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Factorization semigroups and irreducible components of the Hurwitz space. II

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Abstract. We continue the investigation started in [1]. Let $\mathrm{HUR}^{\mathcal{S}_d}_{d,t}(\mathbb{P}^1)$ be the Hurwitz space of coverings of degree d of the projective line $\mathbb{P}^1$ with Galois group $\mathcal{S}_d$ and monodromy type t. The monodromy type is a set of local monodromy types, which are defined as conjugacy classes of permutations σ in the symmetric group $\mathcal{S}_d$ acting on the set $I_d=\{1,\dots,d\}$. We prove that if the type t contains sufficiently many local monodromies belonging to the conjugacy class C of an odd permutation σ which leaves $f_C\geqslant 2$ elements of I_d fixed, then the Hurwitz space $\mathrm{HUR}^{\mathcal{S}_d}_{d,t}(\mathbb{P}^1)$ is irreducible.

Keywords: semigroup, factorizations of an element of a group, irreducible components of the Hurwitz space.

Introduction

This paper is continuation of [1]. Before stating its results, we recall the main definitions and notation used in [1]. A quadruple (S,G,α,ρ), where S is a semigroup, G is a group and $\alpha\colon S\to G$, $\rho\colon G\to \mathrm{Aut}(S)$ are homomorphisms, is called a *semigroup S over a group G* if for all $s_1,s_2\in S$ we have

$$s_1\cdot s_2=\rho(\alpha(s_1))(s_2)\cdot s_1=s_2\cdot\lambda(\alpha(s_2))(s_1), \tag{1}$$

where $\lambda(g)=\rho(g^{-1})$. Let (S_1,G,α_1,ρ_1) and (S_2,G,α_2,ρ_2) be semigroups over G. A homomorphism of semigroups $\varphi\colon S_1\to S_2$ is said to be *defined over* G if $\alpha_1(s)=\alpha_2(\varphi(s))$ and $\rho_2(g)(\varphi(s))=\varphi(\rho_1(g)(s))$ for all $s\in S_1$ and $g\in G$.

A pair (G,O), where O is a subset of G invariant under inner automorphisms of G, is called an *equipped group*. With every equipped group (G,O) one can associate a semigroup $S_O=S(G,O)$ over G (called the *factorization semigroup of elements of G with factors in O*) generated by the elements of the alphabet $X=X_O=\{x_g\mid g\in O\}$ subject to the relations

$$x_{g_1}\cdot x_{g_2}=x_{g_2}\cdot x_{g_2^{-1}g_1g_2}=x_{g_1g_2g_1^{-1}}\cdot x_{g_1} \tag{2}$$

for all $x_{g_1},x_{g_2}\in X$, and if $g_2=\mathbf{1}$, then $x_{g_1}\cdot x_{\mathbf{1}}=x_{g_1}$. We define a map $\alpha\colon X\to G$ by putting $\alpha(x_g)=g$ for every $x_g\in X$. It induces a homomorphism $\alpha\colon S_O\to G$ called

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the *product homomorphism*. The action ρ (on the left) of G on S_O is induced by the following action on the alphabet X:

$$x_a \in X \mapsto \rho(g)(x_a) = x_{gag^{-1}} \in X$$

for $g \in G$. Note that $\alpha(\rho(g)(s)) = g\alpha(s)g^{-1}$ for all $s \in S_O$ and $g \in G$.

Let $O \setminus \{\mathbf{1}\} = C_1 \sqcup \dots \sqcup C_m$ be the decomposition of O into a disjoint union of conjugacy classes of elements of G. Every element $s = x_{g_1} \cdot \ldots \cdot x_{g_n} \in S_O$ determines an element $\tau(s) = n_1 C_1 + \dots + n_m C_m$ of the free Abelian semigroup generated by the symbols $C_1, \dots, C_m$ (the element $\tau(s)$ is called the *type* of s), where n_i is the number of those factors x_{g_j} in the factorization $s = x_{g_1} \cdot \ldots \cdot x_{g_n}$ which satisfy $g_j \in C_i$. The number $n = \sum_{i=1}^{m} n_i$ is called the *length* of s and is denoted by $\ln(s)$. A subsemigroup S of S_G is said to be *stable* if there is an element $s \in S$ (called a *stabilizing element* of S) such that $s_1 \cdot s = s_2 \cdot s$ for all $s_1, s_2 \in S$ satisfying $\alpha(s_1) = \alpha(s_2)$ and $\tau(s_1) = \tau(s_2)$.

For every element $s = x_{g_1} \cdot \ldots \cdot x_{g_n} \in S_O$, let $G_s = \langle g_1, \dots, g_n \rangle$ be the subgroup of G generated by the elements $g_1, \dots, g_n$. Given any (not necessarily proper) subgroups H and Γ of G, one can define subsemigroups $S_O^H = \{s \in S(G,O) \mid G_s = H\}$ and $S_{O,\Gamma} = \{s \in S(G,O) \mid \alpha(s) \in \Gamma\}$. If H and Γ are normal subgroups of G, then $S_{O,\Gamma}$ and S_O^H are semigroups over G. By definition, $S_{O,\Gamma}^H = S_{O,\Gamma} \cap S_O^H$.

