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Factorization semigroups
and irreducible components of the Hurwitz space. II

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Abstract. We continue the investigation started in [1]. Let $HUR_{d,t}^S(\mathbb{P}^1)$ be the Hurwitz space of coverings of degree $d$ of the projective line $\mathbb{P}^1$ with Galois group $S_d$ and monodromy type $t$. The monodromy type is a set of local monodromy types, which are defined as conjugacy classes of permutations $\sigma$ in the symmetric group $S_d$ acting on the set $I_d = \{1, \ldots, d\}$. We prove that if the type $t$ contains sufficiently many local monodromies belonging to the conjugacy class $C$ of an odd permutation $\sigma$ which leaves $f_C \geq 2$ elements of $I_d$ fixed, then the Hurwitz space $HUR_{d,t}^S(\mathbb{P}^1)$ is irreducible.

Keywords: semigroup, factorizations of an element of a group, irreducible components of the Hurwitz space.

Introduction

This paper is continuation of [1]. Before stating its results, we recall the main definitions and notation used in [1]. A quadruple $(S, G, \alpha, \rho)$, where $S$ is a semigroup, $G$ is a group and $\alpha : S \rightarrow G$, $\rho : G \rightarrow \text{Aut}(S)$ are homomorphisms, is called a semigroup $S$ over a group $G$ if for all $s_1, s_2 \in S$ we have

$$s_1 \cdot s_2 = \rho(\alpha(s_1))(s_2) \cdot s_1 = s_2 \cdot \lambda(\alpha(s_2))(s_1),$$

(1)

where $\lambda(g) = \rho(g^{-1})$. Let $(S_1, G, \alpha_1, \rho_1)$ and $(S_2, G, \alpha_2, \rho_2)$ be semigroups over $G$. A homomorphism of semigroups $\varphi : S_1 \rightarrow S_2$ is said to be defined over $G$ if $\alpha_1(s) = \alpha_2(\varphi(s))$ and $\rho_2(g)(\varphi(s)) = \varphi(\rho_1(g)(s))$ for all $s \in S_1$ and $g \in G$.

A pair $(G, O)$, where $O$ is a subset of $G$ invariant under inner automorphisms of $G$, is called an equipped group. With every equipped group $(G, O)$ one can associate a semigroup $S_O = S(G, O)$ over $G$ (called the factorization semigroup of elements of $G$ with factors in $O$) generated by the elements of the alphabet $X = X_O = \{x_g \mid g \in O\}$ subject to the relations

$$x_{g_1} \cdot x_{g_2} = x_{g_2} \cdot x_{g_2^{-1}g_1g_2} = x_{g_1g_2g_2^{-1}g_1} \cdot x_{g_1}$$

(2)

for all $x_{g_1}, x_{g_2} \in X$, and if $g_2 = 1$, then $x_{g_1} \cdot x_1 = x_{g_1}$. We define a map $\alpha : X \rightarrow G$ by putting $\alpha(x_g) = g$ for every $x_g \in X$. It induces a homomorphism $\alpha : S_O \rightarrow G$ called...
the product homomorphism. The action \( \rho \) (on the left) of \( G \) on \( S_O \) is induced by the following action on the alphabet \( X \):

\[
x_a \in X \mapsto \rho(g)(x_a) = x_{ gag^{-1}} \in X
\]

for \( g \in G \). Note that \( \alpha(\rho(g)(s)) = g\alpha(s)g^{-1} \) for all \( s \in S_O \) and \( g \in G \).

Let \( O \setminus \{1\} = C_1 \sqcup \cdots \sqcup C_m \) be the decomposition of \( O \) into a disjoint union of conjugacy classes of elements of \( G \). Every element \( s = x_{g_1} \cdots x_{g_n} \in S_O \) determines an element \( \tau(s) = n_1C_1 + \cdots + n_mC_m \) of the free Abelian semigroup generated by the symbols \( C_1, \ldots, C_m \) (the element \( \tau(s) \) is called the type of \( s \)), where \( n_i \) is the number of those factors \( x_{g_j} \) in the factorization \( s = x_{g_1} \cdots x_{g_n} \) which satisfy \( g_j \in C_i \). The number \( n = \sum_{i=1}^{m} n_i \) is called the length of \( s \) and is denoted by \( \ln(s) \).

A subsemigroup \( S \) of \( S_G \) is said to be stable if there is an element \( s \in S \) (called a stabilizing element of \( S \)) such that \( s_1 \cdot s = s_2 \cdot s \) for all \( s_1, s_2 \in S \) satisfying \( \alpha(s_1) = \alpha(s_2) \) and \( \tau(s_1) = \tau(s_2) \).

