2} d v d u
$$

If $Z$ is a curve, $Z \subset S$ and $C=Z$, then it follows from [1, 2] that

$$
\begin{equation*}
\left.\frac{A_{X}(\mathbf{F})}{S_{X}(\mathbf{F})} \geqslant \min \left\{\frac{1}{S_{X}(S)}, \frac{1}{S\left(W_{\bullet}, \bullet\right.} ; C\right)\right\} \tag{1}
\end{equation*}
$$

Hence, if $Z$ is a curve, $Z \subset S, C=Z$ and $S\left(W_{\bullet, \bullet}^{S}, C\right)<1$, then $\beta(\mathbf{F})>0$, since $S_{X}(S)<1$ by [6].
Lemma 11. Suppose that $Z$ is a curve, $Z \subset E_{\mathbb{P}^{3}}$, and $\pi(Z)$ is not a point. Then $\beta(\mathbf{F})>0$.
Proof. Let $e$ be the invariant of the ruled surface $E_{\mathbb{P}^{3}}$ defined in Proposition 2.8 in [8, Chapter V]. Then $e \geqslant-1$ by [12]. Moreover, there is a section $C_{0}$ of the projection $E_{\mathbb{P}^{3}} \rightarrow C_{5}$ such that $C_{0}^{2}=-e$. Let $\ell$ a fiber of this projection. Then $\left.H_{\mathbb{P}^{3}}\right|_{E_{\mathbb{P}}} \equiv 5 \ell$ and $\left.E_{\mathbb{P}^{3}}\right|_{E_{\mathbb{P}} 3} \equiv-C_{0}+\lambda \ell$ for some integer $\lambda$. Since

$$
-20=-c_{1}\left(N_{C_{5} / \mathbb{P}^{3}}\right)=E_{\mathbb{P}^{3}}^{3}=\left(-C_{0}+\lambda \ell\right)^{2}=-e-2 \lambda,
$$

we get $\lambda=\frac{20-e}{2}$. Then $e$ is even, so $e \geqslant 0$. Moreover, since $3 H_{\mathbb{P}^{3}}-E_{\mathbb{P}^{3}} \sim H_{Q}$ is nef, the divisor

$$
\left.\left(3 H_{\mathbb{P}^{3}}-E_{\mathbb{P}^{3}}\right)\right|_{E_{\mathbb{P}^{3}}} \equiv C_{0}+(15-\lambda) \ell
$$

is also nef. Then $0 \leqslant\left(C_{0}+(15-\lambda) \ell\right) \cdot C_{0}=15-e-\lambda=15-e-\frac{20-e}{2}$, which implies $e \leqslant 10$ and hence we have $e \in\{0,2,4,6,8,10\}$.

Now, we set $S=E_{\mathbb{P}^{3}}$ and $C=Z$. Using (1), we see that to prove that $\beta(\mathbf{F})>0$, it is enough to show that $S\left(W_{\bullet, \bullet}^{S} ; C\right)<1$. Let us estimate $S\left(W_{\bullet, \bullet}^{S} ; C\right)$. It follows from Example 10 that $\tau=\frac{3}{5}$ and

$$
\left.P(u)\right|_{S} \equiv\left\{\begin{array}{l}
(1+u) C_{0}+\left(10+\frac{1}{2} e+\frac{1}{2} u e-10 u\right) \ell \text { for } 0 \leqslant u \leqslant \frac{1}{3} \\
(3-5 u) C_{0}+\left(15+\frac{3}{2} e-25 u-\frac{5}{2} u e\right) \ell \text { for } \frac{1}{3} \leqslant u \leqslant \frac{3}{5}
\end{array}\right.
$$

Moreover, if $0 \leqslant u \leqslant \frac{1}{3}$, then $N(u)=0$. Furthermore, if $\frac{1}{3} \leqslant u \leqslant \frac{3}{5}$, then

$$
\left.N(u)\right|_{S}=\left.(3 u-1) E_{Q}\right|_{S},
$$

and $\left.E_{Q}\right|_{S} \equiv 2 C_{0}+(5+e) \ell$. But it follows from Proposition 2.20 in [8, Chapter V] that

$$
Z \equiv a C_{0}+b \ell
$$

for some integers $a$ and $b$ such that $a \geqslant 0$ and $b \geqslant a e$. Since $\pi(Z)$ is not a point, we also have $a \geqslant 1$. This gives ord ${ }_{C}\left(\left.E_{Q}\right|_{S}\right) \leqslant 2$. Hence, if $\frac{1}{3} \leqslant u \leqslant \frac{3}{5}$, then $d(u) \leqslant 2(3 u-1)$. This gives

$$
\begin{aligned}
& S\left(W_{\bullet, \bullet}^{S} ; C\right)= \frac{3}{24} \int_{\frac{1}{3}}^{\frac{3}{5}} d(u)\left(\left.P(u)\right|_{S}\right)^{2} d u+\frac{3}{24} \int_{0}^{\frac{3}{5}} \int_{0}^{\infty} \operatorname{vol}\left(\left.P(u)\right|_{S}-v C\right) d v d u \leqslant \\
& \leqslant \frac{3}{24} \int_{\frac{1}{3}}^{\frac{3}{5}} 2(3 u-1)\left(\left.P(u)\right|_{S}\right)^{2} d u+\frac{3}{24} \int_{0}^{\frac{3}{5}} \int_{0}^{\infty} \operatorname{vol}\left(\left.P(u)\right|_{S}-v C\right) d v d u= \\
&= \frac{3}{24} \int_{\frac{1}{3}}^{\frac{3}{5}} 2(3 u-1)\left(250 u^{2}-300 u+90\right) d u+\frac{3}{24} \int_{0}^{\frac{3}{5}} \int_{0}^{\infty} \operatorname{vol}\left(\left.P(u)\right|_{S}-v C\right) d v d u= \\
&=\frac{32}{405}+\frac{3}{24} \int_{0}^{\frac{3}{5}} \int_{0}^{\infty} \operatorname{vol}\left(\left.P(u)\right|_{S}-v C\right) d v d u=\frac{32}{405}+\frac{3}{24} \int_{0}^{\frac{3}{5}} \int_{0}^{\infty} \operatorname{vol}\left(\left.P(u)\right|_{S}-v\left(a C_{0}+b \ell\right)\right) d v d u
\end{aligned}
$$

On the other hand, since $a \geqslant 1$, we have

$$
\frac{3}{24} \int_{0}^{\frac{3}{5}} \int_{0}^{\infty} \operatorname{vol}\left(\left.P(u)\right|_{S}-v\left(a C_{0}+b \ell\right)\right) d v d u \leqslant \frac{3}{24} \int_{0}^{\frac{3}{5}} \int_{0}^{\infty} \operatorname{vol}\left(\left.P(u)\right|_{S}-v C_{0}\right) d v d u
$$

Therefore, to show that $S\left(W_{\bullet,}^{S} ; C\right)<1$, we may assume that $Z=C_{0}$. Then

$$
t(u)=\left\{\begin{array}{l}
1+u \text { for } 0 \leqslant u \leqslant \frac{1}{3} \\
3-5 u \text { for } \frac{1}{3} \leqslant u \leqslant \frac{3}{5}
\end{array}\right.
$$

Moreover, if $0 \leqslant u \leqslant \frac{1}{3}$ and $v \in[0, t(u)]$, then

$$
P(u, v)=(1+u-v) C_{0}+\left(10+\frac{1}{2} e+\frac{1}{2} u e-10 u\right) \ell
$$

and the negative part $N(u, v)$ is trivial. Similarly, if $\frac{1}{3} \leqslant u \leqslant \frac{3}{5}$ and $v \in[0, t(u)]$, then

$$
P(u, v)=(3-5 u-v) C_{0}+\left(15+\frac{3}{2} e-25 u-\frac{5}{2} u e\right) \ell
$$

and the negative part $N(u, v)$ is trivial. Using the collected data, we compute

$$
\begin{aligned}
& \frac{3}{24} \int_{0}^{\frac{3}{5}} \int_{0}^{\infty} \operatorname{vol}\left(\left.P(u)\right|_{S}-v C_{0}\right) d v d u=\frac{3}{24} \int_{0}^{\frac{3}{5}} \int_{0}^{t(u)}(P(u, v))^{2} d v d u= \\
& =\frac{3}{24} \int_{0}^{\frac{1}{3}} \int_{0}^{u+1}\left(20+(e-20) v-20 u^{2}-e v^{2}+(e+20) v u\right) d v d u+ \\
& \quad+\frac{3}{24} \int_{\frac{1}{3}}^{\frac{3}{5}} \int_{0}^{3-5 u}\left(90-300 u+(3 e-30) v+250 u^{2}-e v^{2}+(-5 e+50) v u\right) d v d u=\frac{377 e}{25920}+\frac{733}{1296}
\end{aligned}
$$