Let $\mathcal{S}_d$ be the symmetric group acting on the set $I_d = \{1, \dots, d\}$ and let $T_d \subset \mathcal{S}_d$ be the subset of transpositions. We denote the semigroup $S_{\mathcal{S}_d}$ by Σ_d. By Theorem 2.3 in [1], the element

$$h = \left(\prod_{i=1}^{d-1} x_{(i,i+1)} \right)^3$$

is a stabilizing element of Σ_d. Here $(i, i+1) \in T_d$ is the transposition interchanging the elements i and $i+1$ of I_d.

The aim of this paper is to prove that a similar result holds for almost all odd elements of $\mathcal{S}_d$. More precisely, let $C = C_\sigma$ be the conjugacy class of a permutation $\sigma \in \mathcal{S}_d$, n_C the order of $\sigma \in C$, $k_C = |C|$ the number of elements of C, and f_C the number of elements of I_d that remain fixed under the action of $\sigma \in C$ on I_d.

It is known that if σ is an odd permutation, then elements of C generate the whole group $\mathcal{S}_d$ and, in particular, any transposition $(i,j) \in \mathcal{S}_d$ can be written as a product of permutations belonging to C. In the case when $f_C \geqslant 2$, we write m_C for the minimal number (counting multiplicities) of permutations in $C \cap \mathcal{S}_{d-2}$ needed to express $(1,2)$ as a product of elements of $C \cap \mathcal{S}_{d-2}$. We also fix any one of these expressions:

$$(1,2) = \sigma_1 \ldots \sigma_{m_C}, \qquad \sigma_i \in C \cap \mathcal{S}_{d-2}. \tag{3}$$

Theorem 1. *Let C be the conjugacy class of an odd permutation $\sigma \in \mathcal{S}_d$. If $f_C \geqslant 2$, then there is a constant*

$$N = N_C < 3^{d-3}(2d-1)(d-1)m_C + n_C k_C + 1$$

such that every element $s = \widetilde{s} \cdot \overline{s} \in \Sigma_d^{\mathcal{S}_d}$ with $\overline{s} \in S_C$ and $\ln(\overline{s}) \geqslant N$ is uniquely determined by $\tau(s)$ and $\alpha(s)$.

Corollary 1. *Let an equipped symmetric group* $(\mathcal{S}_d, O)$ *be such that the set* O *contains the conjugacy class* C *of an odd permutation* σ, $f_C \geqslant 2$. *Then* $S_O = S(\mathcal{S}_d, O)$ *is a stable semigroup.*

Note that the constant N_C whose existence is asserted in Theorem 1 is generally greater than 1. For example, it is shown in [2] that this is the case when C is the conjugacy class of $\sigma = (1,2)(3,4,5) \in \mathcal{S}_8$.

The proof of Theorem 1 is similar to that of Theorem 2.3 in [1]. It is based on the following theorem.

Theorem 2. *Let* C *be the conjugacy class of an odd permutation* $\sigma \in \mathcal{S}_d$, *and let* $\overline{s}_{(i_1,i_2)} \in S_C$ *be an element with the following properties*:

(i) $\alpha(\overline{s}_{(i_1,i_2)}) = (i_1, i_2)$,

(ii) *there are* $i_3, i_4 \in I_d \setminus \{i_1, i_2\}$ *such that* $\rho((i_3,i_4))(\overline{s}_{(i_1,i_2)}) = \overline{s}_{(i_1,i_2)}$.

Then there is an embedding over $\mathcal{S}_d$ *of the semigroup* $S^{\mathcal{S}_d}_{T_d}$ *in the semigroup* S_C.

Let $\mathrm{HUR}_{d,b}(\mathbb{P}^1)$ (resp. $\mathrm{HUR}^G_{d,b}(\mathbb{P}^1)$) be the Hurwitz space of ramified coverings of degree d of the projective line $\mathbb{P}^1$ (defined over $\mathbb{C}$) branched over b points (resp. with Galois group G). It was shown in [1] that the irreducible components of $\mathrm{HUR}_{d,b}(\mathbb{P}^1)$ are in one-to-one correspondence with the orbits of the action of $\mathcal{S}_d$ by simultaneous conjugation (that is, the action determined by the homomorphism ρ) on the set $\Sigma_{d,\mathbf{1},\mathbf{b}} = \{s \in \Sigma_{d,\mathbf{1}} \mid \ln(s) = b\}$, and if $G = \mathcal{S}_d$, then the irreducible components of $\mathrm{HUR}^{\mathcal{S}_d}_{d,b}(\mathbb{P}^1)$ are in one-to-one correspondence with the elements of $\Sigma^{\mathcal{S}_d}_{d,\mathbf{1}}$ of length b. If an irreducible component of $\mathrm{HUR}^{\mathcal{S}_d}_{d,b}(\mathbb{P}^1)$ corresponds to an element $s \in \Sigma^{\mathcal{S}_d}_{d,\mathbf{1}}$, then $\tau(s)$ is called the *monodromy factorization type* of coverings belonging to this irreducible component. We denote the union of all irreducible components corresponding to the elements $s \in \Sigma^{\mathcal{S}_d}_{d,\mathbf{1}}$ with $\tau(s) = t$ by $\mathrm{HUR}^{\mathcal{S}_d}_{d,t}(\mathbb{P}^1)$.

