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On complete degenerations of surfaces with ordinary singularities in $\mathbb{P}^3$

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Abstract. We investigate the problem of the existence of degenerations of surfaces in $\mathbb{P}^3$ with ordinary singularities into plane arrangements in general position.

Bibliography: 14 titles.

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Introduction

In this article we investigate degenerations of algebraic surfaces in $\mathbb{P}^3$ with ordinary singularities. To begin, consider the classical prototype of this situation, namely, degenerations of plane algebraic curves. As is known, any smooth projective curve can be projected to $\mathbb{P}^2$ onto a nodal curve C — a curve with ordinary double points — nodes (singularities of type A_1). According to Severi's theorem ([1]–[3]) any nodal plane curve of degree m can be degenerated into an arrangement of m lines in general position $\mathscr{L} = \bigcup_{i=1}^{m} L_i \subset \mathbb{P}^2$. Such a degeneration defines a subset S_0 of the set S of double points of the curve $\mathscr{L}$, which consists of the limit double points of the curve C. Conversely, for any subset $S_0 \subset S$ there exists a degeneration of nodal curves (maybe, reducible) for which S_0 is the set of limit double points ([3]), in other words, there exists a smoothing of double points $S \setminus S_0$ of the curve $\mathscr{L}$ (preserving double points lying in S_0).

In dimension 2, surfaces in $\mathbb{P}^3$ with ordinary singularities are the analogue of nodal curves. Only such singularities appear under generic projections of a smooth surface $X \subset \mathbb{P}^r$ to $\mathbb{P}^3$ (see, for example, [4]). Let $Y \subset \mathbb{P}^3$ be a surface of degree m with ordinary singularities. This means that the singular set $\operatorname{Sing} Y$ is the double curve D; the curve D itself is smooth except the finite set of triple points T; the generic point $y \in D$ is nodal on Y (is locally defined by the equation $xy = 0$); the points $y \in T$ are triple points of Y (locally, $xyz = 0$); besides, Y has a finite set of pinch points at which Y is locally defined by the equation $x^2 = y^2 z$.

A maximally degenerate (reducible) surface of degree m with ordinary singularities is an arrangement

$$\mathscr{P} = P_1 \cup \cdots \cup P_m \subset \mathbb{P}^3$$

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of m planes in general position. In this case the double curve $\operatorname{Sing}\mathscr{P}=\mathscr{L}$ consists of $\binom{m}{2}$ lines $L_{i,j}=P_i\cap P_j$,

$$\mathscr{L}=\bigcup_{1\leqslant i<j\leqslant m} L_{i,j},$$

on which $\binom{m}{3}$ triple points lie,

$$\mathscr{T}=\bigcup T_{i,j,k},\quad T_{i,j,k}=P_i\cap P_j\cap P_k,\qquad 1\leqslant i<j<k\leqslant m.$$

A *degeneration of surfaces* (in general, *varieties*) of given type is a one-dimensional family, or for brevity, its zero fibre, the generic fibre of which is a surface of given type (usually smooth). We consider families, all fibres of which are surfaces in $\mathbb{P}^3$ with ordinary singularities. A degeneration is a surface which has 'more' singularities than the generic fibre. More precisely, a *degeneration* of a surface $Y\subset\mathbb{P}^3$ with ordinary singularities is a flat family of embedded surfaces $Y_u\subset\mathbb{P}^3$ with ordinary singularities, parametrized by points $u\in U$ of a smooth curve U (or even of a disc $U\subset\mathbb{C}$), such that

(i) for the generic point $u\in U$ the fibres Y_u have singularities of the same type as $Y=Y_{u_1}$, and the fibre Y_{u_0} over the point $u_0=0$ is called *degenerate*;

(ii) there is a flat family $D_u\subset Y_u$, where D_u is the double curve of the surface Y_u for $u\neq u_0$, and the curve $\mathscr{D}=D_{u_0}\subset\operatorname{Sing}Y_{u_0}$ is called the *limit double curve*;

(iii) there is a flat family T_u, $u\in U$, where $T_u\subset D_u$ is the set of triple points of the surface Y_u for $u\neq u_0$, and $T_{u_0}=\mathscr{T}_3$ is the set of triple points of the curve $\mathscr{D}$.

A degeneration is called *complete* if the degenerate fibre $Y_{u_0}=\mathscr{P}$ is a plane arrangement in general position.

A complete degeneration defines a limit double curve $\mathscr{D}\subset\mathscr{L}$. Conversely, if a line arrangement $\mathscr{D}\subset\mathscr{L}$ is selected and there is a complete degeneration of surfaces $Y_u\subset\mathbb{P}^3$ (not necessary irreducible) with ordinary singularities such that $D_{u_0}=\mathscr{D}$, then we say that the plane arrangement $\mathscr{P}$ *is smoothed outside* $\mathscr{D}$.

As in the case of curves, there are two questions. Can every surface $Y\subset\mathbb{P}^3$ with ordinary singularities be completely degenerated? Can every line arrangement $\mathscr{D}\subset\mathscr{L}$ of a plane arrangement $\mathscr{P}$ be smoothed outside $\mathscr{D}$?

The problem of degeneracy of smooth surfaces $X\subset\mathbb{P}^N$ into plane arrangements was investigated previously, but in another setting, which is analogous to the Zeuthen's problem for curves (the degenerate surfaces are surfaces of Zappa, or so called Zappatics; a survey of obtained results about Zappatic surfaces can be found in [5], see also [6]).

In spite of the fact that in many cases surfaces with ordinary singularities can be completely degenerated (see § 3 and § 6), we shall show in the article that in general the answers to the questions formulated above are negative. The reason why it is impossibile to completely degenerate a surface with ordinary singularities can be the fact that it is impossible to degenerate the double curve of the surface to a line arrangement $\mathscr{D}$ (see § 5), as well as the fact that the corresponding limit pair $(\mathscr{P},\mathscr{D})$ does not exist, although the double curve of the surface can be degenerated into a line arrangement (see § 8.5). In addition we show that there exist pairs $(\mathscr{P}_m, \mathscr{D})$ which cannot be smoothed outside of a fixed line arrangement $\mathscr{D}$. In some cases such pairs cannot be smoothed for any $m = \deg \mathscr{P}_m$ (see § 7.2), and in other cases such pairs can be smoothed only for sufficiently large m (see § 7.1).

§ 1. Expression of numerical characteristics of a surface in terms of degeneration

There are two classical ways to study smooth surfaces $X \subset \mathbb{P}^r$: to consider its generic projection to $\mathbb{P}^3$ or onto $\mathbb{P}^2$, respectively. We consider a slightly more general situation. We begin with a surface $Y \subset \mathbb{P}^3$ with ordinary singularities, and the normalization $\mathfrak{n}\colon X \to Y$ of Y and its composition with a projection of Y onto $\mathbb{P}^2$ define a generic covering $X \to \mathbb{P}^2$. The definition of a complete degeneration makes it possible to express numerical characteristics of the surface Y in terms of its degeneration. Invariants of a surface X can be expressed both in terms of numerical characteristics of the surface Y and in terms of numerical characteristics of a generic covering $X \to \mathbb{P}^2$. This gives an expression of numerical characteristics of the covering in terms of degeneration.

1.1. Numerical characteristics of a surface with ordinary singularities. Let $Y \subset \mathbb{P}^3$ be an irreducible surface with ordinary singularities, $D = D_1 \cup \dots \cup D_k$ the double curve, $g_i = g(D_i)$ its geometric genus, $d_i = \deg D_i$ the degree of an irreducible component D_i, T the set of triple points, Ω the set of pinch points. The main numerical characteristics of the embedding $Y \subset \mathbb{P}^3$ are:

$m = \deg Y$, the degree of the surface;
k, the number of irreducible components of the double curve;
$\overline{g} = \sum_{i=1}^{k} g_i$, the geometric genus of the double curve;
$\overline{d} = \sum_{i=1}^{k} d_i$, the degree of the double curve;
$t = \sharp(T)$, the number of triple points;
$\omega = \sharp(\Omega)$, the number of pinch points.

We call the collection of numerical data

$$\operatorname{type}(Y) = (m, \overline{d}, k, \overline{g}, t)$$

the *type of the surface* Y.

