FIBERS OVER INFINITY OF LANDAU–GINZBURG MODELS

IVAN CHELTSOV AND VICTOR PRZYJALKOWSKI

Abstract. We conjecture that the number of components of the fiber over infinity of Landau–Ginzburg model for a smooth Fano variety \(X\) equals the dimension of the anti-canonical system of \(X\). We verify this conjecture for log Calabi–Yau compactifications of toric Landau–Ginzburg models for smooth Fano threefolds, complete intersections, and some toric varieties.

1. Introduction

Let \(X\) be a smooth Fano variety of dimension \(n\). Then its \textit{Landau–Ginzburg model} is a certain pair \((Y, w)\) that consists of a smooth (quasi-projective) variety \(Y\) of dimension \(n\) and a regular function

\[ w: Y \to \mathbb{A}^1, \]

which is called a superpotential. Its fibers are compact and \(K_Y \sim 0\), so that general fiber of \(w\) is a smooth Calabi–Yau variety of dimension \(n-1\). Homological Mirror Symmetry conjecture predicts that the derived category of singularities of the singular fibers of \(w\) is equivalent to the Fukaya category of the variety \(X\), while the Fukaya–Seidel category of the pair \((Y, w)\) is equivalent to the bounded derived category of coherent sheaves on \(X\).

In short: the geometry of \(X\) should be determined by singular fibers of \(w\).

Often, Landau–Ginzburg models of smooth Fano varieties can be constructed via their toric degenerations (see [P18a]). In this case, the variety \(Y\) contains a torus \((\mathbb{C}^*)^n\), one has \(K_Y \sim 0\), and there exists a commutative diagram

\[
\begin{array}{ccc}
(C^*)^n & \rightarrow & Y \\
\downarrow \scriptstyle p & & \downarrow \scriptstyle w \\
C & \rightarrow & \mathbb{C}
\end{array}
\]

for some Laurent polynomial \(p \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\) which is defined by an appropriate toric degeneration of the variety \(X\). Then \(p\) is said to be a \textit{toric Landau–Ginzburg model} of the Fano variety \(X\), and \((Y, w)\) is said to be its \textit{Calabi–Yau compactification}.

If \((Y, w)\) is a Calabi–Yau compactification of a toric Landau–Ginzburg model, then the number of reducible fibers of the morphism \(w: Y \to \mathbb{C}\) does not depend on the choice of the Calabi–Yau compactification. Likewise, the number of reducible components of each singular fiber of \(w\) does not depend on the compactification either. Therefore, it is natural
to expect that these numbers contain some information about the smooth Fano variety $X$. For instance, we have the following.

**Conjecture 1.2** (see [P13, PS15]). Let $X$ be a smooth Fano variety of dimension $n$, and let $(Y, w)$ be a Calabi–Yau compactification of its toric Landau–Ginzburg model. Then

$$h^{1,n-1}(X) = \sum_{P \in \mathbb{C}^1} (\rho_P - 1),$$

where $\rho_P$ is is the number of irreducible components of the fiber $w^{-1}(P)$.

In some cases this conjecture can be derived from Homological Mirror Symmetry conjecture, cf. [KKP17] and [Ha17]. Recently, Conjecture 1.2 has been verified for Calabi–Yau compactifications of toric Landau–Ginzburg models of smooth Fano complete intersections and smooth Fano threefolds (see [P13, PS15, CP18]).

In all considered cases, the commutative diagram (1.1) can be extended to a commutative diagram

$$
\begin{array}{c}
(C^*)^n \leftarrow Y \leftarrow Z \\
\downarrow \rho \downarrow w \downarrow f \\
\mathbb{C} \leftarrow \mathbb{C} \leftarrow \mathbb{P}^1
\end{array}
$$

such that $Z$ is a smooth proper variety that satisfies certain natural geometric conditions, e.g. the fiber $f^{-1}(\infty)$ is reduced, it has at most normal crossing singularities, and

$$f^{-1}(\infty) \sim -K_Z.$$

Then $(Z, f)$ is called the log Calabi–Yau compactification of the toric Landau–Ginzburg model $p$ (see [P18a, Definition 3.6]). Observe that the number of irreducible components of the fiber $f^{-1}(\infty)$ does not depend on the choice of the log Calabi–Yau compactification, so that one can expect that this number keeps some information about the Fano variety $X$. The following two examples confirm this.

**Example 1.4.** Let $X$ be a smooth del Pezzo surface, and let $(Z, f)$ be a log Calabi–Yau compactification of its toric Landau–Ginzburg model constructed in [AKO06]. Then the fiber $f^{-1}(\infty)$ consists of

$$\chi(O(-K_X)) - 1 = h^0(O_X(-K_X)) - 1 = K_X^2$$

irreducible rational curves.

