

DEGREE OF IRRATIONALITY OF FANO THREEFOLD HYPERSURFACES

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ABSTRACT. We study degree of irrationality of quasismooth anticanonically embedded weighted Fano 3-fold hypersurfaces that have terminal singularities.

Throughout this paper, all varieties are assumed to be projective and defined over \mathbb{C} .

1. INTRODUCTION

Let a_1, \dots, a_5 be positive integers with $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$. Let X_d be a quasi-smooth and well-formed ([16]) hypersurface of degree d in the weighted projective space $\mathbb{P}(a_1, a_2, a_3, a_4, a_5)$. Suppose that

$$d < a_1 + a_2 + a_3 + a_4 + a_5.$$

and let $I = a_1 + a_2 + a_3 + a_4 + a_5 - d$. Then X_d is a Fano threefold with quotient singularities. Note that there exists infinitely many possibilities for the septuple $(a_1, a_2, a_3, a_4, a_5, d, I)$. If $I = 1$, all of them have been found in [19].

Suppose, in addition, that the singularities of the 3-fold hypersurface X_d are terminal. Then it follows from the Birkar's boundedness results ([4, 5]) that there are finitely many possibilities for $(a_1, a_2, a_3, a_4, a_5, d, I)$. In fact, there are 130 such possibilities, excluding weighted projective spaces, that define 130 families of weighted hypersurfaces ([6, 7, 16]). Fano 3-folds in these families have been studied in [1, 8, 9, 10, 11, 12, 13, 14, 20, 25, 27, 28]. In particular, we know that X_d is irrational if $I = 1$, and we know all deformation families with $I \geq 2$ such that their general members are irrational. In the following, we will label these 130 deformation families by $\mathbb{N}^{\circ}1, \mathbb{N}^{\circ}2, \mathbb{N}^{\circ}3, \dots, \mathbb{N}^{\circ}130$ as in [1, 13, 14].

Remark. If X_d is smooth, then it belongs to one of the following deformation families:

$\mathbb{N}^{\circ}1$: $(a_1, a_2, a_3, a_4, a_5, d, I) = (1, 1, 1, 1, 1, 4, 1)$, i.e. X_d is a quartic 3-fold,

$\mathbb{N}^{\circ}3$: $(a_1, a_2, a_3, a_4, a_5, d, I) = (1, 1, 1, 1, 3, 6, 1)$, i.e. X_d is a sextic double solid.

$\mathbb{N}^{\circ}96$: $(a_1, a_2, a_3, a_4, a_5, d, I) = (1, 1, 1, 1, 1, 3, 2)$, i.e. X_d is a cubic 3-fold,

$\mathbb{N}^{\circ}97$: $(a_1, a_2, a_3, a_4, a_5, d, I) = (1, 1, 1, 1, 1, 3, 2)$, i.e. X_d is a quartic double solid,

$\mathbb{N}^{\circ}98$: $(a_1, a_2, a_3, a_4, a_5, d, I) = (1, 1, 1, 2, 3, 6, 2)$, i.e. X_d is a double Veronese cone,

$\mathbb{N}^{\circ}104$: $(a_1, a_2, a_3, a_4, a_5, d, I) = (1, 1, 1, 1, 1, 2, 3)$, i.e. X_d is a quadric 3-fold.

All members of the remaining 124 deformation families are singular, and their singularities are described in [6, 7, 14]. Note that $I = 1$ for the families $\mathbb{N}^{\circ}1, \mathbb{N}^{\circ}2, \mathbb{N}^{\circ}3, \dots, \mathbb{N}^{\circ}94, \mathbb{N}^{\circ}95$.

