# BIRATIONAL RIGIDITY AND ALPHA INVARIANTS OF FANO VARIETIES 

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#### Abstract

We prove that for every $\epsilon>0$, there is a birationally super-rigid Fano variety $X$ such that $\frac{1}{2} \leqslant \alpha(X) \leqslant \frac{1}{2}+\epsilon$. Also we show that for every $\epsilon>0$, there is a Fano variety $X$ and a finite subgroup $G \subset \operatorname{Aut}(X)$ such that $X$ is $G$-birationally super-rigid, and $\alpha_{G}(X)<\epsilon$.


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Throughout this paper, we assume that all varieties are projective, normal and defined over $\mathbb{C}$.

## 1. Introduction

Let $X$ be a Fano variety with terminal singularities. If $\operatorname{rk~} \mathrm{Cl}(X)=1$, then $X$ is a Mori fibre space. In this case, we say that $X$ is birationally rigid if $X$ is not birational to other Mori fibre spaces [8]. Similarly, we say that $X$ is birationally super-rigid if it is birationally rigid and $\operatorname{Bir}(X)=\operatorname{Aut}(X)$. Examples of birationally super-rigid smooth Fano varieties include

- smooth hypersurfaces in $\mathbb{P}^{n+1}$ of degree $n+1 \geqslant 4$ [6, 29, 28, 31, 37, 42, 44, 51];
- smooth weighted hypersuraces in $\mathbb{P}\left(1^{n+1}, n\right)$ of degree $2 n \geqslant 6$ [43].

Note that these examples of smooth Fano varieties are known to be K-stable [3, 7, 14, 16, 27, 30 . One can prove this by using Tian's criterion. Namely, recall from [41, 49] that $X$ is K-stable if

$$
\alpha(X)>\frac{\operatorname{dim}(X)}{\operatorname{dim}(X)+1}
$$

where $\alpha(X)$ is the $\alpha$-invariant of $X$ that can be defined as follows:

$$
\alpha(X)=\sup \left\{\begin{array}{l|l}
\lambda \in \mathbb{Q} & \begin{array}{l}
\text { the log pair }(X, \lambda D) \text { is log canonical } \\
\text { for any effective } \mathbb{Q} \text {-divisor } D \sim_{\mathbb{Q}}-K_{X}
\end{array}
\end{array}\right\} .
$$

If $X$ is smooth, then $X$ is also K-stable in the case when $\alpha(X)=\frac{\operatorname{dim}(X)}{\operatorname{dim}(X)+1}$ and $\operatorname{dim}(X) \geqslant 2$ [30]. On the other hand, if $X$ is a smooth hypersurface in $\mathbb{P}^{n+1}$ of degree $n+1$, then [7, 14] gives

$$
\alpha(X) \geqslant \frac{n}{n+1}=\frac{\operatorname{dim}(X)}{\operatorname{dim}(X)+1}
$$

Similarly, if $X$ is smooth hypersuraces in $\mathbb{P}\left(1^{n+1}, n\right)$ of degree $2 n \geqslant 2$, then [16] gives

$$
\alpha(X) \geqslant \frac{2 n-1}{2 n}>\frac{n}{n+1}=\frac{\operatorname{dim}(X)}{\operatorname{dim}(X)+1} .
$$

This shows that all smooth hypersurfaces in $\mathbb{P}^{n+1}$ of degree $n+1 \geqslant 3$ and all smooth weighted hypersuraces in $\mathbb{P}\left(1^{n+1}, n\right)$ of degree $2 n \geqslant 4$ are K-stable. This gives an evidence for

Conjecture 1.1 ([35]). Let $X$ be a Fano variety with terminal singularities such that $\mathrm{rk} \mathrm{Cl}(X)=1$. Suppose that $X$ is birationally rigid. Then $X$ is $K$-stable.

This conjecture has been already verified for many Fano varieties [10, 11, 12, 35, 15, 24, 51, 47, but it is still open in full generality (cf. [40]). On the other hand, we have the following result:

Theorem 1.2 ([48]). Let $X$ be a Fano variety with terminal singularities such that $\mathrm{rk} \mathrm{Cl}(X)=1$. Suppose that $X$ is birationally super-rigid and $\alpha(X) \geqslant \frac{1}{2}$. Then $X$ is $K$-stable.

This naturally leads to the question:
Question 1.3 ([48]). Is it true that $\alpha(X) \geqslant \frac{1}{2}$ for any birationally super-rigid Fano variety $X$ ?
In this paper we show that the bound $\frac{1}{2}$ is optimal by proving the following theorem.
Theorem 1.4. For every $\epsilon>0$, there exists a singular Fano variety $X$ with terminal singularities such that $\operatorname{rk~} \mathrm{Cl}(X)=1$, the variety $X$ is birationally super-rigid, and

$$
\frac{1}{2} \leqslant \alpha(X) \leqslant \frac{1}{2}+\epsilon
$$

We also answer a natural equivariant version of Question 1.3, which can be stated as follows. Suppose that ${\operatorname{rk~} \mathrm{Cl}^{G}}^{( }(X)=1$ for a finite subgroup $G \subset \operatorname{Aut}(X)$, so that $X$ is a $G$-Mori fibre space. Then $X$ is $G$-birationally rigid if it is not $G$-birational to other $G$-Mori fibre spaces [21, § 3.1.1]. Similarly, the Fano variety $X$ is said to be $G$-birationally super-rigid if $X$ is $G$-birationally rigid, and $X$ does not have non-biregular $G$-birational selfmaps. Finally, we let

$$
\alpha_{G}(X)=\sup \left\{\begin{array}{l|l}
\lambda \in \mathbb{Q} & \begin{array}{l}
\text { the pair }(X, \lambda D) \text { is log canonical for every } \\
\text { effective } G \text {-invariant } \mathbb{Q} \text {-divisor } D \sim_{\mathbb{Q}}-K_{X}
\end{array}
\end{array}\right\} .
$$

If $\alpha_{G}(X)>\frac{\operatorname{dim}(X)}{\operatorname{dim}(X)+1}$, then $X$ is K-polystable by [52, Corollary 1.3].
Question 1.5. Is it true that $\alpha_{G}(X) \geqslant \frac{1}{2}$ for any $G$-birationally super-rigid Fano variety $X$ ?
The answer to this question is positive in dimension two:
Exercise 1.6 ([9, [18, 46]). If $\operatorname{dim}(X)=2$ and $X$ is $G$-birationally super-rigid, then $\alpha_{G}(X) \geqslant \frac{2}{3}$.
In dimension three we still do not know whether our Question 1.5 has a positive answer or not, but many examples suggest that the answer is probably positive.
Example 1.7 ([19, 20, 22]). Suppose that $X=\mathbb{P}^{3}$, and let $G$ be any finite subgroup in $\operatorname{Aut}(X)$. Then $X$ is $G$-birationally super-rigid if and only if the following four conditions are satisfied
(i) $X$ does not have $G$-orbits of length $\leqslant 4$;
(ii) $X$ does not contains $G$-invariant lines;
(iii) $X$ does not contains $G$-invariant pairs of skew lines;
(iv) $G$ is not isomorphic to $\mathfrak{A}_{5}, \mathfrak{S}_{5}, \mathrm{PSL}_{2}\left(\mathbf{F}_{7}\right), \mathfrak{A}_{6}, \boldsymbol{\mu}_{2}^{4} \rtimes \boldsymbol{\mu}_{5}$ and $\boldsymbol{\mu}_{2}^{4} \rtimes \mathrm{D}_{10}$.

Using this criterion and [18], we see that $\alpha_{G}(X) \geqslant \frac{1}{2}$ if $X$ is $G$-birationally super-rigid.
In this paper, we prove that the answer to Question 1.5 is very negative in higher dimensions.

Theorem 1.8. For every $\epsilon>0$, there is a smooth Fano variety $X$ and a finite subgroup $G \subset \operatorname{Aut}(X)$ such that $\mathrm{rkPic}^{G}(X)=1$, the variety $X$ is $G$-birationally super-rigid, and $\alpha_{G}(X)<\epsilon$.

Let us describe the structure of this paper. In Section 2, we prove Theorem 1.4, In Section 3, we study equivariant birational geometry of a smooth quadric threefold $Q \subset \mathbb{P}^{4}$ for the natural action of the symmetric group $\mathfrak{S}_{5}$, which should be interesting for mathematicians working on finite subgroups of the space Cremona group (cf. [50, § 9]). This example inspired Theorem 1.8. In Section 4, we present few results used in the proof of Theorem 1.8, which is done in Section 5.

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## 2. The proof of Theorem 1.4

We fix a positive integer $a \geqslant 2$. Then we let $X$ be a quasi-smooth well-formed singular weighted hypersurfaces of degree $2 a+1$ in $\mathbb{P}\left(1^{a+2}, a\right)$ that is given by the following equation:

$$
y^{2} x_{1}+f_{2 a+1}\left(x_{1}, \ldots, x_{a+2}\right)=0
$$

where each $x_{i}$ is a coordinate of weight $1, y$ is a coordinate of weight $a$, and $f_{2 a+1}$ is a general homogeneous polynomial of degree $2 a+1$. Then

- $X$ is a Fano variety of dimension $N=a+1$,
- the class group of the variety $X$ is of rank 1 ,
- the singularities of $X$ consist of one singular point $O_{y}=(0: \ldots: 0: 1)$, which is a terminal quotient singularity of type $\frac{1}{a}(1, \ldots, 1)$.
Further, it follows from [36] that

$$
\alpha(X) \leqslant \frac{a+1}{2 a+1}=\frac{1}{2}+\frac{1}{4 a+2} .
$$

In this section, we prove the following result, which implies Theorem 1.4.
Theorem 2.1. The Fano variety $X$ is birationally super-rigid.
This theorem also answers positively [36, Question 7.2.3].
Remark 2.2. If $a=2$, then $X$ is known to be birationally super-rigid [15, 24].
Let $\pi: X \longrightarrow \mathbb{P}^{N}$ be the projection from the point $O_{y}$. Then $\pi$ contracts the following divisor:

$$
D=\left\{x_{1}=0, f_{2 a+1}\left(x_{1}, \ldots, x_{a+2}\right)=0\right\} \subset \mathbb{P}\left(1^{a+2}, a\right)
$$

Further, one has the following diagram:

where $f$ is the weighted blow-up of the point $O_{y}$ with weights $(1, \ldots, 1)$, the map $g$ is a morphism, the variety $U$ is a hypersurface in $\mathbb{P}\left(1^{a+2}, a+1\right)$ of degree $2 a+2$ that is given by

$$
z^{2}+x_{1} f_{2 a+1}\left(x_{1}, \ldots, x_{a+2}\right)=0
$$

the morphism $\nu$ is a birational morphism that contracts the strict transform of the divisor $D$, and the morphism $\theta$ is a double cover that is branched over the hypersurface $x_{1} f_{2 a+1}\left(x_{1}, \ldots, x_{a+2}\right)=0$.

Here, we consider $x_{1}, \ldots, x_{a+2}$ as coordinates on $\mathbb{P}^{N}$ and as coordinates of weight 1 on the weighted projective space $\mathbb{P}\left(1^{a+2}, a+1\right)$, where $z$ is a coordinate of weight $a+1$.

Now, let us prove Theorem 2.1. Assume the contrary, i.e. there exist a birational map

$$
\Phi: X \rightarrow W
$$

to a Mori fibre space $W$ that is not isomorphism. Let $\mathcal{M}$ be a birational transform of a very ample complete linear system on $W$ via $\Phi$. Let $\lambda \in \mathbb{Q}_{>0}$ be the positive rational number such that

$$
K_{X}+\lambda \mathcal{M} \sim_{\mathbb{Q}} 0 .
$$

Then, by the Noether-Fano inequality [25], the singularities of the pair $(X, \lambda \mathcal{M})$ are not canonical. Let $Z$ be a center of non-canonical singularities of the $\log$ pair $(X, \lambda \mathcal{M})$.

