# K-MODULI OF FANO THREEFOLDS IN FAMILY №3.10 

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## Abstract. We find all K-polystable limits of smooth Fano threefolds in family №3.10.

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Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

## 1. Introduction

Let $Q$ be a smooth quadric threefold in $\mathbb{P}^{4}$, let $C_{1}$ and $C_{2}$ be disjoint smooth irreducible conics in the quadric $Q$. Then we can choose coordinates $x, y, z, t, w$ on $\mathbb{P}^{4}$ such that

$$
\begin{aligned}
& C_{1}=\left\{x=0, y=0, w^{2}+z t=0\right\}, \\
& C_{2}=\left\{z=0, t=0, w^{2}+x y=0\right\},
\end{aligned}
$$

and $Q$ is one of the following quadrics:
(】) $Q=\left\{w^{2}+x y+z t=a(x t+y z)+b(x z+y t)\right\},(a, b) \in \mathbb{C}^{2}$ such that $a \pm b \neq \pm 1$,
(I) $Q=\left\{w^{2}+x y+z t=a(x t+y z)+x z\right\}, a \in \mathbb{C}$ such that $a \neq \pm 1$,
(7) $Q=\left\{w^{2}+x y+z t=x t+x z\right\}$.

Let $\pi: X \rightarrow Q$ be the blow up of the quadric threefold $Q$ along the conics $C_{1}$ and $C_{2}$. Then $X$ is a smooth Fano threefold in the deformation family №3.10, and all smooth members of this deformation family can be obtained in this way.

Alternatively, we can describe $X$ as a complete intersection in $\mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4}$ of three smooth divisors of degree $(1,0,1),(0,1,1),(0,0,2)$. Namely, if $Q$ is the quadric ( $\beth)$, then
(】) $\quad X=\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t, w^{2}+x y+z t=a(x t+y z)+b(x z+y t)\right\}$ where $\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right]\right)$ are coordinates on $\mathbb{P}^{1} \times \mathbb{P}^{1}$. If $Q$ is the quadric ( $\left.\mathbb{I}\right)$, then

$$
\begin{equation*}
X=\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t, w^{2}+x y+z t=a(x t+y z)+x z\right\} \tag{I}
\end{equation*}
$$

Finally, if $Q$ is the quadric ( 7 ), then

$$
\begin{equation*}
X=\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t, w^{2}+x y+z t=x t+x z\right\} \tag{7}
\end{equation*}
$$

We know all K-polystable smooth Fano threefolds in the deformation family №3.10. Namely, it has been proved in [4] that $X$ is K-polystable $\Longleftrightarrow Q$ is the quadric ( $\beth$ ).

The goal of this paper is prove the following result：
Main Theorem．All K－polystable smoothable Fano threefolds in the family №3．10 are complete intersections in $\mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4}$ which can be described as follows：

$$
\begin{equation*}
\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t, w^{2}+x y+z t=a(x t+y z)+b(x z+y t)\right\}, \tag{】}
\end{equation*}
$$

（巛）$\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t, w^{2}+r(x+y)^{2}+r(z+t)^{2}=(2 s+2)(x t+y z)+(2 s-2)(x z+y t)\right\}$ ， where $(a, b) \in \mathbb{C}^{2}$ and $[r: s] \in \mathbb{P}^{1}$ ．

K－polystable Fano threefolds in the Main Theorem form an irreducible two－dimensional component of the K－moduli of smoothable Fano threefolds，see Section 6 for the global description of this component．Another two－dimensional component of this moduli space has been described in［5］，and its one－dimensional components are described in［1］．

Let us say few words about the threefolds in the Main Theorem．Let $\mathscr{X}$ be one of them． Then，using the natural projection $\mathscr{X} \rightarrow \mathbb{P}^{4}$ ，we get a birational morphism $\varpi: \mathscr{X} \rightarrow \mathscr{Q}$ ， where $\mathscr{Q}$ is a（possibly singular）irreducible quadric in $\mathbb{P}^{4}$ such that either

$$
\begin{equation*}
\mathscr{Q}=\left\{w^{2}+x y+z t=a(x t+y z)+b(x z+y t)\right\} \tag{】}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathscr{Q}=\left\{w^{2}+r(x+y)^{2}+r(z+t)^{2}=(2 s+2)(x t+y z)+(2 s-2)(x z+y t)\right\} \tag{※}
\end{equation*}
$$

where $(a, b) \in \mathbb{C}^{2}$ and $[r: s] \in \mathbb{P}^{1}$ ．The morphism $\varpi$ is a blow up of two conics：

$$
\begin{aligned}
& \mathscr{C}_{1}=\mathscr{Q} \cap\{x=0, y=0\} \\
& \mathscr{C}_{2}=\mathscr{Q} \cap\{z=0, t=0\}
\end{aligned}
$$

These conics are contained in the smooth locus of the quadric $\mathscr{Q}$ ，and
－ $\mathscr{C}_{1}$ and $\mathscr{C}_{2}$ are smooth if $\mathscr{Q}$ is the quadric（ $\left.\beth\right)$ ，
－ $\mathscr{C}_{1}$ and $\mathscr{C}_{2}$ are reduced and singular if $\mathscr{Q}$ is the quadric（ $\aleph$ ）with $[r: s] \neq[0: 1]$ ，
－ $\mathscr{C}_{1}$ and $\mathscr{C}_{2}$ are non－reduced if $\mathscr{Q}$ is the quadric（ $\aleph$ ）with $[r: s]=[0: 1]$ ．
If $\mathscr{Q}$ is the quadric $(\beth)$ ，then $\mathscr{X}$ is singular $\Longleftrightarrow \mathscr{Q}$ is singular $\Longleftrightarrow a \pm b \pm 1=0$ ． In this case，the singularities of the quadric $\mathscr{Q}$ can be described as follows：
－if $a \pm b \pm 1=0$ and $a b \neq 0$ ，then $\mathscr{Q}$ has one singular point，
－if $a \pm b \pm 1=0$ and $a b=0$ ，then $\mathscr{Q}$ is singular along a line．
Similarly，if $\mathscr{Q}$ is the quadric $(\aleph)$ ，then $\mathscr{X}$ is singular，because both $\mathscr{C}_{1}$ and $\mathscr{C}_{2}$ are singular． In this case， $\mathscr{Q}$ is smooth $\Longleftrightarrow[r: s] \neq[ \pm 1: 1]$ ，and $|\operatorname{Sing}(\mathscr{Q})|=1$ when $[r: s]=[ \pm 1: 1]$ ．

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## 2. First Git quotient (bad moduli space)

In this section we describe a compact moduli space for Fano threefolds in family №3.10, by applying GIT techniques to the description of such threefolds as blow-ups of quadrics along two smooth disjoint conics. As in Section 1, we fix two smooth disjoint conics

$$
\begin{aligned}
& C_{1}=\left\{x=0, y=0, w^{2}+z t=0\right\} \subset \mathbb{P}^{4}, \\
& C_{2}=\left\{z=0, t=0, w^{2}+x y=0\right\} \subset \mathbb{P}^{4},
\end{aligned}
$$

where $[x: y: z: t: w] \in \mathbb{P}^{4}$. Let $Q$ be a quadric threefold in $\mathbb{P}^{4}$ that contains $C_{1}$ and $C_{2}$. Then $Q$ is given by an equation of the form

$$
\begin{equation*}
\alpha\left(x y+z t+w^{2}\right)+\beta x z+\gamma x t+\delta y z+\epsilon y t=0 \tag{2.1}
\end{equation*}
$$

where $[\alpha: \beta: \gamma: \delta: \epsilon] \in \mathbb{P}^{4}$. Let $\pi: X \rightarrow Q$ be the blow up of of the conics $C_{1}$ and $C_{2}$. If $Q$ is smooth, then $X$ is a smooth Fano threefold in the deformation family №3.10.

Note that equations of the form (2.1) are preserved by the action of $\left(\mathbb{C}^{*}\right)^{2}$ given by

$$
[x: y: z: t: w] \mapsto\left[\lambda x: \frac{1}{\lambda} y: \mu z: \frac{1}{\mu} t: w\right]
$$

for $(\lambda, \mu) \in\left(\mathbb{C}^{*}\right)^{2}$. This induces a $\left(\mathbb{C}^{*}\right)^{2}$-action on the parameter space $\mathbb{P}^{4}$ given by

$$
[\alpha: \beta: \gamma: \delta: \epsilon] \mapsto\left[\alpha: \lambda \mu \beta: \frac{\lambda}{\mu} \gamma: \frac{\mu}{\lambda} \delta: \frac{1}{\lambda \mu} \epsilon\right]
$$

for $(\lambda, \mu) \in\left(\mathbb{C}^{*}\right)^{2}$. The GIT quotient by this action is isomorphic to $\mathbb{P}^{2}$ with coordinates

$$
\left[\mathbf{x}_{0}: \mathbf{x}_{1}: \mathbf{x}_{2}\right]=\left[\alpha^{2}: \beta \epsilon: \gamma \delta\right] .
$$

We now classify the unstable, semistable, and stable points, and corresponding quadrics.
2.1. Unstable points. The unstable points $[\alpha: \beta: \gamma: \delta: \epsilon] \in \mathbb{P}^{4}$ and the corresponding quadric threefolds can be described as follows:

$$
\begin{aligned}
& \alpha=\beta=\gamma=0 \text { and } Q \\
&\alpha=\beta=\delta y(\delta z+\epsilon t)=0\}, \\
& \alpha=\gamma=\epsilon=0 \text { and } Q=\{t(\gamma x+\epsilon y)=0\}, \\
& \alpha=\delta=\epsilon=0 \text { and } Q=\{z(\beta x+\delta y)=0\}, \\
&\alpha z+\gamma t)=0\} .
\end{aligned}
$$

In these cases, the quadric $Q$ degenerates to a union of two hyperplanes.
2.2. Stable points. A point $[\alpha: \beta: \gamma: \delta: \epsilon] \in \mathbb{P}^{4}$ is stable if and only if its orbit is closed and its stabiliser is finite. This occurs if and only if all of $\beta, \gamma, \delta$ and $\epsilon$ are nonzero, corresponding to points in the GIT quotient $\mathbb{P}^{2}$ away from the lines $\left\{\mathbf{x}_{1}=0\right\}$ and $\left\{\mathbf{x}_{2}=0\right\}$. We separate the corresponding quadric threefolds into three cases.
(S1) Stable orbits with $\alpha \neq 0$ correspond to points in $\mathbb{P}^{2}$ with all of $\mathbf{x}_{0}, \mathbf{x}_{1}, \mathbf{x}_{2}$ nonzero. Taking the affine chart $\mathbf{x}_{0}=1$, each such orbit contains quadrics of the form

$$
w^{2}+x y+z t+a(x t+y z)+b(x z+y t)=0
$$

for $(a, b) \in\left(\mathbb{C}^{*}\right)^{2}$ given by $a= \pm \sqrt{\mathbf{x}_{2}}$ and $b= \pm \sqrt{\mathbf{x}_{1}}$. The quadric is singular if and only if $a \pm b \pm 1=0$, which corresponds to the affine curve

$$
1-2 \mathbf{x}_{1}-2 \mathbf{x}_{2}+\mathbf{x}_{1}^{2}-2 \mathbf{x}_{1} \mathbf{x}_{2}+\mathbf{x}_{2}^{2}=0
$$

(S2) Orbits with $\alpha=0$ and $\gamma \delta \neq \beta \epsilon$ correspond to points in the quotient $\mathbb{P}^{2}$ that satisfy the following conditions: $\mathbf{x}_{0}=0$ and $\mathbf{x}_{1} \neq \mathbf{x}_{2}$. So, taking the affine chart $\mathbf{x}_{1}=1$, we see that each such orbit contains a quadric of the form

$$
a(x t+y z)+x z+y t=0
$$

for $a \in \mathbb{C}^{*} \backslash\{ \pm 1\}$ given by $a= \pm \sqrt{\mathbf{x}_{2}}$. Such quadrics have one singular point.
(S3) There is a unique stable orbit with $\alpha=0$ and $\gamma \delta=\beta \epsilon \neq 0$, corresponding to the point $\left[\mathbf{x}_{0}: \mathbf{x}_{1}: \mathbf{x}_{2}\right]=[0: 1: 1]$ in the quotient. It contains the threefold

$$
(x+y)(z+t)=0,
$$

which is a union of two hyperplanes.
2.3. Strictly semistable points. A point $[\alpha: \beta: \gamma: \delta: \epsilon]$ is strictly semistable if and only if it is not unstable, and at least one of $\beta, \gamma, \delta, \epsilon$ is zero. Such points have non-closed orbits or infinite stabilisers.

Example 2.2. If $\beta \neq 0, \gamma \neq 0$ and $\delta \neq 0$, then the orbit of $[\alpha: \beta: \gamma: \delta: 0]$ contains the point $[\alpha: 0: \gamma: \delta: 0]$ in its closure, and the orbit of $[\alpha: 0: \gamma: \delta: 0]$ is closed but has infinite stabiliser given by $(\lambda, \lambda) \in\left(\mathbb{C}^{*}\right)^{2}$.

The strictly semistable points belonging to minimal orbits (strictly polystable points) are described below, along with their corresponding quadric threefolds.
(SS1) All points of the form $[\alpha: 0: \gamma: \delta: 0]$ or $[\alpha: \beta: 0: 0: \epsilon]$, where $\alpha \beta \gamma \delta \epsilon \neq 0$, and either $\alpha^{2} \neq \gamma \delta$ or $\alpha^{2} \neq \beta \epsilon$, respectively. Such orbits lie over points in the quotient

$$
\begin{aligned}
& \left\{\left[\mathbf{x}_{0}: 0: \mathbf{x}_{2}\right] \mid \mathbf{x}_{0}, \mathbf{x}_{2} \in \mathbb{C}^{*}, \mathbf{x}_{0} \neq \mathbf{x}_{2}\right\} \\
& \left\{\left[\mathbf{x}_{0}: \mathbf{x}_{1}: 0\right] \mid \mathbf{x}_{0}, \mathbf{x}_{1} \in \mathbb{C}^{*}, \mathbf{x}_{0} \neq \mathbf{x}_{1}\right\}
\end{aligned}
$$

The corresponding quadrics are

$$
\begin{aligned}
& \left\{w^{2}+x y+z t+a(x t+y z)=0\right\} \\
& \left\{w^{2}+x y+z t+b(x z+y t)=0\right\}
\end{aligned}
$$

where $a, b \in \mathbb{C}^{*} \backslash\{ \pm 1\}$. They are smooth.
(SS2) The orbit of the point $[1: 0: 0: 0: 0]$. It gives the point $[1: 0: 0]$ in the quotient. The corresponding quadric threefold is $\left\{w^{2}+x y+z t=0\right\}$ - it is smooth.
(SS3) Two orbits of all points $[0: 0: \gamma: \delta: 0]$ and $[0: \beta: 0: 0: \epsilon]$ such that $\beta \gamma \delta \epsilon \neq 0$, which lie over the points $[0: 0: 1]$ and $[0: 1: 0]$ in the quotient $\mathbb{P}^{2}$, respectively. This gives two singular quadrics: $\{x t+y z=0\}$ and $\{x z+y t=0\}$.
(SS4) Two orbits consisting of points of the form $[\alpha: 0: \gamma: \delta: 0]$ or $[\alpha: \beta: 0: 0: \epsilon]$, where $\alpha \beta \gamma \delta \epsilon \neq 0$, and either $\alpha^{2}=\gamma \delta$ or $\alpha^{2}=\beta \epsilon$, respectively. These orbits lie over the points $[1: 0: 1]$ and $[1: 1: 0]$ in the quotient, and the quadrics are

$$
\begin{aligned}
& w^{2}+x y+z t+x t+y z=0 \\
& w^{2}+x y+z t+x z+y t=0
\end{aligned}
$$

Both of them are irreducible and singular along the line $\{w=y-t=x-z=0\}$.
2.4. Quotient space. Note that the space of quadrics of the form (2.1) admits a discrete action of the group $\boldsymbol{\mu}_{2}^{3}$ given by

$$
\begin{aligned}
& \sigma_{1}:[x: y: z: t: w] \mapsto[y: x: z: t: w], \\
& \sigma_{2}:[x: y: z: t: w] \mapsto[z: t: x: y: w], \\
& \sigma_{3}:[x: y: z: t: w] \mapsto[x: y: z: t:-w] .
\end{aligned}
$$

These involutions act on the parameter space $\mathbb{P}^{4}$ by

$$
\begin{aligned}
& \sigma_{1}:[\alpha: \beta: \gamma: \delta: \epsilon] \mapsto[\alpha: \delta: \epsilon: \beta: \gamma], \\
& \sigma_{2}:[\alpha: \beta: \gamma: \delta: \epsilon] \mapsto[\alpha: \beta: \delta: \gamma: \epsilon], \\
& \sigma_{3}:[\alpha: \beta: \gamma: \delta: \epsilon] \mapsto[\alpha: \beta: \gamma: \delta: \epsilon] .
\end{aligned}
$$

The involutions $\sigma_{2}$ and $\sigma_{3}$ act trivially on the GIT quotient, whilst $\sigma_{1}$ acts by

$$
\sigma_{1}:\left[\mathbf{x}_{0}: \mathbf{x}_{1}: \mathbf{x}_{2}\right] \mapsto\left[\mathbf{x}_{0}: \mathbf{x}_{2}: \mathbf{x}_{1}\right] .
$$

Therefore, to get a moduli space from our GIT quotient, we must further quotient it by the involution $\sigma_{1}$ to get a copy of the weighted projective space $\mathbb{P}(1,1,2)$ with coordinates

$$
[\xi: \eta: \zeta]=\left[\mathbf{x}_{0}: \mathbf{x}_{1}+\mathbf{x}_{2}:\left(\mathbf{x}_{1}-\mathbf{x}_{2}\right)^{2}\right]=\left[\alpha^{2}: \beta \epsilon+\gamma \delta:(\beta \epsilon-\gamma \delta)^{2}\right] .
$$

Over this quotient it is easy to describe where the stable and semistable threefolds occur. Namely, the stable threefolds occur where $\eta^{2} \neq \zeta$ as follows:
(S1) the locus with $\xi \neq 0$ and $\eta^{2} \neq \zeta$, and singular quadrics lie over $\xi^{2}-\xi \eta+\zeta=0$,
(S2) the locus $\left\{[0: \eta: \zeta] \mid \eta^{2} \neq \zeta\right.$ and $\left.\zeta \neq 0\right\}$,
(S3) the point $[\xi: \eta: \zeta]=[0: 1: 0]$.
Likewise, the strictly semistable quadrics occur along the curve $\eta^{2}=\zeta$ as follows:
(SS1) the locus where $\eta^{2}=\zeta$ and $[\xi: \eta: \zeta] \notin\{[1: 0: 0],[0: 1: 1],[1: 1: 1]\}$,
(SS2) the point $[\xi: \eta: \zeta]=[1: 0: 0]$,
(SS3) the point $[\xi: \eta: \zeta]=[0: 1: 1]$,
(SS4) the point $[\xi: \eta: \zeta]=[1: 1: 1]$.
Unfortunately, the constructed moduli space is not the moduli space of K-polystable threefolds we are looking for, because the threefold $X$ obtained by blowing up the quadric in the class (S3) is reducible. Moreover, all threefolds obtained by blowing up quadrics in the class (S2) are isomorphic, and the quadrics in the class (S2) give K-unstable threefolds:
Lemma 2.3. Let $Q$ be the quadric $\{a(x t+y z)+x z+y t=0\} \subset \mathbb{P}^{4}$ for $a \in \mathbb{C} \backslash\{0, \pm 1\}$, and let $\pi: X \rightarrow Q$ be the blow up of the quadric $Q$ along $C_{1}$ and $C_{2}$. Then $X$ is K-unstable.
Proof. Observe that both $C_{1}$ and $C_{2}$ do not contain the singular point of the quadric $Q$, the quadric threefold $Q$ is a cone over $\mathbb{P}^{1} \times \mathbb{P}^{1}$ with vertex at the point $[0: 0: 0: 0: 1]$, and there exists the following commutative diagram:

where $\phi$ and $\varphi$ are blow ups of the singular points $\operatorname{Sing}(Q)$ and $\operatorname{Sing}(X), v$ is a $\mathbb{P}^{1}$-bundle, $\varpi$ is the blow up of the preimages of the conics $C_{1}$ and $C_{2}$, and $\nu$ is a conic bundle.