The following theorem is a corollary of Theorem 1.

Theorem 3. *The space* $\mathrm{HUR}^{\mathcal{S}_d}_{d,t}(\mathbb{P}^1)$ *is irreducible if the monodromy factorization type* t *contains more than* N_C *factors belonging to the conjugacy class* C *of an odd permutation* $\sigma \in \mathcal{S}_d$ *with* $f_C \geqslant 2$, *where* N_C *is the number defined in Theorem* 1.

We note that an analogue of Theorem 3 holds for the Hurwitz spaces of d-sheeted coverings of the disc $\Delta = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$ (resp. d-sheeted coverings of the affine line $\mathbb{C}^1$).

§ 1. Proof of Theorem 2

There is no loss of generality in assuming that $(i_1, i_2) = (1,2)$ and $(i_3, i_4) = (3,4)$.

For every transposition $(i,j) \in T_d$ we choose a permutation $\sigma_{i,j} \in \mathcal{S}_d$ such that $(i,j) = \sigma_{i,j}(1,2)\sigma^{-1}_{i,j}$ and put

$$c = \overline{s}^2_{(1,2)} \cdot \overline{s}^2_{(2,3)} \cdot \dots \cdot \overline{s}^2_{(d-1,d)},$$

where $\overline{s}_{(i,j)} = \rho(\sigma_{i,j})(\overline{s}_{(1,2)})$.

Clearly, $\alpha(\overline{s}_{(i,j)}) = (i,j)$ and $\alpha(c) = \mathbf{1}$. Since the transpositions $(1,2), \dots, (d-1,d)$ generate the whole group $\mathcal{S}_d$, we have $c \in S^{\mathcal{S}_d}_{C,\mathbf{1}}$. Therefore, by Proposition 1.1, 2) in [1], the element c is fixed under the conjugation action of $\mathcal{S}_d$ on S_C.

Given any $k \geqslant 4$, we write $Z_k \simeq \mathcal{S}_2 \times \mathcal{S}_{k-2}$ for the subgroup of $\mathcal{S}_d$ generated by the transpositions $(1,2)$ and (i,j), $3 \leqslant i < j \leqslant k$. Note that Z_d is the centralizer of $(1,2)$ in $\mathcal{S}_d$.

Assertion 1. *There is $z_{(1,2)} \in S_C$ such that $\alpha(z_{(1,2)}) = (1,2)$ and $\rho(\sigma)(z_{(1,2)}) = z_{(1,2)}$ for all $\sigma \in Z_d$.*

Proof. We use induction on k to prove the existence of an element $y_{(1,2),k} \in S_C^{\mathcal{S}_d}$ such that $\alpha(y_{(1,2),k}) = (1,2)$ and $\rho(\sigma)(y_{(1,2),k}) = y_{(1,2),k}$ for all $\sigma \in Z_k$. Then $z_{(1,2)} = y_{(1,2),d}$ is the desired element.

Put $y_{(1,2),4} = \overline{s}_{(1,2)} \cdot c$. Moving the first factor $\overline{s}_{(1,2)}$ to the right, we get

$$\begin{aligned} y_{(1,2),4} &= \overline{s}_{(1,2)} \cdot \overline{s}_{(1,2)} \cdot \overline{s}_{(1,2)} \cdot \overline{s}^2_{(2,3)} \cdot \ldots \cdot \overline{s}^2_{(d-1,d)} \\ &= \rho((1,2))(\overline{s}_{(1,2)}) \cdot \overline{s}_{(1,2)} \cdot \overline{s}_{(1,2)} \cdot \overline{s}^2_{(2,3)} \cdot \ldots \cdot \overline{s}^2_{(d-1,d)} \\ &= \rho((1,2))(\overline{s}_{(1,2)}) \cdot c = \rho((1,2))(\overline{s}_{(1,2)}) \cdot \rho((1,2))(c) \\ &= \rho((1,2))(\overline{s}_{(1,2)} \cdot c) = \rho((1,2))(y_{(1,2),4}) \end{aligned}$$

since c is fixed under the conjugation action of $\mathcal{S}_d$.

Using the hypotheses of Theorem 2, we similarly have

$$\begin{aligned} \rho((3,4))(y_{(1,2),4}) &= \rho((3,4))(\overline{s}_{(1,2)} \cdot c) = \rho((3,4))(\overline{s}_{(1,2)}) \cdot \rho((3,4))(c) \\ &= \overline{s}_{(1,2)} \cdot c = y_{(1,2),4}, \end{aligned}$$

whence $\rho(\sigma)(y_{(1,2),4}) = y_{(1,2),4}$ for all $\sigma \in Z_4$.