Now, let $E$ be the $f$-exceptional divisor, and let $\widetilde{\mathcal{M}}$ be the strict transform of the mobile linear system $\mathcal{M}$ on the variety $\widetilde{X}$. Then $E \cong \mathbb{P}^{a}$, and

$$
\begin{aligned}
K_{\tilde{X}} & \sim_{\mathbb{Q}} f^{*}\left(K_{X}\right)+\frac{1}{a} E, \\
\lambda \widetilde{\mathcal{M}} & \sim_{\mathbb{Q}} f^{*}(\lambda \mathcal{M})-\mu E,
\end{aligned}
$$

for some $\mu \in \mathbb{Q}_{\geqslant 0}$. Therefore, we have

$$
K_{\tilde{X}}+\lambda \widetilde{\mathcal{M}} \sim_{\mathbb{Q}} f^{*}\left(K_{X}+\lambda \mathcal{M}\right)+\left(\frac{1}{a}-\mu\right) E .
$$

Thus, if $\mu>\frac{1}{a}$, then $O_{y}$ is a center of non-canonical singularities of the $\log$ pair $(X, \lambda \mathcal{M})$.
Lemma 2.3 (cf. [32] for $a=2$ ). Suppose that $O_{y} \in Z$. Then $\mu>\frac{1}{a}$.
Proof. Suppose that $\mu \leqslant \frac{1}{a}$. Let us seek for a contradiction.
2.1. Case: $Z \neq O_{y}$. Let $\widetilde{Z}$ be the strict transform of $Z$ via $f$. Then $\operatorname{mult}_{\widetilde{Z}}(\widetilde{\mathcal{M}})>\frac{1}{\lambda}$ and hence,

$$
\begin{equation*}
\operatorname{mult}_{P}\left(\left.\widetilde{\mathcal{M}}\right|_{E}\right)>\frac{1}{\lambda} \tag{2.4}
\end{equation*}
$$

for any point $P \in \widetilde{Z} \cap E$. Notice that

$$
\left.\lambda \widetilde{\mathcal{M}}\right|_{E} \sim_{\mathbb{Q}}-\left.\mu E\right|_{E} \sim_{\mathbb{Q}} a \mu H
$$

where $H$ is a hyperplane in $E \cong \mathbb{P}^{a}$. Since $a \mu \leqslant 1$, this contradicts to (2.4).
2.2. Case: $Z=O_{y}$. We write

$$
K_{\tilde{X}}+\lambda \widetilde{\mathcal{M}}+\left(\mu-\frac{1}{a}\right) E \sim_{\mathbb{Q}} f^{*}\left(K_{X}+\lambda \mathcal{M}\right)
$$

Hence, the singularities of the log pair $\left(\widetilde{X}, \lambda \widetilde{\mathcal{M}}+\left(\mu-\frac{1}{a}\right) E\right)$ are not canonical at some point $P \in E$. Then the singularities of the log pair $(\widetilde{X}, \lambda \widetilde{\mathcal{M}})$ are also not canonical at $P$, so that

$$
\operatorname{mult}_{P}(\widetilde{\mathcal{M}})>\frac{1}{\lambda}
$$

Now, we argue as in the previous case to obtain a contradiction.

One the other hand, we have
Lemma 2.5. One has $\mu \leqslant \frac{1}{a}$.

Proof. One has $g^{*}\left(\mathcal{O}_{\mathbb{P}^{N}}(1)\right) \sim_{\mathbb{Q}} f^{*}\left(-K_{X}\right)-\frac{1}{a} E$. Then

$$
\left(f^{*}\left(-K_{X}\right)-\frac{1}{a} E\right) \cdot C=0
$$

for any curve $C$ contracted by $g$. Thus, if $\mu>\frac{1}{a}$, then

$$
\widetilde{M} \cdot C=\frac{1}{\lambda}\left(f^{*}\left(-K_{X}\right)-\mu E\right)<0
$$

for a general divisor $\widetilde{M} \in \widetilde{\mathcal{M}}$. This is a contradiction, because the linear system $\widetilde{\mathcal{M}}$ is mobile, and the curves contracted by $g$ span a divisor in $\widetilde{X}$ - the proper transform of the divisor $D$.

Corollary 2.6. One has $O_{y} \notin Z$.
Thus, we see that $Z$ is contained in the smooth locus of the variety $X$.
Lemma 2.7. One has $\operatorname{dim}(Z)=a-1$.
Proof. Suppose that $\operatorname{dim}(Z)<a-1$. Let $M_{1}$ and $M_{2}$ be sufficiently general divisors in $\mathcal{M}$, and let $P$ be a sufficiently general point in $Z$. Then

$$
\left(M_{1} \cdot M_{2}\right)_{P}>\frac{4}{\lambda^{2}}
$$

by [44] or [26, Corollary 3.4]. Let $\mathcal{L}$ be the linear subsystem in $\left|-K_{X}\right|$ consisting of all divisors that pass through the point $P$, and let $H_{1}, \ldots, H_{N-2}$ be sufficiently general divisors in the system $\mathcal{L}$. If $P \notin D$, then the base locus of $\mathcal{L}$ does not contain curves, which gives

$$
\frac{2 a+1}{a \lambda^{2}}=M_{1} \cdot M_{2} \cdot H_{1} \cdot \ldots \cdot H_{N-2} \geqslant\left(M_{1} \cdot M_{2}\right)_{P}>\frac{4}{\lambda^{2}}
$$

which is a contradiction. Thus, we see that $P \in D$.
Let $L \subset D$ be the curve containing $P$ that is contracted by $\pi$. Then $L$ is the only curve contained in the base locus of the linear system $\mathcal{L}$. After a linear change of coordinates, we can assume that

$$
P=(0: 0: 1: 0: \ldots: 0: 1)
$$

and $H_{i}=X \cap\left\{x_{i+3}=0\right\}$ for $i=1, \ldots, N-2$. Consider the surface $S$ defined as

$$
S=\bigcap_{i=1}^{N-2} H_{i} .
$$

We can identify $S$ with a surface in $\mathbb{P}(1,1,1, a)$ given by

$$
y^{2} x_{1}+f_{2 a+1}\left(x_{1}, x_{2}, x_{3}, 0, \ldots, 0\right)=0
$$

Then $L=S \cap\left\{x_{1}=x_{2}=0\right\}$. Let $\mathcal{M}_{S}=\left.\mathcal{M}\right|_{S}$. Then $\lambda \mathcal{M}_{S}=m L+\lambda \Delta$ for some non-negative rational number $m \in \mathbb{Q} \geqslant 0$ and some mobile linear system $\Delta$ on the surface $S$. Moreover, applying the inversion of adjunction [38, Theorem 5.50], we see that $\left(S, \lambda \mathcal{M}_{S}\right)$ is not $\log$ canonical at $P$.

Let $H$ be a general curve in $\left|\mathcal{O}_{S}(1)\right|$, and let $H_{L}$ be a general curve in $\left|\mathcal{O}_{S}(1)\right|$ that contains $L$. Then $H \cdot L=\frac{1}{a}$ and

$$
S \cap H_{L}=L+R,
$$

where $R$ is a curve in $S$ such that $L \not \subset \operatorname{Supp}(R)$. One can check that $L \cdot R=2$ and $H \cdot R=2$. Thus, using $(L+R) \cdot L=H \cdot L=\frac{1}{a}$, we get

$$
L^{2}=-2+\frac{1}{a}
$$

which can also be shown using the subadjunction formula on $S$.

Now, using Corti's inequality [26, Theorem 3.1], we get

$$
\begin{aligned}
& 4(1-m)<\lambda^{2}\left(\Delta_{1} \cdot \Delta_{2}\right)_{P} \leqslant \lambda^{2} \Delta_{1} \cdot \Delta_{2}= \\
& \quad=(H-m L)^{2}=H^{2}-2 m H \cdot L+m^{2} L^{2}=\frac{2 a+1}{a}-\frac{2 m}{a}+m^{2}\left(-2+\frac{1}{a}\right)
\end{aligned}
$$

which gives

$$
0>\frac{(2 a-1)(m-1)^{2}}{a}
$$

This is a contradiction, since $a \geqslant 2$.
Therefore, we see that $\operatorname{dim}(Z)=\operatorname{dim}(X)-2$. Then

$$
\operatorname{mult}_{Z}(\mathcal{M})>\frac{1}{\lambda}
$$

Let $M_{1}$ and $M_{2}$ be general divisors in $\mathcal{M}$. Then

$$
\frac{3}{\lambda^{2}}>\frac{2 a+1}{\lambda^{2} a}=\left(-K_{X}\right)^{N-2} \cdot M_{1} \cdot M_{2} \geqslant \operatorname{mult}_{Z}^{2}(\mathcal{M})\left(-K_{X}\right)^{N-2} \cdot Z>\frac{1}{\lambda^{2}}\left(-K_{X}\right)^{N-2} \cdot Z
$$

so that $\left(-K_{X}\right)^{N-2} \cdot Z \in\{1,2\}$.
Now, let $H_{1}, \ldots, H_{N-2}$ be general divisors in $\left|-K_{X}\right|$. After a linear change of coordinate system one can assume that $H_{i}=X \cap\left\{x_{i+4}=0\right\}$ for $i=1, \ldots, N-3$. Let $V$ be the threefold defined as

$$
V=\bigcap_{i=1}^{N-3} H_{i}
$$

Then we can identify $V$ with the hypersurface in $\mathbb{P}\left(1^{4}, a\right)$ given by

$$
y^{2} x_{1}+f_{2 a+1}\left(x_{1}, \ldots, x_{4}, 0, \ldots, 0\right)=0
$$

Let $C=V \cap Z, \mathcal{M}_{V}=\left.\mathcal{M}\right|_{V}$, and let $H$ be a general surface in $\left|\mathcal{O}_{V}(1)\right|$. Then

- $C$ is an irreducible curve such that $C \cdot H \in\{1,2\}$,
- $C$ is contained in the smooth locus of the hypersurface $V$,
- $C$ is a center of non-canonical singularities of the $\log \operatorname{pair}\left(V, \lambda \mathcal{M}_{V}\right)$.

We set $\mu=\lambda$ mult $_{C}\left(\mathcal{M}_{V}\right)$. Then $\mu>1$.
Lemma 2.8. One has $C \cdot H \neq 1$.
Proof. Suppose that $C \cdot H=1$. We can choose coordinates on $\mathbb{P}\left(1^{4}, a\right)$ such that

$$
C=\left\{x_{2}=0, x_{3}=0, y+F\left(x_{1}, \ldots, x_{4}\right)=0\right\} \subset \mathbb{P}\left(1^{4}, a\right),
$$

where $F\left(x_{1}, \ldots x_{4}\right)$ is a homogeneous polynomial of degree $a$. Note that $C \cong \mathbb{P}^{1}$.
Now, we let $\beta: \widetilde{V} \rightarrow V$ be the blow-up of the curve $C$, and let $E$ be the $\beta$-exceptional divisor. We claim that $E^{3}=a-1$. Indeed, let $\widetilde{S}_{1}, \widetilde{S}_{2}, \widetilde{S}_{3}$ be the strict transforms on $\widetilde{V}$ of the surfaces that are cut out on $V$ by the equations $y+F\left(x_{1}, \ldots, x_{4}\right)=0, x_{2}=0, x_{3}=0$, respectively. Then

$$
0=\widetilde{S}_{1} \cdot \widetilde{S}_{2} \cdot \widetilde{S}_{3}=\left(a \beta^{*}(H)-E\right) \cdot\left(\beta^{*}(H)-E\right)^{2}=a H^{3}+(a+2) \beta^{*}(H) \cdot E^{2}-E^{3}=a-1-E^{3}
$$ which gives $E^{3}=a-1$ as claimed.

Let $\mathcal{M}_{\widetilde{V}}$ be the strict transform of the linear system $\mathcal{M}_{V}$ on the threefold $\widetilde{V}$. Then

$$
\lambda \mathcal{M}_{\widetilde{V}} \sim_{\mathbb{Q}} \beta^{*}(H)-\mu E
$$

One the other hand, since $\mathcal{M}_{\tilde{V}}$ is mobile and $a \beta^{*}(H)-E$ is nef, we get $0 \leqslant\left(a \beta^{*}(H)-E\right) \cdot\left(\beta^{*}(H)-\mu E\right)^{2}=a H^{3}+\left(2 \mu+a \mu^{2}\right) \beta^{*}(H) \cdot E^{2}-\mu^{2} E^{3}=(\mu-1)^{2}-2 a\left(\mu^{2}-1\right)<0$,
which is a contradiction.
Thus, we see that $C \cdot H=2$. Then we can change coordinates on $\mathbb{P}\left(1^{4}, a\right)$ such that
(A) either

$$
C=\left\{x_{4}=0, x_{1} x_{2}+x_{3}^{2}=0, y+F_{a}\left(x_{1}, \ldots, x_{4}\right)=0\right\} \subset \mathbb{P}\left(1^{4}, a\right)
$$

for some homogeneous polynomial $F_{a}\left(x_{1}, \ldots, x_{4}\right)$ of degree $a$,
(B) or

$$
C=\left\{x_{2}=0, x_{3}=0, y^{2}+F_{2 a}\left(x_{1}, \ldots, x_{4}\right)=0\right\} \subset \mathbb{P}\left(1^{4}, a\right)
$$

for some homogeneous polynomial $F_{2 a}\left(x_{1}, \ldots, x_{4}\right)$ of degree $2 a$.
In case (A), we have $C \cong \mathbb{P}^{1}$. In case (B), the curve $C$ may have singularities.
In both cases, let $\beta: \widetilde{V} \rightarrow V$ be the blow-up of the curve $C$, and let $E$ be the $\beta$-exceptional divisor. Then, arguing as in the proof of Lemma 2.8, we get

$$
E^{3}=\left\{\begin{array}{l}
2 a-4 \text { in case }(\mathrm{A}), \\
-2 \text { in case }(\mathrm{B}) .
\end{array}\right.
$$

Let $\mathcal{M}_{\widetilde{V}}$ be the strict transform of the linear system $\mathcal{M}_{V}$ on the threefold $\widetilde{V}$. Then

$$
\lambda \mathcal{M}_{\tilde{V}} \sim_{\mathbb{Q}} \beta^{*}(H)-\mu E
$$

Moreover, in case (A), the divisor $a \beta^{*}(H)-E$ is nef, so that

$$
\left(a \beta^{*}(H)-E\right) \cdot\left(\beta^{*}(H)-\mu E\right)^{2} \geqslant 0
$$

because $\mathcal{M}_{\widetilde{V}}$ is mobile. But
$\left(a \beta^{*}(H)-E\right) \cdot\left(\beta^{*}(H)-\mu E\right)^{2}=a H^{3}+\left(2 \mu+a \mu^{2}\right) \beta^{*}(H) \cdot E^{2}-\mu^{2} E^{3}=(2 \mu-1)^{2}-2 a\left(2 \mu^{2}-1\right)<0$, because $\mu>1$. Likewise, in case (B), the divisor $2 a \beta^{*}(H)-E$ is nef, which gives

$$
\begin{aligned}
& 0 \leqslant\left(2 a \beta^{*}(H)-E\right) \cdot\left(\beta^{*}(H)-\mu E\right)^{2}=2 a H^{3}+\left(2 \mu+2 a \mu^{2}\right) \beta^{*}(H) \cdot E^{2}-\mu^{2} E^{3}= \\
& =-2(\mu-1)(1-\mu+2 a(1+\mu))<0
\end{aligned}
$$

Thus, we get a contradiction in both cases (A) and (B). This completes the proof of Theorem 2.1.