For every element \( s = x_{g_1} \cdots x_{g_n} \in S_O \), let \( G_s = \langle g_1, \ldots, g_n \rangle \) be the subgroup of \( G \) generated by the elements \( g_1, \ldots, g_n \). Given any (not necessarily proper) subgroups \( H \) and \( \Gamma \) of \( G \), one can define subsemigroups \( S_{O,H}^\Gamma = \{ s \in S(G, O) \mid G_s = H \} \) and \( S_{O,\Gamma} = \{ s \in S(G, O) \mid \alpha(s) \in \Gamma \} \). If \( H \) and \( \Gamma \) are normal subgroups of \( G \), then \( S_{O,\Gamma} \) are semigroups over \( G \). By definition, \( S_{O,H}^\Gamma \) is the subset of transpositions. We denote the semigroup \( S_{S_d} \) by \( \Sigma_d \). By Theorem 2.3 in [1], the element

\[
h = \left( \prod_{i=1}^{d-1} x_{(i,i+1)} \right)^3
\]

is a stabilizing element of \( \Sigma_d \). Here \((i, i+1) \in T_d \) is the transposition interchanging the elements \( i \) and \( i+1 \) of \( I_d \).

The aim of this paper is to prove that a similar result holds for almost all odd elements of \( S_d \). More precisely, let \( C = C_\sigma \) be the conjugacy class of a permutation \( \sigma \in S_d, n_C \) the order of \( C \), \( k_C = |C| \) the number of elements of \( C \), and \( f_C \) the number of elements of \( I_d \) that remain fixed under the action of \( \sigma \) on \( I_d \).

It is known that if \( \sigma \) is an odd permutation, then elements of \( C \) generate the whole group \( S_d \) and, in particular, any transposition \((i,j) \in S_d \) can be written as a product of permutations belonging to \( C \). In the case when \( f_C \geq 2 \), we write \( m_C \) for the minimal number (counting multiplicities) of permutations in \( C \cap S_{d-2} \) needed to express \((1,2)\) as a product of elements of \( C \cap S_{d-2} \). We also fix any one of these expressions:

\[
(1,2) = \sigma_1 \cdots \sigma_{m_C}, \quad \sigma_i \in C \cap S_{d-2}.
\]

**Theorem 1.** Let \( C \) be the conjugacy class of an odd permutation \( \sigma \in S_d \). If \( f_C \geq 2 \), then there is a constant

\[
N = N_C < 3^{d-3}(2d - 1)(d-1)m_C + n_C k_C + 1
\]

such that every element \( s = \tilde{s} \cdot \tilde{\pi} \in \Sigma_d^{S_d} \) with \( \tilde{\pi} \in S_C \) and \( \ln(\tilde{\pi}) \geq N \) is uniquely determined by \( \tau(s) \) and \( \alpha(s) \).
Corollary 1. Let an equipped symmetric group \((S_d, O)\) be such that the set \(O\) contains the conjugacy class \(C\) of an odd permutation \(\sigma\), \(f_C \geq 2\). Then \(S_O = S(S_d, O)\) is a stable semigroup.

Note that the constant \(N_C\) whose existence is asserted in Theorem 1 is generally greater than 1. For example, it is shown in [2] that this is the case when \(C\) is the conjugacy class of \(\sigma = (1, 2)(3, 4, 5) \in S_8\).

The proof of Theorem 1 is similar to that of Theorem 2.3 in [1]. It is based on the following theorem.

**Theorem 2.** Let \(C\) be the conjugacy class of an odd permutation \(\sigma \in S_d\), and let \(\bar{s}_{(i_1, i_2)} \in S_C\) be an element with the following properties:

(i) \(\alpha(\bar{s}_{(i_1, i_2)}) = (i_1, i_2)\),
(ii) there are \(i_3, i_4 \in I_d \setminus \{i_1, i_2\}\) such that \(\rho((i_3, i_4))(\bar{s}_{(i_1, i_2)}) = \bar{s}_{(i_1, i_2)}\).

Then there is an embedding over \(S_d\) of the semigroup \(S_{t_d}^{S_d}\) in the semigroup \(S_C\).

Let \(\text{HUR}_{d,b}(\mathbb{P}^1)\) (resp. \(\text{HUR}_{d,b}^G(\mathbb{P}^1)\)) be the Hurwitz space of ramified coverings of degree \(d\) of the projective line \(\mathbb{P}^1\) (defined over \(\mathbb{C}\)) branched over \(b\) points (resp. with Galois group \(G\)). It was shown in [1] that the irreducible components of \(\text{HUR}_{d,b}(\mathbb{P}^1)\) are in one-to-one correspondence with the orbits of the action of \(S_d\) by simultaneous conjugation (that is, the action determined by the homomorphism \(\rho\) on the set \(\Sigma_{d,1,b} = \{s \in S_d \mid \ln(s) = b\}\), and if \(G = S_d\), then the irreducible components of \(\text{HUR}_{d,b}^S(\mathbb{P}^1)\) are in one-to-one correspondence with the elements of \(\Sigma_{d,1}^{S_d}\) of length \(b\). If an irreducible component of \(\text{HUR}_{d,b}^S(\mathbb{P}^1)\) corresponds to an element \(s \in \Sigma_{d,1}^{S_d}\), then \(\tau(s)\) is called the monodromy factorization type of coverings belonging to this irreducible component. We denote the union of all irreducible components corresponding to the elements \(s \in \Sigma_{d,1}^{S_d}\) with \(\tau(s) = t\) by \(\text{HUR}_{d,t}^{S_d}(\mathbb{P}^1)\).