**Example 1.5.** Let $X$ be a smooth Fano threefold such that the divisor $-K_X$ is very ample, and let $(Z, f)$ be a log Calabi–Yau compactification of its toric Landau–Ginzburg model constructed in [ACGK12, P17, CCGK16]. Then $f^{-1}(\infty)$ consists of

$$\chi(O(-K_X)) - 1 = h^0(O_X(-K_X)) - 1 = \frac{(-K_X)^3}{2} + 2$$

irreducible rational surfaces by [P17, Corollary 35], see also [Ha16, Theorem 2.3.14].

We pose the following conjecture.
**Conjecture 1.6.** Let $X$ be a smooth Fano variety, and let $(Z, f)$ be a log Calabi–Yau compactification of its toric Landau–Ginzburg model. Then $f^{-1}(\infty)$ consists of

$$\chi(O(-K_X)) - 1 = h^0(O_X(-K_X)) - 1$$

irreducible components.

The main result of the paper is the following.

**Theorem 1.7.** Conjecture 1.6 holds for

- §2: rigid maximally-mutable toric Landau–Ginzburg models for smooth Fano threefolds;
- §3: Givental’s toric Landau–Ginzburg models for Fano complete intersections;
- §4: Givental’s toric Landau–Ginzburg models for toric varieties whose dual toric varieties admit crepant resolutions.

**Remark 1.8.** Observe that Conjecture 1.6 together with the conjectural existence of toric Landau–Ginzburg models of smooth Fano threefolds [P18a, Conjecture 3.9] imply that

$$h^0(O_X(-K_X)) \geq 2,$$

which is only known for $\dim(X) \leq 5$ (see [HV11, Theorem 1.7], [HS19, Theorem 1.1.1]). Note also that Kawamata’s [K00, Conjecture 2.1] implies that $h^0(O_X(-K_X)) \geq 1$.

Homological Mirror Symmetry implies that the monodromy around $f^{-1}(\infty)$ is maximally unipotent (see [KKP17, §2.2]). Thus, if the fiber $f^{-1}(\infty)$ in (1.3) is a divisor with simple normal crossing singularities, then its dual intersection complex is expected to be homeomorphic to a sphere of dimension $n - 1$ (see [KoXu16, Question 7]). This follows from [KoXu16, Proposition 8] for $n \leq 5$. However, the following example shows that we cannot always expect $f^{-1}(\infty)$ to be a divisor with simple normal crossing singularities.

**Example 1.9.** Let $X$ be a smooth intersection of two general sextics in $\mathbb{P}(1, 1, 2, 2, 3, 3)$. Then $X$ is a smooth Fano fourfold and $h^0(O_X(-K_X)) - 1 = 2$. A toric Landau–Ginzburg model for $X$ is the following Laurent polynomial:

$$p = \frac{(x + y + 1)^6(z + t + 1)^6}{x^3yz^3t}.$$ 

The change of variables $x = \frac{a^2c}{b^2d}, y = \frac{ac}{b^2d} - \frac{a^2c}{b^2d} - 1, z = c, t = d - c - 1$ gives us a birational map $(\mathbb{C}^*)^4 \to \mathbb{C}^4$ that maps the pencil $p = \lambda$ to the pencil of quintics in $\mathbb{C}^4$ given by

$$d^4 = \lambda(abc - a^3c - b^3d)(d - c - 1),$$

where $\lambda$ is a parameter in $\mathbb{C} \cup \{\infty\}$. Now arguing as in [CP18], one can construct a log Calabi–Yau compactification $(Z, f)$ of the toric Landau–Ginsburg model $p$. Then $f^{-1}(\infty)$ consists of two irreducible divisors, and the monodromy around this fiber is maximally unipotent. All other log Calabi–Yau compactifications differ from $(Z, f)$ by flops, so that their fibers over $\infty$ also consist of two irreducible divisors. If one of them is a divisor with simple normal crossing singularities, then its dual intersection complex must be homeomorphic to a three-dimensional sphere by [KoXu16, Proposition 8], which is impossible by the dimension reasons.
Nevertheless, all toric Landau–Ginsburg models we consider in this paper admit log Calabi–Yau compactifications such their fibers over $\infty$ are divisors with simple normal crossing singularities. For toric Landau–Ginsburg models of smooth Fano threefolds, this follows from the construction of the log Calabi–Yau compactifications given in [P17] except for the families №2.1 and №10.1. For each of these two families, the fiber over $\infty$ does not have simple normal crossing singularities, but one can flop the log Calabi–Yau compactification in several curves contained in this fiber such that the resulting divisor has simple normal crossing singularities.

Let us describe the structure of this paper. In Section 2, we verify Conjecture 1.6 for smooth Fano threefolds. In Section 3, we verify Conjecture 1.6 for smooth Fano complete intersections. In Section 4, we verify Conjecture 1.6 for some smooth toric Fano varieties.