In this paper, we study the degree of irrationality of Fano 3-folds in these 130 families. Recall from [23] that the degree of irrationality $d(V)$ of a variety V is the smallest possible degree of the generically finite dominant rational map $V \dashrightarrow \mathbb{P}^n$, where $n = \dim(V)$. Then

$$\text{the variety } V \text{ is rational} \iff d(V) = 1.$$

If $I \geq 2$, then $d < 3a_5$, so the projection $\mathbb{P}(1, a_2, a_3, a_4, a_5) \dashrightarrow \mathbb{P}(1, a_2, a_3, a_4)$ induces a dominant rational map $X_d \dashrightarrow \mathbb{P}(1, a_2, a_3, a_4)$ such that it is either birational or generically two-to-one, so that $d(X_d) \in \{1, 2\}$ in this case. For $I = 1$, we prove

Main Theorem. *Suppose that $I = 1$. Then the following assertions hold:*

- (1) $d(X_d) \in \{2, 3\}$;
- (2) $d(X_d) = 2$ if the 3-fold X_d is not contained in the deformation families №1, №19, №28, №39, №49, №59, №66, №84;
- (3) $d(X_d) = 3$, if the 3-fold X_d is a general member of one of the eight deformation families №1, №19, №28, №39, №49, №59, №66, №84.

Conjecture. *Suppose that X_d is any quasismooth member in one of the eight deformation families №1, №19, №28, №39, №49, №59, №66, №84. Then $d(X_d) = 3$.*

If X_d is contained in one of the families №1, №19, №28, №39, №49, №59, №66, №84, then the septuple $(a_1, a_2, a_3, a_4, a_5, d, I)$ is described in the following table below.

№	a_1	a_2	a_3	a_4	a_5	d	I
1	1	1	1	1	1	4	1
19	1	2	3	3	4	12	1
28	1	3	3	4	5	15	1
39	1	3	4	5	6	18	1
49	1	3	5	6	7	21	1
59	1	3	6	7	8	24	1
66	1	5	6	7	9	27	1
84	1	7	8	9	12	36	1

It would be interesting to study degree of irrationality of rationally connected threefolds in a more general setting. If a rationally connected 3-fold is birational to a conic bundle, its degree of irrationality is 1 or 2. Similarly, if a rationally connected 3-fold is birational to a fibration into del Pezzo surfaces, it is birational to an elliptic fibration with a section, which implies that the degree of irrationality of the threefold is either 1 or 2.

Proposition. *Let V be a smooth Fano 3-fold that is not a quartic 3-fold. Then $d(V)$ is either 1 or 2.*

Proof. We may assume that V is not birational neither to a conic bundle nor to an elliptic fibration with a section. Then V is sextic double solid. This gives $d(V) \leq 2$. \square

Applying Minimal Model Program, we see that every rationally connected threefold whose degree of irrationality is larger than 2 is birational to a Fano 3-fold with terminal \mathbb{Q} -factorial singularities that has Picard rank 1, which is birational neither to a conic bundle nor to a fibration into del Pezzo surfaces. Such a Fano 3-fold is said to be *solid* ([3]). It would be interesting to study the degree of irrationality of solid Fano 3-folds.

Let us describe the structure of this paper. In Sections 2, 3, and 4, we study weighted Fano 3-fold hypersurfaces in the deformation families №1, №19, №28, №39, №49, №59, №66, №84. Then, in Section 5, we prove out Main Theorem.

From now on, we denote by $\text{Aut}(X)$ the group of automorphisms of a threefold X and by $\text{Bir}(X)$ the group of birational automorphisms of X . We use coordinates x, y, z, t, w for $\mathbb{P}(a_1, a_2, a_3, a_4, a_5)$ with $\text{wt}(x) = a_1, \text{wt}(y) = a_2, \text{wt}(z) = a_3, \text{wt}(t) = a_4, \text{wt}(w) = a_5$.

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2. FAMILY №1

Let X_4 be a smooth quartic 3-fold in \mathbb{P}^4 . Then X_4 is irrational ([18]). Thus, considering a projection $X_4 \dashrightarrow \mathbb{P}^3$ from a point in X_4 , we get $d(V) \in \{2, 3\}$.

Lemma 2.1. *If X_4 is general, then $d(X_4) = 3$.*

Proof. Suppose that $d(X_4) = 2$. Then there exists a rational map $X_4 \dashrightarrow \mathbb{P}^3$ that is generically two-to-one. This map induces a birational involution τ of X_4 . The involution τ is biregular since

$$\text{Bir}(X_4) = \text{Aut}(X_4).$$

by [18]. However, if X_4 is general, the group $\text{Aut}(X_4)$ is trivial ([22]). □

We expect that $d(X_4) = 3$. Indeed, suppose $d(X_4) = 2$. As in the proof of Lemma 2.1, we obtain a biregular involution τ of X_4 such that the quotient $Y = X_4/\tau$ is rational. Then the fixed locus of the involution τ is either

- a smooth quartic surface,
- or a disjoint union of a smooth plane quartic curve and four different points.