As explained above, this gives

$$
\left.S\left(W_{\bullet, \bullet}^{S} ; C\right) \leqslant \frac{32}{405}+\frac{3}{24} \int_{0}^{\frac{3}{5}} \int_{0}^{\infty} \operatorname{vol}\left(\left.P(u)\right|_{S}-v C_{0}\right)\right) d v d u=\frac{377 e}{25920}+\frac{4177}{6480}
$$

Since $e \in\{0,2,4,6,8,10\}$, we conclude that $S\left(W_{\bullet, \bullet}^{S} ; C\right)<1$. Then $\beta(\mathbf{F})>0$ by (11).
Let $f: \widetilde{S} \rightarrow S$ be the blow up of the point $P$, and let $F$ be the $f$-exceptional curve. Write

$$
f^{*}\left(\left.N(u)\right|_{S}\right)=\widetilde{d}(u) F+\tilde{N}^{\prime}(u)
$$

where $\widetilde{N}^{\prime}(u)$ is the strict transform of the divisor $\left.N(u)\right|_{S}$ on the surface $\widetilde{S}$, and $\widetilde{d}(u)=\operatorname{mult}_{P}\left(\left.N(u)\right|_{S}\right)$. For every $u \in[0, \tau]$, we set

$$
\widetilde{t}(u)=\sup \left\{v \in \mathbb{R}_{\geqslant 0} \mid \text { the divisor } f^{*}\left(\left.P(u)\right|_{S}\right)-v F \text { is pseudo-effective }\right\} .
$$

For $v \in[0, \widetilde{t}(u)]$, we let $\widetilde{P}(u, v)$ be the positive part of the Zariski decomposition of $f^{*}\left(\left.P(u)\right|_{S}\right)-v F$, and we let $\widetilde{N}(u, v)$ be its negative part. As above, we let

$$
S\left(W_{\bullet, \bullet}^{S} ; F\right)=\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{\tau} \widetilde{d}(u)\left(\left.P(u)\right|_{S}\right)^{2} d u+\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{\tau} \int_{0}^{\infty} \operatorname{vol}\left(f^{*}\left(\left.P(u)\right|_{S}\right)-v F\right) d v d u
$$

which we can rewrite as

$$
S\left(W_{\bullet, \bullet}^{S} ; F\right)=\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{\tau} \widetilde{d}(u)(P(u, 0))^{2} d u+\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{\tau} \int_{0}^{\tilde{t}(u)}(\widetilde{P}(u, v))^{2} d v d u
$$

For every point $O \in F$, we let

$$
F_{O}\left(W_{\bullet,,, 0}^{\widetilde{S}, F}\right)=\frac{6}{\left(-K_{X}\right)^{3}} \int_{0}^{\tau} \int_{0}^{\tilde{t}(u)}(\widetilde{P}(u, v) \cdot F) \cdot \operatorname{ord}_{O}\left(\left.\widetilde{N}^{\prime}(u)\right|_{F}+\left.\widetilde{N}(u, v)\right|_{F}\right) d v d u
$$

and

$$
S\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F} ; O\right)=\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{\tau} \int_{0}^{\tilde{t}(u)}(\widetilde{P}(u, v) \cdot F)^{2} d v d u+F_{O}\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F}\right)
$$

Then it follows from [1, 2] that

$$
\begin{equation*}
\frac{A_{X}(\mathbf{F})}{S_{X}(\mathbf{F})} \geqslant \min \left\{\frac{1}{S_{X}(S)}, \frac{2}{S\left(W_{\bullet, \bullet}^{S} ; F\right)}, \inf _{O \in F} \frac{1}{S\left(W_{\bullet, \bullet \bullet \bullet}^{\tilde{S}} ; O\right)}\right\} . \tag{2}
\end{equation*}
$$

In the next two lemmas, we show how to apply this inequality to prove that $\beta(\mathbf{F})>0$ under certain generality conditions on the position of the point $P$.
Lemma 12. Let $S$ be a general surface in $\left|H_{\mathbb{P}^{3}}\right|$ such that $P \in S$. Suppose $P \notin E_{\mathbb{P}^{3}},-K_{S}$ is ample, and $P$ is not contained in a $(-1)$-curve in $S$. Then $\beta(\mathbf{F})>0$.
Proof. Observe that $\pi(S)$ is a general plane in $\mathbb{P}^{3}$ that contains $\pi(P)$, and $\pi$ induces a birational morphism $\varpi: S \rightarrow \pi(S)$ that blows up the points $\pi(S) \cap C_{5}$. Let $\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}, \mathbf{e}_{4}, \mathbf{e}_{5}$ be the $\varpi$-exceptional curves. Then $\left.E_{\mathbb{P}^{3}}\right|_{S}=\mathbf{e}_{1}+\mathbf{e}_{2}+\mathbf{e}_{3}+\mathbf{e}_{4}+\mathbf{e}_{5}$.

Let $L=\left.H_{\mathbb{P}^{3}}\right|_{S}$. For $i \in\{1,2,3,4,5\}$, the pencils $\left|L-\mathbf{e}_{i}\right|$ and $\left|2 L+\mathbf{e}_{i}-\mathbf{e}_{1}-\mathbf{e}_{2}-\mathbf{e}_{3}-\mathbf{e}_{4}-\mathbf{e}_{5}\right|$ contain irreducible curves that pass through the point $P$. Denote these curves by $Z_{i}$ and $Z_{i}^{\prime}$, respectively. Then $\varpi\left(Z_{i}\right)$ is the line in $\pi(S)$ that passes through $\varpi(P)$ and $\varpi\left(\mathbf{e}_{i}\right)$, and $\varpi\left(Z_{i}^{\prime}\right)$ is the conic that passes through $\varphi(P)$ and all points among $\varpi\left(\mathbf{e}_{1}\right), \varpi\left(\mathbf{e}_{2}\right), \varpi\left(\mathbf{e}_{3}\right), \varpi\left(\mathbf{e}_{4}\right), \varpi\left(\mathbf{e}_{5}\right)$ except for $\varpi\left(\mathbf{e}_{i}\right)$. Set

$$
\begin{aligned}
Z & =\sum_{i=1}^{5} Z_{i} \sim 5 L-\left(\mathbf{e}_{1}+\mathbf{e}_{2}+\mathbf{e}_{3}+\mathbf{e}_{4}+\mathbf{e}_{5}\right), \\
Z^{\prime} & =\sum_{i=1}^{5} Z_{i}^{\prime} \sim 10 L-4\left(\mathbf{e}_{1}+\mathbf{e}_{2}+\mathbf{e}_{3}+\mathbf{e}_{4}+\mathbf{e}_{5}\right) .
\end{aligned}
$$

Let $\widetilde{Z}$ and $\widetilde{Z}^{\prime}$ be the proper transforms on $\widetilde{S}$ of the curves $Z$ and $Z^{\prime}$, respectively. On the surface $\widetilde{S}$, we have $F^{2}=-1, \widetilde{Z} \cdot \widetilde{Z}^{\prime}=F \cdot \widetilde{Z}=F \cdot \widetilde{Z}^{\prime}=5, \widetilde{Z}^{2}=\left(\widetilde{Z}^{\prime}\right)^{2}=-5$. Using Example 图, we get $\tau=\frac{3}{2}$ and

$$
f^{*}\left(\left.P(u)\right|_{S}\right)-v F \sim_{\mathbb{R}}\left\{\begin{array}{cl}
\frac{3-2 u}{5} \widetilde{Z}+\frac{1+u}{10} \widetilde{Z}^{\prime}+\frac{7-3 u-2 v}{2} F \text { for } 0 \leqslant u \leqslant 1 \\
\frac{3-2 u}{5}\left(\widetilde{Z}+\widetilde{Z}^{\prime}\right)+(6-4 u-v) F \text { for } 1 \leqslant u \leqslant \frac{3}{2} \\
7
\end{array}\right.
$$

This gives

$$
\widetilde{t}(u)= \begin{cases}\frac{7-3 u}{2} & \text { for } 0 \leqslant u \leqslant 1 \\ 6-4 u & \text { for } 1 \leqslant u \leqslant \frac{3}{2}\end{cases}
$$