Let $E_{1}$ and $E_{2}$ be proper transforms on the threefold $\widetilde{X}$ of the $\pi$-exceptional surfaces such that $\pi \circ \phi\left(E_{1}\right)=C_{1}$ and $\pi \circ \phi\left(E_{2}\right)=C_{2}$. Then $v \circ \varpi\left(E_{1}\right)$ and $v \circ \varpi\left(E_{2}\right)$ are disjoint rulings of the surface $\mathbb{P}^{1} \times \mathbb{P}^{1}$, because planes spanned by $C_{1}$ and $C_{2}$ are contained in $Q$. Therefore, we may assume that both $v \circ \varpi\left(E_{1}\right)$ and $v \circ \varpi\left(E_{2}\right)$ are divisors of degree $(1,0)$. Let $H_{1}$ and $H_{2}$ be the pull back on $\widetilde{X}$ of the divisors on $\mathbb{P}^{1} \times \mathbb{P}^{1}$ of degree $(1,0)$ and $(0,1)$, let $S_{1}$ and $S_{2}$ be proper transforms on $\widetilde{X}$ of the planes in $Q$ that contain $C_{1}$ and $C_{2}$, and let $F$ be the $\varphi$-exceptional surface. Then $H_{1} \sim S_{1}+E_{1}$ and $H_{1} \sim S_{2}+E_{2}$, so that

$$
\varphi^{*}\left(-K_{X}\right) \sim 3\left(H_{1}+H_{2}\right)+3 F-E_{1}-E_{2} \sim 3\left(S_{1}+S_{2}\right)+3 F+2\left(E_{1}+E_{2}\right)
$$

Now, take $u \in \mathbb{R}_{\geqslant 0}$. Then
$\varphi^{*}\left(-K_{X}\right)-u F \sim_{\mathbb{R}} 3\left(H_{1}+H_{2}\right)+(3-u) F-E_{1}-E_{2} \sim_{\mathbb{R}} 3\left(S_{1}+S_{2}\right)+(3-u) F+2\left(E_{1}+E_{2}\right)$,
so this divisor is pseudoeffective $\Longleftrightarrow u \leqslant 3$. Moreover, this divisor is nef $\Longleftrightarrow u \leqslant 1$. Furthermore, if $u \in(1,3)$, then its Zariski decomposition can be described as follows:

$$
\varphi^{*}\left(-K_{X}\right)-u F \sim_{\mathbb{R}} \underbrace{\varphi^{*}\left(-K_{X}\right)-u F-\frac{u-1}{2}\left(S_{1}+S_{2}\right)}_{\text {positive part }}+\underbrace{\frac{u-1}{2}\left(S_{1}+S_{2}\right)}_{\text {negative part }} .
$$

Thus, we see that

$$
\begin{aligned}
& \beta(F)=2-\frac{1}{26} \int_{0}^{1}\left(3\left(H_{1}+H_{2}\right)+(3-u) F-E_{1}-E_{2}\right)^{3} d u- \\
& \quad-\frac{1}{26} \int_{2}^{3}\left(3\left(H_{1}+H_{2}\right)+(3-u) F-E_{1}-E_{2}-\frac{u-1}{2}\left(2 H_{1}-E_{1}-E_{2}\right)\right)^{3} d u .
\end{aligned}
$$

To compute these integrals, observe that

$$
\begin{array}{rlrrrr}
H_{1}^{2}=0, & H_{2}^{2}=0, & E_{1} \cdot E_{2}=0, & E_{1} \cdot F=0, & E_{2} \cdot F=0, \\
F^{3}=2, & H_{1} \cdot F^{2}=-1, & H_{2} \cdot F^{2}=-1, & E_{1}^{3}=-4, & E_{2}^{3}=-4, \\
F \cdot H_{1} \cdot H_{2}=1, & H_{1} \cdot E_{1}=0, & H_{1} \cdot E_{2}=0, & H_{2} \cdot E_{1}^{2}=-2, & H_{2} \cdot E_{2}^{2}=-2 .
\end{array}
$$

This gives $\beta(F)=-\frac{3}{52}<0$, which implies that $X$ is K-unstable [6, 7].

## 3. Second GIT quotient (Good moduli space)

In this section, we construct another compact GIT moduli space for Fano threefolds in the deformation family № 3.10. Recall that all K-polystable smooth Fano threefolds in this family are contained in class ( $\beth$ ), and the quadric $Q$ in this case is invariant under the action of $\boldsymbol{\mu}_{2}^{3}$ generated by the following three involutions:

$$
\begin{aligned}
& \tau_{1}:[x: y: z: t: w] \mapsto[y: x: t: z: w], \\
& \tau_{2}:[x: y: z: t: w] \mapsto[z: t: x: y: w], \\
& \tau_{3}:[x: y: z: t: w] \mapsto[x: y: z: t:-w] .
\end{aligned}
$$

In light of this, we take $Q$ to be a quadric invariant under this $\boldsymbol{\mu}_{2}^{3}$-action:

$$
\begin{equation*}
\alpha w^{2}+\frac{\beta}{2}\left(x^{2}+y^{2}+z^{2}+t^{2}\right)+\gamma(x t+y z)+\delta(x z+y t)+\epsilon(x y+z t)=0 \tag{3.1}
\end{equation*}
$$

where $[\alpha: \beta: \gamma: \delta: \epsilon] \in \mathbb{P}^{4}$. As before, we take $C_{1}$ and $C_{2}$ to be the two conics given by the intersection with the planes $\{x=y=0\}$ and $\{z=t=0\}$, namely

$$
\begin{aligned}
& C_{1}=\left\{x=0, y=0, \alpha w^{2}+\frac{\beta}{2}\left(z^{2}+t^{2}\right)+\epsilon z t=0\right\} \\
& C_{2}=\left\{z=0, t=0, \alpha w^{2}+\frac{\beta}{2}\left(x^{2}+y^{2}\right)+\epsilon x y=0\right\}
\end{aligned}
$$

Unlike the construction from Section 2, this general form allows $C_{1}$ and $C_{2}$ to degenerate.
A simple computer calculation shows that equations of the form (3.1) are preserved by a subgroup of $\operatorname{PGL}(5, \mathbb{C})$ isomorphic to $\left(\mathbb{C}^{*}\right)^{2}$, given explicitly by matrices of the form

$$
\left(\begin{array}{ccccc}
\lambda+\mu & \lambda-\mu & 0 & 0 & 0 \\
\lambda-\mu & \lambda+\mu & 0 & 0 & 0 \\
0 & 0 & \lambda-\mu & \lambda+\mu & 0 \\
0 & 0 & \lambda+\mu & \lambda-\mu & 0 \\
0 & 0 & 0 & 0 & -\frac{2}{\lambda^{2} \mu^{2}}
\end{array}\right)
$$

acting on the coordinates $[x: y: z: t: w]$, where $(\lambda, \mu) \in\left(\mathbb{C}^{*}\right)^{2}$ and we have chosen our matrix representative to have determinant $2^{5}=32$ (to avoid having lots of factors of $\frac{1}{2}$ ). This induces a $\left(\mathbb{C}^{*}\right)^{2}$-action on the parameter space $\mathbb{P}^{4}$ given by

$$
\left(\begin{array}{ccccc}
\frac{4}{\lambda^{4} \mu^{4}} & 0 & 0 & 0 & \\
0 & 2\left(\lambda^{2}+\mu^{2}\right) & 0 & 0 & 2\left(\lambda^{2}-\mu^{2}\right) \\
0 & 0 & 2\left(\lambda^{2}-\mu^{2}\right) & 2\left(\lambda^{2}+\mu^{2}\right) & 0 \\
0 & 0 & 2\left(\lambda^{2}+\mu^{2}\right) & 2\left(\lambda^{2}-\mu^{2}\right) & 0 \\
0 & 2\left(\lambda^{2}-\mu^{2}\right) & 0 & 0 & 2\left(\lambda^{2}+\mu^{2}\right)
\end{array}\right)
$$

acting on the coordinates $[\alpha: \beta: \gamma: \delta: \epsilon]$, where $(\lambda, \mu) \in\left(\mathbb{C}^{*}\right)^{2}$.
The ring of invariants for this action is generated by polynomials of the form

$$
\alpha f_{1} f_{2} g_{1} g_{2}
$$

with $f_{i} \in\{(\beta+\epsilon),(\gamma+\delta)\}$ and $g_{i} \in\{(\beta-\epsilon),(\gamma-\delta)\}$. There are nine such polynomials, corresponding to coordinates on $\mathbb{P}^{8}$. One can show that the GIT quotient of the parameter space is isomorphic to $\mathbb{P}^{1} \times \mathbb{P}^{1} \subset \mathbb{P}^{8}$, embedded by the Segre embedding of bidegree $(2,2)$. The isomorphism with $\mathbb{P}^{1} \times \mathbb{P}^{1}$ is given explicitly by

$$
\left(\left[\mathbf{x}_{0}: \mathbf{x}_{1}\right],\left[\mathbf{y}_{0}: \mathbf{y}_{1}\right]\right)=([\beta+\epsilon: \gamma+\delta],[\beta-\epsilon: \gamma-\delta]) .
$$

We now classify the unstable and stable points, and corresponding quadrics. There are no strictly semistable points under this action.
3.1. Unstable points. The unstable points $[\alpha: \beta: \gamma: \delta: \epsilon] \in \mathbb{P}^{4}$ come in two types. The first are the points with $\alpha=0$, which have corresponding quadrics

$$
Q=\left\{\frac{\beta}{2}\left(x^{2}+y^{2}+z^{2}+t^{2}\right)+\gamma(x t+y z)+\delta(x z+y t)+\epsilon(x y+t z)=0\right\} .
$$

In this case, the quadric $Q$ degenerates to a cone over a quadric surface. The second type are those with either

$$
\begin{aligned}
& \beta-\epsilon=\gamma-\delta=0 \text { and } Q=\left\{\alpha w^{2}+\frac{\beta}{2}\left((x+y)^{2}+(z+t)^{2}\right)+\gamma(x+y)(z+t)=0\right\} \\
& \beta+\epsilon=\gamma+\delta=0 \text { and } Q=\left\{\alpha w^{2}+\frac{\beta}{2}\left((x-y)^{2}+(z-t)^{2}\right)+\gamma(x-y)(z-t)=0\right\}
\end{aligned}
$$

In these cases $Q$ is singular along a line and both of the conics $C_{1}$ and $C_{2}$ become reducible.
3.2. Stable points. The stable points are those that are not unstable - there are no strictly semistable points. We separate the corresponding quadric threefolds into 2 cases.
(S1) (Class ( $\beth$ )) Stable orbits with $\alpha \neq 0$ and $\beta \neq \pm \epsilon$ correspond to points with

$$
\mathbf{x}_{0} \mathbf{y}_{0} \neq 0
$$

in the quotient space $\mathbb{P}^{1} \times \mathbb{P}^{1}$. Taking the affine chart with $\mathbf{x}_{0}=1$ and $\mathbf{y}_{0}=-1$, we see that each such orbit contains quadrics of the form

$$
w^{2}+x y+z t+a(x t+y z)+b(x z+y t)=0
$$

for $a, b \in \mathbb{C}^{*}$ given by $a=\frac{1}{2}\left(\mathbf{x}_{1}+\mathbf{y}_{1}\right)$ and $b=\frac{1}{2}\left(\mathbf{x}_{1}-\mathbf{y}_{1}\right)$. This quadric is singular if and only if $a \pm b \pm 1=0$, corresponding to $\mathbf{x}_{1}= \pm 1$ or $\mathbf{y}_{1}= \pm 1$.
(S2) (Class ( $\aleph$ )) Stable orbits with $\alpha \neq 0$ and $\beta=-\epsilon$ or $\beta=\epsilon$, which correspond to points in the quotient $\mathbb{P}^{1} \times \mathbb{P}^{1}$ with $\mathbf{x}_{0}=0$ or $\mathbf{y}_{0}=0$, respectively. Taking the affine chart $\mathbf{x}_{1}=1$ or $\mathbf{y}_{1}=1$, respectively, we see that such orbits contain the quadric

$$
w^{2}+r(x-y)^{2}+r(z-t)^{2}+(2 s+2)(x t+y z)+(2-2 s)(x z+y t)=0
$$

or

$$
w^{2}+r(x+y)^{2}+r(z+t)^{2}+(2 s+2)(x t+y z)+(2 s-2)(x z+y t)=0
$$

for $[r: s] \in \mathbb{P}^{1}$ given by $[r: s]=\left[2 \mathbf{y}_{0}: 2 \mathbf{y}_{1}\right]$ or $[r: s]=\left[2 \mathbf{x}_{0}: 2 \mathbf{x}_{1}\right]$, respectively. Generically this defines a smooth quadric where the conics $C_{1}$ and $C_{2}$ are reduced and singular (pairs of lines). The exceptions are

- $[r: s]=[ \pm 1: 1]$, when the quadric acquires an isolated singularity,
- $[r: s]=[0: 1]$, when $C_{1}$ and $C_{2}$ become non-reduced (double lines).

In Sections 4 and 5, we will show that all Fano threefolds $X$ obtained as blow-ups of quadrics $Q$ from classes ( $\beth$ ) and ( $\aleph$ ) above are K-polystable.
3.3. Quotient space. Observe that the space of quadric threefolds of the form (3.1) admits an additional $\boldsymbol{\mu}_{2}$-action given by

$$
[x: y: z: t: w] \mapsto[x:-y:-z: t: w]
$$

which acts on the parameter space $\mathbb{P}^{4}$ as

$$
[\alpha: \beta: \gamma: \delta: \epsilon] \mapsto[\alpha: \beta: \gamma:-\delta:-\epsilon] .
$$

This involution induces an involution on the GIT quotient given by

$$
\left(\left[\mathbf{x}_{0}: \mathbf{x}_{1}\right],\left[\mathbf{y}_{0}: \mathbf{y}_{1}\right]\right) \mapsto\left(\left[\mathbf{y}_{0}: \mathbf{y}_{1}\right],\left[\mathbf{x}_{0}: \mathbf{x}_{1}\right]\right) .
$$

So, to get a moduli space from our GIT quotient, we must further quotient it by this involution. The result is a copy of $\mathbb{P}^{2}$ with coordinates

$$
[\xi: \eta: \zeta]=\left[\mathbf{x}_{0} \mathbf{y}_{0}: \mathbf{x}_{1} \mathbf{y}_{1}: \mathbf{x}_{0} \mathbf{y}_{1}+\mathbf{x}_{1} \mathbf{y}_{0}\right]=\left[\beta^{2}-\epsilon^{2}: \gamma^{2}-\delta^{2}: 2 \beta \gamma-2 \epsilon \delta\right]
$$

Over this quotient it is easy to describe where each of the types of stable threefolds occur:
(S1) the locus with $\xi \neq 0$, singular quadrics lie over the lines $\xi+\eta= \pm \zeta$.
(S2) the locus with $\xi=0$, singular quadrics lie over the points $[0: 1: 1]$ and $[0:-1: 1]$, and the conics $C_{1}$ and $C_{2}$ become non-reduced at the point $[0: 1: 0]$.
Note that this quotient identifies the two normal forms in class ( $\aleph$ ).

## 4. Threefolds in the class ( $\beth$ )

Let $X$ be the complete intersection

$$
\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t, w^{2}+x y+z t=a(x t+y z)+b(x z+y t)\right\} \subset \mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4}
$$

where $(a, b) \in \mathbb{C}^{2}$, and $\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right)$ are coordinates on $\mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4}$. Then $X$ is singular $\Longleftrightarrow a \pm b \pm 1=0$. If $a \pm b \pm 1=0$ and $a b \neq 0$, then

$$
\operatorname{Sing}(X)=\left\{\begin{array}{l}
([1: 1],[1: 1],[1: 1:-1:-1: 0]) \text { if } a+b+1=0 \\
([1: 1],[1: 1],[1: 1: 1: 1: 0]) \text { if } a+b-1=0 \\
([-1: 1],[-1: 1],[1:-1: 1:-1: 0]) \text { if } a-b+1=0 \\
([-1: 1],[-1: 1],[1:-1:-1: 1: 0]) \text { if } a-b-1=0
\end{array}\right.
$$

If $a \pm b \pm 1=0$ and $a b=0$, then $X$ is singular along the curve

$$
\left\{w=0, x=z, y=t, u_{1} z=v_{1} t, u_{2} z=v_{2} t, v_{1} u_{2}=u_{1} v_{2}\right\} .
$$

Let $Q$ be the quadric threefold in $\mathbb{P}^{4}$ given by $w^{2}+x y+z t=a(x t+y z)+b(x z+y t)$, and let $\pi: X \rightarrow Q$ be the morphism that is induced by the natural projection $X \rightarrow \mathbb{P}^{4}$. Then $\pi$ is a blow up of the following two smooth conics:

$$
\begin{aligned}
& C_{1}=\left\{w^{2}+z t=0, x=0, y=0\right\} \\
& C_{2}=\left\{w^{2}+x y=0, z=0, t=0\right\}
\end{aligned}
$$

The quadric $Q$ is smooth $\Longleftrightarrow a \pm b \pm 1=0$. If $a \pm b \pm 1=0$ and $a b \neq 0$, then

$$
\operatorname{Sing}(Q)=\left\{\begin{array}{l}
{[1: 1:-1:-1: 0] \text { if } a+b+1=0} \\
{[1: 1: 1: 1: 0] \text { if } a+b-1=0} \\
{[1:-1: 1:-1: 0] \text { if } a-b+1=0} \\
{[1:-1:-1: 1: 0] \text { if } a-b-1=0}
\end{array}\right.
$$

If $a \pm b \pm 1=0$ and $a b \neq 0$, then $Q$ is singular along the line $\{w=0, x=z, y=t\}$.
The threefold $X$ admits an action of the group $\boldsymbol{\mu}_{2}^{3}$ given by

$$
\begin{aligned}
& \tau_{1}:\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right) \mapsto\left(\left[v_{1}: u_{1}\right],\left[v_{2}: u_{2}\right],[y: x: t: z: w]\right), \\
& \tau_{2}:\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right) \mapsto\left(\left[u_{2}: v_{2}\right],\left[u_{1}: v_{1}\right],[z: t: x: y: w]\right), \\
& \tau_{3}:\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right) \mapsto\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t:-w]\right) .
\end{aligned}
$$

Let $G$ be the subgroup in $\operatorname{Aut}(X)$ generated by the involutions $\tau_{1}, \tau_{2}, \tau_{3}$. Then

$$
G \cong \boldsymbol{\mu}_{2}^{3}
$$

the blow up $\pi: X \rightarrow Q$ is $G$-equivariant, and the $G$-action on $Q$ induces a $G$-action on the projective space $\mathbb{P}^{4}$. Therefore, we can also consider $G$ as a subgroup in $\mathrm{PGL}_{4}(\mathbb{C})$, where $\tau_{1}, \tau_{2}, \tau_{3}$ act on $\mathbb{P}^{4}$ as in Section 3;

$$
\begin{aligned}
& \tau_{1}:[x: y: z: t: w] \mapsto[y: x: t: z: w], \\
& \tau_{2}:[x: y: z: t: w] \mapsto[z: t: x: y: w], \\
& \tau_{3}:[x: y: z: t: w] \mapsto[x: y: z: t:-w] .
\end{aligned}
$$

Observe that the only $G$-fixed points in $\mathbb{P}^{4}$ are

$$
[1: 1:-1:-1: 0],[1: 1: 1: 1: 0],[1:-1: 1:-1: 0],[1:-1:-1: 1: 0] .
$$

Lemma 4.1. The following two assertions hold:
(1) if $X$ is smooth, then $X$ does not have $G$-fixed points;
(2) if $X$ has one singular point, then $\operatorname{Sing}(X)$ is the only $G$-fixed point in $X$.