The following theorem is a corollary of Theorem 1.

**Theorem 3.** The space \(\text{HUR}_{d,t}^{S_d}(\mathbb{P}^1)\) is irreducible if the monodromy factorization type \(t\) contains more than \(N_C\) factors belonging to the conjugacy class \(C\) of an odd permutation \(\sigma \in S_d\) with \(f_C \geq 2\), where \(N_C\) is the number defined in Theorem 1.

We note that an analogue of Theorem 3 holds for the Hurwitz spaces of \(d\)-sheeted coverings of the disc \(\Delta = \{z \in \mathbb{C} \mid |z| \leq 1\}\) (resp. \(d\)-sheeted coverings of the affine line \(\mathbb{C}^1\)).

\[\xi^1.\text{ Proof of Theorem 2}\]

There is no loss of generality in assuming that \((i_1, i_2) = (1, 2)\) and \((i_3, i_4) = (3, 4)\). For every transposition \((i, j) \in T_d\) we choose a permutation \(\sigma_{i,j} \in S_d\) such that \((i, j) = \sigma_{i,j}(1, 2)\sigma_{i,j}^{-1}\) and put

\[c = \bar{s}_{(1,2)}^2 \cdot \bar{s}_{(2,3)}^2 \cdot \ldots \cdot \bar{s}_{(d-1,d)}^2,\]

where \(\bar{s}_{(i,j)} = \rho(\sigma_{i,j})(\bar{s}_{(1,2)})\).

Clearly, \(\alpha(\bar{s}_{(i,j)}) = (i, j)\) and \(\alpha(c) = 1\). Since the transpositions \((1, 2), \ldots, (d-1,d)\) generate the whole group \(S_d\), we have \(c \in S_{C,d}^{S_d}\). Therefore, by Proposition 1.1, 2) in [1], the element \(c\) is fixed under the conjugation action of \(S_d\) on \(S_C\).
Given any $k \geq 4$, we write $Z_k \simeq S_2 \times S_{k-2}$ for the subgroup of $S_d$ generated by the transpositions $(1, 2)$ and $(i, j)$, $3 \leq i < j \leq k$. Note that $Z_d$ is the centralizer of $(1, 2)$ in $S_d$.

**Assertion 1.** There is $z_{(1,2)} \in S_C$ such that $\alpha(z_{(1,2)}) = (1, 2)$ and $\rho(\sigma)(z_{(1,2)}) = z_{(1,2)}$ for all $\sigma \in Z_d$.

**Proof.** We use induction on $k$ to prove the existence of an element $y_{(1,2),k} \in S_C^{S_d}$ such that $\alpha(y_{(1,2),k}) = (1, 2)$ and $\rho(\sigma)(y_{(1,2),k}) = y_{(1,2),k}$ for all $\sigma \in Z_k$. Then $z_{(1,2)} = y_{(1,2),d}$ is the desired element.

Put $y_{(1,2),4} = \bar{s}_{(1,2)} \cdot c$. Moving the first factor $\bar{s}_{(1,2)}$ to the right, we get

$$y_{(1,2),4} = \bar{s}_{(1,2)} \cdot \bar{s}_{(1,2)} \cdot \bar{s}_{(1,2)} \cdot \bar{s}_{(2,3)} \cdots \bar{s}_{(d-1,d)}$$

$$= \rho((1, 2))(\bar{s}_{(1,2)}) \cdot \bar{s}_{(1,2)} \cdot \bar{s}_{(2,3)} \cdots \bar{s}_{(d-1,d)}$$

$$= \rho((1, 2))(\bar{s}_{(1,2)}) \cdot c = \rho((1, 2))(\bar{s}_{(1,2)}) \cdot \rho((1, 2))(c)$$

$$= \rho((1, 2))(\bar{s}_{(1,2)}) \cdot c = \rho((1, 2))(y_{(1,2),4})$$

since $c$ is fixed under the conjugation action of $S_d$.

Using the hypotheses of Theorem 2, we similarly have

$$\rho((3, 4))(y_{(1,2),4}) = \rho((3, 4))(\bar{s}_{(1,2)}) \cdot c = \rho((3, 4))(\bar{s}_{(1,2)}) \cdot \rho((3, 4))(c)$$

$$= \bar{s}_{(1,2)} \cdot c = y_{(1,2),4},$$

whence $\rho(\sigma)(y_{(1,2),4}) = y_{(1,2),4}$ for all $\sigma \in Z_4$.