Let $\mathfrak{n}\colon X \to Y$ be a normalization of the surface Y. The surface X is smooth. Its invariants are expressed in terms of numerical characteristics of the surface Y by the following formulae (see [8]):

the intersection number of the canonical class is equal to

$$K_X^2 = m(m-4)^2 - (5m-24)\overline{d} + 4(\overline{g} - k) + 9t; \tag{1}$$

the topological Euler number $e(X) = c_2(X)$ is

$$e(X) = m^2(m-4) + 6m - (7m-24)\overline{d} + 8(\overline{g} - k) + 15t. \tag{2}$$

From these formulae and Noether's formula we obtain the Euler characteristic:

$$\chi(\mathscr{O}_X) = \frac{1}{6}m(m^2 - 6m + 11) - (m-4)\overline{d} + (\overline{g} - k) + 2t. \tag{3}$$

The number of pinches is equal to (see [8])

$$\omega = 2\overline{d}(m-4) - 6t - 4(\overline{g} - k). \tag{4}$$

As is known ([7]), the arithmetic genus of a reduced curve C is equal to

$$p_a(C) = g + \delta - r + 1, \tag{5}$$

where g is its geometric genus, δ is the sum of δ-invariants of the singularities and r is the number of irreducible components. For the double curve D, which has only triple points ($\delta = 3$), formula (5) gives

$$p_a(D) = \overline{g} + 3t - k + 1. \tag{6}$$

1.2. Numerical data for description of a curve $\mathscr{D} \subset \mathscr{L}$. Let $\mathscr{P} \subset \mathbb{P}^3$ be an arrangement of m planes in general position, $\mathscr{L} = \bigcup_{1\leqslant i<j\leqslant m} L_{i,j}$ its double curve. Let a curve $\mathscr{D} \subset \mathscr{L}$ be the union of $\overline{d}$ lines $L_{i,j}$ and $\mathscr{R}$ the union of the d remaining lines (the curves $\mathscr{D}$ and $\mathscr{R}$ are also called *line arrangements*),

$$\mathscr{L} = \mathscr{D} \cup \mathscr{R}, \qquad \deg \mathscr{D} = \overline{d}, \quad \deg \mathscr{R} = d, \quad d + \overline{d} = \binom{m}{2}. \tag{7}$$

With respect to the partition $\mathscr{L} = \mathscr{D} \cup \mathscr{R}$ the triple points of the curve $\mathscr{L}$ fall into four types:

$$\mathscr{T} = \mathscr{T}_3 \sqcup \mathscr{T}_2 \sqcup \mathscr{T}_1 \sqcup \mathscr{T}_0,$$

where $\mathscr{T}_3$ consists of those points which are triple on the curve $\mathscr{D}$; $\mathscr{T}_2$ consists of those points which are double on the curve $\mathscr{D}$; $\mathscr{T}_1$ consists of those points which are nonsingular on the curve $\mathscr{D}$ (and are double on the curve $\mathscr{R}$); $\mathscr{T}_0$ consists of those points which do not lie on $\mathscr{D}$ (and are triple on the curve $\mathscr{R}$). Denote by τ_3, τ_2, τ_1 and τ_0 the number of points in the corresponding sets. We have

$$\tau = \tau_3 + \tau_2 + \tau_1 + \tau_0 = \binom{m}{3}. \tag{8}$$

By the formula for the arithmetic genus (5) we have

$$p_a(\mathscr{D}) = \tau_2 + 3\tau_3 - \overline{d} + 1 \tag{9}$$

(the invariant $\delta = 1$ for double points and $\delta = 3$ for triple points).

The numbers τ_i are related as follows. On each line $L_{i,j} = P_i \cap P_j$ there are $m-2$ triple points — the intersection points with planes P_k, $k \neq i, j$. Let us sum up the numbers of triple points of $\mathscr{L}$ lying on $\overline{d}$ lines $L_{i,j} \subset \mathscr{D}$. On the one hand, we get $(m-2)\overline{d}$. On the other hand, we get $3\tau_3 + 2\tau_2 + \tau_1$ because the triple points of $\mathscr{D}$ are counted 3 times, the double points are counted twice and non-singular ones 1 time. We get

$$\tau_1 + 2\tau_2 + 3\tau_3 = (m-2)\overline{d}. \tag{10}$$

An analogous calculation for the curve $\mathscr{R}$ gives

$$\tau_2 + 2\tau_1 + 3\tau_0 = (m-2)d. \tag{11}$$

The data collection type$(\mathscr{P},\mathscr{D}) = (m,\overline{d},k,\tau_2,\tau_3)$ is called the *type of the pair* $(\mathscr{P},\mathscr{D})$. Here k is the number of connected components of the curve $\mathscr{D}\setminus\mathscr{T}_3$, where $\mathscr{T}_3$ is the set of triple points of $\mathscr{D}$.

Note that the type$(\mathscr{P},\mathscr{D})$ does not define the pair $(\mathscr{P},\mathscr{D})$ uniquely up to a homeomorphism: for some types there exist nonhomeomorphic pairs of a given type.

1.3. The graph of a line arrangement. As is known, one can associate with an algebraic variety, and in particular, with a divisor with normal crossings, a polyhedron by the rule: the vertices correspond to the irreducible components; two vertices are connected by an edge if the components have a nonempty intersection; three vertices span a triangle if the corresponding components have a nonempty intersection, and so on. In particular, with an arrangement $\mathscr{P}$ of m planes in general position in $\mathbb{P}^3$ we associate a polyhedron, which is the two-dimensional skeleton of an $(m-1)$-dimensional simplex in $\mathbb{R}^{m-1}$ (or a standard $(m-1)$-simplex in $\mathbb{R}^m$). Denote by $\Gamma(\mathscr{L})$ the graph which is the one-dimensional skeleton of this polyhedron.

We associate with any line arrangement $\mathscr{D}\subset\mathscr{L}$ a graph $\Gamma(\mathscr{D})$, the *graph of the curve* $\mathscr{D}$, which is a subgraph of $\Gamma(\mathscr{L})$, consisting of the union of edges corresponding to the lines $L_{i,j}\subset\mathscr{D}$. If $\mathscr{D}\subset\mathscr{L}$ is a line arrangement, then we denote by $\mathscr{R}$ the complementary line arrangement in $\mathscr{L}$.

We say that a graph Γ is *realizable* if it satisfies the following conditions: it does not contain isolated vertices, it does not contain simple loops (that is, edges with the same source and end-point), and any two vertices are connected by at most one edge. It is obvious that a graph Γ is the graph of a set of double lines $\mathscr{D}$ of a plane arrangement $\mathscr{P}$ in general position if and only if Γ is realizable and the number of its vertices is at most $\deg\mathscr{P}$.

The graph $\Gamma(\mathscr{D})$ codes numerical characteristics of $\mathscr{D}$. The number of vertices of $\Gamma(\mathscr{D})$ is equal to the number $\overline{d}$ of lines composing the curve $\mathscr{D}$. The number τ_3 of triple points of $\mathscr{D}$, obviously, is equal to the number of triangles of $\Gamma(\mathscr{D})$ (by definition, a triangle in a graph is three vertices and three edges connecting these vertices).

We denote by $v(P)$ the valence of a vertex P of the graph $\Gamma(\mathscr{D})$, that is, the number of edges outgoing from P.

Lemma 1. *The number of double points of the curve* $\mathscr{D}$ *is equal to*

$$\tau_2 = \sum_{P\in\Gamma(\mathscr{D})} \frac{1}{2}v(P)(v(P)-1) - 3\tau_3. \tag{12}$$

Proof. A vertex P of valence $v(P)$ corresponds to a plane on which $v(P)$ lines lie. These lines intersect at $\frac{1}{2}v(P)(v(P)-1)$ double points. The sum of the numbers of these double points is the number of double points of $\mathscr{D}$ if $\mathscr{D}$ has no triple points. If $\mathscr{D}$ has triple points, then each triple point $T_{i,j,k} = P_i\cap P_j\cap P_k$, being a double point on each of three planes P_i, P_j, P_k, contributes 3 to the first term of formula (12). To calculate τ_2 we have to remove this contribution, that is, to subtract $3\tau_3$. The proof is complete.

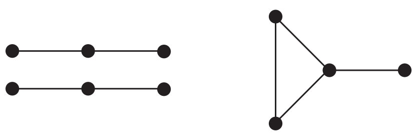
The graph $\bar{\Gamma}(\mathscr{R})$ consisting of all vertices of the graph $\Gamma(\mathscr{L})$ and all edges of the graph $\Gamma(\mathscr{R})$ is called the *augmentation of* $\Gamma(\mathscr{R})$ (*with respect to* $\mathscr{L}$). A graph $\Gamma(\mathscr{R})$ is called *augmented* if $\Gamma(\mathscr{R}) = \bar{\Gamma}(\mathscr{R})$.