Proof. If $Q$ is smooth, it does not contain any $G$-fixed points. Vice versa, if $Q$ is singular, then its singular point is the only $G$-fixed point in $Q$. The claim follows.

We will also need the following technical lemma:
Lemma 4.2. Suppose that $a \pm b \pm 1=0$ and $a b \neq 0$. Let $O$ be the singular point $\operatorname{Sing}(X)$, let $\varphi: \widetilde{X} \rightarrow X$ be the blow up of the point $O$, and let $F$ be the $\varphi$-exceptional surface. Then
(1) $\varphi$ is $G$-equivariant,
(2) the group $G$ acts faithfully on $F$,
(3) $\operatorname{rkPic}^{G}(F)=1$,
(4) $F$ has no $G$-fixed points.

Proof. We only consider the case when $b=1-a$ and $a \notin\{0,1\}$; the other cases are similar. Then

$$
Q=\left\{w^{2}+x y+z t-a(x t+y z)-(1-a)(x z+y t)=0\right\}
$$

and $\pi(O)=[1: 1: 1: 1: 0]$. Let us introduce new coordinates on $\mathbb{P}^{4}$ as follows:

$$
\left\{\begin{array}{l}
\mathbf{x}=x-y \\
\mathbf{y}=x+y-z-t \\
\mathbf{z}=z-t \\
\mathbf{t}=x+y+z+t \\
\mathbf{w}=w
\end{array}\right.
$$

In these coordinates, $Q=\left\{\mathbf{x}^{2}-\mathbf{y}^{2}+\mathbf{z}^{2}=(4 a-2) \mathbf{x z}+4 \mathbf{w}^{2}\right\}$ and $\pi(O)=[0: 0: 0: 1: 0]$. Moreover, the involutions $\tau_{1}, \tau_{2}, \tau_{3}$ acts as follows:

$$
\begin{aligned}
& \tau_{1}:[\mathbf{x}: \mathbf{y}: \mathbf{z}: \mathbf{t}: \mathbf{w}] \mapsto[-\mathbf{x}: \mathbf{y}:-\mathbf{z}: \mathbf{t}: \mathbf{w}] \\
& \tau_{2}:[\mathbf{x}: \mathbf{y}: \mathbf{z}: \mathbf{t}: \mathbf{w}] \mapsto[\mathbf{z}:-\mathbf{y}: \mathbf{x}: \mathbf{t}: \mathbf{w}] \\
& \tau_{3}:[\mathbf{x}: \mathbf{y}: \mathbf{z}: \mathbf{t}: \mathbf{w}] \mapsto[\mathbf{x}: \mathbf{y}: \mathbf{z}: \mathbf{t}:-\mathbf{w}]
\end{aligned}
$$

Furthermore, we can $G$-equivariantly identify $F$ with the projectivization of the tangent cone to $Q$ at the point $\pi(O)$. So, we can consider $F$ as the quadric surface in $\mathbb{P}^{3}$ given by

$$
\mathbf{x}^{2}-\mathbf{y}^{2}+\mathbf{z}^{2}=(4 a-2) \mathbf{x} \mathbf{z}+4 \mathbf{w}^{2}
$$

where now we consider $[\mathbf{x}: \mathbf{y}: \mathbf{z}: \mathbf{w}]$ as coordinates on $\mathbb{P}^{3}$. Now, all assertions are easy to check looking on how the group $G$ acts on the surface $F$.

We have the following $G$-equivariant commutative diagram

where $\eta$ is a conic bundle given by the natural projection $X \rightarrow \mathbb{P}^{1} \times \mathbb{P}^{1}$, and $\rho$ is a rational map that is given by $[x: y: z: t: w] \mapsto([x: y],[z: t])$.
Remark 4.3. The only $G$-fixed points in $\mathbb{P}^{1} \times \mathbb{P}^{1}$ are $([1: 1],[1: 1])$ and $([-1: 1],[-1: 1])$.

Let $\Delta$ be the discriminant curve of the conic bundle $\eta$. Then $\Delta$ is given by

$$
\begin{aligned}
& a^{2} u_{1}^{2} v_{2}^{2}+2 a b u_{1}^{2} u_{2} v_{2}+b^{2} u_{1}^{2} u_{2}^{2}+\left(2 a^{2}+2 b^{2}-4\right) v_{1} v_{2} u_{1} u_{2}+ \\
& \quad+2 a b u_{1} v_{1} u_{2}^{2}+2 a b u_{1} v_{1} v_{2}^{2}+a^{2} u_{2}^{2} v_{1}^{2}+2 a b u_{2} v_{1}^{2} v_{2}+b^{2} v_{1}^{2} v_{2}^{2}=0
\end{aligned}
$$

where we consider $\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right]\right)$ as coordinates on $\mathbb{P}^{1} \times \mathbb{P}^{1}$.
Remark 4.4. If $a \pm b \pm 1 \neq 0$, then $\Delta$ is smooth. If $a \pm b \pm 1=0$ and $a b \neq 0$, then

$$
\operatorname{Sing}(\Delta)=\left\{\begin{array}{l}
([1: 1],[1: 1]) \text { if } a+b \pm 1=0 \\
([-1: 1],[-1: 1]) \text { if } a-b \pm 1=0
\end{array}\right.
$$

Let $H$ be a proper transform on $X$ of a general hyperplane section of the quadric $Q$, and let $E_{1}$ and $E_{2}$ be the $\pi$-exceptional divisors such that $\pi\left(E_{1}\right)=C_{1}$ and $\pi\left(E_{2}\right)=C_{2}$.

Lemma 4.5. Suppose that $X$ has at most isolated singularities. Then

$$
\operatorname{Pic}^{G}(X)=\mathrm{Cl}^{G}(X)=\mathbb{Z}[H] \oplus \mathbb{Z}\left[E_{1}+E_{2}\right]
$$

Proof. If the threefold $X$ is smooth, the assertion is obvious, since $G$ swaps $E_{1}$ and $E_{2}$. If $X$ has one singular point, the assertion follows from Lemma 4.2,

Now, let us describe the cone of $G$-invariant effective divisors on $X$.
Lemma 4.6. Suppose that $X$ has at most isolated singularities. Let $S$ be an $G$-invariant surface in $X$. Then $S \sim n_{1}\left(E_{1}+E_{2}\right)+n_{2}\left(2 H-E_{1}-E_{2}\right)+n_{3} H$ for some $n_{1}, n_{2}, n_{3} \in \mathbb{Z}_{\geqslant 0}$.
Proof. By Lemma 4.5, we see that $S \sim k_{1} H+k_{2}\left(E_{1}+E_{2}\right)$ for some integers $k_{1}$ and $k_{2}$. On the other hand, the conic bundle $\eta: X \rightarrow \mathbb{P}^{1} \times \mathbb{P}^{1}$ is given by $\left|2 H-E_{1}-E_{2}\right|$. Then

$$
S \sim_{\mathbb{Q}} m_{1}\left(E_{1}+E_{2}\right)+m_{2}\left(2 H-E_{1}-E_{2}\right)
$$

for some non-negative rational numbers $m_{1}$ and $m_{2}$. We have $k_{1}=2 m_{2}$ and $m_{1}-m_{2}=k_{2}$. If $k_{1}$ is even, then $m_{1}$ and $m_{2}$ are integers and we are done, since $\operatorname{Pic}(X)$ has no torsion. So, we may assume that $k_{1}=2 n+1$, where $n \in \mathbb{Z}$. Then $m_{2}=n+\frac{1}{2}$ and $m_{1}=k_{2}+n+\frac{1}{2}$, which gives $n \geqslant 0$ and $k_{2}+n \geqslant 0$, since $m_{1} \geqslant 0$ and $m_{2} \geqslant 0$. Hence, we have

$$
S \sim_{\mathbb{Q}}\left(k_{2}+n\right)\left(E_{1}+E_{2}\right)+n\left(2 H-E_{1}-E_{2}\right)+H,
$$

which gives $S \sim\left(k_{2}+n\right)\left(E_{1}+E_{2}\right)+n\left(2 H-E_{1}-E_{2}\right)+H$, since $\operatorname{Pic}(X)$ has no torsion.
Corollary 4.7. Suppose that $X$ has at most isolated singularities. Let $S$ be a $G$-invariant irreducible surface in $X$. Then $\beta(S)>0$.
Proof. We have $\beta(S)=1-S_{X}(S)$, and it follows from Lemma 4.6 that

$$
S_{X}(S)=\frac{1}{26} \int_{0}^{\infty} \operatorname{vol}\left(-K_{X}-u S\right) d u \leqslant \frac{1}{26} \int_{0}^{\infty} \operatorname{vol}\left(-K_{X}-u F\right) d u
$$

where $F$ is one divisor among $H, E_{1}+E_{2}, 2 H-E_{1}-E_{2}$. On the other hand, we have

$$
\frac{1}{26} \int_{0}^{\infty} \operatorname{vol}\left(-K_{X}-u H\right) d u=\frac{1}{26} \int_{0}^{1}\left(-K_{X}-u H\right)^{3} d u=\frac{1}{26} \int_{0}^{1} 2(1-u)\left(u^{2}-8 u+13\right) d u=\frac{21}{52}
$$

Similarly, we have

$$
\frac{1}{26} \int_{0}^{\infty} \operatorname{vol}\left(-K_{X}-u\left(E_{1}+E_{2}\right)\right) d u=\frac{1}{26} \int_{0}^{\frac{1}{2}} 2(2 u-1)\left(2 u^{2}-2 u-13\right) d u=\frac{53}{208}
$$

Finally, we compute

$$
\begin{aligned}
& \frac{1}{26} \int_{0}^{\infty} \operatorname{vol}\left(-K_{X}-u\left(2 H-E_{1}-E_{2}\right)\right) d u=\frac{1}{26} \int_{0}^{1}\left((3-2 u) H+(u-1) E_{1}+(u-1) E_{2}\right)^{3} d u+ \\
& \quad+\frac{1}{26} \int_{1}^{\frac{3}{2}}((3-2 u) H)^{3} d u=\frac{1}{26} \int_{0}^{1} 12 u^{2}-36 u+26 d u+\frac{1}{26} \int_{1}^{\frac{3}{2}} 2(3-2 u)^{3} d u=\frac{49}{104}
\end{aligned}
$$

Thus, we conclude that $\beta(S)>0$, as claimed.
We conclude this section with the following technical result:
Proposition 4.8. Suppose that $X$ has at most isolated singularities, and let $S$ be a smooth surface in one of the linear systems $\left|H-E_{1}\right|$ or $\left|H-E_{2}\right|$. Then

$$
\delta_{P}(X) \geqslant \frac{104}{99}
$$

for every point $P \in S$ such that $P \notin E_{1} \cup E_{2}$.
Proof. We may assume that $S \in\left|H-E_{1}\right|$. Let $u$ be a non-negative real number. Then

$$
-K_{X}-u S \sim_{\mathbb{R}}(2-u) S+\left(H-E_{2}\right)+E_{1} \sim_{\mathbb{R}}(3-u) H-(1-u) E_{1}-E_{2}
$$

Then $-K_{X}-u S$ is nef $\Longleftrightarrow u \in[0,1]$, and $-K_{X}-u S$ is pseudo-effective $\Longleftrightarrow u \in[0,2]$. Moreover, if $u \in[1,2]$, then its Zariski decomposition can be described as follows:

$$
-K_{X}-u S \sim_{\mathbb{R}} \underbrace{(3-u) H-E_{2}}_{\text {positive part }}+\underbrace{(u-1) E_{1}}_{\text {negative part }} .
$$

Thus, for transparency, we let

$$
P(u)=\left\{\begin{array}{l}
(3-u) H-(1-u) E_{1}-E_{2} \text { if } 0 \leqslant u \leqslant 1, \\
(3-u) H-E_{2} \text { if } 1 \leqslant u \leqslant 2,
\end{array}\right.
$$

and

$$
N(u)=\left\{\begin{array}{l}
0 \text { if } 0 \leqslant u \leqslant 1 \\
(u-1) E_{1} \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

Then

$$
(P(u))^{3}=\left\{\begin{array}{l}
26-18 u \text { if } 0 \leqslant u \leqslant 1 \\
40-2 u^{3}+18 u^{2}-48 u \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

Then

$$
S_{X}(S)=\frac{1}{\left(-K_{X}\right)^{3}} \int_{0}^{2} \operatorname{vol}\left(-K_{X}-u S\right) d u=\frac{1}{26} \int_{0}^{2}(P(u))^{3} d u=\frac{3}{4}
$$

so that $\beta(S)=1-S_{X}(S)=\frac{1}{3}$.
Let $P$ be a point in $S$, and let $C$ be an irreducible smooth curve in $S$ such that $P \in C$. Write $\left.N(u)\right|_{S}=N^{\prime}(u)+\operatorname{ord}_{C}\left(\left.N(u)\right|_{S}\right) C$. For every $u \in[0,2]$, set

$$
t(u)=\sup \left\{v \in \mathbb{R}_{\geqslant 0} \mid \text { the divisor }\left.P(u)\right|_{S}-v C \text { is pseudo-effective }\right\} .
$$

Let $v$ be a real number in $[0, t(u)]$, let $P(u, v)$ be the positive part of the Zariski decomposition of the $\mathbb{R}$-divisor $\left.P(u)\right|_{S}-v C$, and let $N(u, v)$ be its negative part. Set

$$
S\left(W_{\bullet, \bullet}^{S} ; C\right)=\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{2}\left(\left.P(u)\right|_{S}\right)^{2} \cdot \operatorname{ord}_{C}\left(\left.N(u)\right|_{S}\right) d u+\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{2} \int_{0}^{t(u)}(P(u, v))^{2} d v d u
$$

and

$$
\begin{aligned}
S\left(W_{\bullet, 0,0}^{S, C} ; P\right)= & \frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{2} \int_{0}^{t(u)}(P(u, v) \cdot C)^{2} d v d u+ \\
& +\frac{6}{\left(-K_{X}\right)^{3}} \int_{0}^{2} \int_{0}^{t(u)}(P(u, v) \cdot C) \cdot \operatorname{ord}_{P}\left(\left.N^{\prime}(u)\right|_{C}+\left.N(u, v)\right|_{C}\right) d v d u
\end{aligned}
$$

Then it follows from [2, 4] that

$$
\delta_{P}(X) \geqslant \min \left\{\frac{1}{S_{X}(S)}, \frac{1}{S\left(W_{\bullet, \bullet}^{S} ; C\right)}, \frac{1}{S\left(W_{\bullet, \bullet, \bullet}^{S, C} ; P\right)}\right\}
$$

But $S_{X}(S)=\frac{3}{4}$. Hence, to complete the proof, it is enough to find an irreducible smooth curve $C \subset S$ such that $P \in C$, and $S\left(W_{\bullet, \bullet}^{S} ; C\right) \leqslant \frac{99}{104} \geqslant S\left(W_{\bullet, \bullet \bullet}^{S, C} ; P\right)$ if $P \notin E_{1} \cup E_{2}$.

Now, suppose that $P \notin E_{1} \cup E_{2}$. In particular, this assumption implies $P \notin \operatorname{Supp}(N(u))$, so that the formulas for $S\left(W_{\bullet, \bullet}^{S} ; C\right)$ and $S\left(W_{\bullet, \bullet, \bullet}^{S, C} ; P\right)$ simplify as follows:

$$
S\left(W_{\bullet,}^{S} ; C\right)=\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{2} \int_{0}^{t(u)}(P(u, v))^{2} d v d u
$$

and

$$
S\left(W_{\bullet, \bullet, \bullet}^{S, C} ; P\right)=\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{2} \int_{0}^{t(u)}(P(u, v) \cdot C)\left((P(u, v) \cdot C)+\operatorname{ord}_{P}\left(\left.N(u, v)\right|_{C}\right)\right) d v d u
$$

To find the required curve $C$, recall that $S$ is a del Pezzo surface of degree 6, and

$$
\left.E_{2}\right|_{S}=\mathbf{e}_{1}+\mathbf{e}_{2},
$$

where $\mathbf{e}_{1}$ and $\mathbf{e}_{2}$ are disjoint ( -1 )-curves in the surface $S$. Let $Z$ be the fiber of the conic bundle $S \rightarrow \mathbb{P}^{1}$ given by $\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right) \mapsto\left[u_{2}: v_{2}\right]$ such that $P \in Z$. Then $Z \cdot \mathbf{e}_{1}=Z \cdot \mathbf{e}_{2}=1$ and

$$
\left.\left.E_{1}\right|_{S} \sim H\right|_{S} \sim Z+\mathbf{e}_{1}+\mathbf{e}_{2}
$$

Let us choose $C$ to be an irreducible component of the fiber $Z$ that contains the point $P$. A priori, we may have the following two possibilities:
(1) $Z$ is an irreducible smooth rational curve and $Z^{2}=0$;
(2) $Z=\ell_{1}+\ell_{2}$ for two (-1)-curves $\ell_{1}$ and $\ell_{2}$ such that $\ell_{1} \cdot \ell_{2}=\ell_{1} \cdot \mathbf{e}_{1}=\ell_{2} \cdot \mathbf{e}_{2}=1$.

In the first case, we have $C=Z$. In the second case, we may assume $P \in \ell_{1}$, so $C=\ell_{1}$. Let us compute $S\left(W_{\bullet, \bullet}^{S} ; C\right)$ and $S\left(W_{\bullet, \bullet, \bullet}^{S, C} ; P\right)$ in each of these cases.