2. Fano threefolds

In this section we prove Conjecture 1.6 for a natural class of toric Landau–Ginsburg models of smooth Fano threefolds — rigid maximally-mutable ones (see [KT]) that correspond one-to-one to smooth Fano threefolds. Let $X$ be a smooth Fano threefold. Then the log Calabi–Yau compactification of its toric Landau–Ginsburg model is given by (1.3), where $p$ is one of the Laurent polynomials explicitly described in [ACGK12, P17, CCGK16]. Let us denote by $[f^{-1}(\infty)]$ the number of irreducible components of the fiber $f^{-1}(\infty)$. We have to show that $[f^{-1}(\infty)] = \frac{(-K_X)^3}{2} + 2$.

The polynomial $p$ is not uniquely determined by $X$. But the number $[f^{-1}(\infty)]$ does not depend on the choice of $p$ provided $p$ is rigid maximally-mutable, so that we may choose $p$ from [CCGK16] among any mirror partners for $X$. Moreover, by Example 1.5, we may assume that $-K_X$ is not very ample, so that $X$ is a smooth Fano threefold №1.1, №1.11, №2.1, №2.2, №2.3, №9.1, or №10.1. Here we use enumeration of families of Fano threefolds from [IP99].

Proposition 2.1. Suppose that $X$ is a Fano threefold №1.1, №2.1, №2.2, №2.3, or №9.1. Then $[f^{-1}(\infty)] = \frac{(-K_X)^3}{2} + 2$.

Proof. It follows from [CP18] that we can choose $p$ such that there is a pencil $S$ of quartic surfaces on $\mathbb{P}^3$ given by $f_4(x, y, z, t) + \lambda g_4(x, y, z, t) = 0$ that expands (1.3) to the following commutative diagram:

\[
\begin{array}{cccccccc}
(C^*)^3 & \rightarrow & Y^\nu & \rightarrow & Z & \leftarrow & X & \rightarrow & V & \rightarrow \mathbb{P}^3 \\
\downarrow p & & \downarrow \phi & & \downarrow f & & \downarrow \chi & & \downarrow \pi & \\
\mathbb{C} & \rightarrow & \mathbb{C}^\nu & \rightarrow & \mathbb{P}^1 & \rightarrow & \mathbb{P}^1 & \rightarrow & \mathbb{P}^1 & \rightarrow \mathbb{P}^3
\end{array}
\]

where $\phi$ is a rational map given by $S$, $V$ is a smooth threefold, $\pi$ is a birational morphism described in [CP18], and $\chi$ is a composition of flops. Here $f_4$ and $g_4$ are homogeneous
quartic polynomials and \( \lambda \in \mathbb{C} \cup \{\infty\} \), where \( \lambda = \infty \) corresponds to the fiber \( f^{-1}(\infty) \). Moreover, it follows from [CP18] that \( \pi \) factors through a birational morphism \( \alpha: U \to \mathbb{P}^3 \) that is uniquely determined by the following three properties:

1. the map \( \alpha^{-1} \) is regular outside of finitely many points in \( X \);
2. the proper transform of the pencil \( \mathcal{S} \) via \( \alpha \), which we denote by \( \mathcal{S} \), is contained in the anticanonical linear system \( |-K_U| \);
3. for every point \( P \in U \), there is a surface in \( \mathcal{S} \) that is smooth at \( P \).

We denote by \( \Sigma \) the (finite) subset in \( X \) consisting of all indeterminacy points of \( \alpha^{-1} \).

Let \( S \) be the quartic surface given by \( g_4(x, y, z, t) = 0 \), let \( \mathcal{S} \) be its proper transform on the threefold \( U \), and let

\[
\mathcal{D} = \mathcal{S} + \sum_{i=1}^{k} a_i E_i,
\]

where \( E_1, \ldots, E_k \) are \( \alpha \)-exceptional surfaces, and \( a_1, \ldots, a_k \) are non-negative integers such that \( \mathcal{D} \sim -K_U \). Then \( \mathcal{D} \in \mathcal{S} \). Moreover, for any \( \mathcal{D}' \in \mathcal{S} \) such that \( \mathcal{D}' \neq \mathcal{D} \), we have

\[
\mathcal{D}_0 \cdot \mathcal{D}_\infty = \sum_{i=1}^{s} m_i \hat{C}_i,
\]

where \( \hat{C}_1, \ldots, \hat{C}_n \) are base curves of the pencil \( \mathcal{S} \), and \( m_1, \ldots, m_s \) are positive numbers. Without loss of generality, we may assume that the base curves of the pencil \( \mathcal{S} \) are the curves \( \alpha(\hat{C}_1), \ldots, \alpha(\hat{C}_r) \) for some \( r \leq n \). Then we let \( C_i = \alpha(\hat{C}_i) \) for every \( i \leq r \).