Suppose that for some $k \geq 4$, $k < d$, we have already constructed an element $y_{(1,2),k} \in S_C^{S_d}$ such that $\alpha(y_{(1,2),k}) = (1, 2)$ and $\rho(\sigma)(y_{(1,2),k}) = y_{(1,2),k}$ for all $\sigma \in Z_k$. Consider the element $y'_{(1,2),k} = \rho((k, k+1))(y_{(1,2),k})$. Clearly, it belongs to $S_C^{S_d}$ and we easily see that $\alpha(y'_{(1,2),k}) = (1, 2)$. Hence the element $y_{(1,2),k} \cdot y'_{(1,2),k}$ belongs to $S_C^{S_d}$ and, therefore, it is fixed under the conjugation action of $S_d$. We claim that $y'_{(1,2),k}$ is fixed under the action of the group $Z'_k$ generated by the transpositions $(i, j) \in Z_{k+1}$, $i, j \neq k$. Indeed, if $(i, j) \in Z'_k$ and $i, j \neq k + 1$, then

$$\rho((i, j))(y'_{(1,2),k}) = \rho((i, j))(\rho((k, k+1))(y_{(1,2),k}))$$

$$= \rho((i, j))(k, k+1))(y_{(1,2),k}) = \rho((k, k+1))(i, j))(y_{(1,2),k})$$

$$= \rho((k, k+1))(\rho((i, j))(y_{(1,2),k})) = \rho((k, k+1))(y_{(1,2),k}) = y'_{(1,2),k}.$$

If $(i, k+1) \in Z'_k$, then

$$\rho((i, k+1))(y'_{(1,2),k}) = \rho((i, k+1))(\rho((k, k+1))(y_{(1,2),k}))$$

$$= \rho((i, k+1))(k, k+1))(y_{(1,2),k}) = \rho((k, k+1))(i, k))(y_{(1,2),k})$$

$$= \rho((k, k+1))(\rho((i, k))(y_{(1,2),k})) = \rho((k, k+1))(y_{(1,2),k}) = y'_{(1,2),k}$$

since $(i, k) \in Z_k$.

Moreover, the elements $y_{(1,2),k}$ and $y'_{(1,2),k}$ commute because

$$y'_{(1,2),k} \cdot y_{(1,2),k} = \rho(\alpha(y'_{(1,2),k}))(y_{(1,2),k}) \cdot y'_{(1,2),k}$$

$$= \rho((1, 2))(y_{(1,2),k}) \cdot y'_{(1,2),k} = y_{(1,2),k} \cdot y'_{(1,2),k}.$$
We put \( y_{1,2, k+1} := y_{1,2}^2 \cdot y_{1,2}' \). Clearly, \( y_{1,2, k+1} \in S_{C,d}^d \) and \( \alpha(y_{1,2, k+1}) = (1, 2) \). We claim that \( \rho(\sigma)(y_{1,2, k+1}) = y_{1,2, k+1} \) for all \( \sigma \in Z_{k+1} \). Indeed, note that the group \( Z_{k+1} \) is generated by the elements of the groups \( Z_k \) and \( Z'_k \). For every \( \sigma \in Z_k \) we have
\[
\rho(\sigma)(y_{1,2, k+1}) = \rho(\sigma)(y_{1,2}, k \cdot y_{1,2}, k \cdot y_{1,2}'_2, k)
\]
\[
= \rho(\sigma)(y_{1,2}, k) \cdot \rho(\sigma)(y_{1,2}, k) \cdot y_{1,2}'_k = y_{1,2, k} \cdot y_{1,2, k} \cdot y_{1,2}'_k
\]
since the element \( y_{1,2, k} \cdot y_{1,2}'_k \in S_{C,d}^d \) is fixed under the conjugation action of \( S_d \).

For every \( \sigma \in Z'_k \) we similarly have
\[
\rho(\sigma)(y_{1,2, k+1}) = \rho(\sigma)(y_{1,2}^2, k \cdot y_{1,2}'_2, k)
\]
\[
= \rho(\sigma)(y_{1,2}^2, k) \cdot \rho(\sigma)(y_{1,2}, k) \cdot y_{1,2}'_k = y_{1,2, k} \cdot y_{1,2, k} \cdot y_{1,2}'_k
\]
since the element \( y_{1,2, k} \cdot y_{1,2}'_k \in S_{C,d}^d \) is fixed under the conjugation action of \( S_d \). The assertion is proved.

Consider the orbit \( X_{T_{C,d}} \) of the element \( z_{(1,2)} \) under the conjugation action of \( S_d \) on \( S_C \), where \( z_{(1,2)} \) is the element constructed in the proof of Assertion 1 with the help of the element \( \tau_{(1,2)} \).