In the former case, the threefold Y is a double cover of \mathbb{P}^3 ramified along a smooth quartic, which is irrational ([29]). Therefore, we conclude that τ fixes disjoint union of smooth plane quartic curve and four points. Now, changing coordinates on \mathbb{P}^4 , we may assume that

$$X_4 = \{h_4(x, y, z) + t^2 a_2(x, y, z) + t w b_2(x, y, z) + w^2 c_2(x, y, z) + g_4(t, w) = 0\} \subset \mathbb{P}^4$$

while τ acts as

$$\tau(x : y : z : t : w) = (x : y : z : -t : -w),$$

where h_i, a_i, b_i, c_i, g_i are homogeneous polynomials of degree i . Then τ fixes the smooth irreducible curve

$$C = \{h_4(x, y, z) = 0, t = 0, w = 0\} \subset \mathbb{P}^4$$

and four points O_1, O_2, O_3, O_4 that are cut out on X_4 by $x = y = z = 0$.

Let $\eta: X_4 \rightarrow Y$ be the quotient map. Then Y is singular along the smooth curve $\eta(C)$, and Y has 4 singular points $\eta(O_1), \eta(O_2), \eta(O_3),$ and $\eta(O_4)$, which are cyclic quotient singular points of type $\frac{1}{2}(1, 1, 1)$. Let \mathcal{H} be the pencil on Y generated by the images of the surfaces cut out on X_4 by $t = 0$ and $w = 0$, respectively. Then we have the following commutative diagram:

$$\begin{array}{ccc} & V & \\ \pi \swarrow & & \searrow \phi \\ Y & \dashrightarrow & \mathbb{P}^1 \end{array}$$

where π is the blow up of the curve $\eta(C)$, the map $Y \dashrightarrow \mathbb{P}^1$ is given by the pencil \mathcal{H} , and ϕ is a Mori fibre space whose general fiber is a smooth del Pezzo surfaces of degree 2. Now, one can try to apply the technique from [2, 21, 24] to show that Y is irrational.

3. FAMILIES №19 AND №28

Before we proceed, let us introduce a lemma which is useful in handling automorphisms for Families №19 and №28.

Lemma 3.1. *Let Σ be a subset in \mathbb{P}^1 that consists of 5 distinct points in general position. Then the group of automorphisms of the pair (\mathbb{P}^1, Σ) is trivial.*

Proof. This is a well-known fact. See, for example, [15, § 6.4] or [26]. \square

Let Y be a general quasismooth hypersurface in the family №19. Then Y is given by quasihomogeneous equation of degree 12

$$f := w^3 + w^2 f_4(x, y, z, t) + w f_8(x, y, z, t) + f_{12}(x, y, z, t) = 0$$

in $\mathbb{P}(1, 2, 3, 3, 4)$, where each f_k is a quasihomogeneous polynomial of degree k in x, y, z, t . We now change the coordinates in the following way:

(1) by using the following type of coordinate change

$$w \mapsto w + *y^2 + *yx^2 + *xz + *xt,$$

we may assume that f has wy^4 but none of $y^6, y^5x^2, y^4xz, y^4xt$;

(2) by using the following type of coordinate change

$$y \mapsto y + *x^2,$$

we may assume that f does not have wy^3x^2 ;

(3) by using the following type of coordinate change

$$z \mapsto z + *xy + *x^3,$$

we may assume that f has neither t^3xy nor t^3x^3 ;