Let $\mathscr{D} \subset \mathscr{L}$ be the limit double curve of a complete degeneration of a surface Y with ordinary singularities in $\mathbb{P}^3$.

Lemma 2. *The number of irreducible components of a surface Y with ordinary singularities in $\mathbb{P}^3$ is equal to the number of connected components of the graph $\overline{\Gamma}(\mathscr{R})$. In particular, a surface Y is irreducible if the graph $\overline{\Gamma}(\mathscr{R})$ is connected.*

Proof. Let $\mathbb{P}^2$ be a plane in generic position with respect to the degenerate fibre $Y_{u_0} = \mathscr{P}$ (and with respect to the fibres sufficiently close to it) of a complete degeneration Y_u of surfaces with ordinary singularities. Consider the restriction of the family Y_u to the plane $\mathbb{P}^2 \subset \mathbb{P}^3$. We obtain a degeneration of a family of nodal curves $C_u = Y_u \cap \mathbb{P}^2$ to an arrangement of m lines $L = C_{u_0}$. The arrangement L consists of m lines $L_i = P_i \cap \mathbb{P}^2$ intersecting at $\binom{m}{2}$ points $\overline{L}_{i,j} = L_{i,j} \cap \mathbb{P}^2$. The line arrangement L defines a graph $\Gamma(L)$, the dual graph of L. Obviously, $\Gamma(L) = \Gamma(\mathscr{L})$. The family of double points $D_u \cap \mathbb{P}^2$ of curves C_u defines a set of limit double points $\widetilde{\mathscr{D}} = \mathscr{D} \cap \mathbb{P}^2$. Let $\widetilde{\mathscr{R}}$ be the set of double points of the arrangement L (the set of smoothed double points of L) complementary to $\widetilde{\mathscr{D}}$. We associate a graph $\Gamma(\widetilde{\mathscr{R}})$ with $\widetilde{\mathscr{R}}$: the vertices of the graph correspond to the lines L_i and two vertices are connected by an edge if the corresponding lines L_i intersect at a point of $\widetilde{\mathscr{R}}$. Obviously, the graphs $\overline{\Gamma}(\mathscr{R})$ and $\Gamma(\widetilde{\mathscr{R}})$ are isomorphic. Furthermore, it follows from Bertini's theorem that the number of irreducible components of the surface Y_u coincides with the number of irreducible components of the curve C_u. On the other hand, it is well known that the number of irreducible components of C_u coincides with the number of connected components of the graph $\Gamma(\widetilde{\mathscr{R}})$. The proof is complete.

In connection with this lemma, a line arrangement $\mathscr{D}$, or more precisely, a pair $(\mathscr{L}, \mathscr{D})$ is called *irreducible* if the graph $\bar{\Gamma}(\mathscr{R})$ is connected.



$\Gamma^{(2,0)^2(4,1)}_{(2,1)^2(1,2,1)}$

Figure 1

In the case of a connected graph we use the symbol $\Gamma^{\overline{d},\tau_3}_{v_1,v_2,\dots}$ for the type of the graph $\Gamma(\mathscr{D})$ (or for the line arrangement $\mathscr{D}$ itself). Here $v_1, v_2, \dots$ are the numbers of vertices of valence 1, 2, and so on, $\overline{d}$ is the degree of $\mathscr{D}$, that is, the number of edges of the graph, τ_3 is the number of triple points of $\mathscr{D}$, that is, the number of triangles in $\Gamma(\mathscr{D})$. If the graph $\Gamma(\mathscr{D})$ is not connected, then we use an analogous notation for connected components and separate data for each component by parentheses. If several components have the same data, we use a multiplicative way of writing. For example, if $\mathscr{D}$ consists of three connected components, two of which are chains of two lines, and the third consists of four lines, three of which intersect at a triple point and the fourth line intersects two of the three lines, then the graph $\Gamma(\mathscr{D})$ has type $\Gamma^{(2,0)^2(4,1)}_{(2,1)^2(1,2,1)}$ (see Fig. 1).

1.4. Numerical characteristics of the limit double curve $\mathscr{D} \subset \mathscr{L}$. The connection between numerical characteristics of double curves $\mathscr{D}$ and D of a complete degeneration Y_u, $u \in U$, is given by the following result.

Proposition 1. *The following equalities hold for the limit double curve $\mathscr{D} \subset \mathscr{L}$ of a complete degeneration Y_u, $u \in U$, of surfaces with ordinary singularities*:

$$\overline{d} = \deg D; \tag{13}$$

$$\tau_3 = t; \tag{14}$$

$$\tau_2 = \overline{d} + \overline{g} - k; \tag{15}$$

$$\omega = 2\tau_1; \tag{16}$$

$$k = \sharp C(\mathscr{D} \setminus \mathscr{T}_3), \tag{17}$$

where $\sharp C(\mathscr{D} \setminus \mathscr{T}_3)$ is the number of connected components of the curve $\mathscr{D} \setminus \mathscr{T}_3$.

Thus, if a surface $Y = Y_{u_1}$ is irreducible, then the types type(Y) *of the surface and* type$(\mathscr{P}, \mathscr{D})$ *of the limit pair are uniquely recovered from each other.*

Proof. Formula (13) is valid because the degree $\deg D_u$ is constant for a flat family D_u. Analogously, formula (14) follows from the flatness of the family T_u. Formula (15) follows from the constancy of the arithmetic genus $p_a(D_u)$ in a flat family D_u and from formulae (6) and (9). To prove (16) we substitute the expression for $(\overline{g} - k)$ from (15) in formula (4). We have

$$\omega = 2\overline{d}(m-4) - 6\tau_3 + 4(\overline{g} - \tau_2) = 2\overline{d}(m-2) - 4\tau_2 - 6\tau_3,$$

and from (10) we obtain $\omega = 2\tau_1$.

Formula (17) follows from the flatness of the family of curves $D_u \setminus T_u$ and from the constancy of the number of connected components in such a family.

The number k of irreducible components of the double curve $D \subset Y$ (formula (17)) can be expressed in terms of the graph $\Gamma(\mathscr{D})$ of the limit curve $\mathscr{D}$ as follows. By definition, a *path* in $\Gamma(\mathscr{D})$ is a sequence of edges for which the initial vertex of the next edge is the end-point of the previous edge. A path is called *prohibited* if two of its successive edges are sides of a triangle in $\Gamma(\mathscr{D})$. We say that two edges of the graph $\Gamma(\mathscr{D})$ are *equivalent* if they are edges of some nonprohibited path, and extend this relation to an equivalence relation on the edge set of the graph $\Gamma(\mathscr{D})$. Immediately one obtains the following lemma.

Lemma 3. *The number k of connected components of the curve D is equal to the number of equivalence classes of edges of the graph $\Gamma(\mathscr{D})$.*

The formulae (14) and (15) permit us to express the invariants of the surface X in terms of numerical characteristics of the pair $(\mathscr{P}, \mathscr{D})$:

$$K_X^2 = m(m-4)^2 - (5m-20)\overline{d} + 4\tau_2 + 9\tau_3, \tag{18}$$

$$e(X) = m^2(m-4) + 6m - (7m-16)\overline{d} + 8\tau_2 + 15\tau_3, \tag{19}$$

or taking (10) into account,

$$K_X^2 = m(m-4)^2 + 10\overline{d} - 5\tau_1 - 6\tau_2 - 6\tau_3, \tag{20}$$

$$e(X) = m^2(m-4) + 6m + 2\overline{d} - 7\tau_1 - 6\tau_2 - 6\tau_3. \tag{21}$$

It follows from Noether's formula and (8) that

$$\chi(\mathscr{O}_X) = \overline{d} + \tau_0 - \frac{m(m-3)}{2}. \tag{22}$$

§ 2. Generic projections onto the plane

2.1. The discriminant of a projection. Let $Y \subset \mathbb{P}^3$ be a surface (not necessarily irreducible) of degree m with ordinary singularities and let $\mathrm{pr}\colon \mathbb{P}^3 \to \mathbb{P}^2$ be a linear projection with centre at a point $o \notin Y$. Choose coordinates in $\mathbb{P}^3$ such that $o = (0:0:0:1)$. Then the projection pr is given by

$$\mathrm{pr}\colon (x_1 : x_2 : x_3 : x_4) \mapsto (x_1 : x_2 : x_3) \in \mathbb{P}^2,$$

and Y has an equation

$$h(x_4) = x_4^m + \sum_{j=0}^{m-1} a_j(x_1, x_2, x_3) x_4^j = 0.$$

The discriminant $\Delta(x_1, x_2, x_3)$ of the polynomial $h(x_4)$ as a polynomial in x_4 is a homogeneous polynomial in the variables x_1, x_2, x_3 of degree $m(m-1)$, and defines in $\mathbb{P}^2$ the *discriminant divisor* Δ of $p = \mathrm{pr}_{|Y}\colon Y \to \mathbb{P}^2$ given by the equation $\Delta(x_1, x_2, x_3) = 0$.