## 3. $\mathfrak{S}_{5}$-INVARIANT QUADRIC THREEFOLD

Let $Q$ be a smooth quadric hypersurface in $\mathbb{P}^{4}$. We can choose coordinates $x_{0}, x_{1}, x_{2}, x_{3}, x_{4}$ on the projective space $\mathbb{P}^{4}$ such that $Q$ is given by the following equation:

$$
\sum_{i=0}^{4} x_{i}^{2}=0
$$

In particular, we see that $Q$ is faithfully acted on by the symmetric group $\mathfrak{S}_{5}$, which permutes the coordinates $x_{0}, x_{1}, x_{2}, x_{3}, x_{4}$. Then $\alpha_{\mathfrak{S}_{5}}(Q) \leqslant \frac{1}{3}$, because $\mathfrak{S}_{5}$ leaves invariant the hyperplane sections of the quadric $Q$ that is cut out by $x_{0}+x_{1}+x_{2}+x_{3}+x_{4}=0$. In fact, arguing as in [18], one can show that $\alpha_{\mathfrak{S}_{5}}(Q)=\frac{1}{3}$.

Keeping in mind the results obtained in [20], one can expect that $Q$ is $\mathfrak{S}_{5}$-birationally rigid. However, this is not the case - the quadric hypersurface $Q$ contains two $\mathfrak{S}_{5}$-orbits of length 5 , and each of them leads to a $G$-birational transformation of the quadric into other $\mathfrak{S}_{5}$-Mori fibre space.

Namely, let $\Sigma_{5}$ be a $\mathfrak{S}_{5}$-orbit of length 5 in $X$, and let $\pi: X \rightarrow Q$ be the blow up of this $\mathfrak{S}_{5}$-orbit. Then there exists the following $\mathfrak{S}_{5}$-equivariant commutative diagram:

where $\zeta$ is a small birational map that flops the proper transforms of 10 conics that contain three points in $\Sigma_{5}, \phi$ is a birational morphism that contracts the proper transforms of 5 hyperplane sections of the quadric $Q$ that pass through four points in $\Sigma_{5}$, and $Y$ is a cubic threefold in $\mathbb{P}^{4}$ such that it has 5 isolated ordinary double points and $\operatorname{rk~} \mathrm{Cl}(Y)=1$. Since $Y$ is a $\mathfrak{S}_{5}$-Mori fibre space, we see that $Q$ is not $\mathfrak{S}_{5}$-birationally rigid. Note that the cubic threefold $Y$ is given in $\mathbb{P}^{4}$ by
$x_{0} x_{1} x_{2}+x_{0} x_{1} x_{3}+x_{0} x_{1} x_{4}+x_{0} x_{2} x_{3}+x_{0} x_{2} x_{4}+x_{0} x_{3} x_{4}+x_{1} x_{2} x_{3}+x_{1} x_{2} x_{4}+x_{1} x_{3} x_{4}+x_{2} x_{3} x_{4}=0$.
This is not difficult to prove [4, 5].
The goal of this section is to prove the following result.
Theorem 3.1. The only $\mathfrak{S}_{5}$-Mori fiber spaces that are $\mathfrak{S}_{5}$-birational to $Q$ are $Q$ and $Y$.
Let us prove Theorem 3.1] Let $\iota \in \operatorname{Aut}(Q)$ be the Galois involution of the double cover $Q \rightarrow \mathbb{P}^{3}$ given by the projection from the point $(1: 1: 1: 1: 1)$. Then $\iota$ commutes with the $\mathfrak{S}_{5}$-action on $Q$. It is well-known [13, 17] that Theorem 3.1 follows from the following technical result:

Theorem 3.2. Let $\mathcal{M}_{Q}$ be any non-empty mobile $\mathfrak{S}_{5}$-invariant linear system on the quadric $Q$, and let $\mathcal{M}_{Y}$ and $\mathcal{M}_{Y}^{\prime}$ be its proper transform on the cubic threefolds $Y$ via $\chi$ and $\chi \circ \iota$, respectively. Choose positive rational numbers $\lambda, \mu, \mu^{\prime}$ such that

$$
\begin{aligned}
\lambda \mathcal{M}_{Q} & \sim_{\mathbb{Q}}-K_{Q} \\
\mu \mathcal{M}_{Y} & \sim_{\mathbb{Q}}-K_{Y} \\
\mu^{\prime} \mathcal{M}_{Y}^{\prime} & \sim_{\mathbb{Q}}-K_{Y^{\prime}}
\end{aligned}
$$

Then one of the log pair $\left(Q, \lambda \mathcal{M}_{Q}\right),\left(Y, \mu \mathcal{M}_{Y}\right)$ or $\left(Y^{\prime}, \mu^{\prime} \mathcal{M}_{Y}^{\prime}\right)$ has canonical singularities.
To prove Theorem 3.2, let us use all notations and assumptions of this theorem. We must prove that at least one of the $\log$ pair $\left(Q, \lambda \mathcal{M}_{Q}\right),\left(Y, \mu \mathcal{M}_{Y}\right)$ or $\left(Y, \mu^{\prime} \mathcal{M}_{Y}^{\prime}\right)$ has canonical singularities. Set $\Sigma_{5}^{\prime}=\iota\left(\Sigma_{5}\right)$. Then $\Sigma_{5}^{\prime}$ is the second $\mathfrak{S}_{5}$-orbit in the quadric $Q$.
Remark 3.3. Let $G$ be a stabilizer in $\mathfrak{S}_{5}$ of a point in $P \in \Sigma_{5} \cup \Sigma_{5}^{\prime}$. Then $G \cong \mathfrak{S}_{4}$ and its induced linear action on the Zariski tangent space $T_{P}(Q)$ is an irreducible representation.

Now using this remark, [1, Lemma 2.4] and [26, Theorem 3.10], we can easily derive the required assertion from the following two propositions, arguing as in the proof of [13, Theorem 1.2].
Proposition 3.4. The log pair $\left(Q, \lambda \mathcal{M}_{Q}\right)$ is canonical away from $\Sigma_{5} \cup \Sigma_{5}^{\prime}$.
Proposition 3.5. The log pairs $\left(Y, \mu \mathcal{M}_{Y}\right)$ and $\left(Y, \mu^{\prime} \mathcal{M}_{Y}^{\prime}\right)$ are canonical away from $\operatorname{Sing}(Y)$.
In the remaining part of this section, we will prove Propositions 3.4 and 3.5. For both proofs, we need the following technical observation, which improves [20, Lemma 2.2].
Remark 3.6. Let $X$ be a variety with terminal singularities, let $D$ be an effective $\mathbb{Q}$-Cartier divisor on the variety $X$, let $\varphi: \widetilde{X} \rightarrow X$ be birational morphism such that $\widetilde{X}$ is normal, let $\widetilde{D}$ be the proper transform on $\widetilde{X}$ of the divisor $D$, and let $E_{1}, \ldots, E_{n}$ be $\varphi$-exceptional divisors. Then

$$
K_{\tilde{X}}+\widetilde{D}+\sum_{i=1}^{n} a\left(E_{i} ; X, D\right) E_{i} \sim_{\mathbb{Q}} \varphi^{*}\left(K_{X}+D\right)
$$

where each $a\left(E_{i} ; X, D\right)$ is a rational number known as the discrepancy of the pair $(X, D)$ along $E_{i}$. Let $E$ be one of the $\varphi$-exceptional divisors. Then

$$
a(E ; X, D)=a(E ; X)-\operatorname{ord}_{E}(D)
$$

where $a(E ; X)$ is the discrepancy of $X$ along $E$. Let $a=a(E ; X)$. If $a(E ; X, D)<0$, then
$a\left(E ; X,\left(1+\frac{1}{a}\right) D\right)=a(E ; X)-\left(1+\frac{1}{a}\right) \operatorname{ord}_{E}(D)<\operatorname{ord}_{E}(D)-\left(1+\frac{1}{a}\right) \operatorname{ord}_{E}(D)=-\frac{\operatorname{ord}_{E}(D)}{a}<-1$, so that the $\log$ pair $\left(X,\left(1+\frac{1}{a}\right) D\right)$ is not $\log$ canonical along $\varphi(E)$. In particular, if $a(E ; X, D)<0$ and $\varphi(E)$ is a smooth point of the variety $X$, then the $\log$ pair

$$
\left(X, \frac{\operatorname{dim}(X)}{\operatorname{dim}(X)-1} D\right)
$$

is not $\log$ canonical at the point $\varphi(E)$.
To prove Proposition [3.4, we have to present few standard basic facts about the $\mathfrak{S}_{5}$-equivariant geometry of the quadric $Q$. Observe that $Q$ contains two $\mathfrak{S}_{5}$-orbits $\Sigma_{10}$ and $\Sigma_{10}^{\prime}$ of length 10 .
Lemma 3.7. If $\Sigma$ is a $\mathfrak{S}_{5}$-orbit in $Q$ with $|\Sigma|<20$, then $\Sigma$ is one of the orbits $\Sigma_{5}, \Sigma_{5}^{\prime}, \Sigma_{10}, \Sigma_{10}^{\prime}$.
Proof. Left to the reader.
Let $H=\left\{x_{0}+x_{1}+x_{2}+x_{3}+x_{4}=0\right\} \subset \mathbb{P}^{4}$ and $S_{2}=H \cap Q$. Then $S_{2}$ is smooth and $\mathfrak{S}_{5}$-invariant. Moreover, the surface $S_{2}$ does not contain $\Sigma_{5}, \Sigma_{5}^{\prime}, \Sigma_{10}, \Sigma_{10}^{\prime}$. Let $\mathcal{B}_{6}$ be the curve in $Q$ given by

$$
\left\{\begin{array}{l}
x_{0}+x_{1}+x_{2}+x_{3}+x_{4}=0, \\
x_{0}^{2}+x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+x_{4}^{2}=0, \\
x_{0}^{3}+x_{1}^{3}+x_{2}^{3}+x_{3}^{3}+x_{4}^{3}=0 .
\end{array}\right.
$$

Then $\mathcal{B}_{6}$ is the unique smooth curve of genus 4 that admits an effective action of the group $\mathfrak{S}_{5}$, which is known as the Bring's curve (see [21, Remark 5.4.2]). Note that $\mathcal{B}_{6} \subset Q \cap H$.

Lemma 3.8. Let $C$ be a $\mathfrak{S}_{5}$-invariant curve in $Q$ such that $\operatorname{deg}(C) \leqslant 6$. Then $C=\mathcal{B}_{6}$.
Proof. We may assume that $C$ is $\mathfrak{S}_{5}$-irreducible, i.e. the symmetric group $\mathfrak{S}_{5}$ acts transitively on the set of its irreducible components. Then $S_{2}$ contains $C$, since otherwise $\left|S_{2} \cap C\right| \leqslant S_{2} \cdot C=12$, which contradicts Lemma 3.7. Thus, if $C \neq \mathcal{B}_{6}$, then

$$
\left|C \cap \mathcal{B}_{6}\right| \leqslant C \cdot \mathcal{B}_{6}=18
$$

which is impossible by Lemma 3.7, since $S_{2}$ does no contain $\Sigma_{5}, \Sigma_{5}^{\prime}, \Sigma_{10}$ and $\Sigma_{10}^{\prime}$.
Corollary 3.9. The log pair $\left(Q, \lambda \mathcal{M}_{Q}\right)$ has log canonical singularities.
Proof. Suppose that the $\log$ pair $\left(Q, \lambda \mathcal{M}_{Q}\right)$ is not $\log$ canonical. Let us seek for a contradiction. If the $\log$ pair $\left(Q, \lambda \mathcal{M}_{Q}\right)$ is $\log$ canonical outside of finitely many points, then it is log canonical outside of a single point by the Kollár-Shokurov connectedness, which must be $\mathfrak{S}_{5}$-invariant point. The latter contradicts Lemma 3.7. Thus, we see that there is a $\mathfrak{S}_{5}$-irreducible curve $C$ such that the $\log$ pair $\left(Q, \lambda \mathcal{M}_{Q}\right)$ is not log canonical at general points of its irreducible components. Then

$$
\left(M_{1} \cdot M_{2}\right)_{C}>\frac{4}{\lambda^{2}}
$$

by [26, Theorem 3.1], where $M_{1}$ and $M_{2}$ are general surfaces in $\mathcal{M}_{Q}$. Using this, we get $\operatorname{deg}(C)<\frac{9}{2}$, which is impossible by Lemma 3.8.