Suppose that for some $k \geqslant 4$, $k < d$, we have already constructed an element $y_{(1,2),k} \in S_C^{\mathcal{S}_d}$ such that $\alpha(y_{(1,2),k}) = (1,2)$ and $\rho(\sigma)(y_{(1,2),k}) = y_{(1,2),k}$ for all $\sigma \in Z_k$. Consider the element $y'_{(1,2),k} = \rho((k,k+1))(y_{(1,2),k})$. Clearly, it belongs to $S_C^{\mathcal{S}_d}$ and we easily see that $\alpha(y'_{(1,2),k}) = (1,2)$. Hence the element $y_{(1,2),k} \cdot y'_{(1,2),k}$ belongs to $S_{C,\mathbf{1}}^{\mathcal{S}_d}$ and, therefore, it is fixed under the conjugation action of $\mathcal{S}_d$. We claim that $y'_{(1,2),k}$ is fixed under the action of the group Z'_k generated by the transpositions $(i,j) \in Z_{k+1}$, $i,j \neq k$. Indeed, if $(i,j) \in Z'_k$ and $i,j \neq k+1$, then

$$\begin{aligned} \rho((i,j))(y'_{(1,2),k}) &= \rho((i,j))\big(\rho((k,k+1))(y_{(1,2),k})\big) \\ &= \rho\big((i,j)(k,k+1)\big)(y_{(1,2),k}) = \rho\big((k,k+1)(i,j)\big)(y_{(1,2),k}) \\ &= \rho((k,k+1))\big(\rho((i,j))(y_{(1,2),k})\big) = \rho((k,k+1))(y_{(1,2),k}) = y'_{(1,2),k}. \end{aligned}$$

If $(i,k+1) \in Z'_k$, then

$$\begin{aligned} \rho((i,k+1))(y'_{(1,2),k}) &= \rho((i,k+1))\big(\rho((k,k+1))(y_{(1,2),k})\big) \\ &= \rho\big((i,k+1)(k,k+1)\big)(y_{(1,2),k}) = \rho\big((k,k+1)(i,k)\big)(y_{(1,2),k}) \\ &= \rho((k,k+1))\big(\rho((i,k))(y_{(1,2),k})\big) = \rho((k,k+1))(y_{(1,2),k}) = y'_{(1,2),k} \end{aligned}$$

since $(i,k) \in Z_k$.

Moreover, the elements $y_{(1,2),k}$ and $y'_{(1,2),k}$ commute because

$$\begin{aligned} y'_{(1,2),k} \cdot y_{(1,2),k} &= \rho\big(\alpha(y'_{(1,2),k})\big)(y_{(1,2),k}) \cdot y'_{(1,2),k} \\ &= \rho((1,2))(y_{(1,2),k}) \cdot y'_{(1,2),k} = y_{(1,2),k} \cdot y'_{(1,2),k}. \end{aligned}$$

We put $y_{(1,2),k+1} := y^2_{(1,2),k} \cdot y'_{(1,2),k}$. Clearly, $y_{(1,2),k+1} \in S_C^{\mathcal{S}_d}$ and $\alpha(y_{(1,2),k+1}) = (1,2)$. We claim that $\rho(\sigma)(y_{(1,2),k+1}) = y_{(1,2),k+1}$ for all $\sigma \in Z_{k+1}$. Indeed, note that the group Z_{k+1} is generated by the elements of the groups Z_k and Z'_k. For every $\sigma \in Z_k$ we have

$$\begin{aligned}\rho(\sigma)(y_{(1,2),k+1}) &= \rho(\sigma)(y_{(1,2),k} \cdot y_{(1,2),k} \cdot y'_{(1,2),k}) \\ &= \rho(\sigma)(y_{(1,2),k}) \cdot \rho(\sigma)(y_{(1,2),k} \cdot y'_{(1,2),k}) = y_{(1,2),k} \cdot y_{(1,2),k} \cdot y'_{(1,2),k}\end{aligned}$$

since the element $y_{(1,2),k} \cdot y'_{(1,2),k} \in S_{C,\mathbf{1}}^{\mathcal{S}_d}$ is fixed under the conjugation action of $\mathcal{S}_d$.

For every $\sigma \in Z'_k$ we similarly have

$$\begin{aligned}\rho(\sigma)(y_{(1,2),k+1}) &= \rho(\sigma)(y^2_{(1,2),k} \cdot y'_{(1,2),k}) \\ &= \rho(\sigma)(y^2_{(1,2),k}) \cdot \rho(\sigma)(y'_{(1,2),k}) = y^2_{(1,2),k} \cdot y'_{(1,2),k} = y_{(1,2),k+1}\end{aligned}$$

since the element $y_{(1,2),k} \cdot y_{(1,2),k} \in S_{C,\mathbf{1}}^{\mathcal{S}_d}$ is fixed under the conjugation action of $\mathcal{S}_d$. The assertion is proved.

Consider the orbit $X_{T_{C,d}}$ of the element $z_{(1,2)}$ under the conjugation action of $\mathcal{S}_d$ on S_C, where $z_{(1,2)}$ is the element constructed in the proof of Assertion 1 with the help of the element $\overline{s}_{(1,2)}$.