First, we note that

$$
\left.P(u)\right|_{S}=\left\{\begin{array}{l}
2 Z+\mathbf{e}_{1}+\mathbf{e}_{2} \text { if } 0 \leqslant u \leqslant 1 \\
(3-u) Z+(2-u)\left(\mathbf{e}_{1}+\mathbf{e}_{2}\right) \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

Let $v$ be a non-negative real number. If $C=Z$, then

$$
\left.P(u)\right|_{S}-v C=\left\{\begin{array}{l}
(2-v) C+\mathbf{e}_{1}+\mathbf{e}_{2} \text { if } 0 \leqslant u \leqslant 1 \\
(3-u-v) C+(2-u)\left(\mathbf{e}_{1}+\mathbf{e}_{2}\right) \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

which implies that

$$
t(u)=\left\{\begin{array}{l}
2 \text { if } 0 \leqslant u \leqslant 1 \\
3-u \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

Furthermore, if $C=Z$ and $u \in[0,1]$, then

$$
P(u, v)=\left\{\begin{array}{l}
(2-v) C+\mathbf{e}_{1}+\mathbf{e}_{2} \text { if } 0 \leqslant v \leqslant 1 \\
(2-v)\left(C+\mathbf{e}_{1}+\mathbf{e}_{2}\right) \text { if } 1 \leqslant v \leqslant 2
\end{array}\right.
$$

and

$$
N(u, v)=\left\{\begin{array}{l}
0 \text { if } 0 \leqslant v \leqslant 1 \\
(v-1)\left(\mathbf{e}_{1}+\mathbf{e}_{2}\right) \text { if } 1 \leqslant v \leqslant 2
\end{array}\right.
$$

which gives

$$
(P(u, v))^{2}=\left\{\begin{array}{l}
6-4 v \text { if } 0 \leqslant v \leqslant 1 \\
2(v-2)^{2} \text { if } 1 \leqslant v \leqslant 2
\end{array}\right.
$$

and

$$
P(u, v) \cdot C=\left\{\begin{array}{l}
2 \text { if } 0 \leqslant v \leqslant 1 \\
4-2 v \text { if } 1 \leqslant v \leqslant 2
\end{array}\right.
$$

Similarly, if $C=Z$ and $u \in[1,2]$, then

$$
P(u, v)=\left\{\begin{array}{l}
(3-u-v) C+(2-u)\left(\mathbf{e}_{1}+\mathbf{e}_{2}\right) \text { if } 0 \leqslant v \leqslant 1, \\
(3-u-v)\left(C+\mathbf{e}_{1}+\mathbf{e}_{2}\right) \text { if } 1 \leqslant v \leqslant 3-u,
\end{array}\right.
$$

and

$$
N(u, v)=\left\{\begin{array}{l}
0 \text { if } 0 \leqslant v \leqslant 1 \\
(v-1)\left(\mathbf{e}_{1}+\mathbf{e}_{2}\right) \text { if } 1 \leqslant v \leqslant 3-u
\end{array}\right.
$$

so that

$$
(P(u, v))^{2}=\left\{\begin{array}{l}
2(2-u)(4-u-2 v) \text { if } 0 \leqslant v \leqslant 1 \\
2(3-u-v)^{2} \text { if } 1 \leqslant v \leqslant 3-u
\end{array}\right.
$$

and

$$
P(u, v) \cdot C=\left\{\begin{array}{l}
4-2 u \text { if } 0 \leqslant v \leqslant 1 \\
6-2 u-2 v \text { if } 1 \leqslant v \leqslant 3-u
\end{array}\right.
$$

In particular, if $C=Z$, then $P \notin \operatorname{Supp}(N(u, v))$, because $P \notin E_{2}$ by our assumption.
Now, integrating, we get $S\left(W_{\bullet, \bullet}^{S} ; C\right)=\frac{3}{4}<\frac{99}{104}$ and $S\left(W_{\bullet, \bullet, \bullet}^{S, C} ; P\right)=\frac{21}{26}<\frac{99}{104}$.
Hence, to complete the proof, we may assume that $Z=\ell_{1}+\ell_{2}$ and $C=\ell_{1}$. Then

$$
\left.P(u)\right|_{S}-v C=\left\{\begin{array}{l}
(2-v) C+2 \ell_{2}+\mathbf{e}_{1}+\mathbf{e}_{2} \text { if } 0 \leqslant u \leqslant 1 \\
(3-u-v) C+(3-u) \ell_{2}+(2-u)\left(\mathbf{e}_{1}+\mathbf{e}_{2}\right) \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

which implies that

$$
t(u)=\left\{\begin{array}{l}
2 \text { if } 0 \leqslant u \leqslant 1 \\
3-u \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

Further, if $u \in[0,1]$, then

$$
P(u, v)=\left\{\begin{array}{l}
(2-v) C+2 \ell_{2}+\mathbf{e}_{1}+\mathbf{e}_{2} \text { if } 0 \leqslant v \leqslant 1 \\
(2-v)\left(C+\mathbf{e}_{1}\right)+(3-v) \ell_{2}+\mathbf{e}_{2} \text { if } 1 \leqslant v \leqslant 2
\end{array}\right.
$$

and

$$
N(u, v)=\left\{\begin{array}{l}
0 \text { if } 0 \leqslant v \leqslant 1 \\
(v-1)\left(\mathbf{e}_{1}+\ell_{2}\right) \text { if } 1 \leqslant v \leqslant 2
\end{array}\right.
$$

which gives

$$
(P(u, v))^{2}=\left\{\begin{array}{l}
6-v^{2}-2 v \text { if } 0 \leqslant v \leqslant 1 \\
(2-v)(4-v) \text { if } 1 \leqslant v \leqslant 2
\end{array}\right.
$$

and

$$
P(u, v) \cdot C=\left\{\begin{array}{l}
1+v \text { if } 0 \leqslant v \leqslant 1 \\
3-v \text { if } 1 \leqslant v \leqslant 2
\end{array}\right.
$$

Likewise, if $u \in[1,2]$, then

$$
P(u, v)=\left\{\begin{array}{l}
(3-u-v) C+(3-u) \ell_{2}+(2-u)\left(\mathbf{e}_{1}+\mathbf{e}_{2}\right) \text { if } 0 \leqslant v \leqslant 2-u \\
(3-u-v) C+(5-2 u-v) \ell_{2}+(2-u)\left(\mathbf{e}_{1}+\mathbf{e}_{2}\right) \text { if } 2-u \leqslant v \leqslant 1 \\
(3-u-v)\left(C+\mathbf{e}_{1}\right)+(5-2 u-v) \ell_{2}+(2-u) \mathbf{e}_{2} \text { if } 1 \leqslant v \leqslant 3-u
\end{array}\right.
$$

and

$$
N(u, v)=\left\{\begin{array}{l}
0 \text { if } 0 \leqslant v \leqslant 2-u \\
(v-2+u) \ell_{2} \text { if } 2-u \leqslant v \leqslant 1 \\
(v-2+u) \ell_{2}+(v-1) \mathbf{e}_{1} \text { if } 1 \leqslant v \leqslant 3-u
\end{array}\right.
$$

so that

$$
(P(u, v))^{2}=\left\{\begin{array}{l}
2 u^{2}+2 u v-v^{2}-12 u-4 v+16 \text { if } 0 \leqslant v \leqslant 2-u \\
(u-2)(3 u+4 v-10) \text { if } 2-u \leqslant v \leqslant 1 \\
(3-u-v)(7-3 u-v) \text { if } 1 \leqslant v \leqslant 3-u
\end{array}\right.
$$

and

$$
P(u, v) \cdot C=\left\{\begin{array}{l}
2-u+v \text { if } 0 \leqslant v \leqslant 2-u \\
4-2 u \text { if } 2-u \leqslant v \leqslant 1 \\
5-2 u-v \text { if } 1 \leqslant v \leqslant 3-u
\end{array}\right.
$$

Now, integrating, we get $S\left(W_{\bullet, 0}^{S} ; C\right)=\frac{99}{104}$. Note that $P \notin \mathbf{e}_{1}$, since $P \notin E_{2}$ by assumption. Therefore, if $P \notin \ell_{2}$, then $P \notin \operatorname{Supp}(N(u, v))$, which implies that $S\left(W_{\bullet,, \bullet ;}^{S, C} ; P\right)=\frac{37}{52}<\frac{99}{104}$.

Similarly, if $P \in \ell_{2}$, then

$$
\begin{aligned}
& S\left(W_{\bullet, 0,0}^{S, C} ; P\right)=\frac{37}{52}+\frac{6}{26} \int_{0}^{2} \int_{0}^{t(u)}(P(u, v) \cdot C) \cdot \operatorname{ord}_{P}\left(\left.N(u, v)\right|_{C}\right) d v d u= \\
& =\frac{37}{52}+\frac{6}{26} \int_{0}^{1} \int_{1}^{2}(P(u, v) \cdot C)(v-1) d v d u+\frac{6}{26} \int_{1}^{2} \int_{2-u}^{3-u}(P(u, v) \cdot C)(v-2+u) d v d u= \\
& =\frac{37}{52}+\frac{6}{26} \int_{0}^{1} \int_{1}^{2}(3-v)(v-1) d v d u+\frac{6}{26} \int_{1}^{2} \int_{2-u}^{1}(4-2 u)(v-2+u) d v d u+ \\
& \quad+\frac{6}{26} \int_{1}^{2} \int_{1}^{3-u}(5-2 u-v)(v-2+u) d v d u=\frac{99}{104} .
\end{aligned}
$$

This completes the proof of Proposition 4.8.
4.1. Smooth threefolds. We continue to use the notation introduced earlier in this section. The goal of this subsection is to give a new proof of the following result.

Theorem 4.9 ([4, Proposition 5.79]). Let $X$ be a smooth Fano threefold from class (ユ). Then $X$ is $K$-polystable.

We begin with a technical lemma that will also be useful in the singular case.
Lemma 4.10. Suppose that $X$ has at worst isolated singularities. Let $\mathbf{F}$ be a $G$-invariant prime divisor $\mathbf{F}$ over the threefold $X$ with

$$
\beta(\mathbf{F})=A_{X}(\mathbf{F})-S_{X}(\mathbf{F}) \leqslant 0
$$

and let $Z$ be its center on $X$. Suppose that $Z$ is not a singular point of the threefold $X$. Then $Z$ is an irreducible curve with $\eta(Z)=([1: 1],[1: 1])$ or $\eta(Z)=([1:-1],[1:-1])$.

Proof. Note first that $Z$ is a $G$-invariant curve, by Lemma 4.1 and Corollary 4.7, and

$$
\delta_{P}(X) \leqslant 1
$$

for every point $P \in Z$. Observe also that $Z \not \subset E_{1} \cup E_{2}$, because the surfaces $E_{1}$ and $E_{2}$ are disjoint and swapped by the action of the group $G$.

Let $S$ be any surface in $\left|H-E_{1}\right|$. If $S \cdot Z \neq 0$, then choosing $S$ to be general enough, we see that $S$ is a smooth del Pezzo surface and $S \cap Z$ contains a point $P \notin E_{1} \cup E_{2}$. This would give $\delta_{P}(X)>1$ by Proposition 4.8, which is a contradiction. So we have

$$
\left(H-E_{1}\right) \cdot Z=0 .
$$

Similarly, we get $\left(H-E_{2}\right) \cdot Z=0$. Therefore, we conclude that $\eta(Z)$ is a point in $\mathbb{P}^{1} \times \mathbb{P}^{1}$. Finally, using Remark 4.3 we get $\eta(Z)=([1: 1],[1: 1])$ or $\eta(Z)=([1:-1],[1:-1])$.
Proof of Theorem 4.9. Suppose for a contradiction that the Fano threefold $X$ is smooth, but $X$ is not K -polystable. It follows from [6, 7, 8] that there exists a $G$-invariant prime divisor $\mathbf{F}$ over $X$ such that $\beta(\mathbf{F}) \leqslant 0$. Let $Z$ be the center of $\mathbf{F}$ on $X$. Then by Lemma4.10, the center $Z$ is a curve with $\eta(Z)=([1: 1],[1: 1])$ or $\eta(Z)=([1:-1],[1:-1])$.

Let $S$ be the unique surface in the pencil $\left|H-E_{1}\right|$ that contains $Z$, and let $\ell=\eta(S)$. If $\eta(Z)=([1: \pm 1],[1: \pm 1])$, then $\ell=\left\{u_{1} \pm v_{1}=0\right\}$. Moreover, one can check that

- $\left\{u_{1}-v_{1}=0\right\}$ intersects $\Delta$ in a single point $\Longleftrightarrow b+a \pm 1=0$,
- $\left\{u_{1}+v_{1}=0\right\}$ intersects $\Delta$ in a single point $\Longleftrightarrow b-a \pm 1=0$.

Since $Q$ is smooth by assumption, we have $b \pm a \pm 1 \neq 0$, and $\ell$ intersects the discriminant curve $\Delta$ in two distinct points. This implies that $S$ is a smooth del Pezzo surface.

Now, applying Proposition 4.8, we get $\delta_{P}(X)>1$ for a general point $P$ in the curve $Z$, which contradicts $\beta(\mathbf{F}) \leqslant 0$. This completes the proof of Theorem 4.9.
4.2. Threefolds with one singular point. We continue to use the notation introduced at the beginning of this section. In this subsection, we will prove the following result.

Theorem 4.11. Suppose that $X$ is a threefold from class (】) with one singular point. Then $X$ is $K$-polystable.

Suppose that the threefold $X$ has one singular point. Then $a \pm b \pm 1=0$ and $a b \neq 0$. Up to a change of coordinates, we may assume that $b=1-a$ and $a \notin\{0,1\}$.

Let $O=\operatorname{Sing}(X)$. Recall that
$X=\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t, w^{2}+x y+z t=a(x t+y z)+(1-a)(x z+y t)\right\} \subset \mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4}$, and $O=([1: 1],[1: 1],[1: 1: 1: 1: 0])$. Similarly, we have

$$
Q=\left\{w^{2}+x y+z t=a(x t+y z)+(1-a)(x z+y t)\right\} \subset \mathbb{P}^{4},
$$

and $\operatorname{Sing}(Q)=\pi(O)=[1: 1: 1: 1: 0]$. Note that $\eta(O)=([1: 1],[1: 1])$.
The fiber of the conic bundle $\eta$ over the point ( $[1: 1],[1: 1]$ ) is a reducible reduced conic $L_{1}+L_{2}$, where $L_{1}$ and $L_{2}$ are two smooth rational curves such that $L_{1} \cap L_{2}=O$, and $\pi\left(L_{1}\right)$ and $\pi\left(L_{2}\right)$ are the only lines in $Q$ that intersect $C_{1}$ and $C_{2}$. We have

$$
\begin{aligned}
& \pi\left(L_{1}\right)=\{x=y, z=t, w+x-z=0\} \\
& \pi\left(L_{2}\right)=\{x=y, z=t, w-x+z=0\}
\end{aligned}
$$

We will use the curves $L_{1}$ and $L_{2}$ in the proofs of the next two lemmas.
Lemma 4.12. Let $\mathbf{F}$ be a $G$-invariant prime divisor $\mathbf{F}$ over the threefold $X$ with $\beta(\mathbf{F}) \leqslant 0$, and let $Z$ be its center on the threefold $X$. Then $Z=O$.

Proof. Suppose that $Z \neq O$. Applying Lemma4.10, we find that $Z$ is an irreducible curve such that $\eta(Z)=([1: 1],[1: 1])$ or $\eta(Z)=([1:-1],[1:-1])$. But $Z \neq L_{1}$ and $Z \neq L_{2}$, since neither of the curves $L_{1}$ nor $L_{2}$ is $G$-invariant, so $\eta(Z)=([1:-1],[1:-1])$.

Let $S$ be the surface in $\left|H-E_{1}\right|$ that is cut out on $X$ by the equation $u_{1}+v_{1}=0$. Then $S$ is smooth and $Z \subset S$, so $\delta_{P}(X)>1$ for a general point $P \in Z$ by Proposition 4.8, This contradicts $\beta(\mathbf{F}) \leqslant 0$.

Lemma 4.13. Let $\mathbf{F}$ be any $G$-invariant prime divisor $\mathbf{F}$ over $X$ such that $C_{X}(\mathbf{F})=O$. Then $\beta(\mathbf{F})>0$.

Proof. Let $\varphi: \widetilde{X} \rightarrow X$ and $\phi: \widetilde{Q} \rightarrow Q$ be blow ups of the points $O$ and $\pi(O)$, respectively. As in the proof of Lemma 2.3, we have the following $G$-equivariant commutative diagram:

where $\varpi$ is the blow up of the proper transforms of the conics $C_{1}$ and $C_{2}, v$ is a $\mathbb{P}^{1}$-bundle, and $\nu$ is a conic bundle. Let $F$ be the $\varphi$-exceptional surface. Then $\beta(F)=2-S_{X}(F)$.

Let $\widetilde{H}=\varphi^{*}(H)$, let $\widetilde{E}_{1}$ and $\widetilde{E}_{2}$ be the proper transforms on $\widetilde{X}$ of the $\pi$-exceptional surfaces $E_{1}$ and $E_{2}$, respectively, let $\widetilde{S}_{1}$ and $\widetilde{S}_{2}$ be the proper transforms of the quadric cones in $Q$ that contain $C_{1}$ and $C_{2}$, respectively. We have $\varpi^{*}\left(-K_{X}\right) \sim 3 \widetilde{H}-\widetilde{E}_{1}-\widetilde{E}_{2}$. We also have $\widetilde{S}_{1} \sim \widetilde{H}-\widetilde{E}_{1}-F$ and $\widetilde{S}_{2} \sim \widetilde{H}-\widetilde{E}_{2}-F$. Take $u \in \mathbb{R}_{\geqslant 0}$. Then

$$
\begin{equation*}
\varpi^{*}\left(-K_{X}\right)-u F \sim_{\mathbb{R}} \frac{3}{2}\left(\widetilde{S}_{1}+\widetilde{S}_{2}\right)+\frac{1}{2}\left(\widetilde{E}_{1}+\widetilde{E}_{2}\right)+(3-u) F \tag{4.14}
\end{equation*}
$$

Hence, intersecting $\varpi^{*}\left(-K_{X}\right)-u F$ with a sufficiently general fiber of the morphism $\nu$, we see that $\varpi^{*}\left(-K_{X}\right)-u F$ is pseudoeffective $\Longleftrightarrow u \in[0,3]$. Moreover, we have

\[

\]

Furthermore, the divisor $\varpi^{*}\left(-K_{X}\right)-u F$ is nef for $u \in[0,1]$. Thus, we have

$$
\begin{aligned}
& S_{X}(F)=\frac{1}{26} \int_{0}^{1}\left(3 \widetilde{H}-\widetilde{E}_{1}-\widetilde{E}_{2}-u F\right)^{3} d u+\frac{1}{26} \int_{1}^{3} \operatorname{vol}\left(\varpi^{*}\left(-K_{X}\right)-u F\right) d u= \\
= & \frac{1}{26} \int_{0}^{1} 26-2 u^{3} d u+\frac{1}{26} \int_{1}^{3} \operatorname{vol}\left(\varpi^{*}\left(-K_{X}\right)-u F\right) d u=\frac{51}{2}+\frac{1}{26} \int_{1}^{3} \operatorname{vol}\left(\varpi^{*}\left(-K_{X}\right)-u F\right) d u
\end{aligned}
$$

If $\ell$ is the preimage on $\widetilde{X}$ of a general ruling of one of the cones $\pi \circ \varphi\left(\widetilde{S}_{1}\right)$ and $\pi \circ \varphi\left(\widetilde{S}_{2}\right)$, then $\left(\varpi^{*}\left(-K_{X}\right)-u F\right) \cdot \ell=2-u$, which shows that $\varpi^{*}\left(-K_{X}\right)-u F$ is not nef for $u>2$.

Let $\widetilde{L}_{1}$ and $\widetilde{L}_{2}$ be the proper transforms on $\widetilde{X}$ of the curves $L_{1}$ and $L_{2}$, respectively. Then $\varpi^{*}\left(-K_{X}\right)-u F$ is not nef for $u>1$, because

$$
\left(\varpi^{*}\left(-K_{X}\right)-u F\right) \cdot \widetilde{L}_{1}=\left(\varpi^{*}\left(-K_{X}\right)-u F\right) \cdot \widetilde{L}_{2}=1-u
$$

Using (4.14), one can show that $\widetilde{L}_{1}$ and $\widetilde{L}_{2}$ are the only curves in $\widetilde{X}$ that have negative intersection with $\varpi^{*}\left(-K_{X}\right)-u F$ for $u \in(1,2]$.