**Assertion 2.** Define a map \( \overline{\alpha} : X_{T_d} \rightarrow X_{T_d} = \{ x_{(i,j)} \mid (i,j) \in T_d \} \) by the formula
\[
\overline{\alpha}(\rho(\sigma)(z_{(1,2)})) = x_{\sigma(1,2)\sigma^{-1}}.
\]
Then this map is a one-to-one correspondence.

**Proof.** The map \( \overline{\alpha} : X_{T_{C,d}} \rightarrow X_{T_d} \) is surjective because for every transposition \( (i,j) \in T_d \) one can find \( \sigma \in S_d \) such that \( (i,j) = \sigma(1,2)\sigma^{-1} \), and this permutation \( \sigma \) satisfies
\[
\alpha(\rho(\sigma)(z_{(1,2)})) = \sigma(1,2)\sigma^{-1} = (i,j),
\]
\[
\alpha(\overline{\alpha}(\rho(\sigma)(z_{(1,2)}))) = \alpha(x_{\sigma(1,2)\sigma^{-1}}) = \sigma(1,2)\sigma^{-1} = (i,j).
\]

The order of the group \( Z_{d} \) is equal to \( 2(d-2)! \). Therefore, by Assertion 1, the number \( |X_{T_{C,d}}| \) of elements in \( X_{T_{C,d}} \) does not exceed \( \frac{d!}{2(d-2)!} = \frac{d(d-1)}{2} = |T_d| \). Hence the map \( \overline{\alpha} : X_{T_{C,d}} \rightarrow X_{T_d} \) is a one-to-one correspondence. The assertion is proved.

We write \( z_{(i,j)} \) for an element \( z \in X_{T_{C,d}} \) such that \( \alpha(z) = (i,j) \). Let \( S_{T_{C,d}} \) be the subsemigroup of \( S_C \) generated by the elements \( z_{(i,j)} \), \( 1 \leq i, j \leq d \), \( i \neq j \). It follows from the construction of the elements \( z_{(i,j)} \) that \( S_{T_{C,d}} \) is a semigroup over \( S_d \).

**Assertion 3.** The subsemigroup \( S_{T_{C,d}} \) of \( S_C \) is a semigroup over \( S_d \). The elements \( z_{(i,j)} \in S_{T_{C,d}}, 1 \leq i, j \leq d, i \neq j \), satisfy the following relations:
\[
z_{(i,j)} = z_{(j,i)} \quad \forall \{i,j\}_{\text{ord}} \subset I_d,
\]
\[
z_{(i_1, i_2)} \cdot z_{(i_1, i_3)} = z_{(i_2, i_3)} \cdot z_{(i_1, i_2)} = z_{(i_1, i_3)} \cdot z_{(i_2, i_3)} \quad \forall \{i_1, i_2, i_3\}_{\text{ord}} \subset I_d,
\]
\[
z_{(i_1, i_2)} \cdot z_{(i_3, i_4)} = z_{(i_3, i_4)} \cdot z_{(i_1, i_2)} \quad \forall \{i_1, i_2, i_3, i_4\}_{\text{ord}} \subset I_d.
\]
Proof. This follows directly from the construction of the elements \( z_{(i,j)} \) and Assertion 1.1 in [1].

Assertion 4. The map \( \overline{\alpha}^{-1}: X_{T_d} \to X_{T_{C,d}} \) can be extended to a surjective homomorphism \( \overline{\alpha}^{-1}: S_{T_d} \to S_{T_{C,d}} \) of semigroups over \( S_d \).

Proof. Substituting \( x_{(i,j)} \) for \( z_{(i,j)} \) in (4), we get the defining relations of the semigroup \( S_{T_d} \). Hence it follows from Assertion 3 that \( \overline{\alpha}^{-1} \) can be extended to a surjective homomorphism of semigroups over \( S_d \). The assertion is proved.

If \( s \in S_{T_{C,d}} \) is a product of \( n \) generators \( z_{(i,j)} \) of the semigroup \( S_{T_{C,d}} \), then we define its \( T \)-length by the formula \( \ln_T(s) = n \). We have \( \ln(s) = \ln_T(\overline{\alpha}^{-1}(s)) \) for \( s \in S_{T_d} \).

Assertion 4 shows that all statements in [1] saying that an element of \( S_{T_d} \) can be represented as a product of some generators \( x_{i,j} \), remain valid for elements of \( S_{T_{C,d}} \) if we replace \( x_{(i,j)} \) by \( z_{(i,j)} \) and lengths by \( T \)-lengths.

We define a subsemigroup \( S^S_{T_{C,d}} \) of \( S_{T_{C,d}} \) by putting

\[
S^S_{T_{C,d}} := \overline{\alpha}^{-1}(S^S_{T_d}).
\]

Theorem 2 follows from the following assertion.