For every \( i \in \{1, \ldots, n\} \), let \( M_i = \text{mult}_{\hat{C}_i}(\mathcal{D}) \) and

\[
\delta_i = \begin{cases} 
0 & \text{if } M_i = 1, \\
m_i - 1 & \text{if } M_i \geq 2. 
\end{cases}
\]

Then it follows from [CP18 (1.10.8)] that

\[
[f^{-1}(\infty)] = [S] + \sum_{i=1}^{r} \delta_i + \sum_{P \in \Sigma} D_P,
\]

where \( [S] \) is the number of irreducible components of the surface \( S \), and \( D_P \) is the defect of the point \( P \in \Sigma \) that is defined as

\[
D_P = A_P + \sum_{i=r+1}^{s} \delta_i,
\]

where \( A_P \) is the total number of indices \( i \in \{1, \ldots, k\} \) such that \( a_i > 0 \) and \( \alpha(E_i) = P \). By [CP18 Lemma 1.12.1], we have \( D_P = 0 \) if the rank of the quadratic form of the (local) defining equation of the surface \( S \) at the point \( P \) is at least 2.

To proceed, we need the following notation: for any subsets \( I, J, \) and \( K \) in \( \{x, y, z, t\} \), we write \( H_I \) for the plane defined by setting the sum of coordinates in \( I \) equal to zero, we write \( L_{I,J} = H_I \cap H_J \), and we write \( P_{I,J,K} = H_I \cap H_J \cap H_K \).
Suppose that $X$ is a Fano threefold $\mathbb{N}1.1$. Then $f_4 = x^4$ and $g_4 = yz(xt - xy - xz - t^2)$, so that the set $\Sigma$ consists of the points $P_{(x),(y),(z)}, P_{(x),(y),(t)}, P_{(x),(z),(t)}, P_{(x),(z),(y,z)}, P_{(y),(t),(y,z)}, P_{(y),(z),(t)}, P_{(z),(y),(z)}$, $r = 3$, $C_1 = L_{(x),(y)}, C_2 = L_{(x),(z)}$ and $C_3 = L_{(x),(t)}$. Moreover, we have $M_1 = 1, M_2 = 1$ and $M_3 = 1$, so that

$$[f^{-1}(\infty)] = 3 + D_{P_{(x),(y),(z)}} + D_{P_{(x),(y),(t)}} + D_{P_{(x),(z),(t)}} + D_{P_{(x),(z),(y,z)}} = 3 = \frac{(-K_X)^3}{2} + 2$$

by (2.3) and $\text{[CP18]}$ Lemma 1.12.1.

Suppose that $X$ is a Fano threefold $\mathbb{N}1.11$. Then $f_4 = x^4$ and $g_4 = yz(xt - xy - t^2)$, so that $\Sigma$ consists of the points $P_{(x),(y),(z)}, P_{(x),(y),(t)}, P_{(x),(z),(t)}, r = 3$, $C_1 = L_{(x),(z)}, C_2 = L_{(x),(t)}$ and $C_3$ is the rational quartic curve given by $y = x^4 + txz^2 - t^2z^2 = 0$. Then $M_1 = 1, M_2 = 1, M_3 = 1, m_1 = 4, m_2 = 8$ and $m_3 = 1$, so that

$$[f^{-1}(\infty)] = 3 + D_{P_{(x),(y),(z)}} + D_{P_{(x),(y),(t)}} + D_{P_{(x),(z),(t)}} + D_{P_{(x),(z),(y,z)}} = 3 + D_{P_{(x),(y),(t)}}$$

by (2.3) and $\text{[CP18]}$ Lemma 1.12.1. To compute $D_{P_{(x),(y),(t)}}$, observe that (locally) $\alpha$ is a blow up of the point $P_{(x),(y),(t)}$. Thus, we may assume that $E_1$ is mapped to $P_{(x),(y),(t)}$. Then $a_1 = 1$, so that $AP_{(x),(y),(t)} = 1$. Moreover, the pencil $\tilde{S}$ has unique base curve in $E_1$, which is a conic in $E_1 \cong \mathbb{P}^2$. We may assume that this curve is $\tilde{C}_4$. Then $M_4 = 2$, which gives $D_{P_{(x),(y),(t)}} = m_4$. On the other hand, we have

$$10 = 8 + \text{mult}_{P_{(x),(y),(t)}}(C) = \text{mult}_{P_{(x),(y),(t)}}(4C_1 + 8C_2 + 3C_3) = 4 + 2m_4,$$

which gives $D_{P_{(x),(y),(t)}} = 3$, so that $[f^{-1}(\infty)] = 6 = \frac{(-K_X)^3}{2} + 2$.