(4) by using the following type of coordinate change

$$t \mapsto t + *xy + *x^3,$$

we may assume that f has neither z^3xy nor z^3x^3 ,

where $*$ means an appropriate constant. Note that our choice of coordinates implies that the sections by $w = 0$ and $y = 0$ are preserved by $\text{Aut}(Y)$. Furthermore, the section by $x = 0$ is also preserved by $\text{Aut}(Y)$, since the 0-dimensional space $|\mathcal{O}_Y(1)|$ is invariant. These imply that all members of $\text{Aut}(Y)$ are given by

$$[x : y : z : t; w] \mapsto [x : A_1y : \alpha_1z + \alpha_2t : \alpha_3z + \alpha_4t : A_2w],$$

where A_1 and A_2 are non-zero constants, and α_i 's are constants such that the matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$$

is invertible. Observe also that

$$f_{12}(x, y, z, t) = y^3g_6(z, t) + y^2h_8(x, z, t) + yh_{10}(x, z, t) + h_{12}(x, z, t),$$

where g_6 is a quasihomogeneous polynomial of degree 6 in z, t , and h_k 's are quasihomogeneous polynomials of degree k in x, z, t . The hypersurface Y has four distinct points

$$g_{12}(z, t) = 0$$

on the curve C defined by $x = y = w = 0$. Moreover, since Y is general, we may assume that $g_6(z, t) = 0$ gives two distinct points on the curve C , and hence

$$g_{12}(z, t)g_6(z, t) = 0$$

defines 6 distinct points Σ in $C \cong \mathbb{P}^1$ which are in general position. Every automorphism of Y induces an automorphism of C that leaves Σ invariant. Then Lemma 3.1 implies that $\alpha_2 = \alpha_3 = 0$ and $\alpha_1 = \alpha_4$. Consequently, every automorphism of Y must be induced by

$$[x : y : z : t : w] \mapsto [x : a_1y : a_2z : a_2t : a_3w]$$

where a_i 's are non-zero constants.

Let W be a general quasismooth hypersurface in the family №28. The hypersurface W is given by quasihomogeneous equation of degree 15

$$w^3 + w^2f_5(x, y, z, t) + wf_{10}(x, y, z, t) + f_{15}(x, y, z, t) = 0$$

in $\mathbb{P}(1, 3, 3, 4, 5)$, where $f_k(x, y, z, t)$ is a quasihomogeneous polynomial of degree k in x, y, z, t . By using the coordinate change

$$w \mapsto w + \frac{1}{3}f_5(x, y, z, t)$$

we may assume that W is defined by quasihomogeneous equation of degree 15

$$w^3 + wf_{10}(x, y, z, t) + f_{15}(x, y, z, t) = 0$$

in $\mathbb{P}(1, 3, 3, 4, 5)$. Since W is quasismooth, we may further change coordinates z and y such that f_{15} has t^3z but not t^3y . Then, we change the coordinates in the following way:

- (1) use the following type of coordinate change

$$z \mapsto z + *x^3$$

in such a way that f_{15} does not have t^3x^3 ;

- (2) use the following type of coordinate change

$$y \mapsto y + *x^3$$

in such a way that f_{15} has y^5 but not y^4x^3 ;

- (3) use the following type of coordinate change

$$t \mapsto t + *zx + *yx + *x^4$$

in such a way that none of t^2z^2x , t^2zyx , t^2zx^4 appear in f_{15} ,

where $*$ means a suitable constant. As above, we see that each member in $\text{Aut}(W)$ is given by

$$[x : y : z : t : w] \mapsto [x : \alpha_1y + \alpha_2z : \alpha_3y + \alpha_4z : A_1t : A_2w]$$

where A_i 's are non-zero constants, and α_i 's are constants such that the matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$$

is invertible. Observe that the hypersurface W has five distinct points on the curve C defined by $x = t = w = 0$, and every automorphism of W induces an automorphism of C that permutes these five points. Since W is general, we may assume that these points are in general position. Thus, applying Lemma 3.1, we see that $\alpha_2 = \alpha_3 = 0$ and $\alpha_1 = \alpha_4$. Consequently, every automorphism of W must be induced by

$$[x : y : z : t : w] \mapsto [x : a_1y : a_1z : a_3t : a_4w]$$

where a_i 's are non-zero constants.

The following table shows all monomials that appear in the quasihomogenous polynomial for each of the families №19 and №28 after required coordinate changes.