Proposition 2. *Let a projection* $\mathrm{pr}\colon \mathbb{P}^3 \to \mathbb{P}^2$ *be such that*

(i) *the composition* $f = p \circ \mathfrak{n}\colon X \to \mathbb{P}^2$ *is unramified over generic points of components of* $\overline{D} = \mathrm{pr}(D)$, *where* $\mathfrak{n}\colon X \to Y$ *is a normalization*;

(ii) *the ramification index of* f *at generic points of its ramification curve* $R \subset X$ *is equal to* 2;

(iii) *the restriction of* f *to the curve* $R \subset X$ *is of degree* 1.

Then the polynomial $\Delta(x_1, x_2, x_3)$ *factors as*

$$\Delta(x_1, x_2, x_3) = \beta(x_1, x_2, x_3) \cdot \rho(x_1, x_2, x_3)^2,$$

where the polynomials $\beta(x_1, x_2, x_3)$ *and* $\rho(x_1, x_2, x_3)$ *have no multiple factors, the equation of the branch curve* $f(R) = B \subset \mathbb{P}^2$ *of* f *is* $\beta(x_1, x_2, x_3) = 0$, *and* $\rho(x_1, x_2, x_3) = 0$ *is the equation of the curve* $\overline{D}$.

Proof. The surface Y is covered by three charts $U_i{:}x_i \neq 0$, $i = 1, 2, 3$, $U_i = \mathbb{C}^3$. In each of these charts the projection pr is of the form $(x, y, z) \mapsto (x, y)$. For example, in the chart U_3: $x_3 \neq 0$, one takes $x_1/x_3 = x$, $x_2/x_3 = y$, $x_4/x_3 = z$. In this chart the surface Y has the equation

$$h(z) = z^m + \sum_{j=0}^{m-1} \overline{a}_j(x, y) z^j = 0,$$

and the equation of the discriminant divisor is $\Delta(x, y, 1) = 0$.

Let $q \in \mathbb{P}^2$ and assume, for instance, that q lies in the chart $\mathbb{C}^2 = \{x_3 \neq 0\}$. Except for a finite set of points (the singular points of B and the images of pinches and triple points of Y), there are an analytic neighbourhood $V \subset \mathbb{C}^2$ of this point and analytic coordinates u, v in it such that the polynomial $h(z)$ is a product of m factors of the form $z - \alpha_j(u, v)$ if $q \notin B$, or of $m - 2$ factors of the same form and one factor of the form $(z - g(u, v))^2 - u$ if $q \in B$, where $\alpha_j(u, v)$, $g(u, v)$ are some analytic functions and $u = 0$ is a local equation of the curve B. Furthermore, as is known, the discriminant of a polynomial $h(z) = \prod_{j=1}^{m}(z - \alpha_j)$ is $\prod_{i<j}(\alpha_i - \alpha_j)^2$. Consequently, the discriminant $\Delta(x, y, 1)$ vanishes only at the points of B and $\overline{D} = \mathrm{pr}(D)$. Moreover, as Y has only ordinary singularities, it follows from the conditions on the projection pr that the equations of the irreducible components of the curve $\overline{D}$ occur in $\Delta(x, y, 1)$ with multiplicity 2, and in the equations of the irreducible components of B they occur with multiplicity 1. The proof is complete.

In particular, if $Y = \mathscr{P}$ is an arrangement of planes in general position, then the discriminant divisor of the restriction of the projection pr to $\mathscr{P}$ equals

$$\Delta = 2\,\mathrm{pr}(\mathscr{L}) = 2\overline{\mathscr{L}}.$$

Moreover, if the projection pr satisfies the conditions of Proposition 2 for each surface Y_u, $u \in U$, of a complete degeneration of surfaces with ordinary singularities, then we obtain three flat families of curves: $\overline{D}_u = \mathrm{pr}(D_u)$, B_u and $\Delta_u = B_u + 2\overline{D}_u$, where $\overline{D}_{u_0} = \mathrm{pr}(\mathscr{D}) = \overline{\mathscr{D}}$ and $B_{u_0} = 2\,\mathrm{pr}(\mathscr{R}) = 2\overline{\mathscr{R}}$.

2.2. Generic coverings of the plane and generic projections of surfaces with ordinary singularities. Let X be a smooth algebraic surface, not necessarily irreducible. Recall ([9]) that a finite covering $f\colon X \to \mathbb{P}^2$ is called *generic* if it is like a generic projection of a projective surface onto the plane, that is, satisfies the following properties

(i) the branch curve $B \subset \mathbb{P}^2$ is cuspidal, that is, has ordinary cusps and nodes as the only singularities;

(ii) $f^*(B) = 2R + C$, where the ramification curve R is smooth, and the curve C is reduced;

(iii) $f_{|R}\colon R \to B$ $f_{|R} : R \to B$ is the normalization of B.

Proposition 3. *Let X be a smooth irreducible projective surface. Then the branch curve $B \subset \mathbb{P}^2$ of a generic covering $f\colon X \to \mathbb{P}^2$ of degree $m \geqslant 2$ is irreducible.*

Proof. The statement is obvious if $\deg f = 2$.

Let $\deg f \geqslant 3$. A generic covering $f\colon X \to \mathbb{P}^2$ branched along a curve B defines (and is defined by) an epimorphism $\mu\colon \pi_1(\mathbb{P}^2 \setminus B) \to \mathscr{S}_m$ to the symmetric group $\mathscr{S}_m$ such that the image $\mu(\gamma)$ of each geometric generator $\gamma \in \pi_1(\mathbb{P}^2 \setminus B)$ (that is, of each simple circuit around the curve B) is a transposition in $\mathscr{S}_m$ (see, for example, [9]). If B splits into the union of two curves B_1 and B_2, then B_1 and B_2 meet each other transversally at nonsingular points since B has only nodes and ordinary cusps as the only singularities. Therefore (see, for example, [10]) the elements of the group Γ_1 generated by simple circuits around B_1 commute with the elements of the group Γ_2 generated by simple circuits around B_2. On the other hand, it is easy to see that the elements of two nontrivial subgroups $\mu(\Gamma_1) \subset \mathscr{S}_m$ and $\mu(\Gamma_2) \subset \mathscr{S}_m$ generated by transpositions cannot generate $\mathscr{S}_m$ if the elements of the group $\mu(\Gamma_1)$ commute with the elements of the group $\mu(\Gamma_2)$. The proof is complete.

As a corollary of Proposition 3 we obtain that the number of irreducible components of the branch curve B of a generic covering $f\colon X \to \mathbb{P}^2$ is equal to the number of irreducible components X_i of X such that $\deg f_{|X_i} \geqslant 2$.

The main numerical characteristics of a generic covering f of the projective plane by an irreducible surface are:

$m = \deg f$, the degree of the covering;

$g = g(B)$, the geometric genus of the branch curve;

$2d = \deg B$, the degree of the branch curve;

n, the number of nodes of B;

c, the number of cusps.

The invariants of a surface X are expressed in terms of numerical characteristics of a generic covering by the following formulae (see [9]):

$$K_X^2 = 9m - 9d + g - 1, \tag{23}$$

$$e(X) = 3m + 2(g-1) - c. \tag{24}$$

Now let $Y \subset \mathbb{P}^3$ be a surface with ordinary singularities and $\mathfrak{n}\colon X \to Y$ its normalization. A projection $\mathrm{pr}\colon \mathbb{P}^3 \to \mathbb{P}^2$ is called *generic with respect to* Y if it is generic for the double curve $D \subset Y$ (and, in particular, $\mathrm{pr}(D)$ has only t triple points and some nodes as singularities) and the composition $f = p \circ \mathfrak{n}\colon X \to \mathbb{P}^2$ is a generic covering of the plane, where $p = \mathrm{pr}_{|Y}$.