Furthermore, if $0 \leqslant u \leqslant 1$, then

$$
\widetilde{P}(u, v)=\left\{\begin{array}{l}
\frac{3-2 u}{5} \widetilde{Z}+\frac{1+u}{10} \widetilde{Z}^{\prime}+\frac{7-3 u-2 v}{2} F \text { for } 0 \leqslant v \leqslant 3-u \\
\frac{18-7 u-5 v}{5} \widetilde{Z}+\frac{1+u}{10} \widetilde{Z}^{\prime}+\frac{7-3 u-2 v}{2} F \text { for } 3-u \leqslant v \leqslant \frac{7-3 u}{2}
\end{array}\right.
$$

and

$$
\tilde{N}(u, v)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant v \leqslant 3-u \\
(v+u-3) \widetilde{Z} \text { for } 3-u \leqslant v \leqslant \frac{7-3 u}{2}
\end{array}\right.
$$

which gives

$$
(\widetilde{P}(u, v))^{2}=\left\{\begin{array}{l}
u^{2}-v^{2}-8 u+11 \text { for } 0 \leqslant v \leqslant 3-u \\
2(4-u-v)(7-3 u-2 v) \text { for } 3-u \leqslant v \leqslant \frac{7-3 u}{2}
\end{array}\right.
$$

and

$$
\widetilde{P}(u, v) \cdot F=\left\{\begin{array}{l}
v \text { for } 0 \leqslant v \leqslant 3-u \\
15-5 u-4 v \text { for } 3-u \leqslant v \leqslant \frac{7-3 u}{2}
\end{array}\right.
$$

If $1 \leqslant u \leqslant \frac{3}{2}$, then $\widetilde{P}(u, v)=\frac{3-2 u}{5}(\widetilde{Z}+\widetilde{Z})+(6-4 u-v) F$ and $\widehat{N}(u, v)=0$ for $v \in[0,6-4 u]$, so that

$$
\widehat{P}(u, v)^{2}=(6-4 u-v)(6-4 u+v)
$$

and $\widehat{P}(u, v) \cdot F=v$ for every $v \in[0,6-4 u]$.
Set $R=\left.E_{Q}\right|_{S}$. Then $R$ is smooth curve, since $S$ is general surface in $\left|H_{\mathbb{P}^{3}}\right|$ that passes through $P$. Let $\widetilde{R}$ be the proper transform of the curve $R$ on the surface $\widetilde{R}$. Then it follows from Example 8 that

$$
\widetilde{N}^{\prime}(u)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant u \leqslant 1 \\
(u-1) \widetilde{R} \text { for } 1 \leqslant u \leqslant \frac{3}{2}
\end{array}\right.
$$

If $0 \leqslant u \leqslant 1$, we have $\widetilde{d}(u)=0$. Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$ and $R$ does not contain $P$, then $\widetilde{d}(u)=0$. Finally, if $1 \leqslant u \leqslant \frac{3}{2}$ and $P \in R$, then $\widetilde{d}(u)=(u-1)$.

Using the data collected above, we can compute $S\left(W_{\bullet, \bullet}^{S} ; F\right)$. Namely, if $P \in E_{Q}$, then

$$
\begin{aligned}
& S\left(W_{\bullet, \bullet}^{S} ; F\right)=\frac{1}{8} \int_{1}^{\frac{3}{2}}(u-1)\left(16 u^{2}-48 u+36\right) d u+\frac{1}{8} \int_{0}^{\frac{3}{2}} \int_{0}^{\widetilde{t}(u)}(\widetilde{P}(u, v))^{2} d v d u= \\
& =\frac{1}{96}+\frac{1}{8} \int_{0}^{1} \int_{0}^{3-u} u^{2}-v^{2}-8 u+11 d v d u+\frac{1}{8} \int_{0}^{1} \int_{3-u}^{\frac{7-3 u}{2}} 2(4-u-v)(7-3 u-2 v) d v d u+ \\
& +\frac{1}{8} \int_{1}^{\frac{3}{2}} \int_{0}^{6-4 u}(6-4 u-v)(6-4 u+v) d v d u=\frac{1}{96}+\frac{655}{384}+\frac{1}{12}=\frac{691}{384}<2 .
\end{aligned}
$$

Similarly, if $P \notin E_{Q}$, then $S\left(W_{\bullet, \bullet}^{S} ; F\right)=\frac{655}{384}+\frac{1}{12}=\frac{229}{8}<2$.

Now, let $O$ be any point in $F$. Then

$$
\begin{aligned}
& S\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F} ; O\right)=\frac{1}{8} \int_{0}^{\frac{3}{2}} \int_{0}^{\widetilde{t}(u)}(\widetilde{P}(u, v) \cdot F)^{2} d v d u+F_{O}\left(W_{\substack{\bullet, \bullet, \bullet}}^{\widetilde{S}, F}\right)=\frac{155}{192}+F_{O}\left(W_{\bullet,, \bullet, \bullet}^{\widetilde{S}, F}\right)= \\
& =\frac{1}{8} \int_{0}^{1} \int_{0}^{3-u} v^{2} d v d u+\frac{1}{8} \int_{0}^{1} \int_{3-u}^{\frac{7-3 u}{2}}(15-5 u-4 v)^{2} d v d u+\frac{1}{8} \int_{1}^{\frac{3}{2}} \int_{0}^{6-4 u} v^{2} d v d u+F_{O}\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F}\right)=\frac{163}{192}+F_{O}\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F}\right) .
\end{aligned}
$$

Moreover, if $O \in \widetilde{R} \cap \widetilde{Z}$, then we compute $F_{O}\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F}\right)$ as follows:

$$
\begin{aligned}
& F_{O}\left(W_{\bullet, \bullet, 0}^{\widetilde{S}, F}\right)=\frac{1}{4} \int_{0}^{\frac{3}{2}} \int_{0}^{\widetilde{t}(u)}(\widetilde{P}(u, v) \cdot F) \cdot \operatorname{ord}_{O}\left(\left.\widetilde{N}^{\prime}(u)\right|_{F}\right) d v d u+\frac{1}{4} \int_{0}^{\frac{3}{2}} \int_{0}^{\widetilde{t}(u)}(\widetilde{P}(u, v) \cdot F) \cdot \operatorname{ord}_{O}\left(\left.\widetilde{N}(u, v)\right|_{F}\right) d v d u= \\
& =\frac{1}{4} \int_{1}^{\frac{3}{2}} \int_{0}^{6-4 u}(\widetilde{P}(u, v) \cdot F)(u-1)(\widetilde{R} \cdot F)_{O} d v d u+\frac{1}{4} \int_{0}^{1} \int_{3-u}^{\frac{7-3 u}{2}}(\widetilde{P}(u, v) \cdot F)(v+u-3)(\widetilde{Z} \cdot F)_{O} d v d u= \\
& =\frac{1}{4} \int_{1}^{\frac{3}{2}} \int_{0}^{6-4 u} v(u-1) d v d u+\frac{1}{4} \int_{0}^{1} \int_{3-u}^{\frac{7-3 u}{2}}(15-5 u-4 v)(v+u-3) d v d u=\frac{1}{96}+\frac{7}{384}=\frac{11}{384},
\end{aligned}
$$

because the curve $\widetilde{R}$ intersects $F$ transversally, and every irreducible component of the curve $\widetilde{Z}$ also intersects $F$ transversally. Hence, if $O \in \widetilde{\widetilde{R}} \cap \widetilde{Z}$, then $S\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}} ; O\right)=\frac{337}{384}<1$. Similar computations imply that $S\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F} ; O\right)<\frac{337}{384}<1$ if $O \notin \widetilde{R}$ or $O \notin \widetilde{Z}$. Thus, using (2), we see that $\beta(\mathbf{F})>0$.
Lemma 13. Let $S$ be a general surface in $\left|H_{\mathbb{P}^{3}}\right|$ such that $P \in S$. Suppose $P \notin E_{\mathbb{P}^{3} 3},-K_{S}$ is ample, and $P$ is contained in a $(-1)$-curve $B \subset S$ such that $\pi(B)$ is a conic. Then $\beta(\mathbf{F})>0$.

Proof. Let us use notations introduced in the proof of Lemma 12, and let $\widetilde{B}$ be the proper transform on the surface $\widetilde{S}$ of the curve $B$. Observe that $\widetilde{B}$ and $\widetilde{Z}$ are disjoint, and $\widetilde{B}=-2$ on the surface $\widetilde{S}$. Moreover, it follows from Example 8 that $\tau=\frac{3}{2}$ and

$$
f^{*}\left(\left.P(u)\right|_{S}\right)-v F \sim_{\mathbb{R}} \begin{cases}\frac{2-u}{3} \widetilde{Z}+\frac{1+u}{3} \widetilde{B}+\frac{11-4 u-3 v}{3} F & \text { for } 0 \leqslant u \leqslant 1 \\ \frac{3-2 u}{3}(\widetilde{Z}+2 \widetilde{B})+\frac{21-14 u-3 v}{3} F & \text { for } 1 \leqslant u \leqslant \frac{3}{2}\end{cases}
$$

This gives

$$
\widetilde{t}(u)=\left\{\begin{array}{l}
\frac{11-4 u}{3} \text { for } 0 \leqslant u \leqslant 1 \\
\frac{21-14 u}{3} \text { for } 1 \leqslant u \leqslant \frac{3}{2}
\end{array}\right.
$$