Assertion 5. The restriction of \( \overline{\alpha}^{-1}: S_{T_d} \to S_{T_{C,d}} \) to \( S^S_{T_d} \),

\[
\overline{\alpha}^{-1}: S^S_{T_d} \to S^S_{T_{C,d}},
\]

is an isomorphism of semigroups over \( S_d \).

Proof. The homomorphism \( \overline{\alpha}^{-1}: S^S_{T_d} \to S^S_{T_{C,d}} \) is injective by Theorem 2.1 in [1].

We also mention the following immediate corollary of Theorem 2.1 in [1] and Assertion 5.

Corollary 2. Every element \( s \) of the semigroup \( S^S_{T_{C,d}} \) is uniquely determined by \( \alpha(s) \) and \( \ln_T(s) \).

§ 2. Proof of Theorem 1

Consider an element \( \overline{z}_{(1,2)} = x_{\sigma_1} \cdots x_{\sigma_m} \), where \( \sigma_1, \ldots, \sigma_m \in C \) are the factors in the factorization (3).

If \( f_C \geq 2 \), we can and will assume that all the permutations \( \sigma_i \) appearing in (3) belong to the subgroup \( S_3 \sim S_{d-2} \) of those elements of \( S_d \) that leave \( 3, 4 \in I_d \) fixed. Then the element \( \overline{z}_{(1,2)} = x_{\sigma_1} \cdots x_{\sigma_m} \) satisfies all the hypotheses of Theorem 2. Hence the elements \( z_{(i,j)} \) constructed in § 1 with the help of \( \overline{z}_{(1,2)} = x_{\sigma_1} \cdots x_{\sigma_m} \) uniquely determine a subsemigroup \( S^S_{T_{C,d}} \) of \( S_C \) isomorphic to \( S^S_{T_d} \) over \( S_d \).

Note that the length of the element \( z_{(1,2)} \) constructed in the proof of Assertion 1 is equal to \( \ln(z_{(1,2)}) = 3^d - 4(2d - 1)m_C \) if we start the construction with \( \overline{z}_{(1,2)} = x_{\sigma_1} \cdots x_{\sigma_m} \).

We put

\[
h_C = (z_{(1,2)} \cdot z_{(2,3)} \cdots z_{(d-1,d)})^3.
\]
Then $h_C$ belongs to $S_{TC,d}^{S_{d,T}}$. We rewrite $h_C$ as a product:

$$h_C = x_{\sigma_1} \cdots x_{\sigma_L}, \quad \sigma_i \in C, \quad i = 1, \ldots, L.$$ 

The length of $h_C$ is easily found to be

$$\ln(h_C) = 3d^{-3}(2d-1)(d-1)m_C := L.$$ 

The following assertion will be used in the proof of Theorem 1.

**Assertion 6.** Under the hypotheses of Theorem 1 suppose that $s = \tilde{s} \cdot \overline{s} \in \Sigma_d^{S_{d,T}}$, where $\overline{s} \in S_C$ has length

$$\ln(\overline{s}) := M \geq 3d^{-3}(2d-1)(d-1)m_C + n_Ck_C.$$ 

Then $s$ can be represented as a product: $s = s' \cdot h_C$.

**Proof.** Write

$$\overline{s} = x_{\sigma_1} \cdots x_{\sigma_M}, \quad \sigma_i \in C. \quad (5)$$

Since $M = \ln(\overline{s}) \geq 3d^{-3}(2d-1)(d-1)m_C + n_Ck_C$, there is a permutation $\sigma \in C$ such that at least $n_C + 1$ factors in (5) are equal to $x_\sigma$. Therefore $\overline{s}$ can be written as $\overline{s} = \tilde{s}' \cdot x_{\sigma_c}^{n_C}$, where $\tilde{s}' \in S_C$ is such that $\tilde{s} \cdot \tilde{s}' \in \Sigma_d^{S_{d,T}}$. By Lemma 1.1 in [1] we have

$$s = \tilde{s} \cdot \tilde{s}' \cdot x_{\sigma_c}^{n_C} = \tilde{s} \cdot \tilde{s}' \cdot x_{\sigma_c}^{n_C} = \tilde{s} \cdot \overline{s}_L \cdot x_{\sigma_L},$$

where $\overline{s}_L = \tilde{s}' \cdot x_{\sigma_c}^{n_C-1}$. Note that $\tilde{s} \cdot \overline{s}_L \in \Sigma_d^{S_{d,T}}$ and $\ln(\overline{s}_L) > n_Ck_C$. Therefore, by the same argument, $\tilde{s} \cdot \overline{s}_L$ can be written as $\tilde{s} \cdot \overline{s}_L = \tilde{s} \cdot \overline{s}_L' \cdot x_{\sigma_c}^{n_C-1} \cdot x_{\sigma_{L-1}}$. We put $\overline{s}_{L-1} = \overline{s}_L' \cdot x_{\sigma_c}^{n_C-1}$. Repeating the same arguments for $\tilde{s} \cdot \overline{s}_{L-1}$, we obtain that $\tilde{s} \cdot \overline{s}_{L-1} = \tilde{s} \cdot \overline{s}_{L-2} \cdot x_{\sigma_{L-1}}$, and so on. At the $L$th step we finally get

$$s = \tilde{s} \cdot \overline{s} = \tilde{s} \cdot \overline{s}_0 \cdot (x_{\sigma_1} \cdots x_{\sigma_L}) = \tilde{s} \cdot \overline{s}_0 \cdot h_C.$$

The assertion is proved.