Assertion 2. *Define a map* $\overline{\alpha}\colon X_{T_{C,d}} \to X_{T_d} = \{x_{(i,j)} \mid (i,j) \in T_d\}$ *by the formula*

$$\overline{\alpha}\big(\rho(\sigma)(z_{(1,2)})\big) = x_{\sigma(1,2)\sigma^{-1}}.$$

Then this map is a one-to-one correspondence.

Proof. The map $\overline{\alpha}\colon X_{T_{C,d}} \to X_{T_d}$ is surjective because for every transposition $(i,j) \in T_d$ one can find $\sigma \in \mathcal{S}_d$ such that $(i,j) = \sigma(1,2)\sigma^{-1}$, and this permutation σ satisfies

$$\alpha\big(\rho(\sigma)(z_{(1,2)})\big) = \sigma(1,2)\sigma^{-1} = (i,j),$$
$$\alpha\big(\overline{\alpha}(\rho(\sigma)(z_{(1,2)}))\big) = \alpha(x_{\sigma(1,2)\sigma^{-1}}) = \sigma(1,2)\sigma^{-1} = (i,j).$$

The order of the group Z_d is equal to $2(d-2)!$. Therefore, by Assertion 1, the number $|X_{T_{C,d}}|$ of elements in $X_{T_{C,d}}$ does not exceed $\frac{d!}{2(d-2)!} = \frac{d(d-1)}{2} = |T_d|$. Hence the map $\overline{\alpha}\colon X_{T_{C,d}} \to X_{T_d}$ is a one-to-one correspondence. The assertion is proved.

We write $z_{(i,j)}$ for an element $z \in X_{T_{C,d}}$ such that $\alpha(z) = (i,j)$. Let $S_{T_{C,d}}$ be the subsemigroup of S_C generated by the elements $z_{(i,j)}$, $1 \leqslant i,j \leqslant d$, $i \neq j$. It follows from the construction of the elements $z_{(i,j)}$ that $S_{T_{C,d}}$ is a semigroup over $\mathcal{S}_d$.

Assertion 3. *The subsemigroup* $S_{T_{C,d}}$ *of* S_C *is a semigroup over* $\mathcal{S}_d$. *The elements* $z_{(i,j)} \in S_{T_{C,d}}$, $1 \leqslant i,j \leqslant d$, $i \neq j$, *satisfy the following relations*:

$$\begin{gathered} z_{(i,j)} = z_{(j,i)} \qquad \forall\,\{i,j\}_{\mathrm{ord}} \subset I_d, \\ z_{(i_1,i_2)} \cdot z_{(i_1,i_3)} = z_{(i_2,i_3)} \cdot z_{(i_1,i_2)} = z_{(i_1,i_3)} \cdot z_{(i_2,i_3)} \qquad \forall\,\{i_1,i_2,i_3\}_{\mathrm{ord}} \subset I_d, \\ z_{(i_1,i_2)} \cdot z_{(i_3,i_4)} = z_{(i_3,i_4)} \cdot z_{(i_1,i_2)} \qquad \forall\,\{i_1,i_2,i_3,i_4\}_{\mathrm{ord}} \subset I_d. \end{gathered} \tag{4}$$

Proof. This follows directly from the construction of the elements $z_{(i,j)}$ and Assertion 1.1 in [1].

Assertion 4. *The map* $\overline{\alpha}^{-1}\colon X_{T_d}\to X_{T_{C,d}}$ *can be extended to a surjective homomorphism* $\overline{\alpha}^{-1}\colon S_{T_d}\to S_{T_{C,d}}$ *of semigroups over* $\mathcal{S}_d$.

Proof. Substituting $x_{(i,j)}$ for $z_{(i,j)}$ in (4), we get the defining relations of the semigroup S_{T_d}. Hence it follows from Assertion 3 that $\overline{\alpha}^{-1}$ can be extended to a surjective homomorphism of semigroups over $\mathcal{S}_d$. The assertion is proved.

If $s\in S_{T_{C,d}}$ is a product of n generators $z_{(i,j)}$ of the semigroup $S_{T_{C,d}}$, then we define its T*-length* by the formula $\ln_T(s)=n$. We have $\ln(s)=\ln_T(\overline{\alpha}^{-1}(s))$ for $s\in S_{T_d}$.

Assertion 4 shows that all statements in [1] saying that an element of S_{T_d} can be represented as a product of some generators $x_{i,j}$, remain valid for elements of $S_{T_{C,d}}$ if we replace $x_{(i,j)}$ by $z_{(i,j)}$ and lengths by T-lengths.

We define a subsemigroup $S^{\mathcal{S}_d,T}_{T_{C,d}}$ of $S_{T_{C,d}}$ by putting

$$S^{\mathcal{S}_d,T}_{T_{C,d}}:=\overline{\alpha}^{-1}(S^{\mathcal{S}_d}_{T_d}).$$

Theorem 2 follows from the following assertion.