№	d	monomials of degree d
19	12	$w^3, zt^3, z^2t^2, z^3t, yt^2w, yztw, yz^2w, y^2w^2, y^3t^2, y^3zt, y^3z^2, y^4w, xyzt^2, xyz^2t,$ $xy^2tw, xy^2zw, xzw^2, xtw^2, x^2t^2w, x^2ztw, x^2z^2w, x^2y^2t^2, x^2y^2zt, x^2y^2z^2,$ $x^3zt^2, x^3z^2t, x^3ytw, x^3yzw, x^3y^3t, x^3y^3z, x^4yt^2, x^4yzt, x^4yz^2, x^4y^2w, x^4y^4,$ $x^4w^2, x^5tw, x^5zw, x^5y^2t, x^5y^2z, x^6t^2, x^6zt, x^6z^2, x^6yw, x^6y^3, x^7yt, x^7yz,$ $x^8w, x^8y^2, x^9t, x^9z, x^{10}y, x^{12}$
28	15	$w^3, zt^3, z^2tw, yt^3, yztw, yz^4, y^2tw, y^2z^3, y^3z^2, y^4z, xz^3w, xyz^2w, xy^2t^2,$ $xy^2zw, xy^3w, x^2t^2w, x^2z^3t, x^2yz^2t, x^2y^2zt, x^2y^3t, x^3ztw, x^3ytw, x^3yz^3,$ $x^3y^2z^2, x^3y^3z, x^4z^2w, x^4yt^2, x^4yzw, x^4y^2w, x^5z^2t, x^5yzt, x^5y^2t, x^6tw, x^6z^3,$ $x^6yz^2, x^6y^2z, x^6y^3, x^7t^2, x^7zw, x^7yw, x^8zt, x^8yt, x^9z^2, x^9yz, x^9y^2, x^{10}w, x^{11}t,$ $x^{12}z, x^{12}y, x^{15}$

Using this, it is not difficult to see that the automorphism group of a general hypersurface in each of the families №19 and №28 is trivial. In particular, it contains no involutions.

Suppose that for a general hypersurface X in the families №19 and №28, $d(X) = 2$. Then $\text{Bir}(X)$ has an involution. However, since X is birationally super-rigid ([13]), we have

$$\text{Bir}(X) = \text{Aut}(X),$$

and hence $\text{Aut}(X)$ has an involution. This is a contradiction. Therefore, $d(X) = 3$.

4. FAMILIES №39, №49, №59, №66, №84

Let X_d be a quasismooth and well-formed hypersurface of degree d in the weighted projective space $\mathbb{P}(a_1, a_2, a_3, a_4, a_5)$. Suppose that X_d belongs to one of the families №39, №49, №59, №66, №84. Since the hypersurface X_d is birationally super-rigid ([13]), we have

$$\text{Bir}(X_d) = \text{Aut}(X_d).$$

It follows from the same argument in Section 3 that in order to have $d(X_d) = 3$, it is enough to show that $\text{Aut}(X_d)$ is trivial.

We have $d = 3a_5$ and the hypersurface is defined by quasihomogeneous equation

$$f(x, y, z, t, w) := w^3 + w^2 f_{a_5}(x, y, z, t) + w f_{2a_5}(x, y, z, t) + f_d(x, y, z, t) = 0,$$

where $f_k(x, y, z, t)$ is a quasihomogeneous polynomial of degree k in x, y, z, t .

By using the coordinate change

$$w \mapsto w + \frac{1}{3}f_{a_5}(x, y, z, t),$$

we may assume that $f_{a_5}(x, y, z, t) = 0$, that is, the hypersurface is defined by quasi-homogeneous equation

$$(4.1) \quad w^3 + wf_{2a_5}(x, y, z, t) + f_d(x, y, z, t) = 0.$$

The section by $x = 0$ is preserved by $\text{Aut}(X_d)$ since the 0-dimensional space $|\mathcal{O}_{X_d}(1)|$ is invariant. Due to the assumption (4.1), the section by $w = 0$ is also invariant under the action of $\text{Aut}(X_d)$.