Note that $\widetilde{L}_{1} \cong \widetilde{L}_{2} \cong \mathbb{P}^{1}$ and their normal bundles are isomorphic to $\mathcal{O}_{\mathbb{P}^{1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(-1)$. Let $\psi: \widehat{X} \rightarrow \widetilde{X}$ be the blow up of $\widetilde{L}_{1}$ and $\widetilde{L}_{2}$, let $G_{1}$ and $G_{2}$ be the $\psi$-exceptional surfaces such that $\psi\left(G_{1}\right)=\widetilde{L}_{1}$ and $\psi\left(G_{2}\right)=\widetilde{L}_{2}$, let $\underset{\sim}{H}, \widehat{E}_{1}, \widehat{E}_{2}, \widehat{S}_{1}, \widehat{S}_{2}, \widehat{F}$, be the proper transforms on the threefold $\widehat{X}$ of the surfaces $\widetilde{H}, \widetilde{E}_{1}, \widetilde{E}_{2}, \widetilde{S}_{1}, \widetilde{S}_{2}, F$, respectively. Then

$$
(\varpi \circ \psi)^{*}\left(-K_{X}\right) \sim 3 \widehat{H}-\widehat{E}_{1}-\widehat{E}_{2}
$$

We have $\widehat{S}_{1} \sim \widehat{H}-\widehat{E}_{1}-\widehat{F}-G_{1}$ and $\widehat{S}_{2} \sim \widehat{H}-\widehat{E}_{2}-\widehat{F}-G_{2}$. This gives

$$
\begin{equation*}
(\varpi \circ \psi)^{*}\left(-K_{X}\right)-u \widehat{F} \sim_{\mathbb{R}} \frac{3}{2}\left(\widehat{S}_{1}+\widehat{S}_{2}\right)+\frac{1}{2}\left(\widehat{E}_{1}+\widehat{E}_{2}\right)+\frac{3}{2}\left(G_{1}+G_{2}\right)+(3-u) \widehat{F} \tag{4.15}
\end{equation*}
$$

For $u \in[0,3]$, the Zariski decomposition of this divisor can be computed on $\widehat{X}$ as follows:

- if $u \in[0,1]$, then the divisor $(\varpi \circ \psi)^{*}\left(-K_{X}\right)-u \widehat{F}$ is nef,
- if $u \in[1,2]$, then the positive (nef) part of the Zariski decomposition is

$$
\begin{aligned}
(\varpi \circ \psi)^{*}\left(-K_{X}\right)-u \widehat{F} & -(u-1)\left(G_{1}+G_{2}\right) \sim_{\mathbb{R}} \\
& \sim_{\mathbb{R}} 3 \widehat{H}-\widehat{E}_{1}-\widehat{E}_{2}-(u-1)\left(G_{1}+G_{2}\right) \sim_{\mathbb{R}} \\
& \sim_{\mathbb{R}} \frac{3}{2}\left(\widehat{S}_{1}+\widehat{S}_{2}\right)+\frac{1}{2}\left(\widehat{E}_{1}+\widehat{E}_{2}\right)+\frac{5-2 u}{2}\left(G_{1}+G_{2}\right)+(3-u) \widehat{F},
\end{aligned}
$$

and the negative part of the Zariski decomposition is $(u-1)\left(G_{1}+G_{2}\right)$,

- if $u \in(2,3]$, then the positive part of the Zariski decomposition is

$$
\begin{aligned}
(\varpi \circ \psi)^{*}\left(-K_{X}\right)-u \widehat{F}-(u-1)\left(G_{1}+G_{2}\right)-(u-2)\left(\widehat{S}_{1}+\widehat{S}_{2}\right) \sim_{\mathbb{R}} \\
\quad \sim_{\mathbb{R}} 3 \widehat{H}-\widehat{E}_{1}-\widehat{E}_{2}-(u-1)\left(G_{1}+G_{2}\right)-(u-2)\left(\widehat{S}_{1}+\widehat{S}_{2}\right) \sim_{\mathbb{R}} \\
\quad \sim_{\mathbb{R}}(7-2 u) \widehat{H}-(3-u)\left(\widehat{E}_{1}+\widehat{E}_{2}+G_{1}+G_{2}\right)+(u-4) F \sim_{\mathbb{R}} \\
\quad \sim_{\mathbb{R}} \frac{7-2 u}{2}\left(\widehat{S}_{1}+\widehat{S}_{2}\right)+\frac{1}{2}\left(\widehat{E}_{1}+\widehat{E}_{2}\right)+\frac{5-2 u}{2}\left(G_{1}+G_{2}\right)+(3-u) \widehat{F},
\end{aligned}
$$

and the negative part is $(u-1)\left(G_{1}+G_{2}\right)+(u-2)\left(\widehat{S}_{1}+\widehat{S}_{2}\right)$.
The intersections of the divisors $\widehat{H}, \widehat{E}_{1}, \widehat{E}_{2}, \widehat{F}, G_{1}, G_{2}$ on $\widehat{X}$ can be computed as follows:

$$
\begin{array}{rlrlrl}
\widehat{H}^{3} & =2, & \widehat{E}_{1}^{3} & =-4, & \widehat{E}_{2}^{3} & =-4, \\
G_{2}^{3} & =2, & \widehat{F}^{3} & =2, & \widehat{H} \cdot \widehat{E}_{1}^{2} & =-2, \\
\widehat{F} \cdot G_{1}^{2} & =-1, & \widehat{F} \cdot G_{2}^{2} & =-1, & \widehat{E_{1}} \cdot G_{1}^{2} & =-1, \\
\widehat{E} \cdot \widehat{E}_{2}^{2} & =-2, \\
\widehat{E}_{1} \cdot G_{2}^{2} & =-1, & \widehat{E}_{2} \cdot G_{2}^{2} & =-1, & \widehat{H} \cdot G_{1}^{2} & =-1, \\
\widehat{E} & G_{1}^{2} & =-1, \\
& \widehat{H} \cdot G_{2}^{2} & =-1,
\end{array}
$$

and other triple intersections are zero. Now we are ready to compute $S_{X}(F)$. We have

$$
\begin{aligned}
S_{X}(F) & =\frac{1}{26} \int_{0}^{1}\left((\pi \circ \varpi)^{*}\left(-K_{X}\right)-u \widehat{F}\right)^{3} d u+\frac{1}{26} \int_{1}^{2}\left(3 \widehat{H}-\widehat{E}_{1}-\widehat{E}_{2}-(u-1)\left(G_{1}+G_{2}\right)\right)^{3} d u+ \\
& +\frac{1}{26} \int_{2}^{3}\left((7-2 u) \widehat{H}-(3-u)\left(\widehat{E}_{1}+\widehat{E}_{2}+G_{1}+G_{2}\right)+(u-4) F\right)^{3} d u= \\
& =\frac{1}{26} \int_{0}^{1} 26-2 u^{3} d u+\frac{1}{26} \int_{1}^{2} 24+6 u-6 u^{2} d u+\frac{1}{26} \int_{2}^{3} 6(2-u)(4-u) d u=\frac{99}{52}
\end{aligned}
$$

Thus, we see that $\beta(F)=\frac{5}{52}>0$.
Now, suppose that $\beta(\mathbf{F}) \leqslant 0$. Let $\widetilde{Z}$ be the center of the divisor $\mathbf{F}$ on the threefold $\widetilde{X}$. Then $\widetilde{Z}$ is a $G$-invariant irreducible proper subvariety of the surface $F$, since $\beta(F)>0$.

Recall from Lemma 4.2 that $G$ acts faithfully on $F$, and $F$ does not have $G$-fixed points. Hence, we conclude that $\widetilde{Z}$ is a curve. Let $\widehat{Z}$ be the proper transform of this curve on $\widehat{X}$. Then $\widehat{Z}=C_{\widehat{X}}(\mathbf{F})$ and $\widehat{Z} \subset \widehat{F}$. Let us apply Abban-Zhuang theory [2] to the flag $\widehat{Z} \subset \widehat{F}$.

For every $u \in[0,3]$, let $P(u)$ be the positive (nef) part of the Zariski decomposition of the divisor $(\varpi \circ \psi)^{*}\left(-K_{X}\right)-u \widehat{F}$ (described above), and let $N(u)$ be its negative part. Observe that $\widehat{Z} \not \subset \operatorname{Supp}(N(u))$, because $\widehat{Z}$ is irreducible and $G$-invariant. Set

$$
t(u)=\sup \left\{v \in \mathbb{R}_{\geqslant 0} \mid \text { the divisor }\left.P(u)\right|_{\widehat{F}}-v \widehat{Z} \text { is pseudo-effective }\right\} .
$$

For $v \in[0, t(u)]$, let $P(u, v)$ be the nef part of the Zariski decomposition of $\left.P(u)\right|_{\widehat{F}}-v \widehat{Z}$, and let $N(u, v)$ be the negative part of this Zariski decomposition. Set

$$
S\left(W_{\bullet, \bullet}^{\widehat{F}} ; \widehat{Z}\right)=\frac{3}{26} \int_{0}^{3} \int_{0}^{t(u)}(P(u, v))^{2} d v d u
$$

Then it follows from [2, Theorem 3.3] and [4, Corollary 1.109] that

$$
1 \geqslant \frac{A_{X}(\mathbf{F})}{S_{X}(\mathbf{F})} \geqslant \min \left\{\frac{A_{X}(\widehat{F})}{S_{X}(\widehat{F})}, \frac{1}{S\left(W_{\bullet} \widehat{\bullet} ; \widehat{Z}\right)}\right\}=\min \left\{\frac{104}{99}, \frac{1}{S\left(W_{\bullet} \widehat{\bullet} ; \widehat{Z}\right)}\right\} .
$$

Therefore, we conclude that $S\left(W_{\bullet, \bullet} \widehat{\widehat{Z}}\right) \geqslant 1$. Let us show that $S\left(W_{\bullet, \bullet} \widehat{F} ; \widehat{Z}\right)<1$.
The morphism $\psi$ induces a $G$-equivariant birational morphism $\sigma: \widehat{F} \rightarrow F$ that blows
 Thus, we see that $\widehat{F}$ is a smooth del Pezzo surface of degree 6 , $\operatorname{rk~}^{\operatorname{Pic}}{ }^{G}(\widehat{F})=2$, and there exists the following $G$-equivariant diagram:

where $\xi$ is a $G$-minimal conic bundle with 2 singular fibers. Set $\mathbf{g}_{1}=\left.G_{1}\right|_{\widehat{F}}$ and $\mathbf{g}_{2}=\left.G_{2}\right|_{\widehat{F}}$. Let $\mathbf{h}$ be a curve on $F$ of degree $(1,1)$, and let $\mathbf{f}$ be a smooth fiber of the morphism $\xi$. Then $\mathbf{f} \sim \sigma^{*}(\mathbf{h})-\mathbf{g}_{1}-\mathbf{g}_{2}$. Since $\widehat{Z}$ is $G$-invariant, we have $\widehat{Z} \sim n \mathbf{f}+m\left(\mathbf{g}_{1}+\mathbf{g}_{2}\right)$ for some non-negative integers $n$ and $m$. Since $\widehat{Z}$ is irreducible, we have $n>0$, so that

$$
S\left(W_{\bullet, \bullet}^{\widehat{F}}, \widehat{Z}\right)=\frac{3}{26} \int_{0}^{3} \int_{0}^{\infty} \operatorname{vol}\left(\left.P(u)\right|_{\widehat{F}}-v \widehat{Z}\right) d v d u \leqslant \frac{3}{26} \int_{0}^{3} \int_{0}^{\infty} \operatorname{vol}\left(\left.P(u)\right|_{\widehat{F}}-v \mathbf{f}\right) d v d u
$$

Hence, to show that $S\left(W_{\bullet, 0} \widehat{\widehat{F}} ; \widehat{Z}\right)<1$, we may assume that $\widehat{Z}=\mathbf{f}$.
We have $\left.G_{1}\right|_{\widehat{F}}=\mathbf{g}_{1},\left.G_{2}\right|_{\widehat{F}}=\mathbf{g}_{2},\left.\widehat{F}\right|_{\widehat{F}} \sim \sigma^{*}(\mathbf{h}) \sim \mathbf{f}+\mathbf{g}_{1}+\mathbf{g}_{2}$ and $\left.\left.\left.\widehat{H}\right|_{\widehat{F}} \sim \widehat{E}_{1}\right|_{\widehat{F}} \sim \widehat{E}_{2}\right|_{\widehat{F}} \sim 0$. This gives

$$
\left.P(u)\right|_{\widehat{F}}-v \mathbf{f} \sim_{\mathbb{R}}\left\{\begin{array}{l}
(u-v) \mathbf{f}+u\left(\mathbf{g}_{1}+\mathbf{g}_{2}\right) \text { if } 0 \leqslant u \leqslant 1 \\
(u-v) \mathbf{f}+\mathbf{g}_{1}+\mathbf{g}_{2} \text { if } 1 \leqslant u \leqslant 2 \\
(4-u-v) \mathbf{f}+\mathbf{g}_{1}+\mathbf{g}_{2} \text { if } 2 \leqslant u \leqslant 3
\end{array}\right.
$$

which implies that

$$
t(u)=\left\{\begin{array}{l}
u \text { if } 0 \leqslant u \leqslant 2 \\
4-u \text { if } 2 \leqslant u \leqslant 3
\end{array}\right.
$$

If $u \in[0,1]$, then $P(u, v)=(u-v)\left(\mathbf{f}+\mathbf{g}_{1}+\mathbf{g}_{2}\right)$ and $N(u, v)=v\left(\mathbf{g}_{1}+\mathbf{g}_{2}\right)$ for $v \in[0, u]$, which gives $(P(u, v))^{2}=2(u-v)^{2}$ for every $(u, v) \in[0,1] \times[0, u]$. If $u \in[1,2]$, then

$$
P(u, v)=\left\{\begin{array}{l}
(u-v) \mathbf{f}+\mathbf{g}_{1}+\mathbf{g}_{2} \text { if } 0 \leqslant v \leqslant u-1 \\
(u-v)\left(\mathbf{f}+\mathbf{g}_{1}+\mathbf{g}_{2}\right) \text { if } u-1 \leqslant v \leqslant u
\end{array}\right.
$$

and

$$
N(u, v)=\left\{\begin{array}{l}
0 \text { if } 0 \leqslant v \leqslant u-1 \\
(v-u+1)\left(\mathbf{g}_{1}+\mathbf{g}_{2}\right) \text { if } u-1 \leqslant u \leqslant u
\end{array}\right.
$$

which gives

$$
(P(u, v))^{2}=\left\{\begin{array}{l}
4 u-4 v-2 \text { if } 0 \leqslant v \leqslant u-1 \\
2(u-v)^{2} \text { if } u-1 \leqslant v \leqslant u
\end{array}\right.
$$

Finally, if $u \in[2,3]$, then

$$
P(u, v)=\left\{\begin{array}{l}
(4-u-v) \mathbf{f}+\mathbf{g}_{1}+\mathbf{g}_{2} \text { if } 0 \leqslant v \leqslant 3-u \\
(4-u-v)\left(\mathbf{f}+\mathbf{g}_{1}+\mathbf{g}_{2}\right) \text { if } 3-u \leqslant v \leqslant 4-u
\end{array}\right.
$$

and

$$
N(u, v)=\left\{\begin{array}{l}
0 \text { if } 0 \leqslant v \leqslant 3-u \\
(v+u-3)\left(\mathbf{g}_{1}+\mathbf{g}_{2}\right) \text { if } 3-u \leqslant u \leqslant 4-u
\end{array}\right.
$$

which gives

$$
(P(u, v))^{2}=\left\{\begin{array}{l}
14-4 u-4 v \text { if } 0 \leqslant v \leqslant 3-u \\
2(4-u-v)^{2} \text { if } 3-u \leqslant v \leqslant 4-u
\end{array}\right.
$$

Now, integrating, we get

$$
\begin{aligned}
S\left(W_{\bullet, \bullet}^{\widehat{F}} ; \mathbf{f}\right)= & \frac{3}{26} \int_{0}^{1} \int_{0}^{u} 2(u-v)^{2} d v d u+\frac{3}{26} \int_{1}^{2} \int_{0}^{u-1} 4 u-4 v-2 d v d u+\frac{3}{26} \int_{1}^{2} \int_{u-1}^{u} 2(u-v)^{2} d v d u+ \\
& +\frac{3}{26} \int_{2}^{3} \int_{0}^{3-u} 14-4 u-4 v d v d u+\frac{3}{26} \int_{2}^{3} \int_{3-u}^{4-u} 2(4-u-v)^{2} d v d u=\frac{29}{52}<1
\end{aligned}
$$

which is a contradiction. Lemma 4.13 is proved.
Now, using the $G$-equivariant valuative criterion for K-polystability [8, Corollary 4.14], we see that Theorem 4.11 follows from Lemmas 4.12 and 4.13 .
4.3. Threefolds with non-isolated singularities. We continue to use the notation introduced at the beginning of this section. Suppose, in addition, that $a \pm b \pm 1=a b=0$. Up to a change of coordinates, we may assume that

$$
X=\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t, w^{2}+(x+z)(y+t)=0\right\} \subset \mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4}
$$

and $Q=\left\{w^{2}+(x+z)(y+t)=0\right\} \subset \mathbb{P}^{4}$. Then $X$ is singular along the curve

$$
C=\left\{w=0, x+z=0, y+t=0, u_{1} z=v_{1} t, u_{2} z=v_{2} t, v_{1} u_{2}=u_{1} v_{2}\right\}
$$

and $Q$ is a quadric in $\mathbb{P}^{4}$ that is singular along the line $L=\{w=0, x+z=0, y+t=0\}$. Recall that $\pi: X \rightarrow Q$ is a blow up of the following two smooth conics:

$$
\begin{aligned}
& C_{1}=\left\{w^{2}+z t=0, x=0, y=0\right\} \\
& C_{2}=\left\{w^{2}+x y=0, z=0, t=0\right\}
\end{aligned}
$$

which both lie in the smooth locus of the quadric $Q$. We have $L=\pi(C)$. In this subsection we will prove the following result.

Theorem 4.16. The threefold $X$ from class ( $\beth$ ) described above, which has non-isolated singularities, is K-polystable.

Let $\Pi_{1}$ and $\Pi_{2}$ be the planes $\{x+z=0, w=0\}$ and $\{y+t=0, w=0\}$ in $\mathbb{P}^{4}$, respectively. Then $\Pi_{1}$ and $\Pi_{2}$ are contained in the quadric $Q$. In fact, we have $\Pi_{1}+\Pi_{1}=Q \cap\{w=0\}$. Now, we let $S_{1}$ and $S_{2}$ be the proper transforms on $X$ of the planes $\Pi_{1}$ and $\Pi_{2}$, respectively. Then $S_{1} \cap S_{2}=X \cap\{w=0\}, \Pi_{1} \cap \Pi_{2}=\operatorname{Sing}(Q)=L$ and $S_{1} \cap S_{2}=\operatorname{Sing}(X)=C$.

Recall that $\tau_{3}$ is the involution in $\operatorname{Aut}(X)$ given by

$$
\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right) \mapsto\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t:-w]\right)
$$

We set $W=X / \tau_{3}$, and let $\theta: X \rightarrow W$ be the quotient map. Then $\theta$ is ramified in $S_{1}+S_{2}$, and $W$ is the smooth Fano threefold in the family №3.25, which can be identified with

$$
\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t\right\} \subset \mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{3},
$$

where we consider $\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t]\right)$ as coordinates on $\mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{3}$. Moreover, we have the following commutative diagram:

where

- $\vartheta$ is the double cover given by $[x: y: z: t: w] \mapsto[x: y: z: t]$,
- $\varpi$ is a blow up of the lines $\vartheta\left(C_{1}\right)=\{x=0, y=0\}$ and $\vartheta\left(C_{2}\right)=\{z=0, t=0\}$,
- the map $Q \xrightarrow{ } \mathbb{P}^{1} \times \mathbb{P}^{1}$ is given by $[x: y: z: t: w] \mapsto([x: y],[z: t])$,
- the map $\mathbb{P}^{3} \rightarrow \mathbb{P}^{1} \times \mathbb{P}^{1}$ is given by $[x: y: z: t] \mapsto([x: y],[z: t])$,
- $\eta$ and $W \rightarrow \mathbb{P}^{1} \times \mathbb{P}^{1}$ are natural projections.

Observe that the double cover $\vartheta$ is ramified in $\Pi_{1}+\Pi_{2}$, the morphism $\eta$ is a conic bundle, and the morphism $W \rightarrow \mathbb{P}^{1} \times \mathbb{P}^{1}$ is a $\mathbb{P}^{1}$-bundle.