Proposition 4. *Let* Y_u, $u \in U$, $\dim U = 1$, *be a complete degeneration of surfaces of degree* m *with ordinary singularities. Then for almost all generic projections* $\mathrm{pr}\colon \mathbb{P}^3 \to \mathbb{P}^2$ *of the plane arrangement* $\mathscr{P} = Y_{u_0}$ *the projections* pr *are generic with respect to* Y_u *for all* $u \in U$, *except perhaps for a finite number of values* u.

Proof. According to the definition of a degeneration the surfaces Y_u, $u \in U$, are the fibres of a restriction to $\mathscr{Y} \subset \mathbb{P}^3 \times U$ of the projection onto the second factor.

By the definition of a generic projection applied to a plane arrangement it follows that there exist a finite covering $\bigcup U_i = \mathbb{P}^2$ by small open balls U_i with centres at points p_i and an open neighbourhood V of $\mathscr{P}$ such that the number of connected components of the intersection $\mathrm{pr}^{-1}(U_i) \cap V$ equals respectively: m if $p_i \notin \overline{\mathscr{L}} = \mathrm{pr}(\mathscr{L})$, $m-1$ if p_i is a nonsingular point of $\overline{\mathscr{L}}$, and $m-2$ if p_i is a singular point of $\overline{\mathscr{L}}$. For u sufficiently close to u_0 the surface $Y_u \subset V$ and hence for such u the restriction of pr to Y_u satisfies the conditions of Proposition 2. It follows from the flatness of the family of branch curves B_u that for almost all $u \in U$ the curves B_u are reduced. According to [11] the restriction of a generic projection to Y_{u_1} (for fixed u_1) is a generic covering of the plane and, in particular, the curve B_{u_1} is cuspidal. Therefore, if the projection is generic simultaneously for $\mathscr{P}$ and for Y_{u_1}, then it follows from the flatness of the family B_u that for almost all u the curves B_u have singularities not worse than the singularities of B_{u_1}, that is, for almost all $u \in U$ the curves B_u are also cuspidal and the restriction of the projection pr to Y_u is a generic covering of the plane. The proof is complete.

2.3. Numerical data for the description of a projection of a curve $\mathscr{D} \subset \mathscr{L}$ onto the plane. Let $\mathrm{pr}\colon \mathbb{P}^3 \to \mathbb{P}^2$ be a generic projection of an arrangement of m planes $\mathscr{P} \subset \mathbb{P}^3$. Then the curve

$$\overline{\mathscr{L}} = \mathrm{pr}(\mathscr{L}) \subset \mathbb{P}^2$$

has τ triple points and

$$\nu = \frac{1}{2}\binom{m}{2}\binom{m-2}{2} = \frac{m(m-1)(m-2)(m-3)}{8} \tag{25}$$

double points — points of intersection of lines $\mathrm{pr}(L_{i,j})$ and $\mathrm{pr}(L_{k,l})$, which are projections of skew lines $L_{i,j}$ and $L_{k,l}$ of the line arrangement $\mathscr{L} \subset \mathbb{P}^3$.

Let $\mathscr{L} = \mathscr{D} \cup \mathscr{R}$ be a partition of $\binom{m}{2}$ lines in $\mathscr{L}$ into two parts, containing $\overline{d}$ and d lines, respectively. We denote $\overline{\mathscr{D}} = \mathrm{pr}(\mathscr{D})$ and $\overline{\mathscr{R}} = \mathrm{pr}(\mathscr{R})$. Then the ν double points of the curve $\overline{\mathscr{L}}$ fall into three sets: ν_2 points which are double on $\overline{\mathscr{D}}$; ν_1 points which are simple on $\overline{\mathscr{D}}$ (they are intersection points of $\overline{\mathscr{D}}$ and $\overline{\mathscr{R}}$); ν_0 points which do not lie on $\overline{\mathscr{D}}$ (they are double points of $\overline{\mathscr{R}}$),

$$\nu = \nu_2 + \nu_1 + \nu_0. \tag{26}$$

The curve $\overline{\mathscr{D}}$, which is an arrangement of $\bar{d}$ lines, has τ_3 triple points and $\tau_2 + \nu_2$ double points, τ_2 of which were on $\mathscr{D}$, and ν_2 appeared under projection of skew lines of the curve $\mathscr{D}$. Applying formula (5) to the curve $\overline{\mathscr{D}}$ we get

$$\frac{1}{2}(\overline{d}-1)(\overline{d}-2) = (\tau_2 + \nu_2) + 3\tau_3 - \overline{d} + 1,$$

or $\frac{1}{2}\overline{d}(\overline{d}-1) = \nu_2 + \tau_2 + 3\tau_3$; hence

$$\nu_2 = \frac{1}{2}\overline{d}(\overline{d}-1) - \tau_2 - 3\tau_3. \tag{27}$$

The analogous formula for the curve $\overline{\mathscr{R}}$ of degree d with τ_0 triple points and $\tau_1 + \nu_0$ double points gives

$$\nu_0 = \frac{1}{2}d(d-1) - \tau_1 - 3\tau_0. \tag{28}$$

2.4. Expression of numerical characteristics of a covering in terms of degeneracy. We expressed the invariants K_X^2 and $e(X)$ of an irreducible surface X in terms of numerical characteristics of a degeneracy of the surface Y. On the other hand, K_X^2 and $e(X)$ can be expressed in terms of numerical characteristics of a generic covering (see (23) and (24)). This gives an expression of the numerical characteristics of the covering in terms of the numerical characteristics of the degeneration.

Proposition 5. *If Y_u, $u \in U$, is a complete degeneration of an irreducible surface Y, then the numerical characteristics of a generic projection $p\colon Y \to \mathbb{P}^2$ are expressed in terms of the numerical characteristics of the degeneration by the formulae*:

$$\deg B = 2d, \tag{29}$$
$$g - 1 = 6\tau_0 + \tau_1 - d, \tag{30}$$
$$c = 6\tau_0 + 3\tau_1, \tag{31}$$
$$n = 4\nu_0. \tag{32}$$

Proof. First we prove (29), that is, that d in the notation of degree $\deg B$ in § 2.2 equals to $d = \deg \mathscr{R}$ in formula (7). Let L be a generic line in $\mathbb{P}^2$, and $\overline{L} = p^{-1}(L)$ the corresponding hyperplane section of Y. Then $\overline{L}$ is an irreducible plane curve of degree m with $\overline{d} = \deg D$ nodes. Hence the geometric genus $g(\overline{L}) = \frac{(m-1)(m-2)}{2} - \overline{d}$. On the other hand, $p\colon \overline{L} \to L$ is a covering of degree m ramified at $\deg B = (B, L)$ points. Therefore, by Hurwitz's formula, we have: $2g(\overline{L}) - 2 = -2m + \deg B$. It follows that

$$\deg B = (m-1)(m-2) - 2\overline{d} - 2 + 2m = m(m-1) - 2\overline{d} = 2d.$$

Let us prove formula (30). From (23) we have $g - 1 = K_X^2 - 9m + 9d$ and by virtue of (18)

$$g - 1 = m(m-4)^2 + 10\overline{d} - 5\tau_1 - 6\tau_2 - 6\tau_3 - 9m + 9d.$$

Substituting $\overline{d} = (m-1)(m-2)/2 - d$ from (7) and $\tau_2 + \tau_3 = \tau - \tau_1 - \tau_0$ from (8) we obtain formula (30):

$$\begin{aligned} g - 1 &= m(m-4)^2 + 5m(m-1) - 10d - 5\tau_1 - m(m-1)(m-2) \\ &\quad + 6\tau_1 + 6\tau_0 - 9m + 9d = 6\tau_0 + \tau_1 - d. \end{aligned}$$

To calculate c we use formula (24): $c = -e(X) + 3m + 2(g-1)$. Substituting the expression for $e(X)$ from (21) and $g - 1$ from (30) we obtain

$$\begin{aligned} c &= -m^2(m-4) - 6m - 2\overline{d} + 7\tau_1 + 6\tau_2 + 6\tau_3 + 3m + 12\tau_0 + 2\tau_1 - 2d \\ &= -m^2(m-4) - 3m - 2(\overline{d} + d) + 6(\tau_0 + \tau_1 + \tau_2 + \tau_3) + 6\tau_0 + 3\tau_1. \end{aligned}$$

Applying formulae (7) and (8) we obtain (31).

To calculate n we use the formula for the genus of a plane curve:

$$\frac{\deg B(\deg B - 3)}{2} = g - 1 + c + n.$$

From here and from formulae (29), (30) and (31) we obtain

$$d(2d-3) = 6\tau_0 + \tau_1 - d + 6\tau_0 + 3\tau_1 + n = 12\tau_0 + 4\tau_1 + n - d,$$

that is, $n - (d-1) - 12\tau_0 - 4\tau_1 = 4\nu_0$ by virtue of formula (28). The proof is complete.