Assertion 5. *The restriction of* $\overline{\alpha}^{-1}\colon S_{T_d}\to S_{T_{C,d}}$ *to* $S^{\mathcal{S}_d}_{T_d}$,

$$\overline{\alpha}^{-1}\colon S^{\mathcal{S}_d}_{T_d}\to S^{\mathcal{S}_d,T}_{T_{C,d}},$$

is an isomorphism of semigroups over $\mathcal{S}_d$.

Proof. The homomorphism $\overline{\alpha}^{-1}\colon S^{\mathcal{S}_d}_{T_d}\to S^{\mathcal{S}_d,T}_{T_{C,d}}$ is injective by Theorem 2.1 in [1].

We also mention the following immediate corollary of Theorem 2.1 in [1] and Assertion 5.

Corollary 2. *Every element* s *of the semigroup* $S^{\mathcal{S}_d,T}_{T_{C,d}}$ *is uniquely determined by* $\alpha(s)$ *and* $\ln_T(s)$.

§ 2. Proof of Theorem 1

Consider an element $\overline{s}_{(1,2)}=x_{\sigma_1}\cdot\dots\cdot x_{\sigma_{m_C}}$, where $\sigma_1,\dots,\sigma_{m_C}\in C$ are the factors in the factorization (3).

If $f_C\geqslant 2$, we can and will assume that all the permutations σ_i appearing in (3) belong to the subgroup $\mathcal{S}_d^{\{3,4\}}\simeq\mathcal{S}_{d-2}$ of those elements of $\mathcal{S}_d$ that leave $3,4\in I_d$ fixed. Then the element $\overline{s}_{(1,2)}=x_{\sigma_1}\cdot\dots\cdot x_{\sigma_{m_C}}$ satisfies all the hypotheses of Theorem 2. Hence the elements $z_{(i,j)}$ constructed in § 1 with the help of $\overline{s}_{(1,2)}=x_{\sigma_1}\cdot\dots\cdot x_{\sigma_{m_C}}$ uniquely determine a subsemigroup $S^{\mathcal{S}_d,T}_{T_{C,d}}$ of S_C isomorphic to $S^{\mathcal{S}_d}_{T_d}$ over $\mathcal{S}_d$.

Note that the length of the element $z_{(1,2)}$ constructed in the proof of Assertion 1 is equal to $\ln(z_{(1,2)})=3^{d-4}(2d-1)m_C$ if we start the construction with $\overline{s}_{(1,2)}=x_{\sigma_1}\cdot\dots\cdot x_{\sigma_{m_C}}$.

We put

$$h_C=(z_{(1,2)}\cdot z_{(2,3)}\cdot\dots\cdot z_{(d-1,d)})^3.$$

Then h_C belongs to $S^{\mathcal{S}_d,T}_{T_{C,d}}$. We rewrite h_C as a product:

$$h_C = x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_L}, \qquad \sigma_i \in C, \quad i = 1, \ldots, L.$$

The length of h_C is easily found to be

$$\ln(h_C) = 3^{d-3}(2d-1)(d-1)m_C := L.$$

The following assertion will be used in the proof of Theorem 1.

Assertion 6. *Under the hypotheses of Theorem 1 suppose that* $s = \widetilde{s} \cdot \overline{s} \in \Sigma^{\mathcal{S}_d}_d$, *where* $\overline{s} \in S_C$ *has length*

$$\ln(\overline{s}) := M \geqslant 3^{d-3}(2d-1)(d-1)m_C + n_C k_C.$$

Then s *can be represented as a product:* $s = \widetilde{s}\,' \cdot h_C$.

Proof. Write

$$\overline{s} = x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_M}, \qquad \sigma_i \in C. \tag{5}$$

Since $M = \ln(\overline{s}) \geqslant 3^{d-3}(2d-1)(d-1)m_C + n_C k_C > n_C k_C$, there is a permutation $\sigma \in C$ such that at least $n_C + 1$ factors in (5) are equal to x_σ. Therefore $\overline{s}$ can be written as $\overline{s} = \overline{s}' \cdot x^{n_C}_\sigma$, where $\overline{s}' \in S_C$ is such that $\widetilde{s} \cdot \overline{s}' \in \Sigma^{\mathcal{S}_d}_d$. By Lemma 1.1 in [1] we have

$$s = \widetilde{s} \cdot \overline{s}' \cdot x^{n_C}_\sigma = \widetilde{s} \cdot \overline{s}' \cdot x^{n_C}_{\sigma_L} = \widetilde{s} \cdot \overline{s}_L \cdot x_{\sigma_L},$$

where $\overline{s}_L = \overline{s}' \cdot x^{n_C-1}_{\sigma_L}$. Note that $\widetilde{s} \cdot \overline{s}_L \in \Sigma^{\mathcal{S}_d}_d$ and $\ln(\overline{s}_L) > n_C k_C$. Therefore, by the same argument, $\widetilde{s} \cdot \overline{s}_L$ can be written as $\widetilde{s} \cdot \overline{s}_L = \widetilde{s} \cdot \overline{s}'_L \cdot x^{n_C-1}_{\sigma_{L-1}} \cdot x_{\sigma_{L-1}}$. We put $\overline{s}_{L-1} = \overline{s}'_L \cdot x^{n_C-1}_{\sigma_{L-1}}$. Repeating the same arguments for $\widetilde{s} \cdot \overline{s}_{L-1}$, we obtain that $\widetilde{s} \cdot \overline{s}_{L-1} = \widetilde{s} \cdot \overline{s}_{L-2} \cdot x_{\sigma_{L-1}}$, and so on. At the Lth step we finally get

$$s = \widetilde{s} \cdot \overline{s} = \widetilde{s} \cdot \overline{s}_0 \cdot (x_{\sigma_1} \cdot \ldots \cdot x_{\sigma_L}) = \widetilde{s} \cdot \overline{s}_0 \cdot h_C.$$

The assertion is proved.