In addition to the finite subgroup $G \cong \boldsymbol{\mu}_{2}^{3}$ described in the beginning of this section, the group $\operatorname{Aut}(X)$ contains a subgroup $\Gamma \cong \mathbb{C}^{*}$ that consists of the automorphisms

$$
\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right) \mapsto\left(\left[\frac{u_{1}}{\lambda}: \lambda v_{1}\right],\left[\frac{u_{2}}{\lambda}: \lambda v_{2}\right],\left[\lambda x: \frac{y}{\lambda}: \lambda z: \frac{t}{\lambda}: w\right]\right)
$$

where $\lambda \in \mathbb{C}^{*}$. Let $\mathbf{G}$ be the subgroup in $\operatorname{Aut}(X)$ generated by the subgroup $\Gamma$ together with the involutions $\tau_{1}, \tau_{2}, \tau_{3}$ described earlier in this section. Note that $\tau_{3} \in \Gamma$, so that

$$
\mathbf{G} \cong\left(\mathbb{C}^{*} \rtimes \boldsymbol{\mu}_{2}\right) \times \boldsymbol{\mu}_{2}
$$

because $\tau_{1} \circ \gamma=\gamma \circ \tau_{1}$ and $\tau_{2} \circ \gamma \circ \tau_{2}=\gamma^{-1}$ for every $\gamma \in \Gamma$. Note also that the commutative diagram (4.17) is G-equivariant, and the curve $C$ is G-invariant. Set

$$
C^{\prime}=\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t, w^{2}+(x+z)(y+t)=0, x=z, t=y\right\} \subset \mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4} .
$$

Then $C^{\prime}$ is a G-invariant irreducible smooth curve such that $\pi\left(C^{\prime}\right)$ is a smooth conic.
Lemma 4.18. The following assertions holds:
(1) $X$ does not have $\mathbf{G}$-fixed points;
(2) the only $\mathbf{G}$-invariant irreducible curves in $X$ are $C$ and $C^{\prime}$.

Proof. Since the diagram (4.17) is G-equivariant and $\tau_{2}$ swaps the $\pi$-exceptional surfaces, it is enough to prove the following assertions for the induced G-action on $\mathbb{P}^{3}$ :
(1) $\mathbb{P}^{3}$ does not have G-fixed points;
(2) the only G-invariant irreducible curves in $\mathbb{P}^{3}$ are the lines $\vartheta(L)$ and $\vartheta \circ \pi\left(C^{\prime}\right)$.

Both these assertions are easy to check, since $\tau_{1}$ and $\tau_{2}$ act on $\mathbb{P}^{3}$ as

$$
\begin{aligned}
& \tau_{1}:[x: y: z: t] \mapsto[y: x: t: z], \\
& \tau_{2}:[x: y: z: t] \mapsto[z: t: x: y],
\end{aligned}
$$

and the subgroup $\Gamma$ acts on $\mathbb{P}^{3}$ as $[x: y: z: t: w] \mapsto\left[\lambda^{2} x: y: \lambda^{2} z: t\right]$, where $\lambda \in \mathbb{C}^{*}$.
Let $Q \longrightarrow \mathbb{P}^{2}$ be the rational map given by $[x: y: z: t: w] \mapsto[x+z: y+t: w]$. This map is undefined along the line $\pi(C)=\operatorname{Sing}(Q)=\{x+z=0, y+t=0, w=0\}$, and the closure of its image is a smooth conic $\mathscr{C} \subset \mathbb{P}^{2}$. Therefore, we have a G-equivariant dominant map $\chi: Q \rightarrow \mathscr{C}$, which fits the following $\mathbf{G}$-equivariant commutative diagram:

where both $\widetilde{X}$ and $\widetilde{Q}$ are smooth threefolds, and

- $\phi$ is the blow of the curve $C$,
- $\widetilde{Q} \rightarrow \underset{Q}{Q}$ is the blow up of the line $L$,
- $\widetilde{X} \rightarrow \widetilde{Q}$ is the blow up of the preimages of the conics $C_{1}$ and $C_{2}$,
- $\mathrm{pr}_{1}$ is the projection to the first factor,
- $\mathrm{pr}_{2,3}$ is the projection to the product of the second and the third factors,
- $\widetilde{Q} \rightarrow \mathscr{C}$ is a $\mathbb{P}^{2}$-bundle,
- $\rho$ is a birational morphism.

To describe $\rho$ more explicitly, let $\mathscr{S}$ be the surface in $Q$ that is cut out by $x t=y z$, and let $\widetilde{\mathscr{S}}$ be its proper transform on $\widetilde{X}$. Then $\rho$ contracts the surface $\widetilde{\mathscr{S}}$ to a smooth curve. Note that the birational map $Q \rightarrow \mathscr{C} \times \mathbb{P}^{1} \times \mathbb{P}^{1}$ in (4.19) is given by

$$
[x: y: z: t: w] \mapsto([x+z: y+t: w],[x: y],[z: t])
$$

Furthermore, there exists a G-equivariant ramified double cover $v: \mathscr{C} \rightarrow \mathbb{P}^{1}$ that fits into the following $\mathbf{G}$-equivariant commutative diagram:

where $\tilde{\theta}$ is the quotient by the involution $\tau_{3}$, the map $\varphi$ is the blow up of the curve $\theta(C)$, and $\varrho$ is the blow up of the tridiagonal. Note that $\widetilde{W}$ is the unique smooth Fano threefold in the deformation family №4.6, and $\varpi \circ \varphi$ blows up the lines $\vartheta(L), \vartheta\left(C_{1}\right), \vartheta\left(C_{2}\right)$.

Let $H$ be the proper transform on $X$ of a general hyperplane section of the quadric $Q$, let $E_{1}$ and $E_{2}$ be the $\pi$-exceptional surfaces such that $\pi\left(E_{1}\right)=C_{1}$ and $\pi\left(E_{2}\right)=C_{2}$. Then

$$
-K_{X} \sim 3 H-E_{1}-E_{2}
$$

Let $\widetilde{H}, \widetilde{E}_{1}, \widetilde{E}_{2}$ be the proper transforms on $\widetilde{X}$ of the surfaces $H, E_{1}, E_{2}$, respectively, and let $F$ be the $\phi$-exceptional surface. Then $-K_{\widetilde{X}} \sim \phi^{*}\left(-K_{X}\right) \sim 3 \widetilde{H}-\widetilde{E}_{1}-\widetilde{E}_{2}$ and

$$
\widetilde{\mathscr{S}} \sim 2 \widetilde{H}-\widetilde{E}_{1}-\widetilde{E}_{2}-F
$$

Note that $F$ and $\widetilde{\mathscr{S}}$ are G-invariant prime divisors in $\widetilde{X}$, which intersect transversally along a smooth irreducible G-invariant curve. Furthermore, using (4.20), we also see that the surface $\widetilde{\theta}(F)$ is $\varphi$-exceptional, and the surface $\widetilde{\theta}(\widetilde{\mathscr{S}})$ is $\varrho$-exceptional.

Lemma 4.21. The following assertions hold:
(a) One has $F \cong \mathbb{P}^{1} \times \mathbb{P}^{1}$, and $F$ contains exactly two $\mathbf{G}$-invariant irreducible curves, which are both smooth and contained in the linear system $\left|\widetilde{\mathscr{S}}_{F}\right|$.
(b) One has $\widetilde{\mathscr{S}} \cong \mathbb{P}^{1} \times \mathbb{P}^{1}$, and $\widetilde{\mathscr{S}}$ contains exactly two $\mathbf{G}$-invariant irreducible curves, the intersection $F \cap \widetilde{\mathscr{S}}$ and the proper transform of the curve $C^{\prime}$, which are both smooth and contained in the linear system $|F|_{\tilde{\mathcal{I}}} \mid$.

Proof. An easy explicit calculation using (4.20) and the formulas for the G-action.
On $\widetilde{X}$, intersections of the divisors $\widetilde{H}, \widetilde{E}_{1}, \widetilde{E}_{2}, F$ can be computed as follows:

$$
\widetilde{H}^{3}=2, \widetilde{E}_{1}^{3}=\widetilde{E}_{2}^{3}=-4, F^{3}=-4, \widetilde{H} \cdot F^{2}=-2, \widetilde{H} \cdot \widetilde{E}_{1}^{2}=\widetilde{H} \cdot \widetilde{E}_{2}^{2}=-2
$$

and all other triple intersections are zero.
Lemma 4.22. Let $S$ be an $\mathbf{G}$-invariant prime divisor in $\widetilde{X}$. Then $\beta(S)>0$.
Proof. Let $\widetilde{S}_{1}$ and $\widetilde{S}_{2}$ be proper transforms on $\widetilde{X}$ of the surfaces $S_{1}$ and $S_{2}$, respectively. Using (4.19), one can show that $S \sim a \widetilde{H}+b\left(\widetilde{E}_{1}+\widetilde{E}_{2}\right)+c\left(\widetilde{S}_{1}+\widetilde{S}_{2}\right)+d \widetilde{\mathscr{S}}+e F$ for some non-negative integers $a, b, c, d, e$. Thus, to prove the lemma, it is enough to show that

$$
\frac{1}{26} \int_{0}^{\infty} \operatorname{vol}\left(-K_{\tilde{X}}-u D\right) d u<1
$$

for each $D \in\left\{\widetilde{H}, \widetilde{E}_{1}+\widetilde{E}_{2}, \widetilde{S}_{1}+\widetilde{S}_{2}, \widetilde{\mathscr{S}}, F\right\}$. The computation in the first three cases is very similar to the proof of Corollary 4.7; we'll illustrate the cases $D=F$ and $D=\widetilde{\mathscr{S}}$.

First, we compute $\beta(F)$. Let $u$ be a non-negative real number. Then

$$
-K_{\tilde{X}}-u F \sim_{\mathbb{R}} 3 \widetilde{H}-\widetilde{E}_{1}-\widetilde{E}_{2}-u F \sim_{\mathbb{R}} \widetilde{\mathscr{S}}+\widetilde{S}_{1}+\widetilde{S}_{2}+(2-u) F
$$

because $2 \widetilde{S}_{1} \sim 2 \widetilde{S}_{2} \sim \widetilde{H}-F$. This shows that $-K_{\tilde{X}}-u F$ is pseudo-effective $\Longleftrightarrow u \leqslant 2$. Similarly, we see that $-K_{\tilde{X}}-u F$ is nef $\Longleftrightarrow u \in[0,1]$. Furthermore, if $u \in[1,2]$, then the Zariski decomposition of this divisor can be described as follows:

$$
-K_{\tilde{X}}-u F \sim_{\mathbb{R}} \underbrace{-K_{\tilde{X}}-u F-(u-1) \widetilde{\mathscr{S}}}_{\text {positive part }}+\underbrace{(u-1) \widetilde{\mathscr{S}}}_{\text {negative part }}
$$

Hence, we have

$$
\begin{gathered}
\beta(F)=1-\frac{1}{26} \int_{0}^{1}\left(-K_{\tilde{X}}-u F\right)^{2} d u-\frac{1}{26} \int_{1}^{2}\left(-K_{\tilde{X}}-u F-(u-1) \widetilde{\mathscr{S}}\right)^{2} d u= \\
=1-\frac{1}{26} \int_{0}^{1}\left(3 \widetilde{H}-\widetilde{E}_{1}-\widetilde{E}_{2}-u F\right)^{2} d u-\frac{1}{26} \int_{1}^{2}\left((5-2 u) \widetilde{H}-(2-u) \widetilde{E}_{1}-(2-u) \widetilde{E}_{2}-F\right)^{2} d u= \\
=1-\frac{1}{26} \int_{0}^{1} 4 u^{3}-18 u^{2}+26 d u-\frac{1}{26} \int_{1}^{2} 12(u-2)^{2} d u=\frac{1}{26}>0 .
\end{gathered}
$$

Now, we compute $\beta(\widetilde{\mathscr{S}})$. As above, we observe that

$$
-K_{\tilde{X}}-u \widetilde{\mathscr{S}} \sim_{\mathbb{R}} \frac{3-2 u}{2} \widetilde{\mathscr{S}}+\frac{1}{2}\left(\widetilde{E}_{1}+\widetilde{E}_{2}\right)+\frac{3}{2} F .
$$

This shows that the divisor $-K_{\tilde{X}}-u \widetilde{\mathscr{S}}$ is pseudo-effective if and only if $u \leqslant \frac{3}{2}$. If $u \in[0,1]$, then the positive part of the Zariski decomposition of this divisor is

$$
\frac{3-2 u}{2}(\widetilde{\mathscr{S}}+F)+\frac{1}{2}\left(\widetilde{E}_{1}+\widetilde{E}_{2}\right)
$$

and the negative part is $u F$. If $1 \leqslant u \leqslant \frac{3}{2}$, the Zariski decomposition is the following:

$$
-K_{\tilde{X}}-u \widetilde{\mathscr{S}} \sim_{\mathbb{R}} \underbrace{\frac{3-2 u}{2}\left(\widetilde{\mathscr{S}}+F+\widetilde{E}_{1}+\widetilde{E}_{2}\right)}_{\text {positive part }}+\underbrace{u F+(u-1)\left(\widetilde{E}_{1}+\widetilde{E}_{2}\right)}_{\text {negative part }}
$$

Note that $\widetilde{\mathscr{S}}+F+\widetilde{E}_{1}+\widetilde{E}_{2} \sim 2 \widetilde{H}$. Integrating, we get $\frac{1}{26} \int_{0}^{\infty} \operatorname{vol}\left(-K_{\tilde{X}}-u \widetilde{\mathscr{S}}\right) d u=\frac{49}{104}$.
Now we are ready to prove that $X$ is K-polystable. Suppose that $X$ is not K-polystable. By [8, Corollary 4.14], there exists a G-invariant prime divisor $\mathbf{F}$ over $X$ with $\beta(\mathbf{F}) \leqslant 0$. Let $Z$ be its center on $\widetilde{X}$. Then $Z$ is a G-invariant curve by Lemmas 4.18 and 4.22,

Lemma 4.23. One has $Z \not \subset F$.
Proof. Let $\mathbf{s}$ be a section of the projection $F \rightarrow C$ such that $\mathbf{s}^{2}=0$, and let $\mathbf{f}$ be its fiber. Then $\widetilde{E}_{1} \cap F=\varnothing$ and $\widetilde{E}_{2} \cap F=\varnothing$. Moreover, we compute $\left.F\right|_{F} \sim-2 \mathbf{s}+\mathbf{f}$ and $\left.\widetilde{H}\right|_{F} \sim \mathbf{f}$, which gives $\widetilde{\mathscr{S}}_{F} \sim 2 \mathbf{s}+\mathbf{f}$.

Suppose that $Z \subset F$. Then $Z \sim 2 \mathbf{s}+\mathbf{f}$, by Lemma4.21(a). For every number $u \in[0,2]$, let $P(u)$ be the positive part of the Zariski decomposition of $-K_{\tilde{X}}-u F$ described in the proof of Lemma 4.22, and let $N(u)$ be its negative part. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$
\left.P(u)\right|_{F}-v Z=\left\{\begin{array}{l}
(2 u-2 v) \mathbf{s}+(3-u-v) \mathbf{f} \text { if } 0 \leqslant u \leqslant 1 \\
(2-2 v) \mathbf{s}+(4-2 u-v) \mathbf{f} \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

and

$$
\left.N(u)\right|_{F}=\left\{\begin{array}{l}
0 \text { if } 0 \leqslant u \leqslant 1, \\
\left.(u-1) \widetilde{\mathscr{S}}\right|_{F} \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

For $u \in[0,2]$, set $t(u)=\sup \left\{v \in \mathbb{R}_{\geqslant 0}|P(u)|_{F}-v Z\right.$ is pseudo-effective $\}$. Then

$$
t(u)=\left\{\begin{array}{l}
u \text { if } 0 \leqslant u \leqslant 1 \\
1 \text { if } 1 \leqslant u \leqslant \frac{3}{2} \\
4-2 u \text { if } \frac{3}{2} \leqslant u \leqslant 2
\end{array}\right.
$$

Moreover, it follows from [2, Theorem 3.3] and [4, Corollary 1.109] that

$$
1 \geqslant \frac{A_{X}(\mathbf{F})}{S_{X}(\mathbf{F})} \geqslant \min \left\{\frac{1}{S_{X}(F)}, \frac{1}{S\left(W_{\bullet, \bullet}^{F} ; Z\right)}\right\},
$$

where

$$
S\left(W_{\bullet, \bullet}^{F} ; Z\right)=\frac{3}{26} \int_{0}^{2}\left(\left.P(u)\right|_{F}\right)^{2} \operatorname{ord}_{Z}\left(\left.N(u)\right|_{F}\right) d u+\frac{3}{26} \int_{0}^{2} \int_{0}^{t(u)}\left(\left.P(u)\right|_{F}-v Z\right)^{2} d v d u
$$

In the proof of Lemma 4.22, we computed $S_{X}(F)=\frac{25}{26}$, so $S\left(W_{\bullet, \bullet}^{F} ; Z\right) \geqslant 1$. But

$$
\begin{aligned}
& S\left(W_{\bullet, \bullet}^{F} ; Z\right)=\frac{3}{26} \int_{1}^{2}(16-8 u)^{2}(u-1) \operatorname{ord}_{Z}\left(\left.\widetilde{\mathscr{S}}\right|_{F}\right) d u+ \\
&+\frac{3}{26} \int_{0}^{1} \int_{0}^{u} 4(3-u-v)(u-v) d v d u+\frac{3}{26} \int_{1}^{\frac{3}{2}} \int_{0}^{1} 4(1-v)(4-2 u-v) d v d u+ \\
&+\frac{3}{26} \int_{\frac{3}{2}}^{2} \int_{0}^{4-2 u} 4(1-v)(4-2 u-v) d v d u=\frac{2}{13} \operatorname{ord}_{Z}\left(\left.\widetilde{\mathscr{S}}\right|_{F}\right)+\frac{33}{104} \leqslant \frac{49}{104}<1,
\end{aligned}
$$

which is a contradiction
Lemma 4.24. One has $Z \not \subset \widetilde{\mathscr{S}}$.
Proof. Set $\mathbf{s}=\left.F\right|_{\tilde{\mathscr{S}}}$. Then $\mathbf{s}$ is a ruling of $\widetilde{\mathscr{S}} \cong \mathbb{P}^{1} \times \mathbb{P}^{1}$. Let $\mathbf{f}$ be a ruling of this surface such that $\mathbf{s} \cdot \mathbf{f}=1$ and $\mathbf{f}^{2}=0$. Then $\left.\left.\widetilde{E}_{1}\right|_{\tilde{\mathscr{S}}} \sim \widetilde{E}_{2}\right|_{\tilde{\mathscr{S}}} \sim \mathbf{s},\left.\widetilde{H}\right|_{\tilde{\mathscr{S}}} \sim \mathbf{s}+2 \mathbf{f},\left.\widetilde{\mathscr{S}}\right|_{\tilde{\mathscr{S}}} \sim-\mathbf{s}+4 \mathbf{f}$.

Suppose $Z \subset \widetilde{\mathscr{S}}$. Then $Z \sim \mathbf{s}$ by Lemma 4.21 (b). For $u \in\left[0, \frac{3}{2}\right]$, let $P(u)$ be the positive part of the Zariski decomposition of $-K_{\tilde{X}}-u \widetilde{\mathscr{S}}$ described in the proof of Lemma 4.22, and let $N(u)$ be its negative part. Take $v \in \mathbb{R}_{\geqslant 0}$. Then

$$
\left.P(u)\right|_{\tilde{\mathscr{I}}}-v Z=\left\{\begin{array}{l}
(1-v) \mathbf{s}+(6-4 u) \mathbf{f} \text { if } 0 \leqslant u \leqslant 1, \\
(3-2 u-v) \mathbf{s}+(6-4 u) \mathbf{f} \text { if } 1 \leqslant u \leqslant \frac{3}{2},
\end{array}\right.
$$

and

$$
\left.N(u)\right|_{\tilde{\mathscr{S}}}=\left\{\begin{array}{l}
u \mathbf{s} \text { if } 0 \leqslant u \leqslant 1, \\
u \mathbf{s}+\left.(u-1) \widetilde{E}_{1}\right|_{\tilde{\mathscr{S}}}+\left.(u-1) \widetilde{E}_{1}\right|_{\tilde{\mathscr{S}}} \text { if } 1 \leqslant u \leqslant \frac{3}{2} .
\end{array}\right.
$$

Note that $Z \not \subset \widetilde{E}_{1} \cup \widetilde{E}_{2}$, because $Z$ is G-invariant.