2.5. Degenerations of cubic surfaces. The geometric meaning of the formulae for c and n. Let $p\colon Y \to \mathbb{P}^2$ be a generic projection of a surface $Y \subset \mathbb{P}^3$, $\deg Y = 3$.

It is known [8] that irreducible surfaces with ordinary singularities of degree $m = 3$ are either smooth cubics or cubics the double curve of which is a line.

If Y is a smooth cubic, then, as is known, the branch curve $B \subset \mathbb{P}^2$ is a curve of degree 6 with six cusps (which in addition lie on a conic).

If the double curve D is a line ($\overline{d} = 1$), then the surface Y has two pinches, $\omega = 2$, and the discriminant curve B is a rational quartic ($2d = 4$) with three cusps ($c = 3$).

If a surface Y is reducible, then either $Y = P \cup Q$ is the union of a plane and a quadric, or Y is the union of three planes.

Now consider a complete degeneration of a cubic Y into an arrangement of three planes $\mathscr{P} = P_1 \cup P_2 \cup P_3$ with the double curve $\mathscr{L} = L_{1,2} \cup L_{2,3} \cup L_{3,1}$. The arrangement $\mathscr{P}$ has one triple point $s = L_{1,2} \cap L_{2,3} \cap L_{3,1}$. Let $\mathscr{D}$ and $\mathscr{R}$ be the limit double curve and the limit ramification curve, respectively. The surface Y is obtained from $\mathscr{P}$ by smoothing $\mathscr{R}$ (smoothing outside $\mathscr{D}$). We have 4 possibilities:

1) if the generic fibre Y is a smooth cubic, then $\mathscr{L} = \mathscr{R}$, $\mathscr{D} = \varnothing$, that is, the double curve $\mathscr{L}$ is smoothed completely; in this case from the triple point $\overline{s} = p(\mathscr{T}_0) \in \overline{\mathscr{R}}$ there appear 6 cusps (and no nodes) on the branch curve B;

2) if the generic fibre Y is an irreducible cubic whose double curve D is a line, then $\mathscr{D} = L_{1,2}$ (for example), and $\mathscr{R} = L_{2,3} \cup L_{3,1}$, that is, two lines in $\mathscr{L}$ are smoothed and one line remains double; the point $s \in \mathscr{T}_1$; in this case from the point s there appear two pinches on Y, and the point $\overline{s}$ gives three cusps on the curve B;

3) if the generic fibre $Y = P \cup Q$, then $\mathscr{R} = L_{1,2}$, $\mathscr{D} = L_{2,3} \cup L_{3,1}$; the point $s \in \mathscr{T}_2$; the curves $2\mathscr{R}$ and $\mathscr{D}$ are smoothed into plane conics, one of which becomes a ramification curve R and the other becomes a double curve D; the curve B is a conic in $\mathbb{P}^2$.

4) Finally, if Y is a union of three planes, then $\mathscr{L} = \mathscr{D}$, $\mathscr{R} = \varnothing$ and $s \in \mathscr{T}_3$; the double curve $\mathscr{L}$ is not smoothed at all.

Now let $\mathscr{P}$ be a complete degeneration of a surface Y of degree m with ordinary singularities, and let $\mathscr{D}$ and $\mathscr{R}$ be the limit double curve and the limit ramification curve. If $s \in \mathscr{T}$ is a triple point on $\mathscr{P}$, then (locally) in a neighbourhood of the point s the smoothing outside $\mathscr{D}$, or the regeneration, looks the same as in the case of the regeneration of a cubic surface, and this explains the previously obtained formulae in the following way.

Formula (31) is explained by the fact that the regeneration in the case $s \in \mathscr{T}_0$ gives 6 cusps on the curve B, and in the case $s \in \mathscr{T}_1$ there appear 3 cusps.

Formula (16) is explained by the fact that pinches (in pairs) appear only from points $s \in \mathscr{T}_1$.

Finally, formula (32) is explained as follows. Let q be one of the ν_0 double points of the curve $\overline{\mathscr{R}}$, $q = \overline{L}_{i,j} \cap \overline{L}_{k,l}$, where $L_{i,j}$ and $L_{k,l}$ is a pair of skew lines. In a neighbourhood of a line $L_{i,j}$ the surface $\mathscr{P}$ is given (locally) by the equation $z^2 - x^2 = 0$ and smoothed surface Y_u is given by an equation $z^2 - x^2 = \varepsilon$. Projecting to the plane $\mathbb{P}^2$, we obtain two branches of the branch curve B_u close ('parallel') to the curve $\overline{L}_{i,j}$. Analogously, for the curve $L_{k,l}$ we obtain two branches of the branch curve B_u close to the line $\overline{L}_{k,l}$. These two pairs of branches meet in four points.

Thus, each of the ν_0 pairs of skew lines of the curve $\mathscr{R}$ under regeneration gives 4 nodes on the curve B, and we obtain the formula $n = 4\nu_0$.

§ 3. Complete degeneracy of quartic surfaces

Let us analyse complete degenerations of surfaces of degree $m = 4$. If a surface Y is reducible, then it is obvious that complete degenerations of components of Y give a complete degeneration of Y. Thus, we can assume the surface Y to be irreducible since the case $m = 2$ is trivial and the case $m = 3$ has been described in § 2.5.

3.1. Degenerations of irreducible quartics with ordinary singularities. The description of all irreducible quartics $Y \subset \mathbb{P}^3$ with ordinary singularities can be found in [8]. There are 6 types of such surfaces in accordance with the types of the double curve $D \subset Y$:

1) Y is smooth, that is, $D = \varnothing$;
2) D is a line;
3) D is a plane conic;
4) D is a pair of skew lines;
5) D is the union of three lines meeting at a point;
6) D is a rational normal curve of degree 3.

We show that in each of these cases there is a complete degeneration of Y. Let $F(x) = 0$ be an equation of Y and let $H_i(x) = 0$, $i = 1, \dots, 4$, be the equations of four planes $P_1, \dots, P_4$ in general position in $\mathbb{P}^3$.

In cases 1), 2), 4) and 5) we can obtain the desired complete degenerations Y_u, $u \in \mathbb{C}$, in the form

$$u \cdot F(x) + (1 - u)H_1(x)H_2(x)H_3(x)H_4(x) = 0,$$

where in case 1) the linear functions $H_1(x), \dots, H_4(x)$ are arbitrary and such that $P_1, \dots, P_4$ are in general position; in case 2) $H_1(x) = H_2(x) = 0$ are the equations of the line D; in case 4) $H_1(x) = H_2(x) = 0$ and $H_3(x) = H_4(x) = 0$ are the equations of the skew lines forming D; in case 5) $H_1(x)H_2(x)H_3(x) = 0$ is the equation of a union of three planes the double curve of which is the union of three lines forming D. In all cases the double curve $\mathscr{L}$ of the arrangement $\mathscr{P}$ of four planes $P_1, \dots, P_4$ contains the double curve $D_u \subset Y_u$ and the limit double curve $\mathscr{D}$ coincides with D.

In the special case 3) the double curve D is a complete intersection. We consider this case in § 6.

Consider case 6). As is known, all smooth space curves of degree 3 are projectively equivalent. Such a curve D has parametrization $x = t$, $y = t^2$, $z = t^3$ in appropriate affine coordinates. It is not a complete intersection and is defined by three equations: $y = x^2$, $xy = z$, $y^2 = xz$. In homogeneous coordinates $(x_1 : x_2 : x_3 : x_4) = (x : y : z : 1)$ in $\mathbb{P}^3$ the curve D is the intersection of three quadrics:

$$x_2x_4 = x_1^2, \qquad x_1x_2 = x_3x_4, \qquad x_2^2 = x_1x_3.$$

Consider the family of curves D_u: $x = t$, $y = ut^2$, $z = u^3t^3$. The curves D_u are given by the three equations $y = ux^2$, $u^2xy = z$, $uy^2 = xz$, or in homogeneous coordinates

$$x_2x_4 - ux_1^2 = 0, \qquad x_3x_4 - u^2x_1x_2 = 0, \qquad x_1x_3 - ux_2^2 = 0.$$

The family D_u defines a degeneration of the curve D (for $u = 1$) to the curve $\mathscr{D}$ (for $u = 0$), which is given by the equations: $x_2x_4 = 0$, $x_3x_4 = 0$, $x_1x_3 = 0$. If P_i is a plane given by the equation $x_i = 0$, $L_{i,j} = P_i \cap P_j$, then $\mathscr{D}$ is the chain of lines $L_{1,4} \cup L_{4,3} \cup L_{3,2}$.