Observe that $S_{2} \cong \mathbb{P}^{1} \times \mathbb{P}^{2}$, and the induced $\mathfrak{S}_{5}$-action on $S_{2}$ is faithful.

Lemma 3.10 (cf. [23, Theorem 7.5]). One has $\alpha_{\mathfrak{S}_{5}}\left(S_{2}\right)=\frac{3}{2}$.
Proof. Observe that $\operatorname{Pic}^{\mathfrak{G}_{5}}\left(S_{2}\right)=\mathbb{Z}\left[\left.H\right|_{S_{2}}\right]$ and $\mathcal{B}_{6} \in|3 H|_{S_{2}} \mid$. But $|H|_{S_{2}} \mid$ and $|2 H|_{S_{2}} \mid$ do not contain any $\mathfrak{S}_{5}$-invariant curves. Hence, we have $\alpha_{\mathfrak{S}_{5}}\left(S_{2}\right)=\frac{3}{2}$ by [9, Lemma 5.1] and Lemma 3.7.

Now we ready to prove Proposition 3.4,
Proof of Proposition 3.4. Suppose $\left(Q, \lambda \mathcal{M}_{Q}\right)$ is not canonical. Denote by $\Sigma$ its non-canonical locus. To complete the proof, we have to show that $\Sigma \subseteq \Sigma_{5} \cup \Sigma_{5}^{\prime}$.

First, let us show that the set $\Sigma$ consists of finitely many points. Indeed, suppose that $\Sigma$ contains a $\mathfrak{S}_{5}$-irreducible curve $C$. Then

$$
\begin{equation*}
\operatorname{mult}_{C}\left(\mathcal{M}_{Q}\right)>\frac{1}{\lambda} \tag{3.11}
\end{equation*}
$$

which easily implies that $\operatorname{deg}(C)<18$. Arguing as in the proof of Lemma 3.8, we see that $C \subseteq S_{2}$. Then (3.11) gives $\operatorname{deg}(C)<6$, which is impossible by Lemma 3.8. Hence, we see that $\Sigma$ is finite.

If $\Sigma \cap S_{2} \neq \varnothing$, then the $\log$ pair $\left(S_{2},\left.\lambda \mathcal{M}_{Q}\right|_{S_{2}}\right)$ is not $\log$ canonical by the inversion of adjunction, which is impossible by Lemma 3.10. Thus, we have $\Sigma \cap S_{2}=\varnothing$.

Applying Remark 3.6, we see that $\left(Q, \frac{3 \lambda}{2} \mathcal{M}_{Q}\right)$ is not log canonical at every point of the set $\Sigma$. Take $\varepsilon \in \mathbb{Q}_{>0}$ such that $\Sigma \subset \operatorname{Nklt}\left(Q, \frac{3 \lambda-\varepsilon}{2} \mathcal{M}_{Q}\right)$. Set $\Omega=\operatorname{Nklt}\left(Q, \frac{3 \lambda-\varepsilon}{2} \mathcal{M}_{Q}\right)$. Then $\Omega$ is $\mathfrak{S}_{5}$-invariant. Moreover, arguing as in the proof of Corollary 3.9, we see that the locus $\Omega$ does not contain curves, so that $\Omega$ is a finite set. Now, applying Nadel vanishing theorem, we get $h^{1}\left(Q, \mathcal{J} \otimes \mathcal{O}_{Q}\left(\left.2 H\right|_{Q}\right)\right)=0$, where $\mathcal{J}$ is the multiplier ideal sheaf of the $\log$ pair $\left(Q, \frac{3-\varepsilon}{2} \lambda \mathcal{M}_{Q}\right)$. This gives

$$
|\Sigma| \leqslant|\Omega| \leqslant h^{0}\left(Q, \mathcal{O}_{Q}\left(\left.2 H\right|_{Q}\right)\right)=14
$$

because $\operatorname{Supp}(\mathcal{J})=\Omega$. Now, using Lemma 3.7, we see that one of the following possibilities holds:

- $\Sigma \subseteq \Sigma_{5} \cup \Sigma_{5}^{\prime}$;
- $\Omega=\Sigma=\Sigma_{10}$;
- $\Omega=\Sigma=\Sigma_{10}^{\prime}$.

If $\Sigma \subseteq \Sigma_{5} \cup \Sigma_{5}^{\prime}$, we are done. Hence, without loss of generality, we may assume that $\Omega=\Sigma=\Sigma_{10}$. Let us show that this assumption leads to a contradiction.

Let $\mathcal{D}$ be the linear subsystem in $|2 H|$ that consists of all surfaces in $|2 H|$ that pass through $\Sigma_{10}$. By counting parameters, we get $\operatorname{dim}(\mathcal{D}) \geqslant 4$. Arguing as in the proof of Lemma 3.8, we see that the base locus of the linear system $\mathcal{D}$ contains no curves. Using [44] or [26, Corollary 3.4], we get

$$
\frac{36}{\lambda^{2}}=D \cdot M_{1} \cdot M_{2} \geqslant \sum_{P \in \Sigma_{10}}\left(M_{1} \cdot M_{2}\right)_{P}>\sum_{P \in \Sigma_{10}} \frac{4}{\lambda^{2}}=\frac{40}{\lambda^{2}},
$$

which is absurd. This completes the proof of Proposition 3.4.
Now, let us present a few facts about the threefold $Y$. Its singular locus consists of five nodes:

$$
\begin{aligned}
& P_{1}=(1: 0: 0: 0: 0), \\
& P_{2}=(0: 1: 0: 0: 0), \\
& P_{3}=(0: 0: 1: 0: 0), \\
& P_{4}=(0: 0: 0: 1: 0), \\
& P_{5}=(0: 0: 0: 0: 1) .
\end{aligned}
$$

Note that $(3: 3: 3: 3:-2) \in Y \backslash \operatorname{Sing}(Y)$. Let $\Theta_{5}$ be the $\mathfrak{S}_{5}$-orbit of this point. Then $\left|\Theta_{5}\right|=5$. For every $1 \leqslant i<j \leqslant 5$, we let $\ell_{i j}$ be the line in $\mathbb{P}^{4}$ that passes through the nodes $P_{i}$ and $P_{j}$. Let $\mathcal{L}_{10}$ be the union of these lines. Then $\mathcal{L}_{10} \subset Y$, and $\mathcal{L}_{10} \cap H$ is a $\mathfrak{S}_{5}$-orbit $\Theta_{10}$ of length 10 . The cubic $Y$ contains two more $\mathfrak{S}_{5}$-orbits of length 10 , which we denote by $\Theta_{10}^{\prime}$ and $\Theta_{10}^{\prime \prime}$.

Lemma 3.12. The orbits $\operatorname{Sing}(Y), \Theta_{5}, \Theta_{10}, \Theta_{10}^{\prime}, \Theta_{10}^{\prime \prime}$ are all $\mathfrak{S}_{5}$-orbit in $Y$ of length $<20$.
Proof. Left to the reader.
Let $S_{3}=Y \cap H$. Then $S_{3}$ is a smooth cubic surface known as the Clebsch diagonal cubic surface. It follows from [21, Lemma 6.3.12] that $\Theta_{10} \subset S_{3}$, but $S_{3}$ does not contain $\operatorname{Sing}(Y), \Theta_{5}, \Theta_{10}^{\prime}, \Theta_{10}^{\prime \prime}$. Observe also that $S_{3}$ contains the curve $\mathcal{B}_{6}$.

Lemma 3.13. Let $C$ be a $\mathfrak{S}_{5}$-invariant curve in $Y$ such that $\operatorname{deg}(C) \leqslant 10$. Then $C=\mathcal{B}_{6}$ or $\mathcal{L}_{10}$.
Proof. If $C \subset S_{3}$, the assertion follows from [21, Theorem 6.3.18]. Hence, we assume that $C \not \subset S_{3}$. Then, arguing as in the proof of Lemma 3.8, we conclude that and $C \cdot H=\Theta_{10}$.

We suppose that the curve $C$ is irreducible. Then $C$ has to be singular at every point $P \in \Theta_{10}$, because the stabilizer in $\mathfrak{S}_{5}$ of the point $P$ acts faithfully on the Zariski tangent space $T_{P}(C)$. Thus, if $C$ is irreducible, then $10=C \cdot H \geqslant 2\left|\Theta_{10}\right|$, which is absurd.

We see that $C$ is reducible and $\operatorname{deg}(C)=10$. Let $C_{1}$ be an irreducible components of the curve $C$, and let $G$ be the stabilizer in $\mathfrak{S}_{5}$ of the curve $C_{1}$. Then one of the following four cases holds:
(1) $G \cong \mathfrak{A}_{5}$ and $C$ is a union of 2 irreducible curves of degree 5 ;
(2) $G \cong \mathfrak{S}_{4}$ and $C$ is a union of 5 irreducible conics;
(3) $G \cong \mathfrak{A}_{4}$ the $C$ is a union of 10 lines;
(4) $G \cong \mathfrak{S}_{3} \times \boldsymbol{\mu}_{2}$ and $C$ is a union of 10 lines.

In the case (1), $\Theta_{10}$ splits as two $G$-orbits of length 5 , which is not the case by [21, Lemma 6.3.12]. In the cases (2) and (3), the only two-dimensional $G$-invariant linear subspace of $\mathbb{P}^{4}$ is contained in the $\mathfrak{S}_{5}$-invariant hyperplane $H$, so that $C_{1}$ is contained in $S_{3}$, which contradicts our assumption. In the case (4), one can easily see that $C=\mathcal{L}_{10}$.

Now, we are ready prove Proposition 3.5,
Proof of Proposition 3.5. It is enough to prove that $\left(Y, \mu \mathcal{M}_{Y}\right)$ is canonical away from $\operatorname{Sing}(Y)$. Suppose that this $\log$ pair is not canonical. Let $\Sigma$ be its non-canonical locus.

First, we claim that $\Sigma$ is a finite set. Indeed, suppose that $\Sigma$ contains a $\mathfrak{S}_{5}$-irreducible curve. Then $\operatorname{mult}_{C}(\mathcal{M})>\frac{1}{\mu}$. If $C \subset S_{3}$, this implies that $\operatorname{deg}(C)<6$, which is impossible by Lemma3.13, Thus, we see that $C \not \subset S_{3}$. Then $\operatorname{deg}(C)<12$, so that $H \cdot C<12$. Using Lemmas 3.12 and 3.13, we conclude that $C=\mathcal{L}_{10}$. Let $H^{\prime}$ be the hyperplane in $\mathbb{P}^{4}$ that contains the nodes $P_{1}, P_{2}, P_{3}, P_{4}$, and let $M$ be a general surface in $\mathcal{M}_{Y}$. Then

$$
H^{\prime} \cdot M=m\left(\ell_{12}+\ell_{13}+\ell_{14}+\ell_{23}+\ell_{24}+\ell_{34}\right)+\Delta
$$

where $a$ is an integer such that $a \geqslant \operatorname{mult}_{C}(\mathcal{M})$, and $\Delta$ is an effective one-cycle whose support does not contains the lines $\ell_{12}, \ell_{13}, \ell_{14}, \ell_{23}, \ell_{24}, \ell_{34}$. Therefore, we have

$$
\frac{6}{\mu}=\frac{2}{\mu} H^{3}=H \cdot H^{\prime} \cdot M=6 m+H \cdot \Delta \geqslant 6 m \geqslant 6 \operatorname{mult}_{C}(\mathcal{M})>\frac{6}{\mu}
$$

which is absurd. Thus, we see that $\Sigma$ is a finite set.
Let $\Sigma_{1}$ be the subset in $\Sigma$ that consists of all smooth points of $Y$. We have to show that $\Sigma_{1}=\varnothing$. If $\Sigma_{1} \cap S_{3} \neq \varnothing$, then the $\log$ pair $\left(S_{3},\left.\mu \mathcal{M}_{Y}\right|_{S_{3}}\right)$ is not log canonical, which implies that $\alpha_{\mathfrak{S}_{5}}\left(S_{3}\right)<2$. The latter contradicts [9, Example 1.11]. Thus, we have $\Sigma_{1} \cap S_{3}=\varnothing$.