For $u \in\left[0, \frac{3}{2}\right]$, set $t(u)=\sup \left\{v \in \mathbb{R}_{\geqslant 0}|P(u)|_{\tilde{\mathscr{S}}}-v Z\right.$ is pseudo-effective $\}$. Then

$$
t(u)=\left\{\begin{array}{l}
1 \text { if } 0 \leqslant u \leqslant 1 \\
3-2 u \text { if } 1 \leqslant u \leqslant \frac{3}{2}
\end{array}\right.
$$

Moreover, it follows from [2, Theorem 3.3] and [4, Corollary 1.109] that

$$
1 \geqslant \frac{A_{X}(\mathbf{F})}{S_{X}(\mathbf{F})} \geqslant \min \left\{\frac{1}{S_{X}(\widetilde{\mathscr{S}})}, \frac{1}{S\left(W_{\bullet, \bullet} ; Z\right)}\right\}
$$

where

$$
S\left(W_{\bullet, \bullet} \tilde{\mathscr{\mathscr { P }}} ; Z\right)=\frac{3}{26} \int_{0}^{\frac{3}{2}}\left(\left.P(u)\right|_{\tilde{\mathscr{I}}}\right)^{2} \operatorname{ord}_{Z}\left(\left.N(u)\right|_{\tilde{\mathscr{I}}}\right) d u+\frac{3}{26} \int_{0}^{\frac{3}{2}} \int_{0}^{t(u)}\left(\left.P(u)\right|_{\tilde{\mathscr{P}}}-v Z\right)^{2} d v d u
$$

We know from the proof of Lemma 4.22 that $S_{X}(\widetilde{\mathscr{S}})=\frac{49}{54}$. Then $S\left(W_{\bullet, \bullet}^{\widetilde{\mathscr{S}}} ; Z\right) \geqslant 1$. But

$$
\begin{aligned}
& S\left(W_{\bullet, \bullet}^{\tilde{\mathscr{I}}} ; Z\right)=\frac{3}{26} \int_{0}^{1} 4 u(3-2 u) \operatorname{ord}_{Z}(\mathbf{s}) d u+\frac{3}{26} \int_{1}^{\frac{3}{2}} 4 u(3-2 u)^{2} \operatorname{ord}_{Z}(\mathbf{s}) d u+ \\
+ & \frac{3}{26} \int_{0}^{1} \int_{0}^{1} 4(1-v)(3-2 u) d v d u+\frac{3}{26} \int_{1}^{\frac{3}{2}} \int_{0}^{3-2 u} 4(3-2 u)(3-2 u-v) d v d u=\frac{49 \operatorname{ord}_{Z}(\mathbf{s})+51}{104},
\end{aligned}
$$

which implies that $S\left(W_{\bullet, \bullet}^{\widetilde{\mathscr{P}}} ; Z\right) \leqslant \frac{25}{26}$. This is a contradiction.
By Lemmas 4.23 and 4.24, we have $Z \not \subset F \cup \widetilde{\mathscr{S}}$. But $Z \subset F \cup \widetilde{\mathscr{S}}$ by Lemma 4.18, The obtained contradiction implies that the Fano threefold $X$ is K-polystable, completing the proof of Theorem 4.16

## 5. Threefolds in the Class ( $\aleph$ )

Let $X$ be the following complete intersection in $\mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4}$ :
$\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t, w^{2}+r(x+y)^{2}+r(z+t)^{2}=(2 s+2)(x t+y z)+(2 s-2)(x z+y t)\right\}$, where $[r: s] \in \mathbb{P}^{1}$, and $\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right)$ are coordinates on $\mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4}$. For every $[r: s]$, the threefold $X$ is singular at the following two points

$$
\begin{aligned}
& ([1: 1],[-1: 1],[0: 0:-1: 1: 0]), \\
& ([-1: 1],[1: 1],[-1: 1: 0: 0: 0]),
\end{aligned}
$$

which are both isolated ordinary double singularities of the threefold $X$ if $[r: s] \neq[0: 1]$. Furthermore, if $[r: s]=[ \pm 1: 1]$, then the threefold $X$ has an additional isolated double singularity at the point

$$
([1: 1],[1: 1],[1: 1: \pm 1: \pm 1: 0])
$$

Finally, if $[r: s]=[0: 1]$, then $\operatorname{Sing}(X)$ consists of the following two curves:

$$
\begin{aligned}
& \left\{x=0, y=0, w=0, u_{1} z+v_{1} t=0, u_{2} z-v_{2} t=0, v_{1} u_{2}+u_{1} v_{2}=0\right\} \\
& \left\{z=0, t=0, w=0, u_{1} x-v_{1} y=0, u_{2} x+v_{2} y=0, v_{1} u_{2}+u_{1} v_{2}=0\right\}
\end{aligned}
$$

This is a very special case (see Section 5.3 for more details).

Let $Q$ be the quadric $\left\{w^{2}+r(x+y)^{2}+r(z+t)^{2}=(2 s+2)(x t+y z)+(2 s-2)(x z+y t)\right\} \subset \mathbb{P}^{4}$. Then $Q$ is singular $\Longleftrightarrow[r: s]=[ \pm 1: 1]$. If $[r: s]=[ \pm 1: 1]$, then

$$
\operatorname{Sing}(Q)=[1: 1: \pm 1: \pm 1: 0]
$$

Let $\pi: X \rightarrow Q$ be the birational morphism that is induced by the projection $X \rightarrow \mathbb{P}^{4}$. Then $\pi$ is a blow up of the following two singular conics:

$$
\begin{aligned}
& C_{1}=\left\{x=0, y=0, w^{2}+r(z+t)^{2}=0\right\}, \\
& C_{2}=\left\{z=0, t=0, w^{2}+r(x+y)^{2}=0\right\} .
\end{aligned}
$$

These conics are contained in the smooth locus of the quadric $Q$, and they are reduced unless $[r: s]=[0: 1]$. If $[r: s]=[0: 1]$, both conics are double lines.

As in Section 4, we see that $\operatorname{Aut}(X)$ contains the following commuting involutions:

$$
\begin{aligned}
& \tau_{1}:\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right) \mapsto\left(\left[v_{1}: u_{1}\right],\left[v_{2}: u_{2}\right],[y: x: t: z: w]\right), \\
& \tau_{2}:\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right) \mapsto\left(\left[u_{2}: v_{2}\right],\left[u_{1}: v_{1}\right],[z: t: x: y: w]\right), \\
& \tau_{3}:\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right) \mapsto\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t:-w]\right) .
\end{aligned}
$$

Note that $\pi$ is $\operatorname{Aut}(X)$-equivariant, and these involutions act on $\mathbb{P}^{4}$ as in Section 3:

$$
\begin{aligned}
& \tau_{1}:[x: y: z: t: w] \mapsto[y: x: t: z: w] \\
& \tau_{2}:[x: y: z: t: w] \mapsto[z: t: x: y: w] \\
& \tau_{3}:[x: y: z: t: w] \mapsto[x: y: z: t:-w]
\end{aligned}
$$

These involutions generate a subgroup in $\operatorname{Aut}(X)$ isomorphic to $\boldsymbol{\mu}_{2}^{3}$. But $\operatorname{Aut}(X)$ is larger. In fact, it is infinite. Indeed, consider the monomorphism $\mathbb{C}^{*} \hookrightarrow \mathrm{PGL}_{4}(\mathbb{C})$ given by

$$
\lambda \mapsto\left(\begin{array}{ccccc}
\frac{1+\lambda}{2} & \frac{1-\lambda}{2} & 0 & 0 & 0 \\
\frac{1-\lambda}{2} & \frac{1+\lambda}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{\lambda+1}{2 \lambda} & \frac{\lambda-1}{2 \lambda} & 0 \\
0 & 0 & \frac{\lambda-1}{2 \lambda} & \frac{\lambda+1}{2 \lambda} & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

Note that this $\mathbb{C}^{*}$-action leaves $Q$ invariant. This gives a monomorphism $\mathbb{C}^{*} \hookrightarrow \operatorname{Aut}(Q)$. Since this $\mathbb{C}^{*}$-action also leaves both planes $\{x=0, y=0\}$ and $\{z=0, t=0\}$ invariant, it leaves both conics $C_{1}$ and $C_{2}$ invariant, because

$$
\begin{aligned}
& C_{1}=Q \cap\{x=0, y=0\}, \\
& C_{2}=Q \cap\{z=0, t=0\} .
\end{aligned}
$$

Thus the $\mathbb{C}^{*}$-action lifts to $X$. Let $\Gamma \cong \mathbb{C}^{*}$ be the corresponding subgroup in $\operatorname{Aut}(X)$.
Let $\mathbf{G}$ be the subgroup in $\operatorname{Aut}(X)$ generated by $\Gamma$ together with $\tau_{1}, \tau_{2}, \tau_{3}$. Then

$$
\mathbf{G} \cong\left(\mathbb{C}^{*} \rtimes \boldsymbol{\mu}_{2}\right) \times \boldsymbol{\mu}_{2}
$$

because $\tau_{1} \in \Gamma$, and $\tau_{3} \circ \lambda=\lambda \circ \tau_{3}$ and $\tau_{2} \circ \lambda \circ \tau_{2}=\lambda^{-1}$ for every $\lambda \in \Gamma$.
Lemma 5.1. If $[r: s] \neq[ \pm 1: 1]$, then the Fano threefold $X$ does not have $\mathbf{G}$-fixed points. If $[r: s]=[ \pm 1: 1]$, then the only $\mathbf{G}$-fixed point in $X$ is the singular point

$$
([1: 1],[1: 1],[1: 1: \pm 1: \pm 1: 0])
$$

Proof. To simplify the G-action, let us introduce new coordinates on $\mathbb{P}^{4}$ as follows:

$$
\left\{\begin{array}{l}
\mathbf{x}=x-y \\
\mathbf{y}=x+y-z-t \\
\mathbf{z}=z-t \\
\mathbf{t}=x+y+z+t \\
\mathbf{w}=w
\end{array}\right.
$$

In new coordinates, the defining equation of the quadric $Q$ simplifies to

$$
2 \mathbf{x z}+\frac{r+s}{2} \mathbf{y}^{2}+\frac{r-s}{2} \mathbf{t}^{2}+\mathbf{w}^{2}=0 .
$$

Moreover, we have $C_{1}=\{\mathbf{x}=0, \mathbf{y}+\mathbf{t}=0\} \cap Q$ and $C_{2}=\{\mathbf{z}=0, \mathbf{y}-\mathbf{t}=0\} \cap Q$. The involutions $\tau_{2}$ and $\tau_{3}$ act as

$$
\begin{aligned}
& \tau_{2}:[\mathbf{x}: \mathbf{y}: \mathbf{z}: \mathbf{t}: \mathbf{w}] \mapsto[\mathbf{z}:-\mathbf{y}: \mathbf{x}: \mathbf{t}: \mathbf{w}] \\
& \tau_{3}:[\mathbf{x}: \mathbf{y}: \mathbf{z}: \mathbf{t}: \mathbf{w}] \mapsto[\mathbf{x}: \mathbf{y}: \mathbf{z}: \mathbf{t}:-\mathbf{w}]
\end{aligned}
$$

and the action of the group $\Gamma \cong \mathbb{C}^{*}$ simplifies to

$$
[\mathbf{x}: \mathbf{y}: \mathbf{z}: \mathbf{t}: \mathbf{w}] \mapsto\left[\lambda \mathbf{x}: \mathbf{y}: \frac{\mathbf{z}}{\lambda}: \mathbf{t}: \mathbf{w}\right]
$$

where $\lambda \in \mathbb{C}^{*}$. In new coordinates, the $\mathbf{G}$-fixed points in $\mathbb{P}^{4}$ are

$$
\begin{aligned}
& {[0: 1: 0: 0: 0],} \\
& {[0: 0: 0: 1: 0],} \\
& {[0: 0: 0: 0: 1] .}
\end{aligned}
$$

Note that $[0: 1: 0: 0: 0] \in Q \Longleftrightarrow r+s=0$, and $[0: 0: 0: 1: 0] \in Q \Longleftrightarrow r-s=0$. Moreover, we have $[0: 0: 0: 0: 1] \notin Q$ for every $[r: s] \in \mathbb{P}^{1}$, so the assertion follows.

Arguing as in the proof of Lemma 5.1, we obtain the following result.
Lemma 5.2. Suppose that $[r: s]=[ \pm 1: 1]$. Let $O=([1: 1],[1: 1],[1: 1: \pm 1: \pm 1: 0])$, let $\varphi: \widetilde{X} \rightarrow X$ be the blow up of the point $O$, and let $F$ be the $\varphi$-exceptional surface. Then
(1) $\varphi$ is G-equivariant,
(2) the group $\mathbf{G}$ acts faithfully on $F$,
(3) $\mathrm{rkPic}^{\mathbf{G}}(F)=1$,
(4) $F$ does not contain $\mathbf{G}$-fixed points,
(5) $F \cong \mathbb{P}^{1} \times \mathbb{P}^{1}$ contains exactly two $\mathbf{G}$-invariant irreducible curves, which have bidegree $(1,1)$.

Proof. Let us only consider the case when $[r: s]=[1: 1]$; the other case is analogous. We use the setup and notation introduced in the proof of Lemma 5.1. It follows from this proof that we can G-equivariantly identify the surface $F$ with the quadric surface

$$
\left\{\mathbf{w}^{2}+2 \mathbf{x z}+\mathbf{y}^{2}=0\right\} \subset \mathbb{P}^{3}
$$

where we consider $[\mathbf{x}: \mathbf{y}: \mathbf{z}: \mathbf{w}]$ as coordinates on $\mathbb{P}^{3}$. Now our claims are easy to check.
For instance, if $Z$ is a G-invariant irreducible curve in $F$, then $\Gamma$ acts faithfully on $Z$, which implies that $\tau_{2}$ fixes a point on $Z$. Then, analyzing $\left\langle\tau_{2}\right\rangle$-fixed points in $F$, we see that either $Z=F \cap\{\mathbf{y}=0\}$ or $Z=F \cap\{\mathbf{w}=0\}$ as claimed.

We have the following $G$-equivariant diagram:

where $\eta$ is the conic bundle that is given by the natural projection $\mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4} \rightarrow \mathbb{P}^{1} \times \mathbb{P}^{1}$. Let $\Delta$ be its discriminant curve in $\mathbb{P}^{1} \times \mathbb{P}^{1}$. Then $\Delta=\Delta_{1}+\Delta_{2}$, where

$$
\begin{aligned}
& \Delta_{1}=\left\{(r-s+1) u_{1} u_{2}+(r-s-1) u_{1} v_{2}+(r-s-1) u_{2} v_{1}+(r-s+1) v_{1} v_{2}=0\right\}, \\
& \Delta_{2}=\left\{(r+s-1) u_{1} u_{2}+(r+s+1) u_{1} v_{2}+(r+s+1) u_{2} v_{1}+(r+s-1) v_{1} v_{2}=0\right\}
\end{aligned}
$$

If $[r: s] \neq[0: 1]$, then they meet transversally at $([1:-1],[1: 1])$ and $([1: 1],[1:-1])$. The curves $\Delta_{1}$ and $\Delta_{2}$ are smooth for $[r: s] \neq[ \pm 1: 1]$. If $[r: s]=[1: 1]$, then

$$
\begin{aligned}
\Delta_{1} & =\left\{\left(u_{1}-v_{1}\right)\left(u_{2}-v_{2}\right)=0\right\} \\
\Delta_{2} & =\left\{u_{1} u_{2}+3 u_{1} v_{2}+3 u_{2} v_{1}+v_{1} v_{2}=0\right\}
\end{aligned}
$$

Similarly, if $[r: s]=[1:-1]$, then

$$
\begin{aligned}
\Delta_{1} & =\left\{3 u_{1} u_{2}+u_{1} v_{2}+u_{2} v_{1}+3 v_{1} v_{2}=0\right\} \\
\Delta_{2} & =\left\{\left(u_{1}-v_{1}\right)\left(u_{2}-v_{2}\right)=0\right\}
\end{aligned}
$$

Finally, if $[r: s]=[0: 1]$, then $\Delta_{1}=\Delta_{2}$. We will deal with this case in Section 5.3,
Remark 5.3. The only G-fixed points in $\mathbb{P}^{1} \times \mathbb{P}^{1}$ are $([1: 1],[1: 1])$ and $([1:-1],[1:-1])$.
Suppose $[r: s] \neq[0: 1]$. Let $E_{1}$ and $E_{2}$ be the $\left\langle\tau_{3}\right\rangle$-irreducible reducible $\pi$-exceptional surfaces such that $\pi\left(E_{1}\right)=C_{1}$ and $\pi\left(E_{2}\right)=C_{2}$. Set $H=\pi^{*}\left(\mathcal{O}_{Q}(1)\right)$. Then

$$
\operatorname{Pic}^{\mathbf{G}}(X)=\mathrm{Cl}^{\mathbf{G}}(X)=\mathbb{Z}[H] \oplus \mathbb{Z}\left[E_{1}+E_{2}\right]
$$

This easily follows from Lemma 5.2, since $C_{1}+C_{2}$ is G-irreducible.
Lemma 5.4. Let $S$ be any $\mathbf{G}$-invariant irreducible surface in $X$. Then $\beta(S)>0$.
Proof. The conic bundle $\eta: X \rightarrow \mathbb{P}^{1} \times \mathbb{P}^{1}$ is given by the linear system $\left|2 H+E_{1}+E_{2}\right|$. Therefore, arguing as in the proof of Lemma 4.6, we see that

$$
S \sim a\left(E_{1}+E_{2}\right)+b\left(2 H-E_{1}-E_{2}\right)+c H
$$

for some non-negative integers $a, b, c$. So, arguing exactly as in the proof of Corollary 4.7, we obtain the required assertion.

We conclude this section with the following technical lemma:
Proposition 5.5. Let $S$ be a smooth surface in $\left|H-E_{1}\right|$ or $\left|H-E_{2}\right|$. Then

$$
\delta_{P}(X) \geqslant \frac{104}{99}
$$

for every point $P \in S$ such that $P \notin E_{1} \cup E_{2}$.
Proof. The proof is identical to the proof of Proposition 4.8.
5.1. Threefolds with two singular points. As before, we continue to use the notation and assumptions introduced earlier in this section. Suppose that

$$
[r: s] \neq[ \pm 1: 1]
$$

and $[r: s] \neq[0: 1]$. The goal of this subsection is to prove the following.
Theorem 5.6. Let $X$ be a singular Fano threefold from class ( $\aleph$ ) with two singular points. Then $X$ is $K$-polystable.

The proof of this theorem follows the method used to prove Theorem4.9. We begin with a technical lemma.

Lemma 5.7. Suppose that $X$ is a Fano threefold from class ( $\aleph$ ) with isolated singularities. Let $\mathbf{F}$ be a $\mathbf{G}$-invariant prime divisor over $X$ with $\beta(\mathbf{F}) \leqslant 0$, and let $Z$ be its center on $X$. Suppose that $Z$ is not a $\mathbf{G}$-fixed singular point of the threefold $X$. Then $Z$ is an irreducible curve such that $\eta(Z)=([1: 1],[1: 1])$ or $\eta(Z)=([1:-1],[1:-1])$.

Proof. The proof is identical to the proof of Lemma 4.10, using Lemmas 5.1 and 5.4, and also Proposition 5.5.

Now we prove Theorem 5.6. Suppose that the singular threefold $X$ is not K-polystable. By [8, Corollary 4.14], there is a G-invariant prime divisor $\mathbf{F}$ over $X$ such that $\beta(\mathbf{F}) \leqslant 0$. Let $Z$ be the center of $\mathbf{F}$ on $X$. We seek a contradiction.

