Formal groups and elliptic functions

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References


Recall classical definitions

An **elliptic function** is a meromorphic function in $\mathbb{C}$ that is doubly periodic:

$$f(z + 2\omega) = f(z), \quad f(z + 2\omega') = f(z), \quad \text{Im} \frac{\omega'}{\omega} > 0.$$ 

**Properties**

For any nonconstant elliptic functions $f(z)$, $g(z)$,
— there exists a polynomial $P(x_1, x_2)$, such that $P(f(z), g(z)) \equiv 0$,
— any elliptic function $h(z)$ is a rational function of $f(z)$ and $f'(z)$.

The **Weierstrass function** $\wp(z; g_2, g_3)$ is the unique elliptic function with periods $2\omega, 2\omega'$ and poles only in lattice points such that

$$\lim_{z \to 0} \left( \wp(z) - \frac{1}{z^2} \right) = 0.$$ 

It defines the uniformization of an elliptic curve $\mathcal{V}$

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$
Relations between parameters and periods

The elliptic curve in standard Weierstrass form

\[ V = \{(x, y)\mid y^2 = 4x^3 - g_2x - g_3\} \].

The parameters \( \omega, \omega', \eta, \eta' \) are determined by the relations

\[
2\omega = \oint_a \frac{dx}{y}, \quad 2\omega' = \oint_b \frac{dx}{y}, \quad 2\eta = -\oint_a \frac{xdx}{y}, \quad 2\eta' = -\oint_b \frac{xdx}{y},
\]

where \( \frac{dx}{y} \) and \( \frac{xdx}{y} \) are a set of holomorphic differentials on \( V \), \( a \) and \( b \) are basis cycles on the curve such that \( \eta\omega' - \omega\eta' = \frac{\pi i}{2} \). Vice versa,

\[
g_2 = \sum_{(n,m)\neq(0,0)} \frac{60}{(2m\omega + 2n\omega')^4}, \quad g_3 = \sum_{(n,m)\neq(0,0)} \frac{140}{(2m\omega + 2n\omega')^6}.
\]
Sigma function

The Weierstrass $\zeta$-function is defined by

$$\zeta(z; g_2, g_3)' = -\wp(z; g_2, g_3), \ \lim_{z \to 0} (z\zeta(z)) = 1.$$ 

We have the relation

$$\eta = \zeta(\omega; g_2, g_3).$$

The Weierstrass $\sigma$-function is defined by

$$\left( \ln \sigma(z; g_2, g_3) \right)' = \zeta(z; g_2, g_3), \ \lim_{z \to 0} \left( \frac{\sigma(z)}{z} \right) = 1.$$ 

It is an entire odd quasiperiodic function homogeneous with respect to the grading $\deg z = 1, \deg g_k = -2k$. We have

$$\sigma(z; g_2, g_3) = z - \frac{g_2z^5}{2 \cdot 5!} - \frac{6g_3z^7}{7!} - \frac{g_2^2z^9}{4 \cdot 8!} - \frac{18g_2g_3z^{11}}{11!} + (z^{13})$$
Elliptic function of level $N$

For a lattice $L$ in $\mathbb{C}$ consider the elliptic function $g(x)$ with divisor $N \cdot 0 - N \cdot z$. We demand that $g(x) = x^N + \ldots$.

Set $f(x) = g(x)^{1/N}$ where $f(x) = x + \ldots$.

$$f(x) = \frac{\sigma(x)\sigma(z)}{\sigma(z-x)} \exp(\alpha x - \zeta(z) x).$$

Periodic properties

$$f(x + 2\omega_k) = f(x) \exp(2\eta_k z + 2\omega_k (\alpha - \zeta(z))).$$

For $z = \frac{2n}{N}\omega_1 + \frac{2m}{N}\omega_2$, $\alpha = -\frac{2n}{N}\eta_1 - \frac{2m}{N}\eta_2 + \zeta(z)$, we get elliptic functions of level $N$.

The function $f(x)$ is elliptic with respect to a sublattice $L'$ of $L$. 

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Formal groups and elliptic functions
Let the bundle $\mathbb{C}P(\xi) \to B$ with fiber $\mathbb{C}P(2)$ be the projectivization of a 3-dimensional complex vector bundle $\xi \to B$. A Hirzebruch genus $L_f : \Omega_U \to \mathbb{R}$ is called $\mathbb{C}P(2)$-multiplicative, if we have $L_f[\mathbb{C}P(\xi)] = L_f[\mathbb{C}P(2)]L_f[B]$.