Furthermore, if $0 \leqslant u \leqslant 1$, then

$$
\widetilde{P}(u, v)=\left\{\begin{array}{c}
\frac{2-u}{3} \widetilde{Z}+\frac{1+u}{3} \widetilde{B}+\frac{11-4 u-3 v}{3} F \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{2-u}{3} \widetilde{Z}+\frac{11-4 u-3 v}{6}(\widetilde{B}+2 F) \text { for } 3-2 u \leqslant v \leqslant 3-u \\
\frac{11-4 u-3 v}{6}(2 L+\widetilde{B}+2 F) \text { for } 3-u \leqslant v \leqslant \frac{11-4 u}{3} \\
9
\end{array}\right.
$$

and

$$
\widetilde{N}(u, v)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{v+2 u-3}{2} \widetilde{B} \text { for } 3-2 u \leqslant v \leqslant 3-u \\
\frac{v+2 u-3}{2} \widetilde{B}+(v+u-3) \widetilde{Z} \text { for } 3-u \leqslant v \leqslant \frac{11-4 u}{3}
\end{array}\right.
$$

which gives

$$
(\widetilde{P}(u, v))^{2}=\left\{\begin{array}{l}
u^{2}-v^{2}-8 u+11 \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{31}{2}-14 u-3 v+3 u^{2}-\frac{v^{2}}{2}+2 v u \text { for } 3-2 u \leqslant v \leqslant 3-u \\
\frac{(11-4 u-3 v)^{2}}{2} \text { for } 3-u \leqslant v \leqslant \frac{11-4 u}{3}
\end{array}\right.
$$

and

$$
\widetilde{P}(u, v) \cdot F=\left\{\begin{array}{l}
v \text { for } 0 \leqslant v \leqslant 3-2 u, \\
\frac{3-2 u+v}{2} \text { for } 3-2 u \leqslant v \leqslant 3-u, \\
\frac{33-12 u-9 v}{2} \text { for } 3-u \leqslant v \leqslant \frac{11-4 u}{3} .
\end{array}\right.
$$

Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$, then

$$
\widetilde{P}(u, v)= \begin{cases}\frac{3-2 u}{3}(\widetilde{Z}+2 \widetilde{B})+\frac{21-14 u-3 v}{3} F & \text { for } 0 \leqslant v \leqslant 3-2 u \\ \frac{3-2 u}{3} \widetilde{Z}+\frac{21-14 u-3 v}{6}(\widetilde{B}+2 F) & \text { for } 3-2 u \leqslant v \leqslant 6-4 u \\ \frac{21-14 u-3 v}{6}(2 \widetilde{Z}+\widetilde{B}+2 F) \text { for } 6-4 u \leqslant v \leqslant \frac{21-14 u}{3}\end{cases}
$$

and

$$
\widetilde{N}(u, v)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant v \leqslant 3-2 u, \\
\frac{v+2 u-3}{2} \widetilde{B} \text { for } 3-2 u \leqslant v \leqslant 6-4 u, \\
\frac{v+2 u-3}{2} \widetilde{B}+(v+4 u-6) \widetilde{Z} \text { for } 6-4 u \leqslant v \leqslant \frac{21-14 u}{3},
\end{array}\right.
$$

which gives

$$
(\widetilde{P}(u, v))^{2}=\left\{\begin{array}{l}
(6-4 u-v)(6-4 u+v) \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{81}{2}-54 u-3 v+18 u^{2}-\frac{v^{2}}{2}+2 v u \text { for } 3-2 u \leqslant v \leqslant 6-4 u \\
\frac{(21-14 u-3 v)^{2}}{2} \text { for } 6-4 u \leqslant v \leqslant \frac{21-14 u}{3},
\end{array}\right.
$$

and

$$
\widetilde{P}(u, v) \cdot F=\left\{\begin{array}{l}
v \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{3-2 u+v}{2} \text { for } 3-2 u \leqslant v \leqslant 6-4 u \\
\frac{63-42 u-9 v}{2} \text { for } 6-4 u \leqslant v \leqslant \frac{21-14 u}{3}
\end{array}\right.
$$

Thus, as in the proof of Lemma 12, we compute

$$
S\left(W_{\bullet, \bullet}^{S}, F\right)=\left\{\begin{array}{l}
\frac{523}{288} \text { if } P \in E_{Q} \\
\frac{65}{36} \text { if } P \notin E_{Q}
\end{array}\right.
$$

so that $S\left(W_{\bullet, \bullet}^{S} ; F\right)<2$. Similarly, if $O$ is a point in $F$, then

$$
S\left(W_{\bullet, 0,0}^{\widetilde{S}, F} ; O\right)=\left\{\begin{array}{l}
\frac{257}{288} \text { if } O \in \widetilde{B} \cap \widetilde{R}, \\
\frac{119}{144} \text { if } O \in \widetilde{Z} \cap \widetilde{R}, \\
\frac{127}{144} \text { if } O \in \widetilde{B} \text { and } O \notin \widetilde{R}, \\
\frac{235}{288} \text { if } O \in \widetilde{Z} \text { and } O \notin \widetilde{R}, \\
\frac{307}{384} \text { if } O \notin \widetilde{B} \cup \widetilde{Z} \text { and } O \in \widetilde{R} \\
\frac{101}{128} \text { if } O \notin \widetilde{B} \cup \widetilde{Z} \cup \widetilde{R}
\end{array}\right.
$$

Therefore, using (2), we see that $\beta(\mathbf{F})>0$.
On the other hand, we have the following purely geometric result.
Lemma 14. Suppose that $P \notin E_{\mathbb{P}^{3}}$. Let $S$ be a general surface in $\left|H_{\mathbb{P}^{3}}\right|$ such that $S$ passes through $P$. Then $-K_{S}$ is ample. Further, if $P$ is contained in a $(-1)$-curve $B \subset S$, then $\pi(B)$ is a smooth conic.

Proof. The surface $\pi(S)$ is a general plane in $\mathbb{P}^{3}$ that contains the point $\pi(P)$. Write

$$
\pi(S) \cap C_{5}=P_{1} \cup P_{2} \cup P_{3} \cup P_{4} \cup P_{5}
$$

where $P_{1}, P_{2}, P_{3}, P_{4}, P_{5}$ are distinct points. Then $\pi$ induces a birational morphism $\varpi: S \rightarrow \pi(S)$, which is a blow up of the intersection points $P_{1}, P_{2}, P_{3}, P_{4}, P_{5}$. Thus, to prove that $-K_{S}$ is ample, we must show that at most two points among these five are contained in a line.

If three points among $P_{1}, P_{2}, P_{3}, P_{4}, P_{5}$ are contained in a line $\ell$, it is a trisecant of the curve $C_{5}$, the line $\ell$ is contained in $\pi\left(E_{Q}\right)$, and its proper transform on $X$ is a fiber of the projection $E_{Q} \rightarrow C_{5}^{\prime}$. However, the planes containing $\pi(P)$ and a trisecant of the curve $C_{5}$ form a one-dimensional family. Hence, a general plane in $\mathbb{P}^{3}$ that passes through $\pi(P)$ does not contain trisecants of the curve $C_{5}$, so that at most two points among $P_{1}, P_{2}, P_{3}, P_{4}, P_{5}$ are contained in a line. Thus, $-K_{S}$ is ample.

Now, we suppose that $P$ is contained in a ( -1 )-curve $B \subset S$. If $\pi(B)$ is not a conic, it must be a secant of the curve $C_{5}$ that contains $\pi(P)$. Let $\phi: \mathbb{P}^{3} \rightarrow \mathbb{P}^{2}$ be the linear projection from $\pi(P)$. Since $\pi(P) \notin C_{5}, \phi$ induces a birational morphism $C_{5} \rightarrow \phi\left(C_{5}\right)$, and $\phi\left(C_{5}\right)$ is a singular irreducible curve of degree 5 . Moreover, if $\ell$ is a secant of the curve $C_{5}$ that contains $\pi(P)$, then $\phi(\ell)$ is a singular point of the curve $\phi\left(C_{5}\right)$. Since this curve has finitely many singular points, we conclude that there are finitely many secants of the curve $C_{5}$ that passes through $\pi(P)$. This shows that $\pi(S)$ does not contain secants of the curve $C_{5}$ that pass through $\pi(P)$, because $\pi(S)$ is a general plane in $\mathbb{P}^{3}$ that contains the point $\pi(P)$. So, we conclude that $\pi(B)$ must be a conic.