To complete the proof of Theorem 1, we recall that the proof of Theorem 2.3 in [1] consists of two parts. In the first part it is proved that every element $s = \tilde{s} \cdot \overline{s} \in \Sigma_d^{S_{d,T}}$ with $\overline{s} \in S_{T_d}$ and $\ln(\overline{s}) \geq 3(d-1)$ admits another factorization $s = \tilde{s}_1 \cdot \overline{s}_1$ such that $\overline{s}_1 \in S_{T_d}^{S_{d,T}}$ and $\ln(\overline{s}_1) = 3(d-1)$. In this case, the element $\overline{s}_1$ is uniquely determined by its product $\alpha(\overline{s}_1) = \alpha(\overline{s}_1)^{-1} \alpha(s)$.

In the second part of the proof of Theorem 2.3 in [1] it was proved that every such element $s = \tilde{s}_1 \cdot \overline{s}_1$ may be rewritten as $s = \tilde{s}_2 \cdot \overline{s}_2$, where $\overline{s}_2 \in S_{T_d}^{S_{d,T}}$ is still of length $\ln(\overline{s}_2) = 3(d-1)$ and $\overline{s}_2$ is uniquely determined by the type $\tau(\overline{s}_1)$. Here we have only used properties of the semigroup $S_{T_d}$ and the relations (1) in the factorization semigroups. Therefore, by Assertions 5 and 6, the end of the proof of Theorem 1 coincides almost verbatim with the second part of the proof of Theorem 2.3 in [1]. We need only replace the elements $x_{(i,j)}$ by $y_{(i,j)}$, the lengths of elements by the $T$-lengths, the element $h_{d,g}$ by $\overline{\alpha}^{-1}(h_{d,g})$, the semigroup $S_{T_d}^{S_{d,T}}$ by $S_{TC_d}^{S_{d,T}}$ and the homomorphism $r$ by $\overline{r} = \overline{\alpha}^{-1} \circ r$.

However, at the request of the referee, we give this proof again. To do this, we introduce the notation $h_{C,d,g} = \overline{\alpha}^{-1}(h_{d,g})$ for the image of the Hurwitz element $h_{d,g} = x_{(1,2)}^2 \cdot x_{(1,2)}^2 \cdots x_{(d-1,d)}^2.$
Lemma 1. For every disjoint union \( \{i_1,1, \ldots , i_{k,1}\} \sqcup \cdots \sqcup \{i_{1,n}, \ldots , i_{k,n}\} \) of ordered subsets of \( I_d \), the Hurwitz element \( h_{C,d,0} \) can be represented as a product

\[
h_{C,d,0} = (z(i_{1,1},i_{2,1}) \cdots z(i_{k-1,1},i_{k,1})) \cdots (z(i_{1,n},i_{2,n}) \cdots z(i_{k-1,n},i_{k,n})) \cdot \overline{h},
\]

where \( \overline{h} \) is an element of \( S_{T_{C,d}}^{S_d} \).

Proof. This follows directly from Lemma 2.9 in [1] and Assertion 5.

By Assertion 6, the element \( s \) can be represented as a product \( s = \overline{s}' \cdot \overline{s} \), where \( \overline{s} \) is an element of \( S_{T_{C,d}}^{S_d} \) of \( T \)-length \( k \geq 3(d-1) \) (in our case \( \overline{s} = h_{C} \) and \( k = 3(d-1) \)) and \( \overline{s}' = x_{\sigma_1} \cdot \cdots \cdot x_{\sigma_m} \). By Proposition 2.4 in [1] and Assertion 5 we have \( \overline{s} = h_{C,d,0} \cdot \overline{s}' \).

To complete the proof of Theorem 1, we use induction on \( m \). If \( m = 0 \) (that is, \( s \in S_{T_{C,d}} \)), then Theorem 1 follows from Proposition 2.4 in [1] and Assertion 5.