Now, using Remark 3.6, we see that $\left(Y, \frac{3 \mu}{2} \mathcal{M}_{Y}\right)$ is not $\log$ canonical at every point of the set $\Sigma_{1}$. Moreover, arguing exactly as in the proof of Corollary 3.9 and using Lemma 3.13, we see that each point of the subset $\Sigma_{1}$ is an isolated center of non-log canonical singularities of the pair $\left(Y, \frac{3 \mu}{2} \mathcal{M}_{Y}\right)$. Now, using Nadel vanishing theorem as we did in the proof of Proposition 3.4, we see that $\left|\Sigma_{1}\right| \leqslant 5$. Therefore, we have $\Sigma_{1}=\Theta_{5}$ by Lemma 3.12,

Let $M_{1}$ and $M_{2}$ be general surfaces in $\mathcal{M}_{Y}$. Then it follows from [44] or [26, Corollary 3.4] that

$$
\left(M_{1} \cdot M_{2}\right)_{Q}>\frac{4}{\mu^{2}},
$$

for every point $Q \in \Theta_{5}$. Let $Q_{1}, Q_{2}$ and $Q_{3}$ be three points in $\Theta_{5}$, let $\Pi$ be the plane in $\mathbb{P}^{4}$ that contains these three points, and let $\mathcal{C}=\left.Y\right|_{\Pi}$. Then $\mathcal{C}$ is a smooth irreducible cubic curve. Write

$$
M_{1} \cdot M_{2}=\epsilon \mathcal{C}+\Omega
$$

where $\epsilon$ is a non-negative rational number, and $\Omega$ is an effective one-cycle whose support does not contain $\mathcal{C}$. Let $H^{\prime}$ be a general hyperplane section of the cubic hypersurface $Y$ that contains $\mathcal{C}$. Then $H^{\prime}$ does not contain any irreducible component of the one-cycle $\Omega$. Thus, we have

$$
\frac{12}{\mu^{2}}-3 \epsilon=H^{\prime} \cdot \Omega \geqslant \operatorname{mult}_{Q_{1}}(\Omega)+\operatorname{mult}_{Q_{2}}(\Omega)+\operatorname{mult}_{Q_{3}}(\Omega)>3\left(\frac{4}{\mu^{2}}-\epsilon\right)=\frac{12}{\mu^{2}}-3 \epsilon
$$

which is absurd. This completes the proof of Proposition 3.5,
This completes the proof of Theorem [3.1, which also implies that $Q$ is $\mathfrak{S}_{5}$-solid [2, 13, 17].

## 4. Preliminary results

In this section, we prove a few results that will be used towards the proof of Theorem 1.8,
Let $X$ be a variety with at most Kawamata log terminal singularities that is faithfully acted on by a finite group $G$. The following result is a consequence of the technique developed in [45, §3].

Lemma 4.1. Suppose $X$ is smooth. Let $Z$ be a $G$-irreducible subvariety of $X$ of codimension $m$, let $H$ be an ample divisor on $X$, and let $D_{1}, D_{2}, \ldots, D_{m}$ be effective divisors on $X$ such that

$$
D_{1} \sim_{\mathbb{Q}} D_{2} \sim_{\mathbb{Q}} \cdots \sim_{\mathbb{Q}} D_{m} \sim_{\mathbb{Q}} H,
$$

and $Z$ is a $G$-irreducible component of the intersection $\cap_{i=1}^{m} \operatorname{Supp}\left(D_{i}\right)$. Let $Y \subset X$ be an effective cycle of codimension $c \leqslant m$. Then

$$
\frac{\operatorname{mult}_{Z}(Y)}{\operatorname{deg}(Y)} \leqslant\left(\operatorname{deg}(Z) \cdot \min _{S} \prod_{i \in S} \operatorname{mult}_{Z}\left(D_{i}\right)\right)^{-1}
$$

where the minimum is taken over all subsets $S \subseteq\{1, \ldots, m\}$ of cardinality $m-c$.
Proof. We may assume that $Y$ is irreducible and $Z \subseteq Y$. We construct a sequence of irreducible subvarieties $Y_{c}, \ldots, Y_{m}$ and a permutation $D_{1}^{\prime}, \ldots, D_{m}^{\prime}$ of $D_{1}, \ldots, D_{m}$ such that

- $Y_{c}=Y$;
- $\operatorname{codim}_{X}\left(Y_{i}\right)=i$;
- $Y_{i} \not \subset \operatorname{Supp}\left(D_{i+1}^{\prime}\right)$;
- $Y_{i+1}$ is a component of $Y_{i} \cdot D_{i+1}^{\prime}$ that contains $Z$;
- for all $c \leqslant i \leqslant m-1$ one has

$$
\frac{\operatorname{mult}_{Z}\left(Y_{i+1}\right)}{\operatorname{deg}\left(Y_{i+1}\right)} \geqslant \operatorname{mult}_{Z}\left(D_{i+1}^{\prime}\right) \cdot \frac{\operatorname{mult}_{Z}\left(Y_{i}\right)}{\operatorname{deg}\left(Y_{i}\right)} .
$$

Once this is done, the lemma follows immediately from the trivial equality $Y_{m}=Z$.
Suppose that $Y_{c}, \ldots, Y_{i}$ and $D_{c+1}^{\prime}, \ldots, D_{i}^{\prime}$ have been constructed for some $i<m$. Then

$$
Y_{i} \subseteq \bigcap_{j=c+1}^{i} \operatorname{Supp}\left(D_{j}\right)
$$

Since $\cap_{i=1}^{m} \operatorname{Supp}\left(D_{i}\right)$ has codimension $m$ in a neighbourhood of $Z$ by assumption and

$$
\operatorname{codim}_{X}\left(Y_{i}\right)=i<m
$$

then there exists some $D_{j}$, which is necessary different from $D_{c+1}^{\prime}, \ldots, D_{i}^{\prime}$, which gives $Y_{i} \nsubseteq D_{j}$. We may then take $D_{i+1}^{\prime}=D_{j}$ and $Y_{i+1}$ an irreducible component of $\left(Y_{i} \cdot D_{j}\right)$ such that

$$
\frac{\operatorname{mult}_{Z}\left(Y_{i+1}\right)}{\operatorname{deg}\left(Y_{i+1}\right)} \geqslant \frac{\operatorname{mult}_{Z}\left(Y_{i} \cdot D_{j}\right)}{\operatorname{deg}\left(Y_{i} \cdot D_{j}\right)} \geqslant \operatorname{mult}_{Z} D_{j} \cdot \frac{\operatorname{mult}_{Z}\left(Y_{i}\right)}{\operatorname{deg}\left(Y_{i}\right)}
$$

By induction this finishes the construction.
Now, let $D$ be either an effective $\mathbb{Q}$-divisor on $X$ (a boundary), or a movable (mobile) boundary:

$$
D=\sum_{i=1}^{r} a_{i} \mathcal{M}_{i}
$$

where each $a_{i} \in \mathbb{Q}_{\geqslant 0}$, and each $\mathcal{M}_{i}$ is a linear system on $X$ that does not have fixed components. Suppose, in addition, that $D$ is $G$-invariant.

Lemma 4.2. Suppose that $(X, D)$ is not log canonical, and $D$ is ample. Then there exists positive rational number $\epsilon<1$ such that the following assertions hold:

- if $D$ is a $\mathbb{Q}$-divisor, there exists a $G$-invariant effective $\mathbb{Q}$-divisor $D^{\prime} \sim_{\mathbb{Q}}(1-\epsilon) D$ such that the log pair $\left(X, D^{\prime}\right)$ has log canonical singularities, and $\operatorname{Nklt}\left(X, D^{\prime}\right)$ is a non-empty disjoint union of minimal log canonical centers of the log pair $\left(X, D^{\prime}\right)$,
- if $D$ is a mobile boundary, there exists a $G$-invariant mobile boundary $D^{\prime} \sim_{\mathbb{Q}}(1-\epsilon) D$ such that the log pair $\left(X, D^{\prime}\right)$ has log canonical singularities, and $\operatorname{Nklt}\left(X, D^{\prime}\right)$ is a non-empty disjoint union of minimal log canonical centers of the log pair $\left(X, D^{\prime}\right)$.
Furthermore, irreducible components of $\operatorname{Nklt}\left(X, D^{\prime}\right)$ are normal, and $G$ transitively permutes them.
Proof. This is an equivariant version of the tie breaking. See [21, Lemma 2.4.10] or [33, 34].
Lemma 4.3. Let $H$ be a very ample divisor in $\operatorname{Pic}(X)$, and let $L$ be a divisor in $\operatorname{Pic}(X)$ such that the divisor $L-\left(K_{X}+D+\operatorname{dim}(X) H\right)$ is ample. Then $|L|$ contains a non-empty $G$-invariant linear subsystem $\mathcal{L}$ such that $\operatorname{Nklt}(X, D)=\operatorname{Bs}(\mathcal{L})$.
Proof. Let $\mathcal{J}=\mathcal{J}(X, D)$ be the multiplier ideal. Then the support of $\mathcal{O}_{X} / \mathcal{J}$ is exactly $\operatorname{Nklt}(X, D)$. By [39, Proposition 9.4.26], $\mathcal{J} \otimes \mathcal{O}_{X}(L)$ is generated by global sections. The $G$-invariant linear system $\mathcal{L}=\left|\mathcal{J} \otimes \mathcal{O}_{X}(L)\right|$ then satisfies the statement of the lemma.

Now, we fix $d, n \in \mathbb{Z}_{>0}$. Let $\mathbb{W}$ be the subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ consisting of all permutation matrices, let $\mathbb{T}$ be the subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ consisting of diagonal matrices whose (non-zero) entries are $d$-th roots of unity, and let $\mathbb{G}$ be the subgroup in $\mathrm{GL}_{n+1}(\mathbb{C})$ generated by $\mathbb{T}$ and $\mathbb{W}$. Then $\mathbb{W} \cong \mathfrak{S}_{n+1}, \mathbb{T} \cong \boldsymbol{\mu}_{d}^{n+1}$, and

$$
\mathbb{G} \cong \mathbb{T} \rtimes \mathbb{W} \cong \boldsymbol{\mu}_{d}^{n+1} \rtimes \mathfrak{S}_{n+1}
$$

Let $W, T, G$ be the images in $\mathrm{PGL}_{n+1}(\mathbb{C})$ via the quotient map of the groups $\mathbb{W}, \mathbb{T}$, $\mathbb{G}$, respectively. Then $W \cong \mathfrak{S}_{n+1}, T \cong \boldsymbol{\mu}_{d}^{n}$, and $G \cong T \rtimes W$. Note that $G$ leaves invariant the Fermat hypersurface

$$
X_{d}:=\left\{\sum_{i=0}^{d} x_{i}^{d}=0\right\} \subset \mathbb{P}^{n}
$$

where $x_{0}, \ldots, x_{n}$ are homogeneous coordinates on $\mathbb{P}^{n}$. If $n \geqslant 2$ and $d \geqslant 3$, then $G=\operatorname{Aut}\left(\mathbb{P}^{n}, X_{d}\right)$.
The examples for Theorem 1.8 are complete intersections in $\mathbb{P}^{n}$ of some Fermat hypersurfaces. The main result of this section is the following proposition. We will use it in the next section.

Proposition 4.4. Let $\mathcal{M}$ be a $W$-invariant linear subsystem in $\left|\mathcal{O}_{\mathbb{P}^{n}}(m)\right|$, let $Z$ be an irreducible component of the intersection $\operatorname{Bs}(\mathcal{M})$, and let $\mathscr{Z}$ be the $W$-irreducible subvariety in $\mathbb{P}^{n}$, whose irreducible component is $Z$. Then at least one of the following two cases holds:
(1) a general point in $Z$ has at most d different coordinates, and $\operatorname{dim}(Z) \leqslant m-1$;
(2) the subvariety $Z$ is an irreducible component of a set-theoretic intersection of $W$-invariant hypersurfaces of degree at most $m$, and $\operatorname{dim}(Z) \geqslant n-m$.
Moreover, in case (1), if $m \leqslant n$ and $n \geqslant 4$, then either $\mathscr{Z}$ has at least $n+1$ irreducible components, or $\mathscr{Z}=Z=(1: 1: \ldots: 1)$.

In particular, the base locus of a $W$-invariant linear subsystem in $\left|\mathcal{O}_{\mathbb{P}^{n}}(m)\right|$ either has dimension at most $m-1$, or has codimension at most $m$. This can be illustrated by the following example.

Example 4.5. In the assumptions and notations of Proposition 4.4, suppose $m=1$ and $Y=\mathbb{P}^{n}$. Then either $\mathcal{M}=\left|\mathcal{O}_{\mathbb{P}^{n}}(1)\right|$, so it is base point free, or one of the following two cases holds:
(1) $\mathcal{M}$ is the linear system of hyperplanes containing the $W$-invariant point $(1: 1: \ldots: 1)$;
(2) $\mathcal{M}$ is the $W$-invariant hyperplane $X_{1}=\left\{x_{0}+\ldots+x_{n}=0\right\} \subset \mathbb{P}^{n}$.

In case (1), we have $\mathscr{Z}=Z=(1: 1: \ldots: 1)$. In case (2), we have $\mathscr{Z}=Z=X_{1}$.
To prove Proposition 4.4, we need to prove a few auxiliary results.
Lemma 4.6. Fix $s \in\{1, \ldots, n\}$, and take positive integers $a_{1}, \ldots, a_{s}$ such that $n=a_{1}+\ldots+a_{s}$. Let $N$ be the number of unordered partitions of the set $\{1, \ldots, n\}$ into subsets of $a_{1}, \ldots, a_{s}$ elements, respectively. Then $N \geqslant n$ unless $s=1, s=n$, or $n=4, s=2, a_{1}=a_{2}=2$.