To complete the proof of Theorem 1, we recall that the proof of Theorem 2.3 in [1] consists of two parts. In the first part it is proved that every element $s = \widetilde{s} \cdot \overline{s} \in \Sigma^{\mathcal{S}_d}_d$ with $\overline{s} \in S_{T_d}$ and $\ln(\overline{s}) \geqslant 3(d-1)$ admits another factorization $s = \widetilde{s}_1 \cdot \overline{s}_1$ such that $\overline{s}_1 \in S^{\mathcal{S}_d}_{T_d}$ and $\ln(\overline{s}_1) = 3(d-1)$. In this case, the element $\overline{s}_1$ is uniquely determined by its product $\alpha(\overline{s}_1) = \alpha(\widetilde{s}_1)^{-1}\alpha(s)$.

In the second part of the proof of Theorem 2.3 in [1] it was proved that every such element $s = \widetilde{s}_1 \cdot \overline{s}_1$ may be rewritten as $s = \widetilde{s}_2 \cdot \overline{s}_2$, where $\overline{s}_2 \in S^{\mathcal{S}_d}_{T_d}$ is still of length $\ln(\overline{s}_2) = 3(d-1)$ and $\widetilde{s}_2$ is uniquely determined by the type $\tau(\widetilde{s}_1)$. Here we have only used properties of the semigroup S_{T_d} and the relations (1) in the factorization semigroups. Therefore, by Assertions 5 and 6, the end of the proof of Theorem 1 coincides almost verbatim with the second part of the proof of Theorem 2.3 in [1]. We need only replace the elements $x_{(i,j)}$ by $z_{(i,j)}$, the lengths of elements by the T-lengths, the element $h_{d,g}$ by $\overline{\alpha}^{\,-1}(h_{d,g})$, the semigroup $S^{\mathcal{S}_d}_{T_d}$ by $S^{\mathcal{S}_d,T}_{T_{C,d}}$ and the homomorphism r by $\overline{r} = \overline{\alpha}^{\,-1} \circ r$.

However, at the request of the referee, we give this proof again. To do this, we introduce the notation $h_{C,d,g} = \overline{\alpha}^{\,-1}(h_{d,g})$ for the image of the Hurwitz element $h_{d,g} = x^{2g}_{(1,2)} \cdot x^2_{(1,,2)} \cdot \ldots \cdot x^2_{(d-1,d)}$.

Lemma 1. *For every disjoint union* $\{i_{1,1},\dots,i_{k_1,1}\}\sqcup\dots\sqcup\{i_{1,n},\dots,i_{k_n,n}\}$ *of ordered subsets of* I_d*, the Hurwitz element* $h_{C,d,0}$ *can be represented as a product*

$$h_{C,d,0}=(z_{(i_{1,1},i_{2,1})}\cdot\dots\cdot z_{(i_{k_1-1,1},i_{k_1,1})})\cdot\dots\cdot(z_{(i_{1,n},i_{2,n})}\cdot\dots\cdot z_{(i_{k_n-1,n},i_{k_n,n})})\cdot\overline{h},$$

where $\overline{h}$ *is an element of* $S^{\mathcal{S}_d,T}_{T_{C,d}}$.

Proof. This follows directly from Lemma 2.9 in [1] and Assertion 5.

By Assertion 6, the element s can be represented as a product $s=\widetilde{s}'\cdot\overline{s}$, where $\overline{s}$ is an element of $S^{\mathcal{S}_d,T}_{T_{C,d}}$ of T-length $k\geqslant 3(d-1)$ (in our case $\overline{s}=h_C$ and $k=3(d-1)$) and $\widetilde{s}'=x_{\sigma'_1}\cdot\dots\cdot x_{\sigma'_m}$. By Proposition 2.4 in [1] and Assertion 5 we have $\overline{s}=h_{C,d,0}\cdot\overline{s}'$.

To complete the proof of Theorem 1, we use induction on m. If $m=0$ (that is, $s\in S_{T_{C,d}}$), then Theorem 1 follows from Proposition 2.4 in [1] and Assertion 5.