By Lemma 5.1, the threefold $X$ does not contain G-fixed singular points. Furthermore, the fibers of the conic bundle $\eta$ over the points $([1: 1],[1: 1])$ and $([1:-1],[1:-1])$ are smooth. So, applying Lemma 4.10, we see that $Z$ is the fiber of the conic bundle $\eta$ over one of the points $([1: 1],[1: 1])$ or $([1:-1],[1:-1])$.

Let $S$ be the unique surface in $\left|H-E_{1}\right|$ that contains $Z$. Observe the following:
(1) if $\eta(Z)=([1: 1],[1: 1])$, then $S$ is cut out on $X$ by $u_{1}-v_{1}=0$, and

$$
\operatorname{Sing}(S)=([1: 1],[-1: 1],[0: 0:-1: 1: 0]) \in \operatorname{Sing}(X)
$$

the surface $\pi(S)$ is a quadric cone with vertex $[0: 0:-1: 1: 0]=\operatorname{Sing}\left(C_{1}\right)$, the intersection $\pi(S) \cap C_{2}$ consists of two distinct smooth points of the conic $C_{2}$, and $\pi$ induces a birational morphism $S \rightarrow \pi(S)$ that blows up these two points;
(2) if $\eta(Z)=([1:-1],[1:-1])$, then $S$ is cut out on $X$ by $u_{1}+v_{1}=0$, and

$$
\operatorname{Sing}(S)=([1:-1],[1,1],[-1: 1: 0: 0: 0]) \in \operatorname{Sing}(X)
$$

the quadric surface $\pi(S)$ is smooth, $\operatorname{Sing}\left(C_{2}\right) \in \pi(S)$, and $\pi$ induces a birational morphism $S \rightarrow \pi(S)$, which is a weighted blow up of the point $\pi(S) \cap C_{2}$.
In both cases, the surface $S$ is a singular sextic del Pezzo surface with one ordinary double point, and the curve $Z$ is contained in the smooth locus of the surface $S$.

Let us apply Abban-Zhuang theory [2, [4] to the flag $Z \subset S$. Take $u \in \mathbb{R}_{\geqslant 0}$. Set

$$
P(u)=\left\{\begin{array}{l}
(3-u) H-(1-u) E_{1}-E_{2} \text { if } 0 \leqslant u \leqslant 1 \\
(3-u) H-E_{2} \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

and

$$
N(u)=\left\{\begin{array}{l}
0 \text { if } 0 \leqslant u \leqslant 1 \\
(u-1) E_{1} \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

Then it follows from the proof of Proposition 4.8 that $P(u)$ and $N(u)$ are the positive and the negative parts of the Zariski decomposition of $-K_{X}-u S$ for $u \in[0,2]$, respectively. Moreover, if $u>2$, then $-K_{X}-u S$ is not pseudo-effective. This gives $S_{X}(S)=\frac{3}{4}$.

Now, we take $v \in \mathbb{R}_{\geqslant 0}$ and consider the divisor $\left.P(u)\right|_{S}-v Z$. For $u \in[0,2]$, set

$$
t(u)=\sup \left\{v \in \mathbb{R}_{\geqslant 0} \mid \text { the divisor }\left.P(u)\right|_{S}-v Z \text { is pseudo-effective }\right\} .
$$

Let $P(u, v)$ be the positive part of the Zariski decomposition of $\left.P(u)\right|_{S}-v Z$ for $v \leqslant t(u)$. Note that $Z \not \subset \operatorname{Supp}(N(u))$ for every $u \in[0,2]$. Thus, it follows from [2, 4] that

$$
\frac{A_{X}(\mathbf{F})}{S_{X}(\mathbf{F})} \geqslant \min \left\{\frac{1}{S_{X}(S)}, \frac{1}{S\left(W_{\bullet, \bullet}^{S} ; Z\right)}\right\}
$$

where

$$
S\left(W_{\bullet, \bullet}^{S} ; Z\right)=\frac{3}{\left(-K_{X}\right)^{3}} \int_{0}^{2} \int_{0}^{t(u)}(P(u, v))^{2} d v d u .
$$

Since $S_{X}(S)=\frac{3}{4}$ and $\beta(\mathbf{F})=A_{X}(\mathbf{F})-S_{X}(\mathbf{F}) \leqslant 0$, we conclude that $S\left(W_{\bullet, 0}^{S} ; Z\right) \geqslant 1$.
Let us compute $S\left(W_{\bullet, \bullet}^{S} ; Z\right)$. Set $\mathbf{e}=\left.E_{2}\right|_{S}$. Then $\mathbf{e}^{2}=-2$ and $Z \cdot \mathbf{e}=2$. We have

$$
\left.P(u)\right|_{S}-v Z=\left\{\begin{array}{l}
(2-v) Z+\mathbf{e} \text { if } 0 \leqslant u \leqslant 1 \\
(3-u-v) Z+(2-u) \mathbf{e} \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

which gives

$$
t(u)=\left\{\begin{array}{l}
2 \text { if } 0 \leqslant u \leqslant 1 \\
3-u \text { if } 1 \leqslant u \leqslant 2
\end{array}\right.
$$

because $\operatorname{Supp}(\mathbf{e})$ is contractible. Furthermore, if $u \in[0,1]$, then

$$
P(u, v)=\left\{\begin{array}{l}
(2-v) Z+\mathbf{e} \text { if } 0 \leqslant v \leqslant 1 \\
(2-v)(Z+\mathbf{e}) \text { if } 1 \leqslant v \leqslant 2
\end{array}\right.
$$

which gives

$$
(P(u, v))^{2}=\left\{\begin{array}{l}
6-4 v \text { if } 0 \leqslant v \leqslant 1 \\
2(v-2)^{2} \text { if } 1 \leqslant v \leqslant 2
\end{array}\right.
$$

Similarly, if $u \in[1,2]$, then

$$
P(u, v)=\left\{\begin{array}{l}
(3-u-v) Z+(2-u) \mathbf{e} \text { if } 0 \leqslant v \leqslant 1 \\
(3-u-v)(Z+\mathbf{e}) \text { if } 1 \leqslant v \leqslant 3-u
\end{array}\right.
$$

which gives

$$
(P(u, v))^{2}=\left\{\begin{array}{l}
2(2-u)(4-u-2 v) \text { if } 0 \leqslant v \leqslant 1 \\
2(3-u-v)^{2} \text { if } 1 \leqslant v \leqslant 3-u
\end{array}\right.
$$

Integrating, we obtain $S\left(W_{\bullet, \bullet}^{S} ; Z\right)=\frac{3}{4}$. But we already proved earlier that $S\left(W_{\bullet, \bullet}^{S} ; Z\right) \geqslant 1$. This is a contradiction. Hence, we see that $X$ is K-polystable.
5.2. Threefolds with three singular points. We continue to use the notation and assumptions introduced at the beginning of this section. Suppose that $[r: s]=[ \pm 1: 1]$. The goal of this subsection is to prove the following.

Theorem 5.8. Let $X$ be a singular threefold from class ( $(\underset{)}{ }$ with three singular points. Then $X$ is $K$-polystable.

The proof follows the method used to prove Theorem 4.11. After changing coordinates, we may assume that $[r: s]=[1: 1]$. Then

$$
X=\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t, w^{2}+(x+y)^{2}+(z+t)^{2}=4(x t+y z)\right\} \subset \mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4}
$$

and the singular locus of the threefold $X$ consists of the following three points:

$$
\begin{gathered}
([1: 1],[-1: 1],[0: 0:-1: 1: 0]), \\
([-1: 1],[1: 1],[-1: 1: 0: 0: 0]), \\
\quad([1: 1],[1: 1],[1: 1: 1: 1: 0]),
\end{gathered}
$$

which are isolated ordinary double singularities of the threefold $X$. We have

$$
Q=\left\{w^{2}+(x+y)^{2}+(z+t)^{2}=4(x t+y z)\right\} \subset \mathbb{P}^{1} \times \mathbb{P}^{4}
$$

the surface $Q$ is a cone with vertex $[1: 1: 1: 1: 0]$, and $\pi$ is a blow up of the conics

$$
\begin{aligned}
& C_{1}=\left\{x=0, y=0, w^{2}+(z+t)^{2}=0\right\} \\
& C_{2}=\left\{z=0, t=0, w^{2}+(x+y)^{2}=0\right\} .
\end{aligned}
$$

Note that $C_{1}$ and $C_{2}$ are reduced singular conics that do not contain $\operatorname{Sing}(Q)$.
The fiber of the conic bundle $\eta$ over the point ( $[1: 1],[1: 1]$ ) is a reducible conic $L_{1}+L_{2}$, where $L_{1}$ and $L_{2}$ are smooth irreducible curves such that

$$
\begin{aligned}
& \pi\left(L_{1}\right)=\{x=y, z=t, w+2 i(x-z)=0\} \\
& \pi\left(L_{2}\right)=\{x=y, z=t, w-2 i(x-z)=0\}
\end{aligned}
$$

Note that $\pi\left(L_{1}\right)$ and $\pi\left(L_{2}\right)$ are the only lines in $Q$ that intersect both conics $C_{1}$ and $C_{2}$, and each of them intersects exactly one irreducible component of the conics $C_{1}$ and $C_{2}$. Observe that the intersection $L_{1} \cap L_{2}$ is the singular point ([1:1], [1:1], $[1: 1: 1: 1: 0]$ ). We denote this singular point by $O$.

Lemma 5.9. Let $\mathbf{F}$ be a $\mathbf{G}$-invariant prime divisor $\mathbf{F}$ over the threefold $X$ with $\beta(\mathbf{F}) \leqslant 0$, and let $Z$ be its center on the threefold $X$. Then $Z=O$.

Proof. By Lemma 5.1, $O$ is the unique G-fixed singular point in $X$. Thus, by Lemma 5.7, we see that one of the following three cases holds:
(1) either $Z=O$,
(2) or $Z$ is an irreducible component of the fiber of $\eta$ over ([1:1], [1:1]),
(3) or $Z$ is an irreducible component of the fiber of $\eta$ over $([1:-1],[1:-1])$.

In the second case, either $Z=L_{1}$ or $Z=L_{2}$, which is impossible, because neither of these curves is G-invariant. Moreover, the fiber of $\eta$ over ( $[1:-1],[1:-1]$ ) is irreducible and smooth. So, either $Z=O$ or $Z$ is the fiber of the conic bundle $\eta$ over ( $[1:-1],[1:-1])$.

Suppose that $Z$ is the fiber of $\eta$ over $([1:-1],[1:-1])$. Set $S=X \cap\left\{u_{1}+v_{1}=0\right\}$. Then $S$ is a singular sextic del Pezzo surface that contains $Z$. Moreover, we have

$$
\operatorname{Sing}(S)=([1:-1],[1,1],[-1: 1: 0: 0: 0]) \in \operatorname{Sing}(X)
$$

this point is an ordinary double point of the surface $S$, the quadric surface $\pi(S)$ is smooth, and $\pi$ induces a weighted blow up $S \rightarrow \pi(S)$ of the single intersection point $\pi(S) \cap C_{2}$. Now, arguing as in the end of Section 5.1, we obtain a contradiction.

Now we are ready to prove that $X$ is K-polystable. Suppose for a contradiction that the singular Fano threefold $X$ is not K-polystable. Then there is a G-invariant prime divisor $\mathbf{F}$ over $X$ such that $\beta(\mathbf{F}) \leqslant 0$. By Lemma 5.9, its center on $X$ is the point $O$.

Let $\varphi: \widetilde{X} \rightarrow X$ be the blow up of the point $O$, and let $F$ be the $\varphi$-exceptional divisor, and let $\widetilde{Z}$ be the center on $\widetilde{X}$ of the prime divisor $\mathbf{F}$. Then $\varphi$ is G-equivariant, and
(1) either $\widetilde{Z}=F$ and $\beta(F)=\beta(\mathbf{F})$, or
(2) $\widetilde{Z}$ is a G-invariant irreducible curve in $F$, or
(3) $\widetilde{Z}$ is a G-fixed point in $F$.

Case (3) is impossible by Lemma 5.2. Moreover, in case (2) we have two choices for $Z$, as described in the proof of Lemma 5.2. Now, arguing exactly as in the proof of Lemma 4.13, we obtain a contradiction. This proves that $X$ is K-polystable.
5.3. Threefolds with non-isolated singularities. It remains to study the singular threefold $X$ from class ( $\aleph$ ) with $[r: s]=[0: 1]$. Recall that $X$ is the complete intersection

$$
\left\{u_{1} x=v_{1} y, u_{2} z=v_{2} t, w^{2}=4(x t+y z)\right\} \subset \mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4}
$$

where $\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right)$ are coordinates on $\mathbb{P}^{1} \times \mathbb{P}^{1} \times \mathbb{P}^{4}$. The goal of this subsection is to prove the following.
Theorem 5.10. The threefold $X$ from class ( $(\aleph)$ described above, which has non-isolated singularities, is K-polystable.

Let

$$
Q=\left\{w^{2}=4(x t+y z)\right\} \subset \mathbb{P}^{4}
$$

and let $\pi: X \rightarrow Q$ be the birational morphism that is induced by the projection $X \rightarrow \mathbb{P}^{4}$. Then $Q$ is a smooth quadric threefold, and the morphism $\pi$ is a weighted blow up with weights $(1,2)$ of the lines $L_{1}=\{x=0, y=0, w=0\}$ and $L_{2}=\{z=0, t=0, w=0\}$. Note that the singular locus of the threefold $X$ consists of two disjoint smooth curves

$$
\begin{aligned}
& \left\{v_{1} u_{2}+u_{1} v_{2}=0, u_{1} z+v_{1} t=0, u_{2} z=v_{2} t, x=0, y=0, w=0\right\} \\
& \left\{v_{1} u_{2}+u_{1} v_{2}=0, u_{2} x+v_{2} y=0, u_{1} x=v_{1} y, z=0, t=0, w=0\right\}
\end{aligned}
$$

Let $S$ be the strict transform on $X$ of the surface in $Q$ that is cut out on $Q$ by $w=0$, let $E_{1}$ and $E_{2}$ be the two $\pi$-exceptional surfaces such that $\pi\left(E_{1}\right)=L_{1}$ and $\pi\left(E_{2}\right)=L_{2}$, and let $\iota$ be the involution in $\operatorname{Aut}(X)$ given by

$$
\iota:\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t: w]\right) \mapsto\left(\left[u_{1}: v_{1}\right],\left[u_{2}: v_{2}\right],[x: y: z: t:-w]\right)
$$

We set $W=X / \iota$ and let $\theta: X \rightarrow W$ be the quotient map. Then $W$ is a smooth threefold and $\theta$ is ramified in $S \cup E_{1} \cup E_{2}$. Moreover, we have the following commutative diagram:

where $\vartheta$ is the double cover ramified in $\pi(S)$ given by $[x: y: z: t: w] \mapsto[x: y: z: t]$ and $\varpi$ is the blow up of the disjoint lines: $\{x=y=0\}$ and $\{z=t=0\}$. Note that $W$ is the unique smooth Fano threefold in the family №3.25.

The threefold $X$ can be also obtained via the following commutative diagram:

where $\alpha$ blows up the lines $L_{1}$ and $L_{2}, \beta$ blows up the $(-1)$-curves in the $\alpha$-exceptional surfaces (which are both isomorphic to $\mathbb{F}_{1}$ ), and $\gamma$ contracts the $\alpha$-exceptional surfaces. Note that $V$ is the smooth Fano threefold in the family №3.20. One can check that

$$
\operatorname{Aut}(X) \cong \operatorname{Aut}(V) \cong\left(\mathbb{C}^{*} \rtimes \boldsymbol{\mu}_{2}\right) \times \mathrm{PGL}_{2}(\mathbb{C})
$$

and $S$ is the only $\operatorname{Aut}(X)$-invariant prime divisor over $X$.
Set $H=\pi^{*}\left(\mathcal{O}_{Q}(1)\right)$, and observe that $-K_{X} \sim 3 H-2 E_{1}-2 E_{2}$ and $S \sim H-E_{1}-E_{2}$. Then, arguing as in the proof of Lemma 4.22, we get

$$
\begin{aligned}
& \beta(S)= 1-\frac{1}{26} \int_{0}^{2}\left(-K_{X}-u S\right)^{3} d u-\frac{1}{26} \int_{2}^{3}\left(-K_{X}-u S-(u-2)\left(E_{1}+E_{2}\right)\right)^{3} d u= \\
&=1-\frac{1}{26} \int_{0}^{2}\left((3-u) H+(u-2)\left(E_{1}+E_{2}\right)\right)^{3} d u-\frac{1}{26} \int_{2}^{3}((3-u) H)^{3} d u= \\
&=1-\frac{1}{26} \int_{0}^{2} 3 u^{2}-18 u+26 d u-\frac{1}{26} \int_{2}^{3} 2(3-u)^{3} d u=1-\frac{49}{52}=\frac{3}{52}>0 .
\end{aligned}
$$

Thus it follows from [8, Corollary 4.14] that $X$ is K-polystable.

## 6. Proof of Main Theorem

In Section 3 we presented the parameter space $T=\mathbb{P}^{4}$ whose open part parametrizes all K-polystable smooth Fano threefolds in the family № 3.10, and described a $\left(\mathbb{C}^{*}\right)^{2}$-action on $T$ such that the GIT quotient $T / /\left(\mathbb{C}^{*}\right)^{2} \cong \mathbb{P}^{1} \times \mathbb{P}^{1}$ parametrizes the Fano threefolds from the Main Theorem, which include all K-polystable smooth Fano threefolds in this family. Now, arguing as in [5, Section 6], we obtain the assertion of the Main Theorem.

Namely, our construction implies that there is a $\mathbb{Q}$-Gorenstein family of Fano varieties over the GIT moduli stack $\left[\stackrel{\circ}{T} /\left(\mathbb{C}^{*}\right)^{2}\right]$ that contains all K-polystable smooth members of the family №3.10, where $\stackrel{\circ}{T}$ is the GIT semistable open subset of the parameter space $T$.

In Sections 4 and 5, we proved that all GIT-polystable objects in our family are in fact K-polystable Fano threefolds. So, we obtain a morphism of stacks

$$
\left[\stackrel{\circ}{T} /\left(\mathbb{C}^{*}\right)^{2}\right] \longrightarrow \mathcal{M}_{3,26}^{\mathrm{Kss}}
$$

where $\mathcal{M}_{3,26}^{\mathrm{Kss}}$ is the K-moduli stack parametrizing K-semistable Fano 3-folds of degree 26 . By [3, Theorem 6.6], this morphism descends to good moduli spaces

$$
\stackrel{\circ}{T} /\left(\mathbb{C}^{*}\right)^{2} \longrightarrow M_{3,26}^{\mathrm{Kps}}
$$

where points of $M_{3,26}^{\mathrm{Kps}}$ parameterize K-polystable Fano threefolds of degree 26. Since both good moduli spaces are known to be proper, we see that the image is an irreducible component of the K-moduli space. This implies the Main Theorem.

We further note that the morphism $\stackrel{\circ}{T} /\left(\mathbb{C}^{*}\right)^{2} \rightarrow M_{3,26}^{\text {Kps }}$ factors as

$$
\stackrel{\circ}{T} /\left(\mathbb{C}^{*}\right)^{2} \xrightarrow{\text { quotient by an involution }} \mathbb{P}^{2} \xrightarrow{\varsigma} M_{3,26}^{\mathrm{Kps}},
$$

where the map $\stackrel{\circ}{T} /\left(\mathbb{C}^{*}\right)^{2} \longrightarrow \mathbb{P}^{2}$ is described in Section 3.3. Then $\varsigma\left(\mathbb{P}^{2}\right)$ is the irreducible component of the moduli space $M_{3,26}^{\mathrm{Kps}}$ that contains K-polystable smooth Fano threefolds in the deformation family №3.10.

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