Hence, applying Lemmas 11, 12, 13, 14, we obtain
Corollary 15. If $\beta(\mathbf{F}) \leqslant 0$, then $Z$ is a fiber of the projection $E_{\mathbb{P}^{3}} \rightarrow C_{5}$.
Remark 16. By [13, Corollary 4.14], Corollary 15 implies both Corollaries 2 and Corollaries 5 ,

To complete the proof of the Main Theorem, we may assume $Z$ is a fiber of the projection $E_{\mathbb{P}^{3}} \rightarrow C_{5}$.
Note that $Z \not \subset E_{Q}$, since $E_{Q}$ is irrational. Thus, we can choose $P \in Z$ such that $P \notin E_{Q}$ either. Now, let $S$ be a general surface in $\left|H_{Q}\right|$ such that $P \in S$. Then the surface $q(S)$ is a general hyperplane section of the quadric $Q$ that contains $q(P)$, so $q(S) \cong \mathbb{P}^{1} \times \mathbb{P}^{1}$ and $q(S) \cap C_{5}^{\prime}=P_{1} \cup P_{2} \cup P_{3} \cup P_{4} \cup P_{5}$, where $P_{1}, P_{2}, P_{3}, P_{4}, P_{5}$ are distinct points in $C_{5}^{\prime}$. Then the morphism $q: X \rightarrow Q$ induces a birational morphism $S \rightarrow q(S)$ that blows up the points $P_{1}, P_{2}, P_{3}, P_{4}, P_{5}$.

Lemma 17. The divisor $-K_{S}$ is ample.
Proof. Observe that $q(S)$ is a general hyperplane section of the quadric $Q$ that passes through $q(P)$, and it follows from the adjunction formula that $-\left.K_{S} \sim H_{\mathbb{P}^{3}}\right|_{S}$. Thus, if $-K_{S}$ is not ample, $S$ contains a fiber $\ell$ of the projection $E_{\mathbb{P}^{3}} \rightarrow C_{5}$ such that $q(\ell)$ is a secant line of the curve $C_{5}^{\prime}$. On the other hand, hyperplane sections of $Q$ that contain $q(P)$ and a secant line of the curve $C_{5}^{\prime}$ form a two-dimensional family. So, we may assume that $q(S)$ is not one of them, which implies that $-K_{S}$ is ample.

Since $-K_{S}$ is ample, the morphism $\pi$ induces an isomorphism $S \cong \pi(S)$, and $\pi(S)$ is a smooth cubic surface in $\mathbb{P}^{3}$ that contains the curve $C_{5}$. Let us identify $S$ with the smooth cubic surface $\pi(S)$. Using this identification, we see that $C_{5}=E_{\mathbb{P}^{3}} \cap S$. Then $P \in C_{5}$, since $P \in E_{\mathbb{P}^{3}}$.

Let $L_{1}$ and $L_{2}$ be the proper transforms on $S$ of two rulings of the surface $q(S) \cong \mathbb{P}^{1} \times \mathbb{P}^{1}$ that pass through the point $q(P)$. Then $L_{1}$ and $L_{2}$ are conics in $S$, because $q\left(L_{1}\right)$ and $q\left(L_{2}\right)$ do not contain any of the points $P_{1}, P_{2}, P_{3}, P_{4}, P_{5}$, since we assume that $q(S)$ is a general hyperplane section of the quadric $Q$ that contains the point $q(P)$. Moreover, it follows from Example 9 that

$$
\left.P(u)\right|_{S} \sim_{\mathbb{R}}\left\{\begin{array}{l}
-K_{S}+(1-u)\left(L_{1}+L_{2}\right) \text { for } 0 \leqslant u \leqslant 1 \\
(4-3 u)\left(-K_{S}\right) \text { for } 1 \leqslant u \leqslant \frac{4}{3}
\end{array}\right.
$$

and

$$
\left.N(u)\right|_{S}=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant u \leqslant 1 \\
(u-1) C_{5} \text { for } 1 \leqslant u \leqslant \frac{4}{3}
\end{array}\right.
$$

Let $T_{P}$ be the unique curve in the linear system $\left|-K_{S}\right|$ that is singular at $P$. Then $T_{P}$ is cut out by the hyperplane in $\mathbb{P}^{3}$ that is tangent to $S$ at the point $P$. In particular, the curve $T_{P}$ is reduced.

Lemma 18. We have the following five possible cases:

- $T_{P}$ is an irreducible cubic curve,
- $T_{P}=\ell+C_{2}$, where $\ell$ is a line, $C_{2}$ is a smooth conic such that $C_{2} \neq L_{1}$ and $C_{2} \neq L_{2}$,
- $T_{P}=\ell_{1}+\ell_{2}+\ell_{3}$, where $\ell_{1}, \ell_{2}, \ell_{3}$ are lines such that $P=\ell_{1} \cap \ell_{2}$ and $P \notin \ell_{3}$.

Proof. A priori, since $T_{P}$ is a reduced cubic curve, we may have the following cases:
(1) $T_{P}$ is an irreducible curve,
(2) $T_{P}=\ell+L_{1}$, where $\ell$ is a line,
(3) $T_{P}=\ell+L_{2}$, where $\ell$ is a line,
(4) $T_{P}=\ell+C_{2}$, where $\ell$ is a line, $C_{2}$ is a smooth conic such that $C_{2} \neq L_{1}$ and $C_{2} \neq L_{2}$,
(5) $T_{P}=\ell_{1}+\ell_{2}+\ell_{3}$, where $\ell_{1}, \ell_{2}, \ell_{3}$ are lines such that $P=\ell_{1} \cap \ell_{2}$ and $P \notin \ell_{3}$,
(6) $T_{P}=\ell_{1}+\ell_{2}+\ell_{3}$, where $\ell_{1}, \ell_{2}, \ell_{3}$ are lines such that $P=\ell_{1} \cap \ell_{2} \cap \ell_{3}$.

If $T_{P}=\ell_{1}+\ell_{2}+\ell_{3}$ for three lines $\ell_{1}, \ell_{2}, \ell_{3}$ such that $P=\ell_{1} \cap \ell_{2} \cap \ell_{3}$, then

$$
2=-K_{S} \cdot L_{1}=\left(\ell_{1}+\ell_{2}+\ell_{3}\right) \cdot L_{1} \geqslant \sum_{i=1}^{3}\left(\ell_{1} \cdot L_{1}\right)_{P} \geqslant 3
$$

which is absurd. Thus, we see that the last case is impossible. To complete the proof, we must show that the second and the third cases are also impossible.

Suppose that $T_{P}=\ell+L_{1}$ for some line $\ell$. Then $q(\ell)$ is a twisted cubic curve in $Q$ that contains all intersection points $P_{1}, P_{2}, P_{3}, P_{4}, P_{5}$. In particular, we see that $E_{Q} \cap \ell \geqslant 5$. On the other hand, the curve $q(\ell)$ is not contained in the surface $q\left(E_{\mathbb{P}^{3}}\right)$, because the only rational curves in the ruled irrational surface $E_{\mathbb{P}^{3}}$ are fibers of the natural projection $E_{\mathbb{P}^{3}} \rightarrow C_{5}$, which are mapped to lines by $q$. Therefore, since $P \in E_{\mathbb{P}^{3}}$ by assumption, we have

$$
1 \leqslant\left(E_{\mathbb{P}^{3}} \cdot \ell\right)_{P} \leqslant E_{\mathbb{P}^{3}} \cdot \ell=\left(5 H_{Q}-3 E_{Q}\right) \cdot \ell=15-3 E_{Q} \cdot \ell \leqslant 0
$$

which is absurd. This shows that the conic $L_{1}$ cannot be an irreducible component of the curve $T_{P}$. Similarly, we see $L_{2}$ is also not an irreducible component of the curve $T_{P}$.

From Example [, we know that $S_{X}(S)<1$. So, it follows from (2) that $\beta(\mathbf{F}) \leqslant 0$ if $S\left(W_{\bullet, \bullet}^{S} ; F\right)<2$, and $S\left(W_{\mathbf{\bullet}, \boldsymbol{\bullet}, \boldsymbol{\bullet}}^{\widetilde{ }} ; O\right)<1$ for every point $O \in F$. Let us check these conditions.