Suppose that $X$ is a smooth Fano threefold $\mathbb{N}2.2$. Then $f_4 = xz^2 - (zt - xy - yz - t^2)z^2$ and $g_4 = xy(zt - xy - yz - t^2)$, so that the set $\Sigma$ consists of the points $P_{(x),(y),(z)}, P_{(x),(y),(t)}, P_{(x),(z),(t)}, r = 5$, $C_1 = L_{(x),(z)}, C_2 = L_{(y),(z)}$, and $C_3, C_4, C_5$ are the conics given by $x = zt - yz - t^2 = 0, y = xz - zt + t^2 = 0$ and $z = xy - t^2$ respectively. Then $M_1 = 1, M_2 = 1, M_3 = 2, M_4 = 1, M_5 = 1, m_1 = 2, m_2 = 2, m_3 = 2, m_4 = 1$ and $m_5 = 3$, so that

$$[f^{-1}(\infty)] = [S] + 1 + D_{P_{(x),(y),(z)}} + D_{P_{(x),(y),(t)}} + D_{P_{(x),(z),(t)}} + D_{P_{(x),(z),(y,z)}} = 4 + D_{P_{(y),(z),(t)}}$$

by (2.3) and $\text{[CP18]}$ Lemma 1.12.1. To compute $D_{P_{(y),(z),(t)}}$, observe that $AP_{(y),(z),(t)} = 0$, because $S$ has a double point at $P_{(y),(z),(t)}$. Moreover, locally near the point $P_{(y),(z),(t)}$, the pencil $S$ is given by

$$\lambda y^2 + z^3 + z^3t - yz^2 - \lambda yzt + \lambda y^2z + \lambda yt^2 - yz^3 - z^2t^2 = 0,$$

where $P_{(y),(z),(t)} = (0,0,0)$. Let $\alpha_1 : U_1 \rightarrow \mathbb{P}^3$ be the blow up of the point $P_{(y),(z),(t)}$, and let $S^1$ be the proper transform on $U_1$ of the surface $S$. A chart of the blow up $\alpha_1$ is given by the coordinate change $y_1 = \frac{y}{t}, z_1 = \frac{z}{t}, t_1 = t$. In this chart, the proper transform of the pencil $S$ is given by

$$\lambda y_1(t_1 + y_1) - \lambda_1 y_1 z_1 + (\lambda t_1 y_1^2 z_1 - t_1^2 z_1^2 - t_1 y_1 z_1^2 - t_1^2 z_1^3 + t_1^2 z_1^3 - t_1 y_1 z_1^3 = 0,$$

and $S^1$ is given by $y_1(t_1 + y_1 - t_1 z_1 + t_1 y_1 z_1) = 0$. Note that the $\alpha_1$-exceptional surface contains unique base curve of the proper transform of the pencil $S$. Without loss of
generality, we may assume that its proper transform on \( U \) is the curve \( \hat{C}_6 \). Then \( M_6 = 2 \).

Arguing as in the proof of [CP18, Lemma 1.12.1], we get

\[ 4 + m_6 = \operatorname{mult}_{\mu} \left( \frac{P(y), (z), (t)}{2} \right) \left( 3C_5 + 2C_1 + 2C_2 + 2C_3 + C_4 \right) = 6, \]

so that \( D_{\mu} = 1 \), which gives \([f^{-1}(\infty)] = 5 = \frac{(K_X)^3}{2} + 2\).

Suppose that \( X \) is a smooth Fano threefold \( N \). Then \( f = x^3y + y(x^3 + xt - t^2) \) and \( g = z(x^3 + xt - t^2) \). In this case, the set \( \Sigma \) consists of the points \( P(x), (y), (z), P(x), (t), \), \( r = 5 \), \( C_1 = L(x), (t) \), \( C_2 = L(y), (z) \), \( C_3 = L(x), (y), z \), the curve \( C_4 \) is given by \( z = x^3 + xy - yt^2 = 0 \), and \( C_5 \) is the conic \( y = x^3 + xt + t^2 = 0 \). Then \( M_1 = 1 \), \( M_2 = 2 \), \( M_3 = 1 \), \( M_4 = 1 \), \( M_5 = 1 \), \( m_1 = 6 \), \( m_2 = 2 \), \( m_3 = 3 \), \( m_4 = 1 \), and \( m_5 = 1 \), so that

\[ [f^{-1}(\infty)] = [S] + 1 + D_{\mu} + D_{\mu} + D_{\mu} = 4 + D_{\mu}, \]

by (2.3) and [CP18, Lemma 1.12.1]. Arguing as in the case \( N \), we get \( D_{\mu} = 2 \), so that \([f^{-1}(\infty)] = 6 = \frac{(K_X)^3}{2} + 2\).

Finally, we consider the case when \( X \) is a smooth Fano threefold \( N \). In this case, we have \( f = x^3y + y(x^3 + xt - t^2) \) and \( g = z(x^3 + xt - t^2) \). Then \( \Sigma \) consists of the points \( P(x), (y), (z), P(x), (t), \), \( r = 4 \), \( C_1 = L(x), (t) \), \( C_2 = L(y), (z) \), \( C_3 = L(x), (y), z \), the curve \( C_4 \) is given by \( y = x^3 + xt - t^2 = 0 \), and \( C_5 \) is given by \( z = x^3 + yt(x + t) = 0 \). Observe that \( M_1 = 1 \), \( M_2 = 2 \), \( M_3 = 2 \), \( M_4 = 1 \), \( m_1 = 6 \), \( m_2 = 3 \), \( m_3 = 2 \), \( m_4 = 1 \). Thus, using (2.3) and [CP18, Lemma 1.12.1], we get

\[ [f^{-1}(\infty)] = 6 + D_{\mu} + D_{\mu} = 6 + D_{\mu}. \]

Arguing as in the case \( N \), we get \( D_{\mu} = 2 \), so that \([f^{-1}(\infty)] = \frac{(K_X)^3}{2} + 2\).