Consider the family of surfaces Y_u given by the equation

$$u(x_2x_4 - ux_1^2)(x_3x_4 - u^2x_1x_2) + u(x_3x_4 - u^2x_1x_2)(x_1x_3 - ux_2^2) \\ + (x_2x_4 - ux_1^2)(x_1x_3 - ux_2^2) = 0.$$

For $u \neq 0$ the double curves are smooth space cubic curves D_u, and for $u = 0$ the surface Y degenerates to the plane arrangement $Y_0 = \{x_1x_2x_3x_4 = 0\}$ and the limit double curve is $\mathscr{D}$.

3.2. Description of possible arrangements of limit double curves. In the case $m = 4$ the graph $\Gamma(\mathscr{L})$ consists of the union of all edges of a tetrahedron. By Lemma 2, a curve $\mathscr{D} \subset \mathscr{L}$ can be a limit double curve of a degeneration of an irreducible quartic with ordinary singularities if the graph $\overline{\Gamma}(\mathscr{R})$ is connected. The graph $\overline{\Gamma}(\mathscr{R})$ is obtained from $\Gamma(\mathscr{L})$ by removing some edges corresponding to lines $L_{i,j} \subset \mathscr{D}$. It is obvious that the graph $\overline{\Gamma}(\mathscr{R})$ remains connected only in the following cases:

1) we remove nothing from $\Gamma(\mathscr{L})$;
2) we remove one edge from $\Gamma(\mathscr{L})$;
3) we remove two adjacent edges;
4) we remove two skew edges;
5) we remove three edges going out from one vertex;
6) we remove a chain of three edges.

Removing four edges leads to a nonisolated vertex, that is, to a disconnected graph $\overline{\Gamma}(\mathscr{R})$.

Thus, up to renumbering the planes $P_1, \dots, P_4$, the curve $\mathscr{D}$ is one of the following:

1) $\mathscr{D} = \varnothing$;
2) $\mathscr{D} = L_{1,2}$;
3) $\mathscr{D} = L_{1,2} \cup L_{1,3}$;
4) $\mathscr{D} = L_{1,2} \cup L_{3,4}$;
5) $\mathscr{D} = L_{1,2} \cup L_{1,3} \cup L_{2,3}$;
6) $\mathscr{D} = L_{1,2} \cup L_{2,3} \cup L_{3,4}$.

In § 3.1 we showed that all such $\mathscr{D}$ are realized as limit double curves. Thus, we obtain the following result.

Theorem 1. *For any surface of degree 4 with ordinary singularities in $\mathbb{P}^3$ there exists a complete degeneration.*

If $\mathscr{P}$ is an arrangement of four planes in general position in $\mathbb{P}^3$, $\mathscr{D} \subset \mathscr{L}$ is any line arrangement, then there exists a complete degeneration of surfaces of degree 4 with ordinary singularities for which $\mathscr{D}$ is the limit double curve.

§ 4. Existence of line arrangements which are not limit

In the previous section we showed that in the case of surfaces with ordinary singularities of degree $m \leqslant 4$ the situation is as good as in the case of plane curves. The answers to both questions put in the introduction are affirmative. But if $m \geqslant 5$, the situation is not so good. In this section we show that there are at least seven arrangements of double lines $\mathscr{D}$ in plane arrangements $\mathscr{P}$, $\deg \mathscr{P} = 5$, such that $\mathscr{P}$ cannot be smoothed outside $\mathscr{D}$.

Theorem 2. *The line arrangements $\mathscr{D}$ with graphs depicted in Fig. 2 are not limit for complete degenerations of surfaces with ordinary singularities of degree* 5.

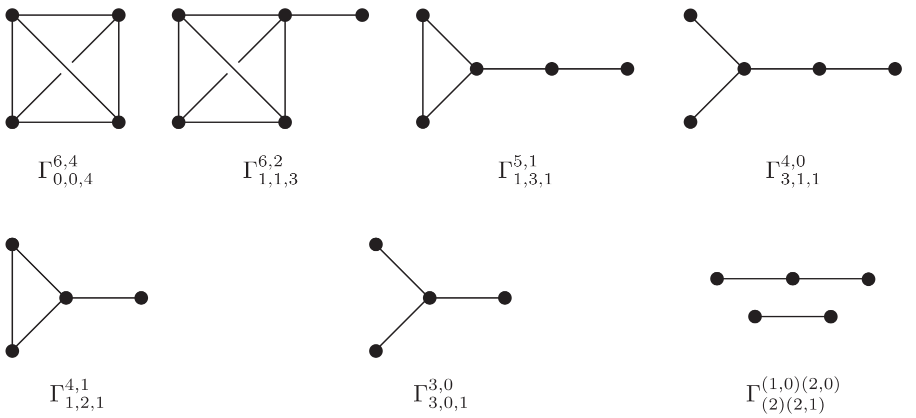


Figure 2

Proof. Assume that the line arrangements with graphs depicted in Fig. 2 are limit double curves for complete degenerations of surfaces Y of degree $m = 5$ with ordinary singularities.

In all cases the graph $\overline{\Gamma}(\mathscr{R})$ is connected and by Lemma 2 the surface Y for which $\mathscr{D}$ is the limit double curve is irreducible.

We can obtain numerical characteristics of curves $\mathscr{D}$. In our case $m = 5$ and therefore, $d + \overline{d} = \binom{5}{2}$, $\tau = \binom{5}{2}$. The number τ_3 is equal to the number of triangles in $\Gamma(\mathscr{D})$. To calculate τ_2 we use formula (12); τ_1 and τ_0 are obtained from formulae (10), (11) and (8):

$$\tau_1 + 2\tau_2 + 3\tau_3 = 3\overline{d}, \qquad \tau_2 + 2\tau_1 + 3\tau_0 = 3d, \qquad \tau_0 + \tau_1 + \tau_2 + \tau_3 = \tau.$$

By (30)–(32) we can find numerical characteristics of the double curve $D \subset Y$ and of the branch curve B: the numbers $\overline{g}$, t, k we find from Proposition 1, and the numbers g, c, n we find from Proposition 5 by applying formulae (30)–(32) and (28).

We consider each of the line arrangements $\mathscr{D}$ with graphs depicted in Fig. 2 and show that in all cases we obtain a contradiction with the assumption made above that these arrangements be limit.

For a line arrangement of type $\Gamma^{3,0}_{3,0,1}$ we have: $\overline{d} = 3$, $\tau_3 = 0$ and $\tau_2 = 3$. By Proposition 1, the double curve $D \subset Y$ is an irreducible smooth curve of degree $\overline{d} = 3$ and genus $\overline{g} = 1$. Hence D is a plane cubic. Suppose D lies in a plane P. Then for any line $L \subset P$ the intersection number $(L, Y)_{\mathbb{P}^2} \geqslant 6$, and since $\deg Y = 5$, we have $L \subset Y$. Thus, $P \subset Y$ and the surface Y is reducible: a contradiction. The same arguments can be applied for line arrangements of the following two types.

For a line arrangement of type $\Gamma^{(1,0)(2,0)}_{(2)(2,1)}$ we have: $\overline{d} = 3$, $\tau_3 = 0$ and $\tau_2 = 1$. By Proposition 1 we have $k = 2$, $\overline{g} = 0$ and the double curve $D \subset Y$ is a disconnected union of a line L_1 and a plane conic Q. Suppose Q lies in a plane P and the point $A = L_1 \cap P$. As in the previous case, we see that any line $L \subset P$ passing through A lies in the surface Y and, consequently, $P \subset Y$. Again we obtain a contradiction with the irreducibility of Y.

For a line arrangement of type $\Gamma^{4,1}_{1,2,1}$ we have: $\overline{d} = 4$, $\tau_3 = 1$ and $\tau_2 = 2$. By Proposition 1 we have $k = 2$, $\overline{g} = 0$, $t = 1$. Hence $D = L \cup C$ consists of two irreducible components, where L is a line and C is a curve of degree 3 with one node. Consequently, the curve C lies in a plane P. As in the previous cases, we obtain that $P \subset Y$ and this gives a contradiction with the irreducibility of Y.

We consider a line arrangement of type $\Gamma^{4,0}_{3,1,1}$ in Theorem 5, where it is shown that the curve $\mathscr{D}$ can not be the limit double curve of a complete degeneration of a surface Y of any degree m, and not only $m = 5$.