Let $\widetilde{L}_{1}, \widetilde{L}_{2}, \widetilde{T}_{P}$ be proper transforms on $\widetilde{S}$ of the curves $L_{1}, L_{2}, T_{P}$. Then

$$
f^{*}\left(\left.P(u)\right|_{S}\right)-v F \sim_{\mathbb{R}}\left\{\begin{array}{l}
\widetilde{T}_{P}+(1-u)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right)+(4-2 u-v) F \text { for } 0 \leqslant u \leqslant 1,  \tag{3}\\
(4-3 u) \widetilde{T}_{P}+(8-6 u-v) F \text { for } 1 \leqslant u \leqslant \frac{4}{3} .
\end{array}\right.
$$

Hence, if $0 \leqslant u \leqslant 1$, then $\widetilde{t}(u)=4-2 u$. Similarly, if $1 \leqslant u \leqslant \frac{4}{3}$, then $\widetilde{t}(u)=8-6 u$. Moreover, since the curve $C_{5}=E_{\mathbb{P}^{3}} \cap S$ is smooth, we have $\widetilde{d}(u)=0$ for $0 \leqslant u \leqslant 1$, and $\widetilde{d}(u)=u-1$ for $1 \leqslant u \leqslant \frac{4}{3}$. Finally, let $\widetilde{C}_{5}$ be the proper transform on $\widetilde{S}$ of the curve $C_{5}$. Then

$$
\widetilde{N}^{\prime}(u)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant u \leqslant 1 \\
(u-1) \widetilde{C}_{5} \text { for } 1 \leqslant u \leqslant \frac{4}{3}
\end{array}\right.
$$

Now, we can compute $S\left(W_{\bullet, \bullet}^{S} ; F\right)$ and $S\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F} ; O\right)$ for every point $O \in F$.
Lemma 19. If $T_{P}$ is irreducible, then $S\left(W_{\bullet, \bullet}^{S} ; F\right)<2$ and $S\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F} ; O\right)<1$ for every point $O \in F$.
Proof. Suppose that the curve $T_{P}$ is irreducible. Then $\widetilde{S}$ is a smooth del Pezzo surface of degree 2. Note that $\widetilde{L}_{1}, \widetilde{L}_{1}, \widetilde{T}_{P}$ are disjoint (-1)-curves on $\widetilde{S}$. If $0 \leqslant u \leqslant \frac{1}{2}$, it follows from (3) that

$$
\widetilde{P}(u, v)=\left\{\begin{array}{l}
\widetilde{T}_{P}+(1-u)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right)+(4-2 u-v) F \text { for } 0 \leqslant v \leqslant 3-u \\
\widetilde{T}_{P}+(4-2 u-v)\left(\widetilde{L}_{1}+\widetilde{L}_{2}+F\right) \text { for } 3-u \leqslant v \leqslant \frac{7-4 u}{2} \\
(4-2 u-v)\left(2 \widetilde{T}_{P}+\widetilde{L}_{1}+\widetilde{L}_{2}+F\right) \text { for } \frac{7-4 u}{2} \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

and

$$
\tilde{N}(u, v)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant v \leqslant 3-u \\
(v+u-3)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right) \text { for } 3-u \leqslant v \leqslant \frac{7-4 u}{2} \\
(v+u-3)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right)+(2 v+4 u-7) \widetilde{T}_{P} \text { for } \frac{7-4 u}{2} \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

which gives

$$
(\widetilde{P}(u, v))^{2}=\left\{\begin{array}{l}
2 u^{2}-v^{2}-12 u+13 \text { for } 0 \leqslant v \leqslant 3-u \\
4 u^{2}+4 u v+v^{2}-24 u-12 v+31 \text { for } 3-u \leqslant v \leqslant \frac{7-4 u}{2}, \\
5(2 u+v-4)^{2} \text { for } \frac{7-4 u}{2} \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

and

$$
\widetilde{P}(u, v) \cdot F=\left\{\begin{array}{l}
v \text { for } 0 \leqslant v \leqslant 3-u \\
6-2 u-v \text { for } 3-u \leqslant v \leqslant \frac{7-4 u}{2} \\
20-10 u-5 v \text { for } \frac{7-4 u}{2} \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

Likewise, if $\frac{1}{2} \leqslant u \leqslant 1$, then it follows from (3) that

$$
\widetilde{P}(u, v)=\left\{\begin{array}{l}
\widetilde{T}_{P}+(1-u)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right)+(4-2 u-v) F \text { for } 0 \leqslant v \leqslant \frac{7-4 u}{2} \\
(4-2 u-v)\left(2 \widetilde{T}_{P}+F\right)+(1-u)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right) \text { for } \frac{7-4 u}{2} \leqslant v \leqslant 3-u \\
(4-2 u-v)\left(2 \widetilde{T}_{P}+\widetilde{L}_{1}+\widetilde{L}_{2}+F\right) \text { for } 3-u \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

and

$$
\widetilde{N}(u, v)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant v \leqslant \frac{7-4 u}{2} \\
(2 v+4 u-7) \widetilde{T}_{P} \text { for } \frac{7-4 u}{2} \leqslant v \leqslant 3-u \\
\left.(v+u-3)\left(\widetilde{L}_{1}\right)+\widetilde{L}_{2}\right)+(2 v+4 u-7) \widetilde{T}_{P} \text { for } 3-u \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

which gives

$$
(\widetilde{P}(u, v))^{2}=\left\{\begin{array}{l}
2 u^{2}-v^{2}-12 u+13 \text { for } 0 \leqslant v \leqslant \frac{7-4 u}{2} \\
18 u^{2}+16 u v+3 v^{2}-68 u-28 v+62 \text { for } \frac{7-4 u}{2} \leqslant v \leqslant 3-u \\
5(2 u+v-4)^{2} \text { for } 3-u \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

and

$$
\widetilde{P}(u, v) \cdot F=\left\{\begin{array}{l}
v \text { for } 0 \leqslant v \leqslant \frac{7-4 u}{2} \\
14-8 u-3 v \text { for } \frac{7-4 u}{2} \leqslant v \leqslant 3-u \\
20-10 u-5 v \text { for } 3-u \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

Similarly, if $1 \leqslant u \leqslant \frac{4}{3}$, then

$$
\widetilde{P}(u, v)=\left\{\begin{array}{l}
(4-3 u) \widetilde{T}_{P}+(8-6 u-v) F \text { for } 0 \leqslant v \leqslant \frac{12-9 u}{2} \\
(8-6 u-v)\left(2 \widetilde{T}_{P}+F\right) \text { for } \frac{12-9 u}{2} \leqslant v \leqslant 8-6 u
\end{array}\right.
$$

and

$$
\tilde{N}(u, v)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant v \leqslant \frac{12-9 u}{2}, \\
(2 v+9 u-12) \widetilde{T}_{P} \text { for } \frac{12-9 u}{2} \leqslant v \leqslant 8-6 u,
\end{array}\right.
$$

which gives

$$
(\widetilde{P}(u, v))^{2}=\left\{\begin{array}{c}
27 u^{2}-v^{2}-72 u+48 \text { for } 0 \leqslant v \leqslant \frac{12-9 u}{2} \\
3(6 u+v-8)^{2} \text { for } \frac{12-9 u}{2} \leqslant v \leqslant 8-6 u
\end{array}\right.
$$

and

$$
\widetilde{P}(u, v) \cdot F=\left\{\begin{array}{l}
v \text { for } 0 \leqslant v \leqslant \frac{12-9 u}{2} \\
24-18 u-3 v \text { for } \frac{12-9 u}{2} \leqslant v \leqslant 8-6 u
\end{array}\right.
$$

As in the proof of Lemma 12, we get $S\left(W_{\bullet, \bullet}^{S} ; F\right)=\frac{1103}{576}$. Likewise, if $O$ is a point in $F$, then

$$
S\left(W_{\bullet, \bullet, \bullet} \widetilde{S} ; O\right)=\left\{\begin{array}{l}
\frac{131}{144} \text { if } O \in \widetilde{T}_{P}, T_{P} \text { has a node at } P, \text { and } O \in \widetilde{C}_{5}, \\
\frac{29}{32} \text { if } O \in \widetilde{T}_{P}, T_{P} \text { has a node at } P, \text { and } O \notin \widetilde{C}_{5}, \\
\frac{277}{288} \text { if } O \in \widetilde{T}_{P}, T_{P} \text { has a cusp at } P \text {, and } O \in \widetilde{C}_{5}, \\
\frac{23}{24} \text { if } O \in \widetilde{T}_{P}, T_{P} \text { has a cusp at } P \text { and } O \notin \widetilde{C}_{5}, \\
\frac{1152}{347} \text { if } O \in \widetilde{L}_{1} \cup \widetilde{L}_{2} \text { and } O \in \widetilde{C}_{5}, \\
\frac{247}{288} \text { if } O \notin \widetilde{L}_{1} \cup \widetilde{L}_{2} \text { and } O \notin \widetilde{C}_{5}, \\
\frac{41}{48} \text { if } O \notin \widetilde{L}_{1} \cup \widetilde{L}_{2} \cup \widetilde{T}_{2} \cup \widetilde{T}_{P} \text { and } O \in \widetilde{C}_{5}, \\
\end{array}\right.
$$

The lemma is proved.
Lemma 20. Suppose $T_{P}=\ell+C_{2}$, where $\ell$ is a line, $C_{2}$ is an smooth conic such that $L_{1} \neq C_{2} \neq L_{2}$. Then $S\left(W_{\bullet, \bullet}^{S} ; F\right)<2$ and $S\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F} ; O\right)<1$ for every point $O \in F$.