**Proposition 2.4.** Suppose that \( X \) is a Fano threefold \( N \) or a Fano threefold \( N \). Then \([f^{-1}(\infty)] = \frac{(K_X)^3}{2} + 2\).

**Proof.** It follows from [CP18] that the following commutative diagram exists:

\[
\begin{array}{cccccc}
V & \xrightarrow{\pi} & \mathbb{P}^2 \times \mathbb{P}^1 & \xrightarrow{\gamma} & \mathbb{C}^3 & \rightarrow & \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* & \rightarrow & Y \rightarrow & Z \\
\downarrow{\phi} & & \downarrow{q} & & \downarrow{p} & & \downarrow{w} & & \downarrow{f} \\
\mathbb{P}^1 & \xrightarrow{\mu} & \mathbb{C}^1 & \rightarrow & \mathbb{C}^1 & \rightarrow & \mathbb{C}^1 & \rightarrow & \mathbb{P}^1 \\
\end{array}
\]

where \( q \) is a surjective morphism, \( \pi \) is a birational morphism, \( V \) is a smooth threefold, the map \( g \) is a surjective morphism such that \(-K_X \sim g^{-1}(\infty)\), and \( \phi \) is a rational map that is given by the pencil \( S \) given by

\[
f_{2,3}(x, y, a, b, c) + \lambda g_{2,3}(x, y, a, b, c) = 0,
\]

where \((x : y, a : b : c)\) is a point in \( \mathbb{P}^1 \times \mathbb{P}^2 \), both \( f_{2,3} \) and \( g_{2,3} \) are bi-homogeneous polynomials of bi-degree \((2, 3)\), and \( \lambda \in \mathbb{C} \cup \{\infty\} \). The diagram (2.5) is similar to (2.2), so that we will follow the proof of Proposition 2.1 and use its notation. The only difference is that \( \mathbb{P}^3 \) is now replaced by \( \mathbb{P}^1 \times \mathbb{P}^2 \), and \( S \) is the surface given by \( g_{2,3}(x, y, a, b, c) = 0 \).
Suppose that \( X \) is a Fano threefold \( \#2.1 \). Then \( f_{2,3} = x(x + y)c^3 - y^2(abc - b^2c - a^3) \) and \( g_{2,3} = y(x + y)(abc - b^2c - a^3) \), the set \( \Sigma \) consists of the point \( P_{\{y\},\{a\},\{c\}} \), and the base locus of the pencil \( \mathcal{S} \) consists of the curve \( C_1 \) given by \( x + y = abc - b^2c - a^3 = 0 \), the curve \( C_2 \) given by \( x = abc - b^2c - a^3 = 0 \), the curve \( C_3 \) given by \( y = c = 0 \), and the curve \( C_4 \) given by \( a = c = 0 \). Hence, we have

\[
[f^{-1}(\infty)] = 4 + D_{P_{\{y\},\{a\},\{c\}}} = 4 = \frac{(-K_X)^3}{2} + 2
\]

by (2.3) and [CP18, Lemma 1.12.1], since \( M_1 = 2 \), \( M_2 = 1 \), \( M_3 = 1 \), \( M_4 = 1 \) and \( m_1 = 2 \).

Suppose that \( X \) is a Fano threefold \( \#10.1 \). Then \( f_{2,3} = xyc^3 + (x^2 + y^2)(abc - b^2c - a^3) \) and \( g_{2,3} = xy(abc - b^2c - a^3) \), so that \( \Sigma = \emptyset \), and the base locus of the pencil \( \mathcal{S} \) consists of the curve \( C_1 \) given by \( x = abc - b^2c - a^3 = 0 \), the curve \( C_2 \) given by \( y = abc - b^2c - a^3 = 0 \), and the curve \( C_3 \) given by \( a = c = 0 \). Then \( [f^{-1}(\infty)] = 5 = \frac{(-K_X)^3}{2} + 2 \) by (2.3), because \( M_1 = 2 \), \( M_2 = 2 \), \( M_3 = 1 \), \( m_1 = 2 \) and \( m_2 = 2 \). \( \square \)