**Theorem (V.M. Buchstaber, E.Yu. Bunkova 2014)**

Let $L_f$ be a $\mathbb{C}P(2)$-multiplicative genus. If $L_f[\mathbb{C}P(2)] \neq 0$, then $L_f$ is the two-parametric Todd genus, and

$$f(x) = \frac{e^{\alpha x} - e^{\beta x}}{\alpha e^{\alpha x} - \beta e^{\beta x}},$$

If $L_f[\mathbb{C}P(2)] = 0$, then $L_f$ is level 3 elliptic genus and

$$f(x) = -\frac{2 \phi(x) + \frac{a^2}{2}}{\phi'(x) - a\phi(x) + b - \frac{a^3}{4}}.$$

Here $g_2 = -\frac{1}{4}(8b - 3a^3)a$, $g_3 = \frac{1}{24}(8b^2 - 12a^3b + 3a^6)$. 

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*Formal groups and elliptic functions*
2-nd Hirzebruch functional equation

\[
\frac{1}{f(x_1 - x_2)f(x_1 - x_3)} + \frac{1}{f(x_2 - x_1)f(x_2 - x_3)} + \frac{1}{f(x_3 - x_1)f(x_3 - x_2)} = C.
\]

Solutions are formal series \( f(x) \) such that \( f(0) = 0, f'(0) = 1 \).

If \( C \neq 0 \), then

\[
f(x) = \frac{e^{\alpha x} - e^{\beta x}}{\alpha e^{\alpha x} - \beta e^{\beta x}}.
\]

If \( C = 0 \), then \( f(x) \) is elliptic function of level 3 and

\[
f(x) = -\frac{2\wp(x) + \frac{a^2}{2}}{\wp'(x) - a\wp(x) + b - \frac{a^3}{4}}.
\]

Here \( g_2 = -\frac{1}{4}(8b - 3a^3)a \), \( g_3 = \frac{1}{24}(8b^2 - 12a^3b + 3a^6) \).
n-th Hirzebruch functional equation

\[
\sum_{j=1}^{n+1} \prod_{i \neq j} \frac{1}{f(x_i - x_j)} = C
\]

Solutions are formal series \( f(x) \) such that \( f(0) = 0, f'(0) = 1 \).

The elliptic genus of level \( N \) is strictly multiplicative in fibre bundles with a manifold \( M_d \) with \( c_1(M_d) \equiv 0 \mod N \) as fibre and a compact connected Lie group \( G \) of automorphisms of \( M_d \) as structure group.

(F. Hirzebruch comparing a paper by P.S. Landweber with a paper by S. Ochanine)

\( \Rightarrow \) Elliptic function of level \( n + 1 \) is a (two-parametric) solution to \( n \)-th Hirzebruch equation (with \( C = 0 \)).
Let the bundle $\mathbb{C}P(\xi) \to B$ with fiber $\mathbb{C}P(2)$ be the projectivization of a 3-dimensional complex vector bundle $\xi \to B$. A Hirzebruch genus $L_f : \Omega_U \to \mathbb{R}$ is called $\mathbb{C}P(2)$-multiplicative, if we have $L_f[\mathbb{C}P(\xi)] = L_f[\mathbb{C}P(2)]L_f[B]$.

**Theorem (V.M. Buchstaber, E.Yu. Bunkova 2014)**

Let $L_f$ be a $\mathbb{C}P(2)$-multiplicative genus.

If $L_f[\mathbb{C}P(2)] \neq 0$, then $L_f$ is the two-parametric Todd genus, and

$$f(x) = \frac{e^{\alpha x} - e^{\beta x}}{\alpha e^{\alpha x} - \beta e^{\beta x}},$$

If $L_f[\mathbb{C}P(2)] = 0$, then $L_f$ is level 3 elliptic genus and

$$f(x) = -\frac{2\wp(x) + \frac{a^2}{2}}{\wp'(x) - a\wp(x) + b - \frac{a^3}{4}}.$$

Here $g_2 = -\frac{1}{4}(8b - 3a^3)a$, $g_3 = \frac{1}{24}(8b^2 - 12a^3b + 3a^6)$. 
Theorem

Let $\Delta$ and $J$ be the discriminant and $J$-invariant of an elliptic curve with generators $(2\omega_1, 2\omega_2)$. Then the discriminant and $J$-invariant of elliptic curves with generators $(2\hat{\omega}_1 = 2\omega_1, 2\hat{\omega}_2 = \frac{2}{3}\omega_2)$ and $(2\hat{\omega}_1 = \frac{2}{3}\omega_1, 2\hat{\omega}_2 = 2\omega_2)$ can be expressed as follows:

Let us introduce formal parameters $p, s$ such that

\[
\Delta = -\frac{1}{27}p^3s, \quad J = -\frac{1}{64}\frac{(p - s)(p - 9s)^3}{p^3s},
\]

then

\[
\hat{\Delta} = -27s^3p, \quad \hat{J} = -\frac{1}{64}\frac{(s - p)(s - 9p)^3}{s^3p}.
\]
1-st Hirzebruch functional equation

\[ \frac{1}{f(x_1 - x_2)} + \frac{1}{f(x_2 - x_1)} = C. \]

Elliptic function of level 2 is Jacobi’s elliptic sine \( sn(x) \).

\[ sn'(x)^2 = 1 - 2\delta sn(x)^2 + \varepsilon sn(x)^4. \]

\( sn(x) \) is the exponential of \textit{universal} formal group of the form

\[ F(u, v) = \frac{u^2 - v^2}{uB(v) - vB(u)}. \]

Thus \( B(u)^2 = 1 - 2\delta u^2 + \varepsilon u^4. \)
Formal group

Let $R$ be a commutative ring with unity 1. A commutative one-dimensional formal group over $R$ is a formal series

$$F(u, v) = u + v + \sum_{i,j > 0} a_{i,j} u^i v^j, \quad a_{i,j} \in R,$$

with conditions

$$F(v, u) = F(u, v), \quad F(u, F(v, w)) = F(F(u, v), w).$$

It’s exponential $f(x) \in R \otimes \mathbb{Q}[[x]]$, $f(0) = 0$, $f'(0) = 1$, is determined by the addition law

$$f(x + y) = F(f(x), f(y)).$$
From the addition law

\[ F(f(x), f(y)) = f(x + y) \]

we get the relations

\[ \left. \frac{\partial}{\partial v} F(u, v) \right|_{v=0} = w(u), \]

\[ \left. \frac{\partial^2}{\partial v^2} F(u, v) \right|_{v=0} = w(u)(w'(u) - w_1), \]

\[ \left. \frac{\partial^3}{\partial v^3} F(u, v) \right|_{v=0} = w(u)^2 w''(u) - 2w(u)w_2 + \\
+ w(u)(w'(u) - w_1)(w'(u) - 2w_1). \]
1-st Hirzebruch functional equation

\[ \frac{1}{f(x_1 - x_2)} + \frac{1}{f(x_2 - x_1)} = C. \]

Elliptic function of level 2 is Jacobi’s elliptic sine \( sn(x) \).

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\( sn(x) \) is the exponential of universal formal group of the form

\[ F(u, v) = \frac{u^2 - v^2}{uB(v) - vB(u)}. \]

Thus \( B(u)^2 = 1 - 2\delta u^2 + \varepsilon u^4 \).
The exponential of the universal formal group of the form

\[ F(u, v) = \frac{u^2 C(v) - v^2 C(u)}{uC(v)^2 - vC(u)^2} \]

is the elliptic function of level 3.
Krichever genus

$$\mathcal{F}_B(u, v) = \frac{u^2 A(v) - v^2 A(u)}{uB(v) - vB(u)}, \quad A(0) = B(0) = 1. \quad (1)$$

Theorem (V.M. Buchstaber, 1990)

The exponential of the universal formal group of the form (1) is

$$f(x) = \frac{\exp(\alpha x)}{\Phi(x)} = \frac{\sigma(x)\sigma(z)}{\sigma(z - x)} e^{\alpha x - \zeta(z)x}.$$
Two-parametric Todd genus, elliptic genera of level 2 and 3

\[
\frac{u^2 A(v) - v^2 A(u)}{uA(v) - vA(u)}, \quad \frac{u^2 - v^2}{uB(v) - vB(u)}, \quad \frac{u^2 C(v) - v^2 C(u)}{uC(v)^2 - vC(u)^2},
\]

\[
\mathbb{Z}[\mu_1, \mu_2], \quad \mathbb{Z}[\mu_2, \mu_4], \quad \mathbb{Z}[\mu_1, \mu_3, \mu_6]/\{\mu_3 = -\mu_6\},
\]

\[
\frac{e^{\alpha x} - e^{\beta x}}{\alpha e^{\alpha x} - \beta e^{\beta x}}, \quad sn(x), \quad \frac{\sigma(x)\sigma(z)}{\sigma(z-x)} \exp(\alpha x - \zeta(z)x),
\]

\[
\frac{-2(\wp(x) - \lambda)}{\wp'(x) - \mu_1 \wp(x) + \mu_1 \lambda}, \quad \frac{-2(\wp(x) - \frac{\delta}{3})}{\wp'(x)}, \quad \frac{-2(\wp(x) + \frac{a^2}{4})}{\wp'(x) - a\wp(x) + b - \frac{a^3}{4}}.
\]
Tate elliptic formal group