Suppose that $m=1$. For the canonical representative $\sigma_{m,0}$ of type $t(\sigma_m)$ (see [1] for a definition of the canonical representative) there is an element $\overline{\sigma}_m\in\mathcal{S}_d$ such that $\sigma_{m,0}=\overline{\sigma}_m^{-1}\sigma'_m\overline{\sigma}_m$. The permutation $\overline{\sigma}_m$ can be factorized into a product of cyclic permutations, and each cyclic permutation can be factorized into a product of transpositions:

$$\overline{\sigma}_m=\big((i_{1,1},i_{2,1})\dots(i_{k_1-1,1},i_{k_1,1})\big)\dots\big((i_{1,n},i_{2,n})\dots(i_{k_n-1,n},i_{k_n,n})\big).$$

Consider the element

$$\overline{r}(x_{\overline{\sigma}_m})=(z_{(i_{1,1},i_{2,1})}\cdot\dots\cdot z_{(i_{k_1-1,1},i_{k_1,1})})\cdot\dots\cdot(z_{(i_{1,n},i_{2,n})}\cdot\dots\cdot z_{(i_{k_n-1,n},i_{k_n,n})})\in S_{T_{C,d}}.$$

By Lemma 1 we have

$$h_{C,d,0}=\overline{r}(x_{\overline{\sigma}_m})\cdot\overline{h}_m,$$

where $\overline{h}_m$ is an element of $S^{\mathcal{S}_d,T}_{T_{C,d}}$. Therefore

$$\begin{aligned}s&=x_{\sigma'_m}\cdot h_{d,0}\cdot\overline{s}'=x_{\sigma'_m}\cdot\overline{r}(x_{\overline{\sigma}_m})\cdot\overline{h}_m\cdot\overline{s}'\\&=\overline{r}(x_{\overline{\sigma}_m})\cdot x_{\sigma_{m,0}}\cdot\overline{h}_m\cdot\overline{s}'=x_{\sigma_{m,0}}\cdot\overline{r}(x_{\overline{\sigma}'_m})\cdot\overline{h}_m\cdot\overline{s}',\end{aligned}$$

where $x_{\overline{\sigma}'_m}=\lambda(\sigma_{m,0})(x_{\overline{\sigma}_m})$. We have $\overline{s}'_1=\overline{r}(x_{\overline{\sigma}'_m})\cdot\overline{h}_m\cdot\overline{s}'\in S^{\mathcal{S}_d,T}_{T_{C,d}}$ and $\alpha(\overline{s}'_1)=\sigma_{m,0}^{-1}\alpha(s)$. Theorem 2.4 in [1] and Assertion 5 imply that $\overline{s}'_1=\overline{r}(x_\sigma)\cdot h_{C,d,g}$, where $\sigma=\alpha(\overline{s}'_1)=\sigma_{m,0}^{-1}\alpha(s)$ and $g=\frac{k-\ln_t(x_\sigma)}{2}-d+1$.

We now assume that Theorem 1 is true for all $m<m_0$ and consider an element

$$s=x_{\sigma_1}\cdot\dots\cdot x_{\sigma_{m_0}}\cdot\overline{s}_1,$$

where the T-length of $\overline{s}_1\in S^{\mathcal{S}_d,T}_{T_{C,d}}$ is equal to $k\geqslant 3(d-1)$. We have

$$\begin{aligned}s&=x_{\sigma_1}\cdot\dots\cdot x_{\sigma_{m_0}}\cdot\overline{s}_1=x_{\sigma'_2}\cdot\dots\cdot x_{\sigma'_{m_0}}\cdot x_{\sigma_1}\cdot\overline{s}_1\\&=x_{\sigma'_2}\cdot\dots\cdot x_{\sigma'_{m_0}}\cdot x_{\sigma_{1,0}}\cdot\overline{s}'_1=x_{\sigma_{1,0}}\cdot x_{\sigma''_2}\cdot\dots\cdot x_{\sigma''_{m_0}}\cdot\overline{s}'_1,\end{aligned}$$

where $\sigma'_j = \sigma_1\sigma_j\sigma_1^{-1}$ and $\sigma''_j = \sigma_{1,0}^{-1}\sigma'_j\sigma_{1,0}$ for $j = 2,\dots,m$, and the element $\overline{s}'_1 \in S_{T_{C,d}}^{\mathcal{S}_d,T}$ satisfies $\ln_T(\overline{s}'_1) = k$. By the induction hypothesis we have

$$s = x_{\sigma_{1,0}} \cdot (x_{\sigma''_2} \cdot \dots \cdot x_{\sigma''_{m_0}} \cdot \overline{s}'_1) = x_{\sigma_{1,0}} \cdot (x_{\sigma_{2,0}} \cdot \dots \cdot x_{\sigma_{m_0,0}} \cdot \overline{s}''_1),$$

where $\overline{s}''_1 \in S_{T_{C,d}}^{\mathcal{S}_d,T}$ and $\ln_T(\overline{s}''_1) = k$. By Proposition 2.4 in [1] and Assertion 5 we have $\overline{s}''_1 = \overline{r}(x_\sigma) \cdot h_{C,d,g}$, where $\sigma = \alpha(\overline{s}''_1) = (\sigma_{1,0}\dots\sigma_{m,0})^{-1}\alpha(s)$ and $g = \frac{k-\ln_t(x_\sigma)}{2} - d + 1$.

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