Proof. We may assume $a_{1} \leqslant a_{2} \leqslant \ldots \leqslant a_{k}$. If $a_{1}=\ldots=a_{i}<a_{i+1}$ for some $i \in\{1, \ldots, k-1\}$, then

$$
N \geqslant\binom{ n}{i a_{1}} \geqslant n .
$$

Hence, we may assume that $a_{1}=\ldots=a_{s}=r$ for some $r \in\{2, \ldots, n-1\}$. Then $s=\frac{n}{r} \geqslant 2$ and

$$
N=\frac{n!}{(r!)^{s} \cdot s!} \geqslant \frac{n(n-1) \cdot \ldots \cdot(n-r+1)}{r!\cdot s} .
$$

Thus, since $n \geqslant 2 s$, we get

$$
N \geqslant(n-1) \cdot \frac{n}{2 s} \cdot \prod_{j=0}^{r-3} \frac{n-2-j}{r-j} \geqslant n-1
$$

with equality only if $n=2 s$ and $n-2=r$, i.e. when $r=s=2$. Since $N$ is a positive integer, the assertion follows.

For the second result, we need the following two conventions. A color set is a finite multiset, where elements (i.e. colors) may appear with multiplicities. If $K=(V, E)$ is a graph and $\mathscr{C}$ is a color set, then a coloring of the graph $K$ by $\mathscr{C}$ is a map $\phi: V \rightarrow \mathscr{C}$ such that

- every color is used at most once, i.e. we have $\left|\phi^{-1}(c)\right| \leqslant 1$ for every $c \in \mathscr{C}$,
- and every pair of adjacent vertices have different color, i.e. we have $\phi(u) \neq \phi(v)$ as integers whenever $(u, v) \in E$.

Lemma 4.7. Let $K=(V, E)$ be a graph such that $K$ contains at least $s \geqslant 1$ connected components, and let $\mathscr{C}$ be a color set of size at least $|V|$ such that $\mathscr{C}$ has at least $|V|-s+1$ different colors. Then there exists a coloring of the graph $K$ by $\mathscr{C}$.

Proof. We use induction on $|V|-s \geqslant 0$. The result is clear when $|V|-s=0$, since in this case there are no edges in $K$. Suppose now that the result has been proved for smaller values of $|V|-s$. We can assume that every connected component of $K$ contains at least 2 vertices, since we can assign any color to isolated points. In particular, the number $|V|-s$ drops if we remove connected components from $V$. It is also clear that we may assume $s \geqslant 2$ and at least one of the colors has multiplicity $\geqslant 2$ (otherwise there are already $|V|$ different colors).

Now, we let $K_{1}=\left(V_{1}, E_{1}\right)$ be a connected component of the graph $K$, and we set $r=\left|V_{1}\right| \geqslant 2$. Let $K^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ be the subgraph of the graph $K$ that is obtained by removing the component $K_{1}$. We may choose a subset $\mathscr{C}_{1} \subseteq \mathscr{C}$ that consists of $r$ distinct colors (each with multiplicity 1) such that the complement $\mathscr{C} \backslash \mathscr{C}_{1}$ has at least $|V|-s+2-r$ different color (here we use the assumption that at least one color in $\mathscr{C}$ has multiplicity $\geqslant 2$ ). Note that we can color the graph $K_{1}$ by $\mathscr{C}_{1}$. By induction hypothesis, we can also color $K^{\prime}$ by $\mathscr{C} \backslash \mathscr{C}_{1}$, since $|V|-s+2-r=\left|V^{\prime}\right|-(s-1)+1$. This gives us a coloring of $K$ by $\mathscr{C}$.

Let us identify $H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(m)\right)$ with the subspace in $\mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$ consisting of all homogeneous polynomials of degree $m$. For $f \in H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(m)\right)$ and a (possibly reducible) subvariety $Y \subset \mathbb{P}^{n}$, we define $\left.f\right|_{Y}$ to be the image of the polynomial $f$ in $H^{0}\left(Y,\left.\mathcal{O}_{\mathbb{P}^{n}}(m)\right|_{Y}\right)$ via the restriction morphism. For any $f \in H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(m)\right)$, we denote by $\mathcal{M}_{f}$ the linear subsystem in $\left|\mathcal{O}_{\mathbb{P}^{n}}(m)\right|$ that is given by the supspace in $H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(m)\right)$ spanned by $\tau^{*}(f)$ for all $\tau \in \mathbb{W}$. Finally, we fix $V=\{0,1, \ldots, n\}$. For every graph $K=(V, E)$, let $c(K)$ be the number of its connected components, and let

$$
f_{K}= \pm \prod_{(i, j) \in E}\left(x_{i}-x_{j}\right) \in H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(|E|)\right)
$$

Lemma 4.8. Let $Y$ be an intersection in $\mathbb{P}^{n}$ of some $W$-invariant hypersurfaces, and fix $\ell \in \mathbb{Z}_{>0}$. Take some $g \in H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(\ell)\right)$ such that $\left.g\right|_{Y}$ is not $\mathbb{W}$-invariant, and let $K=(V, E)$ be a graph. Then there exists a graph $K^{\prime}=\left(V, E^{\prime}\right)$ containing $K$ as a subgraph and $g^{\prime} \in H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(\ell-1)\right)$ such that $c\left(K^{\prime}\right) \geqslant c(K)-1,\left.g^{\prime}\right|_{Y} \neq 0$ and

$$
\operatorname{Bs}\left(\mathcal{M}_{h}\right) \subseteq \operatorname{Bs}\left(\mathcal{M}_{h^{\prime}}\right)
$$

for $h=f_{K} g$ and $h^{\prime}=f_{K^{\prime}} g^{\prime}$.
Proof. Since $\left.g\right|_{Y}$ is not $\mathbb{W}$-invariant, there is a transposition $\tau=(i j) \in \mathbb{W}$ such that $\left.\tau^{*}(g)\right|_{Y} \neq\left. g\right|_{Y}$. Then $\tau^{*}(g)-g=\left(x_{i}-x_{j}\right) g^{\prime}$ for some $g^{\prime} \in H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(\ell-1)\right)$ such that $\left.g^{\prime}\right|_{Y} \neq 0$, since otherwise we would have $\left.\tau^{*}(g)\right|_{Y}=\left.g\right|_{Y}$.

Let $\tau(K)$ be the graph obtained from $K$ by switching the labeling of the vertices $i$ and $j$ without changing any edges, and let $K^{\prime}$ be the graph obtained by adding the edge (ij) to $K \cup \tau(K)$ (take the union of edges). Then $c\left(K^{\prime}\right) \geqslant c(K)-1$.

Let $h=f_{K} g$ and $h^{\prime}=f_{K^{\prime}} g^{\prime}$. Then $\tau^{*}(h)=f_{\tau(K)} \tau^{*}(g)$, and $\operatorname{Bs}\left(\mathcal{M}_{h}\right) \subseteq \operatorname{Bs}\left(\mathcal{M}_{h^{\prime}}\right)$, because $h^{\prime}$ has the same factors (ignoring multiplicities) as

$$
f_{K \cup \tau(K)} \cdot\left(x_{i}-x_{j}\right) g^{\prime}=f_{K \cup \tau(K)}\left(\tau^{*}(g)-g\right)=f_{K-\tau(K)} \tau^{*}(h)-f_{\tau(K)-K} h,
$$

where $K-\tau(K)$ is the graph obtained by removing from $K$ the edges of $\tau(K)$.
Corollary 4.9. Let $Y$ be an intersection in $\mathbb{P}^{n}$ of $W$-invariant hypersurfaces, let $f$ be a polynomial in $H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(m)\right)$ such that $\left.f\right|_{Y} \neq 0$. Then there are $r \in\{0,1, \ldots, m\}$, a graph $K=(V, E)$, and $a \mathbb{W}$-invariant polynomial $g_{0} \in H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(m-r)\right)$ such that $\left.g_{0}\right|_{Y} \neq 0, c(K) \geqslant n+1-r$, and

$$
\begin{equation*}
\operatorname{Bs}\left(\mathcal{M}_{f}\right) \cap Y \subseteq \operatorname{Bs}\left(\mathcal{M}_{g}\right) \tag{4.10}
\end{equation*}
$$

for $g=f_{K} g_{0}$.

Proof. Let us apply Lemma 4.8 repeatedly starting with the graph $(V, \varnothing)$ and $g=f$. This process must stop after at most $m$ steps. Therefore, we obtain a graph $K(V, E)$ and a polynomial $g=f_{K} h$ such that $\operatorname{deg}(h)=m-r$ for $r \leqslant m, c(K) \geqslant n+1-r$, the restriction $\left.h\right|_{Y}$ is $\mathbb{W}$-invariant, and

$$
\operatorname{Bs}\left(\mathcal{M}_{f}\right) \subseteq \operatorname{Bs}\left(\mathcal{M}_{g}\right)
$$

Then we can replace $h$ by a $W$-invariant polynomial $g_{0}$ of the same degree such that $\left.g_{0}\right|_{Y}=\left.h\right|_{Y}$.
Proof of Proposition 4.4. The assertions on $\operatorname{dim}(Z)$ and the assertion on the number of irreducible components of the subvariety $\mathscr{Z}$ follow from Lemma 4.6. Thus, we have to prove that
(1) either a general point in $Z$ has at most $m$ different coordinates,
(2) or the subvariety $Z$ is an irreducible component of a set-theoretic intersection of $W$-invariant hypersurfaces of degree at most $m$.
Let $D_{1}, \ldots, D_{k}$ be $W$-invariant hypersurfaces of degree at most $m$ that contain $Z$, and let $Y$ be the set theoretic intersection $D_{1} \cap \cdots \cap D_{k}$ (if there exist no such hypersurfaces, we set $Y=X$ ). We may assume $Z \varsubsetneqq Y$ (otherwise (2) clearly holds). Hence, there is $f \in \mathcal{M}$ such that $\left.f\right|_{Y} \neq 0$. Note that this gives $Z \subseteq \operatorname{Bs}\left(\mathcal{M}_{f}\right) \cap Y$.

By Corollary 4.9, we find a graph $K=(V, E)$ with $c(K) \geqslant n+1-r$ and a $\mathbb{W}$-invariant polynomial $g_{0} \in H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(m-r)\right)$ such that $\left.g_{0}\right|_{Y} \neq 0$ and (4.10) holds, where $r \in\{0,1, \ldots, m\}$. By the construction of $Y$, we see that $\left.g_{0}\right|_{Z} \neq 0$, thus (4.10) gives

$$
Z \subseteq \operatorname{Bs}\left(\mathcal{M}_{f_{K}}\right)
$$

Pick a general point $z \in Z$ with coordinates $\left[z_{0}: \ldots: z_{n}\right]$ and consider the color set $\mathcal{C}=\left\{z_{0}, \ldots, z_{n}\right\}$. If (1) does not hold, then there are at least $m \geqslant r$ different colors in $\mathcal{C}$. By Lemma 4.7, we may color the graph $K$ by $\mathcal{C}$. After unwinding the definitions, this implies that there is $\sigma \in \mathbb{W}$ such that $\sigma^{*}\left(f_{K}\right)$ does not vanish on $Z$. But this is a contradiction as $\sigma^{*}\left(f_{K}\right) \in \mathcal{M}_{f_{K}}$. So, we conclude that (1) holds in this case and this completes the proof of the proposition.

Let us apply Proposition 4.4. Recall that $X_{d}$ is the Fermat hypersurface in $\mathbb{P}^{n}$ of degree $d$.
Proposition 4.11. If $d \leqslant n \leqslant 3 d-1$, then $\alpha_{G}(X) \geqslant 1$. If $n \geqslant 3 d$, then $\alpha_{G}(X)=\frac{2 d}{n+1-d}$.
Proof. We suppose that $n \geqslant d$. Let $H$ be a hyperplane section of $X_{d}$, let $r=\min \{2 d, n+1-d\}$, and let $D$ be a $G$-invariant effective divisor on $X_{d}$ such that $D \sim_{\mathbb{Q}} r H$. We have $\alpha_{G}(X) \leqslant \frac{2 d}{n+1-d}$, where the right hand side is computed by the $G$-invariant Fermat hypersurface $X_{2 d}$. Hence, both statements of the proposition would follow once we prove that the $\log$ pair $(X, D)$ is $\log$ canonical. Suppose that $(X, D)$ is not $\log$ canonical. Let us seek for a contradiction.

Let $\lambda=\operatorname{lct}(X, D)$ and $Z=\operatorname{Nklt}(X, \lambda D)$. Then $(X, \lambda D)$ is $\log$ canonical, $\lambda<1$ and $Z \neq \varnothing$. Applying Lemma 4.2, we may assume that $Z$ is a disjoint union of irreducible normal subvarieties. But, on the other hand, since $-\left(K_{X}+\lambda D\right)$ is ample, applying Kollár-Shokurov's connectedness, we conclude that $Z$ is an irreducible subvariety. By Lemma 4.3, there exists a $G$-invariant linear subsystem $\mathcal{L} \subset|(3 d-2) H|$ such that $Z=\operatorname{Bs}(\mathcal{L})$.