We now suppose that X_d is general. Using the monomials in the first column, we perform appropriate coordinate changes described in the third column of Table 4.2 so that we might assume that the monomials in the second column do not appear in the defining equation (4.1). For instance, if $f = \dots + at^3y + bt^3x^3 + \dots$ in the first line of №39, where $a \neq 0$, b are constants, then we perform the coordinate change

$$y \mapsto y + \frac{b}{a}x^3$$

so that we could remove the monomial t^3x^3 from f . If $f = \dots + at^3y + bt^2yzx + ct^2y^2x^2 + dt^2yx^5 + \dots$ in the second line of №39, where $a \neq 0$, b, c, d are constants, then we use the coordinate change

$$t \mapsto t + \frac{1}{3a}(bxz + cx^2y + dx^5)$$

to get rid of the monomials $t^2yzx, t^2y^2x^2, t^2yx^5$ from f .

№	f contains	f does not contain	coordinate changes
39	t^3y t^3y z^3w	t^3x^3 $t^2yzx, t^2y^2x^2, t^2yx^5$ z^2wyz, z^2wx^4	$y \mapsto y + *x^3$ $t \mapsto t + *xz + *x^2y + *x^5$ $z \mapsto z + *xy + *x^4$
49	t^3y y^5t z^4x	t^3x^3 $y^5xz, y^5x^6, y^6x^3, y^7$ $z^3x^3y, z^3x^6,$	$y \mapsto y + *x^3$ $t \mapsto t + *xz + *x^3y + *x^6 + *y^2$ $z \mapsto z + *x^2y + *x^5$
59	t^3y t^3y y^6z	t^3x^3 $t^2y^3x, t^2y^2xz, t^2yx^7$ y^6x^6, y^7x^3, y^8	$y \mapsto y + *x^3$ $t \mapsto t + *xy^2 + *xz + *x^7$ $z \mapsto z + *y^2 + *yx^3 + x^6$
66	t^3z y^4t y^4t	t^3xy, t^3x^6 y^4xz, y^4x^7, y^5x^2 y^3tx^5	$z \mapsto z + *xy + *x^6$ $t \mapsto t + *xz + *x^2y + *x^7$ $y \mapsto y + *x^5$
84	y^4z y^4z t^4	y^4x^8, y^5x y^3zx^7 t^3xz, t^3x^2y, t^3x^9	$z \mapsto z + *xy + *x^8$ $y \mapsto y + *x^7$ $t \mapsto t + *xz + *x^2y + *x^9$

(4.2)

where $*$ means an appropriate constant. It then follows that $\text{Aut}(X_d)$ fixes the sections defined by $y = 0$, by $z = 0$, and by $t = 0$. Consequently, all members of $\text{Aut}(X_d)$ are given by

$$[x : y : z : t : w] \mapsto [x : a_1y : a_2z : a_3t : a_4w]$$

where a_i 's are non-zero constants.

The following table shows all the monomials that appear in the quasihomogenous polynomial (4.1) for each of the families №39, №49, №59, №66, №84. Note that all the assumptions of Table 4.2 for (4.1) are applied to the table below.