3. Fano complete intersections

Let \( X \) be a Fano complete intersection in \( \mathbb{P}^N \) of hypersurfaces of degrees \( d_1, \ldots, d_k \), let \( i_X \) be its Fano index, and let \( p \) be the Laurent polynomial

\[
\prod_{i=1}^{k} (x_{i,1} + \cdots + x_{i,d_i-1} + 1)^{d_i} \prod_{j=1}^{\nu} x_{i,j} \prod_{j=1}^{\nu} y_j + y_1 + \cdots + y_{\nu-1} \in \mathbb{C}[x_{i,j}, y_j \pm 1],
\]

which we consider as a regular function on the torus \( (\mathbb{C}^*)^\nu \), where \( \nu = \dim(X) \). Let \( \Delta \) be the Newton polytope of \( p \) in \( \mathcal{N} = \mathbb{Z}^N \), let \( T_X \) be the toric Fano variety whose fan polytope (convex hull of generators of rays of the fan of \( T_X \)) is \( \Delta \), and let

\[
\nabla = \left\{ x \mid \langle x, y \rangle \geq -1 \text{ for all } y \in \Delta \right\} \subset M_\mathbb{R} = \mathcal{N}^\vee \otimes \mathbb{R}
\]

be the dual to \( \Delta \) polytope. Then \( \nabla \) and \( \Delta \) are reflexive (see [P18b]). Let \( M \) be the matrix

\[
\begin{pmatrix}
i_X & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & \cdots & -1 \\
0 & i_X & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & i_X & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & \cdots & -1 \\
-i_X & -i_X & \cdots & -i_X & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & i_X & \cdots & 0 & \cdots & -1 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & i_X & \cdots & -1 & \cdots & -1 \\
0 & 0 & \cdots & 0 & \cdots & -i_X & -i_X & \cdots & -i_X & \cdots & -1 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & i_X - 1 & \cdots & -1 \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & -1 & \cdots & i_X - 1 \\
\end{pmatrix}.
\]
which is formed from \( k \) blocks of sizes \((d_i-1) \times d_i\) and one last block of size \((i_X-1) \times (i_X-1)\). Then it follows from [P18b] that the vertices of \( \nabla \) are the rows of the matrix \( M \). Note that there is a mistake in the size of the last block in [P18b].

It has been shown in [ILP13, P18b] that \( p \) is a toric Landau–Ginzburg model of the variety \( X \) that admits a log Calabi–Yau compactification \((Z, f)\). Moreover, the number of irreducible components of the fiber \( f^{-1}(\infty) \) is the number of integral points in the boundary \( \partial \nabla \) by [P18b, Theorem 1], which we denote by \( r_X \).

**Theorem 3.1** (cf. [P18b Problem 11]). One has \( r_X = h^0(\mathcal{O}_X(-K_X)) - 1 \).

**Proof.** Shift the polytope \( \nabla \) by the vector \((0,0,\ldots,0,1,1,\ldots,1) \) \( \underbrace{\ldots}_{i_X-1 \text{ places}} \).

Denote the shifted polytope by \( \nabla' \). Then the number of integral points in \( \partial \nabla \) is the same as the number of integral points in the boundary \( \partial \nabla' \).

For each \( j \in \{1,\ldots,i_X - 1\} \), let \( v_{0,j} \) be the \((d_1 + \ldots + d_k + j)\)-th row of the matrix \( M \). For each \( i \in \{1,\ldots,k\} \) and \( j \in \{1,\ldots,d_i\} \), let \( v_{i,j} \) be the \((d_1 + \ldots + d_{i-1} + j)\)-th row of \( M \). Then, for each \((i, j) \neq (i, d_i)\), the vertex \( v_{i,j} \in \nabla \) has a unique non-vanishing coordinate. Moreover, this coordinate equals \( i_X \). But the vertex \( v_{i,d_i} \) has \( d_i \) non-vanishing coordinates. They are consequent, and all are equal to \(-i_X\).

Let \( \mathbf{v} \) be an integral vector in \( \nabla' \). Then

\[
\mathbf{v} = \sum_{\alpha} \frac{a_{i,j}v_{i,j}}{\alpha},
\]

where \( \alpha = \sum a_{i,j} \), and \( a_{i,j} \) are some (non-uniquely determined) non-negative integers such that not all of them vanish. Observe also that

\[
\begin{align*}
v_{1,1} + \ldots + v_{1,d_1} &= 0, \\
v_{2,1} + \ldots + v_{2,d_2} &= 0, \\
\vdots \\
v_{k,1} + \ldots + v_{k,d_k} &= 0.
\end{align*}
\]

(3.2)

where \( \mathbf{0} \) is a zero-vector. Using this, we can we can rewrite the expression for \( \mathbf{v} \) as

\[
\mathbf{v} = \frac{a\mathbf{0} + \sum a_{i,j}v_{i,j}}{\alpha},
\]

where \( \alpha = a + \sum a_{i,j} \), and \( a \) and \( a_{i,j} \) are non-negative integers such that not all of them vanish, and for every \( i \in \{1,\ldots,k\} \), there is \( j(i) \) such that \( \alpha_{i,j(i)} = 0 \).

We may assume that \( a, \alpha, \) and \( a_{i,j} \) are altogether coprime. We claim that \( \alpha \) divides \( i_X \). Indeed, write \( i_X = p^\ell q \), where \( p \) is a prime, and \( q \) is an integer that is not divisible by \( p \). Then at least one of the numbers \( \alpha_{i,j} \) is coprime to \( p \). We may assume that it is \( \alpha_{i_0,j_0} \).