Proof. Let $\widetilde{\ell}$ and $c$ be the proper transforms on the surface $\widetilde{S}$ of the curves $\ell$ and $C_{2}$, respectively. Then $\widetilde{\ell}$ is a ( -2 -curve, $\widetilde{C}_{2}$ is a $(-1)$-curve, and the intersection $\widetilde{\ell} \cap \widetilde{C}_{2}$ consists of a single point. Note also that $\widetilde{\ell} \cap \widetilde{C}_{2} \in F \Longleftrightarrow \ell$ and $C_{2}$ are tangent at $P$. If $0 \leqslant u \leqslant \frac{2}{3}$, it follows from (3) that

$$
\widetilde{P}(u, v)=\left\{\begin{array}{l}
\widetilde{\ell}+\widetilde{C}_{2}+(1-u)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right)+(4-2 u-v) F \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{5-2 u-v}{2} \widetilde{\ell}+\widetilde{C}_{2}+(1-u)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right)+(4-2 u-v) F \text { for } 3-2 u \leqslant v \leqslant 3-u, \\
\frac{5-2 u-v}{2} \widetilde{\ell}+\widetilde{C}_{2}+(4-2 u-v)\left(\widetilde{L}_{1}+\widetilde{L}_{2}+F\right) \text { for } 3-u \leqslant v \leqslant \frac{11-6 u}{3} \\
(4-2 u-v)\left(2 \widetilde{\ell}+3 \widetilde{C}_{2}+\widetilde{L}_{1}+\widetilde{L}_{2}+F\right) \text { for } \frac{11-6 u}{3} \leqslant v \leqslant 4-2 u,
\end{array}\right.
$$

and

$$
\widetilde{N}(u, v)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant v \leqslant 3-2 u, \\
\frac{v+2 u-3}{2} \widetilde{\ell} \text { for } 3-2 u \leqslant v \leqslant 3-u, \\
\frac{v+2 u-3}{2} \widetilde{\ell}+(v+u-3)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right) \text { for } 3-u \leqslant v \leqslant \frac{11-6 u}{3}, \\
(4 u+2 v-7) \widetilde{\ell}+(6 u+3 v-11) \widetilde{C}_{2}+(v+u-3)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right) \text { for } \frac{11-6 u}{3} \leqslant v \leqslant 4-2 u,
\end{array}\right.
$$

which gives

$$
(\widetilde{P}(u, v))^{2}=\left\{\begin{array}{l}
2 u^{2}-v^{2}-12 u+13 \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{35}{2}-18 u-3 v+4 u^{2}-\frac{v^{2}}{2}+2 u v \text { for } 3-2 u \leqslant v \leqslant 3-u \\
\frac{71}{2}-30 u-15 v+6 u^{2}+\frac{3 v^{2}}{2}+6 u v \text { for } 3-u \leqslant v \leqslant \frac{11-6 u}{3} \\
6(4-2 u-v)^{2} \text { for } \frac{11-6 u}{3} \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

and

$$
\widetilde{P}(u, v) \cdot F=\left\{\begin{array}{l}
v \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{3-2 u+v}{2} \text { for } 3-2 u \leqslant v \leqslant 3-u \\
\frac{15-6 u-3 v}{2} \text { for } 3-u \leqslant v \leqslant \frac{11-6 u}{3} \\
24-12 u-6 v \text { for } \frac{11-6 u}{3} \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

Likewise, if $\frac{2}{3} \leqslant u \leqslant 1$, then it follows from (3) that

$$
\widetilde{P}(u, v)=\left\{\begin{array}{l}
\widetilde{\ell}+\widetilde{C}_{2}+(1-u)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right)+(4-2 u-v) F \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{5-2 u-v}{2} \widetilde{\ell}+\widetilde{C}_{2}+(1-u)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right)+(4-2 u-v) F \text { for } 3-2 u \leqslant v \leqslant \frac{11-6 u}{3} \\
(4-2 u-v)\left(2 \widetilde{\ell}+3 \widetilde{C}_{2}+F\right)+(1-u)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right) \text { for } \frac{11-6 u}{3} \leqslant v \leqslant 3-u \\
(4-2 u-v)\left(2 \widetilde{\ell}+3 \widetilde{C}_{2}+\widetilde{L}_{1}+\widetilde{L}_{2}+F\right) \text { for } 3-u \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

and

$$
\widetilde{N}(u, v)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant v \leqslant 3-2 u, \\
\frac{v+2 u-3}{2} \widetilde{\ell} \text { for } 3-2 u \leqslant v \leqslant \frac{11-6 u}{3}, \\
(4 u+2 v-7) \widetilde{\ell}+(6 u+3 v-11) \widetilde{C}_{2} \text { for } \frac{11-6 u}{3} \leqslant v \leqslant 3-u, \\
(4 u+2 v-7) \widetilde{\ell}+(6 u+3 v-11) \widetilde{C}_{2}+(v+u-3)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right) \text { for } 3-u \leqslant v \leqslant 4-2 u,
\end{array}\right.
$$

which gives

$$
(\widetilde{P}(u, v))^{2}=\left\{\begin{array}{l}
2 u^{2}-v^{2}-12 u+13 \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{35}{2}-18 u-3 v+4 u^{2}-\frac{v^{2}}{2}+2 u v \text { for } 3-2 u \leqslant v \leqslant \frac{11-6 u}{3} \\
22 u^{2}+20 u v+4 v^{2}-84 u-36 v+78 \text { for } \frac{11-6 u}{3} \leqslant v \leqslant 3-u \\
6(4-2 u-v)^{2} \text { for } 3-u \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

and

$$
\widetilde{P}(u, v) \cdot F=\left\{\begin{array}{l}
v \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{3-2 u+v}{2} \text { for } 3-2 u \leqslant v \leqslant \frac{11-6 u}{3} \\
18-10 u-4 v+18 \text { for } \frac{11-6 u}{3} \leqslant v \leqslant 3-u \\
24-12 u-6 v \text { for } 3-u \leqslant v \leqslant 4-2 u \\
16
\end{array}\right.
$$

Similarly, if $1 \leqslant u \leqslant \frac{4}{3}$, then

$$
\widetilde{P}(u, v)=\left\{\begin{array}{l}
(4-3 u)\left(\widetilde{\ell}+\widetilde{C}_{2}\right)+(8-6 u-v) F \text { for } 0 \leqslant v \leqslant 4-3 u \\
\frac{12-9 u-v}{2} \widetilde{\ell}+(4-3 u) \widetilde{C}_{2}+(8-6 u-v) F \text { for } 4-3 u \leqslant v \leqslant \frac{20-15 u}{2}, \\
(8-6 u-v)\left(2 \widetilde{\ell}+3 \widetilde{C}_{2}+F\right) \text { for } \frac{20-15 u}{2} \leqslant v \leqslant 8-6 u
\end{array}\right.
$$

and

$$
\tilde{N}(u, v)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant v \leqslant 4-3 u \\
\frac{v+3 u-4}{2} \widetilde{\ell} \text { for } 4-3 u \leqslant v \leqslant \frac{20-15 u}{2}, \\
(9 u+2 v-12) \widetilde{\ell}+(15 u+3 v-20) \widetilde{C}_{2} \text { for } \frac{20-15 u}{2} \leqslant v \leqslant 8-6 u
\end{array}\right.
$$

which gives

$$
(\widetilde{P}(u, v))^{2}=\left\{\begin{array}{l}
27 u^{2}-v^{2}-72 u+48 \text { for } 0 \leqslant v \leqslant 4-3 u \\
56-84 u-4 v+\frac{63 u^{2}}{2}-\frac{v^{2}}{2}+3 u v \text { for } 4-3 u \leqslant v \leqslant \frac{20-15 u}{2}, \\
4(6 u+v-8)^{2} \text { for } \frac{20-15 u}{2} \leqslant v \leqslant 8-6 u
\end{array}\right.
$$

and

$$
\widetilde{P}(u, v) \cdot F=\left\{\begin{array}{l}
v \text { for } 0 \leqslant v \leqslant 4-3 u \\
\frac{4-3 u+v}{2} \text { for } 4-3 u \leqslant v \leqslant \frac{20-15 u}{2} \\
32-24 u-4 v \text { for } \frac{20-15 u}{2} \leqslant v \leqslant 8-6 u
\end{array}\right.
$$

Now, we can compute

$$
S\left(W_{\bullet, \bullet}^{S} ; F\right)=\frac{1}{8} \int_{1}^{\frac{4}{3}}(u-1)\left(27 u^{2}-72 u+48\right) d u+\frac{1}{8} \int_{0}^{\frac{4}{3}} \int_{0}^{\widetilde{t}(u)}(\widetilde{P}(u, v))^{2} d v d u=\frac{1661}{864} .
$$