Let $V$ be the vector subspace in $H^{0}\left(X, \mathcal{O}_{X}((3 d-2) H)\right)$ that corresponds to the linear system $\mathcal{L}$. Then $V$ is a $\mathbb{G}$-subrepresentation in $H^{0}\left(X, \mathcal{O}_{X}((3 d-2) H)\right)$. As $\mathbb{T}$-representation, we have

$$
V=\bigoplus_{\chi} V_{\chi},
$$

where the summand runs over all characters $\chi$ of the group $\mathbb{T}$. For each $\chi$, we have $V_{\chi}=\mathbf{x}_{\chi} \cdot W_{\chi}$, where $\mathbf{x}_{\chi}$ is a monomial of degree at most $d-1$ in each homogeneous coordinate $x_{0}, x_{1}, \ldots, x_{n}$, while $\mathbb{T}$ acts trivially on $W_{\chi}$. Each $W_{\chi}$ is the image in $H^{0}\left(X, \mathcal{O}_{X}(m d H)\right)$ of a subspace of

$$
\operatorname{Sym}^{m}(U) \subseteq H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(m d)\right)
$$

where $U=\operatorname{span}\left(x_{0}^{d}, \ldots, x_{n}^{d}\right) \subseteq H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(d)\right)$ and $m \leqslant 2$, because $m d=\operatorname{deg}\left(W_{\chi}\right) \leqslant 3 d-2$.
Since the action of the group $\mathbb{G}$ on the vector space $H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(1)\right)$ is irreducible, we see that the subvariety $Z$ is not contained in a hyperplane, so $Z$ is not contained in $\left\{\mathbf{x}_{\chi}=0\right\}$ for any $\chi$. Then $Z$ is a set-theoretic intersection of zeroes of all polynomials in all $W_{\chi}$. Since $Z$ is invariant under the $G$-action, we see that $\sigma^{*}(f)$ vanishes on $Z$ for any $f \in W_{\chi}$ and any $\sigma \in \mathbb{W} \cong \mathfrak{S}_{n+1}$.

Now, let us consider a morphism $v: \mathbb{P}^{n} \rightarrow \mathbb{P}^{n}$ defined as

$$
v\left(x_{0}: \ldots: x_{n}\right)=\left(x_{0}^{d}: \ldots: x_{n}^{d}\right) .
$$

Then the induced action of $G$ on $\operatorname{Im}(f)=\mathbb{P}^{n}$ is isomorphic (as an action) to the permutational action of the group $W \cong \mathfrak{S}_{n+1}$ on $\mathbb{P}^{n}$. Further, we observe that

$$
(1: \ldots: 1) \notin v\left(X_{d}\right) .
$$

Moreover, since $Z$ is connected and $W$-invariant, so is $v(Z)$. Thus, from the previous discussion, we conclude that $Z$ is the base locus of some $W$-invariant linear system of degree at most $2 d$, generated by all the polynomials in $W_{\chi}$. Then $v(Z)$ is the base locus of some $W$-invariant linear system of degree at most 2. Applying Proposition 4.4 to $v(Z)$, we see that $Z$ is an irreducible component of a set-theoretic intersection of $G$-invariant hypersurfaces of degree at most $2 d$. Then

$$
Z=X_{d} \cap X_{2 d} .
$$

On the other hand, since the $\log$ pair $\left(X_{d}, D\right)$ is not $\log$ canonical along $Z$, we have $\operatorname{mult}_{Z}(D)>1$. This contradicts to $Z \sim_{\mathbb{Q}} 2 d H$ and $r \leqslant 2 d$.

Similarly, we prove the following result.
Proposition 4.12. Let $X=X_{d} \cap X_{2 d} \cap \ldots \cap X_{r d}$ for $r \geqslant 1$, let $H$ be a hyperplane section of $X$, and let $D$ be a $G$-invariant effective $\mathbb{Q}$-divisor on $X$ such that $D \sim_{\mathbb{Q}} q H$ for a positive rational number $q<(r+1)$ d. Suppose, in addition, that $\operatorname{dim}(X) \geqslant 1, n \geqslant 4$, and $d H-\left(K_{X}+D\right)$ is nef. Then the log pair $(X, D)$ is log canonical.

Proof. Replacing $q$ by $\lceil q\rceil$, and $D$ by $\frac{\lceil q\rceil}{q} D$, we may assume that the number $q$ is actually integer. Suppose that $(X, D)$ is not $\log$ canonical. Let us seek for a contradiction.

Let $\lambda=\operatorname{lct}(X, D)$ and $Z=\operatorname{Nklt}(X, \lambda D)$. Then $(X, \lambda D)$ is $\log$ canonical, $\lambda<1$ and $Z \neq \varnothing$. Applying Lemma 4.2, we may assume that $Z$ is a disjoint union of irreducible normal subvarieties, and $Z$ is $G$-irreducible. Moreover, using Lemma 4.3, we see that $Z=\operatorname{Bs}(\mathcal{L})$ for some $G$-invariant linear subsystem $\mathcal{L} \subset|a H|$, where

$$
a=\frac{r(r+1)}{2} d+q-(r-1)
$$

that satisfies $K_{X}+D+(n-r) H \sim_{\mathbb{Q}} a H$.
Now, let $V$ be the vector subspace in $H^{0}\left(X, \mathcal{O}_{X}(a H)\right)$ that corresponds to the linear system $\mathcal{L}$. Then $V$ is a $\mathbb{G}$-subrepresentation in $H^{0}\left(X, \mathcal{O}_{X}(a H)\right)$. As before, we have

$$
V=\bigoplus_{\chi} V_{\chi},
$$

where the summand runs over all characters $\chi$ of the group $\mathbb{T}$. For each $\chi$, we have $V_{\chi}=\mathbf{x}_{\chi} \cdot W_{\chi}$, where $\mathbf{x}_{\chi}$ is a monomial of degree at most $d-1$ in each homogeneous coordinate $x_{0}, x_{1}, \ldots, x_{n}$, and each $W_{\chi}$ is the image in $H^{0}\left(X, \mathcal{O}_{X}(\ell d H)\right)$ of a subspace of

$$
\operatorname{Sym}^{\ell}(U) \subseteq H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(\ell d)\right)
$$

where $U=\operatorname{span}\left(x_{0}^{d}, \ldots, x_{n}^{d}\right) \subseteq H^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(d)\right)$ and $\ell \leqslant\left\lfloor\frac{a}{d}\right\rfloor$.

Now, let $Z_{1}, \ldots, Z_{s}$ be the $T$-irreducible components of the locus $Z$. Then we claim that $s \leqslant n$. Indeed, using Nadel vanishing theorem and the nefness of the divisor $d H-\left(K_{X}+D\right)$, we get

$$
H^{1}\left(X, \mathcal{J}(X, \lambda D) \otimes \mathcal{O}_{X}(d H)\right)=0
$$

where $\mathcal{J}(X, \lambda D)$ is the multiplier ideal sheaf of the $\log$ pair $(X, \lambda D)$. Now, let $\Upsilon$ be the subscheme defined by the multiplier ideal sheaf $\mathcal{J}(X, \lambda D)$ of the log pair $(X, \lambda D)$, and let $\Upsilon_{i}$ be its irreducible component supported on $Z_{i}$ for $i \in\{1, \ldots, s\}$. Then the natural restriction

$$
H^{0}\left(X, \mathcal{O}_{X}(d H)\right) \rightarrow H^{0}\left(\Upsilon, \mathcal{O}_{\Upsilon}\left(\left.d H\right|_{\Upsilon}\right)\right)
$$

is surjective. Taking the $\mathbb{T}$-invariant parts, we see that

$$
\begin{aligned}
s \leqslant \sum_{i=1}^{s} \operatorname{dim}\left(H^{0}\left(\Upsilon_{i}, \mathcal{O}_{\Upsilon_{i}}\left(\left.d H\right|_{\Upsilon_{i}}\right)\right)^{\mathbb{T}}\right) & \leqslant \operatorname{dim}\left(H^{0}\left(\Upsilon, \mathcal{O}_{\Upsilon}\left(\left.d H\right|_{\Upsilon}\right)\right)^{\mathbb{T}}\right) \leqslant \\
& \leqslant \operatorname{dim}\left(H^{0}\left(X, \mathcal{O}_{X}(d H)\right)^{\mathbb{T}}\right)=\operatorname{dim}(U)-1=n
\end{aligned}
$$

Here, the first inequality holds because $H^{0}\left(\Upsilon_{i}, \mathcal{O}_{\Upsilon_{i}}\left(\left.d H\right|_{\Upsilon_{i}}\right)\right)^{\mathbb{T}}$ contains $\left.U\right|_{\Upsilon_{i}} \neq 0$.
We claim that no $T$-irreducible components of $Z$ are contained in coordinate hyperplanes. Indeed, otherwise, such component would be contained in the (unique) minimal $T$-invariant linear subspace in $\mathbb{P}^{n}$, which would imply that $Z$ has at least $n+1 T$-irreducible components.

Let $m=\left\lfloor\frac{a}{d}\right\rfloor$. Arguing as in the proof of Proposition 4.11, we see that $Z$ is the base locus of the $W$-invariant subsystem of $\left|\mathcal{O}_{\mathbb{P}^{n}}(m d)\right|$ generated by $\operatorname{Bs}\left(W_{\chi}\right)$ and hypersurfaces containing $X$. Now, using Proposition 4.4 and the same morphism $v: \mathbb{P}^{n} \rightarrow \mathbb{P}^{n}$ as in Proposition 4.11, we conclude (as in the proof of Proposition 4.11) that $Z$ is a $G$-irreducible component of the set-theoretic intersection of some $G$-invariant hypersurfaces of degree at most $m d$, because the other possibility in Proposition 4.4 is excluded, since $v(Z)$ has at most $n$ irreducible components and

$$
\left\{x_{0}^{d}=x_{1}^{d}=\ldots=x_{n}^{d}\right\} \not \subset X
$$

In particular, we see that the pair $(X, D)$ is not log canonical along $Y=X_{d} \cap X_{2 d} \cap \ldots \cap X_{m d}$, and hence $\operatorname{mult}_{Y}(D)>1$. But as $n>m$ under our assumption (we leave to the reader to verify this), we see that $Y$ is irreducible. Now, applying Lemma 4.1 to $D_{k}=\frac{1}{k d}\left(X_{k d} \cdot X\right)$ for $k=r+1, \ldots, m$, we get

$$
\operatorname{mult}_{Y}(D) \leqslant \frac{\operatorname{deg}(D)}{\operatorname{deg}(Y)} \prod_{i=k+2}^{m} k d \leqslant \frac{\operatorname{deg}(H)}{\operatorname{deg}(Y)} \prod_{k=r+1}^{m} k d=1
$$

which is a contradiction.

## 5. The proof of Theorem 1.8

Let us use all assumptions and notations of Section 4. Let $X$ be the complete intersection in the projective space $\mathbb{P}^{n}$ of the Fermat hypersurfaces $X_{2 d}, X_{3 d}, \ldots, X_{r d}$ for some integer $r \geqslant 2$, and let $H$ be a hyperplane section of the variety $X$. Suppose that

$$
-K_{X} \sim q H
$$

for some $q \leqslant \frac{(r+1) d}{2}$. Then $\alpha_{G}(X) \leqslant \frac{d}{q}$, since $-\left.K_{X} \sim_{\mathbb{Q}} \frac{q}{d} X_{d}\right|_{X}$. So, we can make $\alpha_{G}(X)$ arbitrarily small by choosing $q=\left\lfloor\frac{1}{2}(r+1) d\right\rfloor$ and letting $r \gg 0$. Therefore, to prove Theorem 1.8, it remains to show that $X$ is $G$-birationally super-rigid.

In order to prove this, we use a similar strategy as in Proposition 4.12, Let $\mathcal{M}$ be a $G$-invariant mobile linear system, and let $\lambda$ be a positive rational number such that

$$
K_{X}+\lambda \mathcal{M} \sim_{\mathbb{Q}} 0
$$

As in the proof of the Theorem 1.4 we need to show that $(X, \lambda \mathcal{M})$ has canonical singularities. Suppose the singularities of the pair $(X, \lambda \mathcal{M})$ are non-canonical. Let us seek for a contradiction.

Let $B$ be a center of non-canonical singularities of the $\log$ pair $(X, \lambda \mathcal{M})$. Let us create some non-log canonical behaviour using the center $B$. In the following Lemma 5.1, we first treat the case when $B$ is contained in some special divisor $Y \subset X$, so that $\left(Y,\left.\lambda \mathcal{M}\right|_{Y}\right)$ is not $\log$ canonical by the inversion of adjunction. As in the proof of Proposition 4.12, we will use Nadel vanishing to get an estimate of the possible number of irreducible components of the non-log canonical locus, and then use Proposition 4.4 to derive a contradiction.

Lemma 5.1. Let $r, d \geqslant 2, n \geqslant 4$ be integers, let $H$ be a divisor in $\left|\mathcal{O}_{\mathbb{P}^{n}}(1)\right|_{X} \mid$, and set $Y=X \cap X_{d}$. Assume that $D \sim_{\mathbb{Q}} l H$ is a $G$-invariant effective divisor on $X$ for some $l \leqslant(r+1) d$ such that the divisor $H-\left(K_{X}+D\right)$ is nef. Assume also that $\operatorname{dim} X \geqslant 2$. Then
(1) if $D$ does not contain $Y$ in its support, then $(X, D)$ is log canonical;
(2) the non-log canonical locus of $(X, D)$ is contained in $Y$.