№	d	monomials of degree d
39	18	$w^3, z^2t^2, z^3w, yt^3, yztw, y^2z^3, y^3zt, y^4w, y^6, xz^3t, xy^2tw, xy^3z^2, xy^4t, x^2t^2w, x^2z^4, x^2yz^2t, x^2y^2zw, x^2y^4z, x^3ztw, x^3yz^3, x^3y^2zt, x^3y^3w, x^3y^5, x^4zt^2, x^4ytw, x^4y^2z^2, x^4y^3t, x^5z^2t, x^5yzw, x^5y^3z, x^6z^3, x^6yzt, x^6y^2w, x^6y^4, x^7tw, x^7yz^2, x^7y^2t, x^8t^2, x^8zw, x^8y^2z, x^9zt, x^9yw, x^9y^3, x^{10}z^2, x^{10}yt, x^{11}yz, x^{12}w, x^{12}y^2, x^{13}t, x^{14}z, x^{15}y, x^{18}$
49	21	$w^3, z^3t, yt^3, yztw, y^2z^3, y^3t^2, y^3zw, y^5t, xz^4, xyzt^2, xyz^2w, xy^3zt, x^2t^2w, x^2yz^2t, x^2y^2tw, x^2y^3z^2, x^2y^4w, x^3ztw, x^3y^2t^2, x^3y^2zw, x^3y^4t, x^4zt^2, x^4z^2w, x^4y^2zt, x^4y^4z, x^5z^2t, x^5ytw, x^5y^2z^2, x^5y^3w, x^6yt^2, x^6yzw, x^6y^3t, x^7yzt, x^7y^3z, x^8tw, x^8yz^2, x^8y^2w, x^9t^2, x^9zw, x^9y^2t, x^9y^4, x^{10}zt, x^{10}y^2z, x^{11}z^2, x^{11}yw, x^{12}yt, x^{12}y^3, x^{13}yz, x^{14}w, x^{15}t, x^{15}y^2, x^{16}z, x^{18}y, x^{21}$
59	24	$w^3, z^4, yt^3, yztw, y^2z^3, y^3tw, y^4z^2, y^6z, xyz^2w, xy^3zw, xy^5w, x^2t^2w, x^2yz^2t, x^2y^3zt, x^2y^5t, x^3ztw, x^3yz^3, x^3y^2tw, x^3y^3z^2, x^3y^5z, x^4zt^2, x^4z^2w, x^4y^2t^2, x^4y^2zw, x^4y^4w, x^5z^2t, x^5y^2zt, x^5y^4t, x^6z^3, x^6ytw, x^6y^2z^2, x^6y^4z, x^7yzw, x^7y^3w, x^8yzt, x^8y^3t, x^9tw, x^9yz^2, x^9y^3z, x^9y^5, x^{10}t^2, x^{10}zw, x^{10}y^2w, x^{11}zt, x^{11}y^2t, x^{12}z^2, x^{12}y^2z, x^{12}y^4, x^{13}yw, x^{14}yt, x^{15}yz, x^{15}y^3, x^{16}w, x^{17}t, x^{18}z, x^{18}y^2, x^{21}y, x^{24}$
66	27	$w^3, zt^3, z^3w, yztw, y^3z^2, y^4t, xz^2t^2, xyz^2w, xy^2tw, x^2z^3t, x^2yzt^2, x^2y^2zw, x^3z^4, x^3yz^2t, x^3y^2t^2, x^3y^3w, x^4t^2w, x^4yz^3, x^4y^2zt, x^5ztw, x^5y^2z^2, x^6z^2w, x^6ytw, x^6y^3z, x^7zt^2, x^7yzw, x^8z^2t, x^8yt^2, x^8y^2w, x^9z^3, x^9yzt, x^{10}yz^2, x^{10}y^2t, x^{11}tw, x^{11}y^2z, x^{12}zw, x^{12}y^3, x^{13}t^2, x^{13}yw, x^{14}zt, x^{15}z^2, x^{15}yt, x^{16}yz, x^{17}y^2, x^{18}w, x^{20}t, x^{21}z, x^{22}y, x^{27}$
84	36	$w^3, t^4, z^3w, yztw, y^4z, xyz^2w, xy^2tw, x^2z^2t^2, x^2y^2zw, x^3z^3t, x^3yzt^2, x^3y^3w, x^4z^4, x^4yz^2t, x^4y^2t^2, x^5yz^3, x^5y^2zt, x^6t^2w, x^6y^2z^2, x^6y^3t, x^7ztw, x^8z^2w, x^8ytw, x^9yzw, x^{10}zt^2, x^{10}y^2w, x^{11}z^2t, x^{11}yt^2, x^{12}z^3, x^{12}yzt, x^{13}yz^2, x^{13}y^2t, x^{14}y^2z, x^{15}tw, x^{15}y^3, x^{16}zw, x^{17}yw, x^{18}t^2, x^{19}zt, x^{20}z^2, x^{20}yt, x^{21}yz, x^{22}y^2, x^{24}w, x^{27}t, x^{28}z, x^{29}y, x^{36}$

Using this, we see that the automorphism group of a general hypersurface in each of the families №39, №49, №59, №66, №84 is trivial. In particular, it does not contain any involutions. Therefore, $d(X) = 3$.