General Weierstrass model of elliptic curve in Tate coordinates:

\[ s = u^3 + \mu_1 us + \mu_2 u^2 s + \mu_3 s^2 + \mu_4 us^2 + \mu_6 s^3. \]

\[ \mathcal{F}_T(u, v) = ((u + v)(1 - \mu_3 k - \mu_6 k^2) - uv(\mu_1 + \mu_3 m + \mu_4 k + 2\mu_6 mk)) \times \]
\[ \frac{(1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3)}{(1 + \mu_2 n + \mu_4 n^2 + \mu_6 n^3)(1 - \mu_3 k - \mu_6 k^2)^2} \]

\[ m = \frac{s(u) - s(v)}{u - v}, \quad k = \frac{us(v) - vs(u)}{u - v}, \quad n = m + uv \frac{(1 + \mu_2 m + \mu_4 m^2 + \mu_6 m^3)}{(1 - \mu_3 k - \mu_6 k^2)}. \]

\[ \mathcal{F}_T(u, v) \in \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6][[u, v]]. \]

\[ f(x) = -2 \frac{\varphi(x) - \frac{1}{12}(\mu_1^2 + 4\mu_2)}{\varphi'(x) - \mu_1 \varphi(x) + \frac{1}{12} \mu_1 (\mu_1^2 + 4\mu_2) - \mu_3}. \]
Two-parametric Todd genus, elliptic genera of level 2 and 3

\[
\frac{u^2 A(v) - v^2 A(u)}{uA(v) - vA(u)}, \quad \frac{u^2 - v^2}{uB(v) - vB(u)}, \quad \frac{u^2 C(v) - v^2 C(u)}{uC(v)^2 - vC(u)^2},
\]

\[\mathbb{Z} [\mu_1, \mu_2], \quad \mathbb{Z} [\mu_2, \mu_4], \quad \mathbb{Z} [\mu_1, \mu_3, \mu_6]/\{\mu_3^2 = -\mu_6\},\]

\[
\frac{e^{\alpha x} - e^{\beta x}}{\alpha e^{\alpha x} - \beta e^{\beta x}}, \quad \text{sn}(x), \quad \frac{\sigma(x)\sigma(z)}{\sigma(z-x)} \exp (\alpha x - \zeta(z)x),
\]

\[
\frac{-2(\varphi(x) - \lambda)}{\varphi'(x) - \mu_1 \varphi(x) + \mu_1 \lambda}, \quad \frac{-2(\varphi(x) - \frac{\delta}{3})}{\varphi'(x)}, \quad \frac{-2(\varphi(x) + \frac{a^2}{4})}{\varphi'(x) - a \varphi(x) + b - \frac{a^3}{4}}.
\]
Rings of coefficients

\[ \mathcal{F}_B(u, v) = \frac{u^2 A(v) - v^2 A(u)}{u B(v) - v B(u)}, \quad \mathbb{Z}[a_1, b_2, a_3, b_3, \ldots]/\mathcal{J}_a \]

\[ \mathcal{F}_T(u, v) = \ldots, \quad \mathbb{Z}[\mu_1, \mu_2, \mu_3, \mu_4, \mu_6]. \]
Tate and Buchstaber formal groups

\[ \mathcal{F}_1(u, v) = \frac{u + v - \mu_1 uv}{1 + \mu_2 uv}, \quad \mathbb{Z}[\mu_1, \mu_2] \]

\[ \mathcal{F}_2(u, v) = \frac{u^2 - v^2}{u \sqrt{(1 - \mu_2 v^2)^2 - 4\mu_4 v^4} - v \sqrt{(1 - \mu_2 u^2)^2 - 4\mu_4 u^4}}, \quad \mathbb{Z}[\mu_2, \mu_4] \]

\[ \mathcal{F}_3(u, v) = \frac{u^2(1 - \mu_1 v - \mu_3 s(v)) - v^2(1 - \mu_1 u - \mu_3 s(u))}{u(1 - \mu_1 v - \mu_3 s(v))^2 - v(1 - \mu_1 u - \mu_3 s(u))^2}, \quad \mathbb{Z}[\mu_1, \mu_3, \mu_6] / \{3\mu_6 = -\mu_3^2\} \]

\[ \mathcal{F}_4(u, v) = \frac{u^2(1 + \mu_6 s(v)^2) - v^2(1 + \mu_6 s(u)^2)}{u(1 + \mu_2 v^2 + \mu_6 s(v)^2) - v(1 + \mu_2 u^2 + \mu_6 s(u)^2)}, \quad \mathbb{F}_2[\mu_2, \mu_4, \mu_6] \]
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