On the other hand, the coordinates of the vector \( \alpha \mathbf{v} \) are equal either to

\[(\alpha_{i,j} - \alpha_{i,d_i})i_X\]
for $i \in \{1, \ldots, k\}$, or to $\alpha_{0,i} i_X$. Since $\alpha_{i,j(i)} = 0$ for every $i \in \{1, \ldots, k\}$ by our assumption, we conclude that one of coordinates of the vector $\alpha \mathbf{v}$ is $\gamma i_X$ for some $\gamma \in \mathbb{Z}$ coprime to $p$. This coordinate corresponds to the $i_0$-th block and is equal either to

$$(\alpha_{i_0,j_0} - \alpha_{i_0,d_{i_0}}) i_X$$

if $\alpha_{i_0,d_{i_0}}$ is divisible by $p$, or to

$$(\alpha_{i_0,j_0} - \alpha_{i_0,d_{i_0}}) i_X = -\alpha_{i_0,d_{i_0}} i_X$$

if $\alpha_{i_0,d_{i_0}}$ is not divisible by $p$. This shows that $i_X$ is divisible by $p^r$, so that $\alpha$ divides $i_X$.

Thus, one can consider every integral point in $\nabla'$ as an expression

$$a \mathbf{0} + \sum a_{i,j} v_{i,j}$$

with $a + \sum a_{i,j} = i_X$, where $a$ and $a_{i,j}$ are non-negative integers. This expression is unique modulo the relations (3.2). Hence, we can identify the integral points in $\nabla'$ with monomials of degree $i_X$ in $N + 1$ variables modulo the monomials that are divisible by the monomials corresponding to the vectors $v_{i,1} + \ldots + v_{i,d_i}$ for $i \in \{1, \ldots, k\}$. Then $\nabla'$ has $h^0(\mathcal{O}_X(-K_X))$ integral points. On the other hand, since $\nabla$ is reflexive, its only interior integral point is the origin, so that $r_X = h^0(\mathcal{O}_X(-K_X)) - 1$ as required. \qed

4. Toric Fano varieties

Let $X$ be a smooth toric Fano variety of dimension $n$, let $\Delta$ be its fan polytope, and let $\nabla$ be the dual (integral) polytope, and let $X^\vee$ be the dual toric variety, i.e. the Fano variety whose fan polytope is $\nabla$. Note that $X^\vee$ can be singular. Suppose that $X^\vee$ admits a crepant (toric) resolution $\tilde{X}^\vee \to X^\vee$. Let $p$ be the Laurent polynomial given by the sum of monomials corresponding to vertices of $\Delta$. Then it follows from [P17] that $p$ defines a toric Landau–Ginzburg model of the Fano variety $X$ that admits a log Calabi–Yau compactification $(Z, f)$ such that the existing the following commutative diagram:

$$
\begin{array}{cccccc}
\tilde{X}^\vee & \xrightarrow{\phi} & (\mathbb{C}^*)^n & \xrightarrow{f} & Z \\
\downarrow & & \downarrow & & \\
\mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{C}^* & \xrightarrow{p} & \mathbb{P}^1
\end{array}
$$

where $\phi$ is a rational map given by an anticanonical pencil $\mathcal{S}$ on the (weak Fano) variety $\tilde{X}^\vee$. Note that the toric boundary divisor $\tilde{X}^\vee \setminus (\mathbb{C}^*)^n$ is contained in $\mathcal{S}$.

**Proposition 4.1.** The fiber $f^{-1}(\infty)$ consists of $h^0(\mathcal{O}_X(-K_X)) - 1$ irreducible components.

**Proof.** Since $X$ is smooth, every irreducible toric boundary divisor of $\tilde{X}^\vee$ is isomorphic to a projective space, and the restriction of base locus of the pencil $\mathcal{S}$ on this divisor is a hyperplane that does not contain torus invariant points. Thus, to obtain $Z$, we can blow up (consecutively) irreducible components of the base locus of the pencil $\mathcal{S}$, which implies that $f^{-1}(\infty)$ is the proper transform of the the toric divisor $\tilde{X}^\vee \setminus (\mathbb{C}^*)^n$. In particular, the number of irreducible components of the fiber $f^{-1}(\infty)$ equals the number of integral
points of $\nabla$ minus one. This number is exactly $h^0(\mathcal{O}_X(-K_X))-1$, which can be described as a linear system of Laurent polynomials supported by $\nabla$.

\begin{thebibliography}{99}


Ivan Cheltsov
School of Mathematics, The University of Edinburgh, Edinburgh, UK.
Laboratory of Algebraic Geometry, NRU HSE, 6 Usacheva street, Moscow, Russia, 119048.
I.Cheltsov@ed.ac.uk

Victor Przyjalkowski
Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina street, Moscow, Russia.
victorprz@mi-ras.ru, victorprz@gmail.com