Suppose that \( m = 1 \). For the canonical representative \( \sigma_{m,0} \) of type \( t(\sigma_m) \) (see [1] for a definition of the canonical representative) there is an element \( \overline{\sigma}_m \in S_d \) such that \( \sigma_{m,0} = \overline{\sigma}_m^{-1} \sigma'_m \sigma_m \). The permutation \( \overline{\sigma}_m \) can be factorized into a product of cyclic permutations, and each cyclic permutation can be factorized into a product of transpositions:

\[
\sigma_m = ((i_{1,1},i_{2,1}) \cdots (i_{k-1,1},i_{k,1})) \cdots ((i_{1,n},i_{2,n}) \cdots (i_{k-1,n},i_{k,n})).
\]

Consider the element

\[
\overline{\tau}(x_{\sigma_m}) = (z(i_{1,1},i_{2,1}) \cdots z(i_{k-1,1},i_{k,1})) \cdots (z(i_{1,n},i_{2,n}) \cdots z(i_{k-1,n},i_{k,n})) \in S_{T_{C,d}}.
\]

By Lemma 1 we have

\[
h_{C,d,0} = \overline{\tau}(x_{\sigma_m}) \cdot \overline{h}_m,
\]

where \( \overline{h}_m \) is an element of \( S_{T_{C,d}}^{S_d} \). Therefore

\[
s = x_{\sigma_m} \cdot h_{d,0} \cdot \overline{s}' = x_{\sigma_m} \cdot \overline{\tau}(x_{\sigma_m}) \cdot \overline{h}_m \cdot \overline{s}'
\]

\[
= \overline{\tau}(x_{\sigma_m}) \cdot x_{\sigma_{m,0}} \cdot \overline{h}_m \cdot \overline{s}' = x_{\sigma_{m,0}} \cdot \overline{\tau}(x_{\sigma_m}) \cdot \overline{h}_m \cdot \overline{s}',
\]

where \( x_{\sigma_m} = \lambda(\sigma_{m,0})(x_{\sigma_m}) \). We have \( \overline{s}'_1 = \overline{\tau}(x_{\sigma_m}) \cdot \overline{h}_m \cdot \overline{s}' \in S_{T_{C,d}}^{S_d} \) and \( \alpha(\overline{s}'_1) = \sigma_{m,0}^{-1} \alpha(s) \). Theorem 2.4 in [1] and Assertion 5 imply that \( \overline{s}'_1 = \overline{\tau}(x_{\sigma}) \cdot h_{C,d,g} \), where \( \sigma = \alpha(\overline{s}'_1) = \sigma_{m,0}^{-1} \alpha(s) \) and \( g = \frac{k-\text{Int}(x_{\sigma})}{2} - d + 1 \).

We now assume that Theorem 1 is true for all \( m < m_0 \) and consider an element

\[
s = x_{\sigma_1} \cdot \cdots \cdot x_{\sigma_{m_0}} \cdot \overline{s}_1,
\]

where the \( T \)-length of \( \overline{s}_1 \in S_{T_{C,d}}^{S_d} \) is equal to \( k \geq 3(d-1) \). We have

\[
s = x_{\sigma_1} \cdot \cdots \cdot x_{\sigma_{m_0}} \cdot \overline{s}_1 = x_{\sigma_2'} \cdot \cdots \cdot x_{\sigma_{m_0}'} \cdot x_{\sigma_1} \cdot \overline{s}_1
\]

\[
= x_{\sigma_2'} \cdot \cdots \cdot x_{\sigma_{m_0}'} \cdot x_{\sigma_{1,0}} \cdot \overline{s}'_1 = x_{\sigma_1,0} \cdot x_{\sigma_2'} \cdot \cdots \cdot x_{\sigma_{m_0}'} \cdot \overline{s}'_1,
\]
where \( \sigma_j' = \sigma_1 \sigma_j \sigma_1^{-1} \) and \( \sigma_j'' = \sigma_1^{-1} \sigma_j' \sigma_1 \) for \( j = 2, \ldots, m \), and the element \( \bar{s}_1' \in S_{T_{C,d}}^{S_d,T} \) satisfies \( \ln_T(\bar{s}_1') = k \). By the induction hypothesis we have

\[
s = x_{\sigma_1,0} \cdot (x_{\sigma_2', \ldots, \sigma_{m_0}', \bar{s}_1'}) = x_{\sigma_1,0} \cdot (x_{\sigma_2,0} \cdot \ldots \cdot x_{\sigma_{m_0},0} \cdot \bar{s}_1'),
\]

where \( \bar{s}_1'' \in S_{T_{C,d}}^{S_d,T} \) and \( \ln_T(\bar{s}_1'') = k \). By Proposition 2.4 in [1] and Assertion 5 we have

\[
\bar{s}_1'' = \mathcal{R}(x_\sigma) \cdot h_{C,d,g}, \quad \text{where} \quad \sigma = \alpha(\bar{s}_1'') = (\sigma_1,0 \ldots \sigma_m,0)^{-1} \alpha(s) \quad \text{and} \quad g = \frac{k - \ln_T(x_\sigma)}{2} - d + 1.
\]

**Bibliography**