5. PROOF OF MAIN THEOREM

Let us use assumptions and notations of Section 1. Suppose, in addition, that $I = 1$. If X_d is a general member of the families №1, №19, №28, №39, №49, №59, №66, №84, then it follows from the results proved earlier in Sections 2, 3, and 4 that $\text{deg}(X_d) = 3$. Thus, to prove Main Theorem, we may assume that X_d is not contained in the deformation families №1, №19, №28, №39, №49, №59, №66, №84. Let us show that $d(X_d) = 2$.

We know from [13, 14, 17] that X_d is irrational. Thus, we have $d(X_d) \geq 2$. Moreover, the natural projection

$$\mathbb{P}(1, a_2, a_3, a_4, a_5) \dashrightarrow \mathbb{P}(1, a_2, a_3, a_4)$$

induces a dominant map $X_d \dashrightarrow \mathbb{P}(1, a_2, a_3, a_4)$ that is generically two-to-one if $d < 3a_5$. Thus, if $d < 3a_5$, then $d(X_d) = 2$.

Hence, to complete the proof of Main Theorem, we may further assume that $d \geq 3a_5$. Then the 3-fold X_d is contained in one of the deformation families №9, №17, №27, and the septuple $(a_1, a_2, a_3, a_4, a_5, d, I)$ is described in the following table below.

№	a_1	a_2	a_3	a_4	a_5	d	I
9	1	1	2	3	3	9	1
17	1	1	3	4	4	12	1
27	1	2	3	5	5	15	1

In each of these three cases, we can change quasi-homogeneous coordinates on $\mathbb{P}(1, a_2, a_3, a_4, a_5)$ such that the hypersurface X_d is given by

$$wt(w - t) + w^2 f_{a_5}(x, y, z) + t^2 g_{a_5}(x, y, z) + w t h_{a_5}(x, y, z) + w f_{2a_5}(x, y, z) + t g_{2a_5}(x, y, z) + h_d(x, y, z) = 0,$$

where f_ℓ, g_ℓ, h_ℓ are quasihomogeneous polynomials of degrees ℓ in x, y, z . Note that X_d has three singular points of type $\frac{1}{a_5}(1, a_2, a_3)$ at $[0 : 0 : 0 : 0 : 1]$, $[0 : 0 : 0 : 1 : 0]$, and $[0 : 0 : 0 : 1 : 1]$. The map $X_d \dashrightarrow \mathbb{P}(1, a_2, a_3, a_4)$ given by $(x : y : z : t : w) \mapsto (x : y : z : t)$ is a dominant rational map that is undefined in $[0 : 0 : 0 : 0 : 1]$. Moreover, this map is generically two-to-one, which implies that $d(X_d) = 2$.

Remark 5.1. Let $\eta: \mathbb{P}(1, a_2, a_3, a_4, a_5) \dashrightarrow \mathbb{P}(1, a_2, a_3)$ be the natural projection. Then, arguing as in [8], we can resolve the indeterminacy of the rational map η via the following commutative diagram

$$\begin{array}{ccccc} U_1 & \xleftarrow{\pi_2} & U_2 & \xleftarrow{\pi_3} & U \\ \pi_1 \downarrow & & & & \downarrow \zeta \\ X_d & \dashrightarrow & \dashrightarrow & \dashrightarrow & \mathbb{P}(1, a_2, a_3) \end{array}$$

where π_1 is the weighted blow up of the point $[0 : 0 : 0 : 0 : 1]$ with weights $(1, a_2, a_3)$, π_2 is the weighted blow up with weights $(1, a_2, a_3)$ of the point in U_1 corresponding to $[0 : 0 : 0 : 1 : 0]$, and π_3 is the weighted blow up with weights $(1, a_2, a_3)$ of the point in U_2 corresponding to $[0 : 0 : 0 : 1 : 1]$. Then ζ is an elliptic fibration, since the fibre of η over

a general point on $\mathbb{P}(1, a_2, a_3)$ is isomorphic to a smooth cubic curve on \mathbb{P}^2 . Furthermore, the exceptional surface of the weighted blow up π_i , which is isomorphic to $\mathbb{P}(1, a_2, a_3)$, yields a section of the elliptic fibration. This again implies that $d(X_d) = 2$.

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