Proof. As in the proof of Proposition 4.12, we may assume that $l \in \mathbb{N}$. Suppose that (1) is proved. To prove (2), write

$$
D=t \cdot \frac{l}{d} Y+(1-t) D_{0}
$$

for some $0 \leqslant t \leqslant 1$ and $D_{0} \sim_{\mathbb{Q}} l H$ such that $Y \nsubseteq \operatorname{Supp}\left(D_{0}\right)$. Then $\left(X, D_{0}\right)$ is $\log$ canonical by (1). Hence, every non-log canonical center of $(X, D)$ is a non-log canonical center of the pair $\left(X, \frac{l}{d} Y\right)$. In particular, the non-log canonical locus of $(X, D)$ is contained in $Y$. This proves (2).

Now, let us prove (1). Suppose that $Y \nsubseteq \operatorname{Supp}(D)$, and the $\log$ pair $(X, D)$ is not log canonical. Let us seek for a contradiction. Let $\lambda=\operatorname{lct}(X, D)$ and $Z=\operatorname{Nklt}(X, \lambda D)$. By Lemmas 4.2 and 4.3, we may further assume that $Z$ is $G$-irreducible, $Z$ is a disjoint union of its irreducible components, and $Z=\operatorname{Bs}(\mathcal{L})$ for a $G$-invariant linear system $\mathcal{L} \subset|a H|$, where

$$
a=\left(\frac{r(r+1)}{2}-1\right) d+l-r
$$

satisfies

$$
K_{X}+D+(n-r+1) H \sim_{\mathbb{Q}} a H
$$

Let $s$ be the number of irreducible components of $Z$, and let $Z_{1}, \ldots, Z_{s}$ be these components. By Nadel vanishing applied to the multiplier ideal sheaf $\mathcal{J}(X, \lambda D)$, we have a surjection

$$
H^{0}\left(X, \mathcal{O}_{X}(H)\right) \rightarrow H^{0}\left(Z, \mathcal{O}_{Z}\left(\left.H\right|_{Z}\right)\right)=\bigoplus_{i=1}^{s} H^{0}\left(Z_{i}, \mathcal{O}_{Z_{i}}\left(\left.H\right|_{Z_{i}}\right)\right)
$$

so $s \leqslant h^{0}\left(X, \mathcal{O}_{X}(H)\right)=n+1$. But $h^{0}\left(Z_{i}, \mathcal{O}_{Z_{i}}\left(\left.H\right|_{Z_{i}}\right)\right) \geqslant 1$ with strict inequality for $\operatorname{dim}\left(Z_{i}\right)>0$. Thus, if $s=n+1$, then

$$
n+1=h^{0}\left(Z, \mathcal{O}_{Z}\left(\left.H\right|_{Z}\right)\right)=\sum_{i=1}^{s} h^{0}\left(Z_{i}, \mathcal{O}_{Z_{i}}\left(\left.H\right|_{Z_{i}}\right)\right)
$$

which gives $\operatorname{dim}(Z)=0$, so that we obtain a contradiction $n+1 \geqslant|Z| \geqslant d(n+1)>n+1$ as $d \geqslant 2$, since the length of a $G$-orbit in $X$ is at least $d(n+1)$. This shows that $s \leqslant n$.

Arguing as in the proof of Proposition 4.12, we see that $Z$ is a component of the set-theoretic intersection of some $G$-invariant hypersurfaces of degree at most $m d$, where $m=\left\lfloor\frac{a}{d}\right\rfloor<n$. Set

$$
R=X_{d} \cap X_{2 d} \cap \ldots \cap X_{m d}
$$

Then the pair $(X, D)$ is not $\log$ canonical along $R$. So, since one has $Y \nsubseteq \operatorname{Supp}(D)$, we have

$$
\operatorname{mult}_{R}\left(\left.D\right|_{Y}\right) \geqslant \operatorname{mult}_{R}(D)>1
$$

On the other hand, as in the proof of Proposition 4.12, we obtain $\operatorname{mult}_{R}\left(\left.D\right|_{Y}\right) \leqslant 1$ by Lemma 4.1, The obtained contradiction completes the proof of the lemma.

Finally we treat the general case. Here the main observation is that if the center of non-canonical singularities is not contained in the special divisors, then, as a consequence of Proposition 4.4, it has small dimension, and in this case we can prove the $G$-birational super-rigidity by a similar application of the method of [51].

Theorem 5.2. Let $d \gg 0, r \geqslant 2$ be integers. Assume that $-K_{X} \sim q H$ where $H$ is the hyperplane class and $1 \leqslant q \leqslant \frac{(r+1) d}{2}$. Then $X$ is $G$-birationally super-rigid.

Proof. Assume the contrary. Then, using Noether-Fano inequality [8], we obtain a non-canonical $\log$ pair $(X, \lambda \mathcal{M})$ such that $\mathcal{M}$ is a mobile linear system, and $\lambda \in \mathbb{Q}>0$ such that $K_{X}+\lambda \mathcal{M} \sim_{\mathbb{Q}} 0$. Let $B$ be a center of non-canonical singularities of the pair $(X, \lambda \mathcal{M})$. Let us seek for a contradiction.

Observe that the center $B$ is not contained in the Fermat hypersurface $X_{d}$. Indeed, otherwise, by the inverse of adjunction, the $\log$ pair $\left(Y,\left.M\right|_{Y}\right)$ is not $\log$ canonical, where $Y=X \cap X_{d}$. But

$$
d H-\left(K_{Y}+\left.\lambda \mathcal{M}\right|_{Y}\right)=-\left.\left(K_{X}+\lambda \mathcal{M}\right)\right|_{Y} \sim_{\mathbb{Q}} 0
$$

which is impossible by Proposition 4.12,
We claim that $B$ is not contained in any $T$-invariant hyperplane. Indeed, suppose $B \subseteq\left\{x_{i}=0\right\}$. Let $X^{\prime}=X \cap\left\{x_{i}=0\right\}$, and let $G^{\prime}$ be the stabilizer subgroup in $G$ of the hyperplane $\left\{x_{i}=0\right\}$. Then $G^{\prime} \cong \boldsymbol{\mu}_{d}^{n} \rtimes \mathfrak{S}_{n}$, and $X^{\prime}$ is $G^{\prime}$-invariant. Let $\mathcal{M}^{\prime}=\left.\mathcal{M}\right|_{X^{\prime}}$. Then ( $X^{\prime}, \lambda \mathcal{M}^{\prime}$ ) is not log canonical along $B$ by the inverse of adjunction. But $K_{X^{\prime}}+\lambda \mathcal{M}^{\prime} \sim_{\mathbb{Q}} H$, which gives $B \subset X_{d}$ by Lemma 5.1. However, we already proved that $B \not \subset X_{d}$.

Now by Remark 3.6, the $\log$ pair $(X, 2 \lambda \mathcal{M})$ is not $\log$ canonical along $B$. Let $\mu$ be the smallest positive rational number such that $B \subset \operatorname{Nklt}(X, \mu \mathcal{M})$, and let $Z$ be an irreducible component of the locus $\operatorname{Nklt}(X, \mu \mathcal{M})$ containing $B$. Then $\mu<2 \lambda$, and it follows from [26, Theorem 3.1] that

$$
\operatorname{mult}_{B}\left(M_{1} \cdot M_{2}\right)>4 / \mu^{2},
$$

for general divisors $M_{1}$ and $M_{2}$ in the linear system $\mathcal{M}$. Moreover, it follows from Lemma 4.3 that the subvariety $Z$ is a component of $\operatorname{Bs}(\mathcal{L})$ for some $G$-invariant linear system $\mathcal{L} \subseteq\left|\mathcal{O}_{X}(a)\right|$, where

$$
a=\left(\frac{r(r+1)}{2}-1\right) d+2 q-r
$$

satisfies

$$
K_{X}+2 \lambda \mathcal{M}+(n-r+1) H \sim_{\mathbb{Q}} a H
$$

Furthermore, we know that $Z$ is not contained in any $T$-invariant hyperplane, because $B$ is not contained in any $T$-invariant hyperplane.

Set $m=\left\lfloor\frac{a}{d}\right\rfloor$. Then $m<n$. Now, arguing as in the proof of Proposition 4.11 or Proposition4.12, and using Proposition 4.4, we see that either the subvariety $Z$ is a component of a set-theoretic intersection of $G$-invariant hypersurfaces of degree at most $m d$, or $\operatorname{dim}(B) \leqslant \operatorname{dim}(Z) \leqslant m-1$.

Suppose that the subvariety $Z$ is a component of a set-theoretic intersection of $G$-invariant hypersurfaces of degree at most $m d$. Let $\Gamma^{\prime}$ be an irreducible component of $M_{1} \cdot M_{2}$ such that

$$
\frac{\operatorname{mult}_{Z}\left(\Gamma^{\prime}\right)}{\operatorname{deg}\left(\Gamma^{\prime}\right)} \geqslant \frac{\operatorname{mult}_{Z}\left(M_{1} \cdot M_{2}\right)}{\operatorname{deg}\left(M_{1} \cdot M_{2}\right)}>\frac{1}{\mu^{2} \operatorname{deg}\left(M_{1} \cdot M_{2}\right)}
$$

Since $Z \not \subset X_{d}$, we observe that $\Gamma=\Gamma^{\prime} \cdot X_{d}$ is a codimension 2 cycle on $Y=X \cap X_{d}$ such that

$$
\frac{\operatorname{mult}_{Z^{\prime}}(\Gamma)}{\operatorname{deg}(\Gamma)} \geqslant \frac{\operatorname{mult}_{Z}\left(\Gamma^{\prime}\right)}{\operatorname{deg}\left(\Gamma^{\prime}\right)}>\frac{1}{\mu^{2} \operatorname{deg}\left(M_{1} \cdot M_{2}\right)}
$$

where $Z^{\prime}=X_{d} \cap \ldots \cap X_{m d} \subseteq Z$. On the other hand, let $H_{1}$ and $H_{2}$ be general divisors in $|H|$. Then, as in the proof of Proposition 4.12, we get

$$
\frac{\operatorname{mult}_{Z^{\prime}}(\Gamma)}{\operatorname{deg}(\Gamma)} \leqslant \frac{1}{\operatorname{deg} Z^{\prime}} \prod_{k=r+3}^{m} k d=\frac{1}{d^{2}(r+1)(r+2) \operatorname{deg}\left(H_{1} \cdot H_{2}\right)}<\frac{1}{\mu^{2} \operatorname{deg}\left(M_{1} \cdot M_{2}\right)}
$$

by Lemma 4.1 applied to the divisors $Y \cdot X_{k d}$, for $r+1 \leqslant k \leqslant m$ on the complete intersection $Y$, where the last inequality follows from $2 q \leqslant(r+1) d$. This gives us a contradiction.

Thus, we have $\operatorname{dim} B \leqslant \operatorname{dim} Z \leqslant m-1$. Let $P$ be a sufficiently general point in the center $B$, and let $Y$ be a general codimension $m$ linear section of the complete intersection $X$ containing $P$. Set $\mathcal{M}_{Y}=\left.\mathcal{M}\right|_{Y}$. Then the pair $\left(Y, \lambda \mathcal{M}_{Y}\right)$ is not $\log$ canonical at $P$ by the inverse of adjunction, but the pair $\left(Y, 2 \lambda \mathcal{M}_{Y}\right)$ is $\log$ canonical in a punctured neighbourhood of the point $P$. Note that

$$
K_{Y}+2 \lambda \mathcal{M}_{Y} \sim_{\mathbb{Q}}(m+q) H
$$

Using [51, Corollary 1.8] and the lower bound in [37, Paragraph 56] for the number of lattice points, we obtain the following inequality:

$$
\begin{equation*}
h^{0}\left(Y, \mathcal{O}_{Y}\left(\left.(m+q) H\right|_{Y}\right)\right)>\frac{1}{n-m-r}\binom{2(n-m-r)}{n-m-r} \geqslant \frac{1}{(n-m-r)^{2}} 4^{n-m-r} \tag{5.3}
\end{equation*}
$$

On the other hand, since $Y$ is a complete intersection in $\mathbb{P}^{n-m}$, we also have

$$
\begin{equation*}
h^{0}\left(Y, \mathcal{O}_{Y}\left(\left.(m+q) H\right|_{Y}\right)\right) \leqslant h^{0}\left(\mathbb{P}^{n-m}, \mathcal{O}_{\mathbb{P}^{n-m}}(m+q)\right)=\binom{n+q}{m+q}<2^{n+q} \tag{5.4}
\end{equation*}
$$

Recall that $m=\left\lfloor\frac{a}{d}\right\rfloor \leqslant \frac{r(r+1)}{2}+2 r$ is bounded by a constant (that does not depend on $d$ ), and, by assumption, we have

$$
n=\left(\frac{r(r+1)}{2}-1\right) d+q-1 \geqslant \frac{1}{r+1}\left(\frac{r(r+1)}{2}-1\right) q+q-1 \geqslant \frac{5}{3} q-1 .
$$

Therefore, using (5.4), we obtain

$$
h^{0}\left(Y, \mathcal{O}_{Y}((m+q) H)\right)<2^{1.6 n+1}<\frac{1}{(n-m-r)^{2}} 4^{n-m-r}
$$

when $n \gg 0$, which is equivalent to $d \gg 0$. This contradicts to (5.3). The proof is complete.

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