Let $O$ be a point in $F$. If $O \in \widetilde{\ell} \cap \widetilde{C}_{2} \cap \widetilde{C}_{5}$, then $O \notin \widetilde{L}_{1} \cup \widetilde{L}_{2}$, so that

$$
\begin{aligned}
& F_{O}\left(W_{\bullet,, \bullet \bullet}^{\widetilde{S}, F}\right)=\frac{1}{4} \int_{1}^{\frac{4}{3}} \int_{0}^{8-6 u}(u-1)(\widetilde{P}(u, v) \cdot F) d v d u+\frac{1}{4} \int_{0}^{\frac{4}{3}} \int_{0}^{\widetilde{t}(u)}(\widetilde{P}(u, v) \cdot F) \cdot \operatorname{ord}{ }_{O}\left(\left.\widetilde{N}(u, v)\right|_{F}\right) d v d u= \\
& =\frac{1}{576}+\frac{1}{4} \int_{0}^{\frac{2}{3}} \int_{3-2 u}^{\frac{11-6 u}{3}}(\widetilde{P}(u, v) \cdot F) \frac{v+2 u-3}{2} d v d u+\frac{1}{4} \int_{0}^{\frac{2}{3}} \int_{\frac{11-6 u}{3}}^{4-2 u}(\widetilde{P}(u, v) \cdot F)((6 u+3 v-11)+(4 u+2 v-7)) d v d u+ \\
& +\frac{1}{4} \int_{\frac{2}{3}}^{1} \int_{3-2 u}^{\frac{11-6 u}{3}}(\widetilde{P}(u, v) \cdot F) \frac{v+2 u-3}{2} d v d u+\frac{1}{4} \int_{0}^{\frac{2}{3}} \int_{\frac{11-6 u}{4}}^{4-2 u}(\widetilde{P}(u, v) \cdot F)((6 u+3 v-11)+(4 u+2 v-7)) d v d u+ \\
& +\frac{1}{4} \int_{1}^{\frac{4}{3}} \int_{4-3 u}^{\frac{20-15 u}{3}}(\widetilde{P}(u, v) \cdot F) \frac{v+2 u-3}{2} d v d u+\frac{1}{4} \int_{1}^{\frac{4}{3}} \int_{\frac{20-15 u}{3}}^{8-6 u}(\widetilde{P}(u, v) \cdot F)((6 u+3 v-11)+(4 u+2 v-7)) d v d u=\frac{235}{1728},
\end{aligned}
$$

so that

$$
S\left(W_{\bullet,, \bullet, 0}^{\widetilde{S}} ; O\right)=\frac{1}{8} \int_{0}^{\frac{4}{3}} \int_{0}^{\widetilde{t}(u)}(\widetilde{P}(u, v) \cdot F)^{2} d v d u+\frac{235}{1728}=\frac{1685}{1728} .
$$

Similarly, if $O \in \widetilde{\ell} \cup \widetilde{C}_{2}$, then $\underset{\sim}{S}\left(W_{\bullet, 0,0}^{\widetilde{L}, F} ; O\right) \leqslant \frac{1685}{1728}$. If $O \in \widetilde{L}_{1} \cup \widetilde{L}_{2}$, then $O \notin \widetilde{\ell} \cup \widetilde{C}_{2}$, and $O$ is contained in exactly one of the curves $\widetilde{L}_{1}$ or $\widetilde{L}_{2}$. In this case, we have

$$
S\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F} ; O\right)= \begin{cases}\frac{515}{576} & \text { if } O \in \widetilde{C}_{5} \\ \frac{257}{288} & \text { if } O \notin \widetilde{C}_{5}\end{cases}
$$

The lemma is proved.
Lemma 21. Suppose $T_{P}=\ell_{1}+\ell_{2}+\ell_{3}$, where $\ell_{1}, \ell_{2}, \ell_{3}$ are lines such that $P=\ell_{1} \cap \ell_{2}$ and $P \notin \ell_{3}$. Then $S\left(W_{\bullet, \bullet}^{S} ; F\right)<2$ and $S\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F} ; O\right)<1$ for every point $O \in F$.
Proof. Let $\widetilde{\ell}_{1}, \widetilde{\ell}_{2}, \widetilde{\ell}_{3}$ be the proper transforms on $\widetilde{S}$ of the lines $\ell_{1}, \ell_{2}, \ell_{3}$, respectively. If $0 \leqslant u \leqslant 1$, then it follows from (3) that

$$
\widetilde{P}(u, v)=\left\{\begin{array}{l}
\widetilde{\ell}_{1}+\widetilde{\ell}_{2}+\widetilde{\ell}_{3}+(1-u)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right)+(4-2 u-v) F \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{5-2 u-v}{2}\left(\widetilde{\ell}_{1}+\widetilde{\ell}_{2}\right)+\widetilde{\ell}_{3}+(1-u)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right)+(4-2 u-v) F \text { for } 3-2 u \leqslant v \leqslant 3-u, \\
\frac{5-2 u-v}{2}\left(\widetilde{\ell}_{1}+\widetilde{\ell}_{2}\right)+\widetilde{\ell}_{3}+(4-2 u-v)\left(\widetilde{L}_{1}+\widetilde{L}_{2}+F\right) \text { for } 3-u \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

and

$$
\widetilde{N}(u, v)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant v \leqslant 3-2 u \\
\frac{v+2 u-3}{2}\left(\widetilde{\ell}_{1}+\widetilde{\ell}_{2}\right) \text { for } 3-2 u \leqslant v \leqslant 3-u \\
\frac{v+2 u-3}{2}\left(\widetilde{\ell}_{1}+\widetilde{\ell}_{2}\right)+(v+u-3)\left(\widetilde{L}_{1}+\widetilde{L}_{2}\right) \text { for } 3-u \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

which gives

$$
(\widetilde{P}(u, v))^{2}=\left\{\begin{array}{l}
2 u^{2}-v^{2}-12 u+13 \text { for } 0 \leqslant v \leqslant 3-2 u \\
6 u^{2}+4 u v-24 u-6 v+22 \text { for } 3-2 u \leqslant v \leqslant 3-u \\
8 u^{2}+8 u v+2 v^{2}-36 u-18 v+40 \text { for } 3-u \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

and

$$
\widetilde{P}(u, v) \cdot F=\left\{\begin{array}{l}
v \text { for } 0 \leqslant v \leqslant 3-2 u \\
3-2 u \text { for } 3-2 u \leqslant v \leqslant 3-u \\
9-4 u-2 v \text { for } 3-u \leqslant v \leqslant 4-2 u
\end{array}\right.
$$

Similarly, if $1 \leqslant u \leqslant \frac{3}{2}$, then

$$
\widetilde{P}(u, v)=\left\{\begin{array}{l}
(4-3 u)\left(\widetilde{\ell}_{1}+\widetilde{\ell}_{2}+\widetilde{\ell}_{3}\right)+(8-6 u-v) F \text { for } 0 \leqslant v \leqslant 4-3 u \\
\frac{12-9 u-v}{2}\left(\widetilde{\ell}_{1}+\widetilde{\ell}_{2}\right)+(4-3 u) \widetilde{\ell}_{3}+(8-6 u-v) F \text { for } 4-3 u \leqslant v \leqslant 8-6 u,
\end{array}\right.
$$

and

$$
\tilde{N}(u, v)=\left\{\begin{array}{l}
0 \text { for } 0 \leqslant v \leqslant 4-3 u \\
\frac{v+3 u-4}{2}\left(\widetilde{\ell}_{1}+\tilde{\ell}_{2}\right) \text { for } 4-3 u \leqslant v \leqslant 8-6 u, \\
18
\end{array}\right.
$$

which gives

$$
(\widetilde{P}(u, v))^{2}=\left\{\begin{array}{l}
27 u^{2}-v^{2}-72 u+48 \text { for } 0 \leqslant v \leqslant 4-3 u \\
2(4-3 u)(8-6 u-v) \text { for } 4-3 u \leqslant v \leqslant 8-6 u
\end{array}\right.
$$

and

$$
\widetilde{P}(u, v) \cdot F=\left\{\begin{array}{l}
v \text { for } 0 \leqslant v \leqslant 4-3 u \\
4-3 u \text { for } 4-3 u \leqslant v \leqslant 8-6 u
\end{array}\right.
$$

Since $P \in C_{5}$ and $C_{5}$ is smooth, we compute $S\left(W_{\bullet, \bullet}^{S} ; F\right)=\frac{31}{16}$. Similarly, if $O$ is a point in $F$, then

$$
S\left(W_{\bullet, \bullet, \bullet}^{\widetilde{S}, F} ; O\right)=\left\{\begin{array}{l}
\frac{329}{384} \text { if } O \in \widetilde{\ell}_{1} \cup \widetilde{\ell}_{2} \\
\frac{161}{192} \text { if } O \in \widetilde{L}_{1} \cup \widetilde{L}_{2}, \\
\frac{155}{192} \text { if } O \notin \widetilde{\ell}_{1} \cup \widetilde{\ell}_{2} \cup \widetilde{L}_{1} \cup \widetilde{L}_{2}
\end{array}\right.
$$

The lemma is proved.
Thus, we see that $S\left(W_{\bullet, \bullet}^{S} ; F\right)<2$ and $S\left(W_{\bullet, 0,0}^{\widetilde{S}, F} ; O\right)<1$ for every point $O \in F$. Hence, using (2), we conclude that $\beta(\mathbf{F})>0$. The Main Theorem is proved.
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    Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