For a line arrangement of type $\Gamma^{5,1}_{1,3,1}$ we have: $\overline{d} = 5$, $\tau_3 = 1$, $\tau_2 = 3$ and $\tau_1 = 6$. By (18) and (19) we obtain $K^2_X = 1$ and $e(X) = -1$. It is known from the classification of algebraic surfaces that if $K^2_X > 0$, then $e(X) > 0$. Thus such surfaces do not exist.

A line arrangement of type $\Gamma^{6,4}_{0,0,4}$ is a special case (for $m = 5$) of line arrangements which are considered below in Proposition 9.

For a line arrangement of type $\Gamma^{6,2}_{1,1,3}$ we have: $\overline{d} = 6$, $d = 4$, $\tau_3 = 2$, $\tau_2 = 4$, $\tau_1 = 4$, $\tau_0 = 0$. The branch curve B is cuspidal and has numerical characteristics: $\deg B = 8$, $g = 1$, $c = 12$, $n = 8$. Let us show that the curve $\widehat{B}$ dual to B also is cuspidal. By Plücker's formula we obtain $\deg \widehat{B} = 4$. Since $g(\widehat{B}) = 1$ and a curve of degree 4 has arithmetic genus $p_a(\widehat{B}) = 3$, $\widehat{B}$ has either two singular points with $\delta = 1$, or one singular point with $\delta = 2$. The Milnor number μ and δ are connected by the formula $\mu = 2\delta - r + 1$, where r is the number of branches. Singularities with $\delta = 1$ are either singularities of type A_1, nodes, or of type A_2, cusps, therefore, in the first case the curve $\widehat{B}$ is cuspidal. The second case is impossible since the singularity dual to A_3 is A_3, and the one dual to A_4 is E_8: $x^3 + y^5 = 0$. But the curve dual to $\widehat{B}$ is the curve B, which has only nodes and cusps. Thus, the curve $\widehat{B}$ is cuspidal. In this case it follows from Plücker's formula that $3 \deg B - c = 3 \deg \widehat{B} - \widehat{c}$, where $\widehat{c}$ is the number of cusps of $\widehat{B}$. Therefore, $\widehat{c} = 0$ and, consequently, the curve $\widehat{B}$ is nodal and the number of nodes of $\widehat{B}$ is $\widehat{n} = 2$. For such a curve $\widehat{B}$ there exists a generic covering of the plane of degree $4 = \deg \widehat{B}$ branched along B (see [9]). On the other hand, a generic projection of a surface with ordinary singularities in $\mathbb{P}^3$ onto $\mathbb{P}^2$ is a generic covering, in our case of degree 5. As is shown in [13], generic coverings, which are generic projections, are uniquely defined by their branch curves always, except the case of surfaces of degree 4 with singularities consisting of three lines intersecting in a point.[1] We obtain a contradiction with the uniqueness of the covering, which completes the proof of the theorem.

§ 5. Existence of surfaces not possessing complete degenerations

In this section we show that there exist surfaces in $\mathbb{P}^3$ with ordinary singularities which can not be degenerated into plane arrangements in general position.

5.1. Zeuthen's problem. The problem of the existence of a complete degeneration for any surface with ordinary singularities is closely connected with the problem of degeneration of its double curve D into a line arrangement. In the case of a smooth space curve D it is the famous Zeuthen problem. In [12], a negative solution of Zeuthen's problem is obtained, that is, it is shown that there exist smooth projective curves $D \subset \mathbb{P}^3$ which cannot be degenerated into a line arrangement with double points. One of such curves has degree 30 and genus 113. We call it the *Hartshorne curve*.

Theorem 3. *There exist surfaces in $Y \subset \mathbb{P}^3$ with ordinary singularities which cannot be degenerated into a plane arrangement in general position.*

Proof. Let D be the Hartshorne curve. In the next subsection we prove Proposition 6 that there exists a surface $Y \subset \mathbb{P}^3$ with ordinary singularities the double curve of which is D. By [12] the curve D cannot be degenerated into a line arrangement with double points and, consequently, Y cannot be degenerated into a plane arrangement in general position, which is the required result.

5.2. Existence of a surface with ordinary singularities, the double curve of which is any smooth curve. Let D be a smooth (not necessarily irreducible) curve in $\mathbb{P}^3$. Then for any point $x \in D$ there exist a Zariski open set U_x in $\mathbb{P}^3$ and two rational functions $f_{x,1}$ and $f_{x,2}$, regular in U_x and such that $f_{x,1}$ and $f_{x,2}$ are local parameters in U_x and the curve $D \cap U_x$ is defined by equations $f_{x,1} = f_{x,2} = 0$. Let $f_{x,j} = F_{x,j}/G_{x,j}$, where $F_{x,j}$ and $G_{x,j}$ are relatively prime homogeneous polynomials of degree $M_{x,j}$. The curve D being compact, we can choose a finite covering of D by open sets $U_{x_1}, \dots, U_{x_k}$ such that the polynomials $F_{x_i,j}$, $1 \leqslant i \leqslant k$, $j = 1, 2$, generate the homogeneous ideal of D. Set

$$M(D) = 2 \max_{1 \leqslant i \leqslant k} \Bigl(\max_{j=1,2} M_{x_i,j} \Bigr) + 1. \tag{33}$$

Proposition 6. *For any smooth projective curve $D \subset \mathbb{P}^3$ and for any natural number $m \geqslant M = M(D)$, where $M(D)$ is the number defined in (33), there exists a projective surface Y with ordinary singularities of degree m in $\mathbb{P}^3$ such that $D = \operatorname{Sing} Y$.*

Proof. Let $\sigma \colon \widetilde{\mathbb{P}}^3 \to \mathbb{P}^3$ be a monoidal transformation with centre at D, $E = \sigma^{-1}(D)$ its exceptional divisor and $\widetilde{P} = \sigma^*(P)$ the total inverse image of the plane $P \subset \mathbb{P}^3$.

[1] In [13] this statement was formulated only for surfaces obtained as a generic linear projection onto $\mathbb{P}^3$ of smooth surfaces embedded in some $\mathbb{P}^N$. But the proof of this statement used only the assumption that surfaces in $\mathbb{P}^3$ have only ordinary singularities.

The surfaces $Y_{i,j} = \{F_{x_i,j} = 0\} \subset \mathbb{P}^3$ and $\deg Y_{i,j} = \deg F_{x_i,j} = M_{x_i,j}$, $j = 1, 2$, meet transversally along D in U_{x_i}. It is obvious that for any plane P in $\mathbb{P}^3$ the divisors

$$Y_{i,1} + Y_{i,2} + (m - M_{x_i,1} - M_{x_i,2})\widetilde{P}$$

are the zeros of sections of the sheaf $\mathscr{O}_{\widetilde{\mathbb{P}}^3}(m\widetilde{P} - 2E)$ for $m \geqslant M$. From this it is easy to see that the linear system $|m\sigma^*(P) - 2E|$ is nonempty, does not have fixed components and base points, and for any points $x, y \in \widetilde{\mathbb{P}}^3$ there exists a divisor of this system such that it does not go through the point y and is a smooth reduced surface at x. According to Bertini's theorem the generic member X of the linear system $|m\sigma^*(P) - 2E|$ is a smooth surface, and it is easy to see that its image $Y = \sigma(X)$ is a surface in $\mathbb{P}^3$ of degree m with ordinary singularities along D. The proof is complete.

§ 6. Complete degeneracy of surfaces with complete-intersection double curves

As has been shown in the previous section, in general the answer to the question about the existence of a complete degeneration is negative even in the case of a smooth double curve. In this section we show that the answer to this question is affirmative in the case when the double curve is a complete intersection.

6.1. Equations of surfaces with complete-intersection double curves. The following proposition gives a description of equations of surfaces, double curves of which are complete intersections.

Proposition 7. *Suppose an irreducible surface $Y \subset \mathbb{P}^3$ with ordinary singularities,* $\deg Y = m$, *given by equation $F(x) = 0$, has a smooth double curve $D = Y_1 \cap Y_2$, which is a complete intersection of surfaces Y_1 and Y_2 of degrees m_1 and m_2. Then the polynomial F can be written in the form*

$$F = AF_1^2 + BF_1F_2 + CF_2^2, \tag{34}$$

where $F_1(x) = 0$ and $F_2(x) = 0$ are equations of the surfaces Y_1 and Y_2, respectively, and A, B and C are homogeneous polynomials.

Conversely, if F is written in the form (34) and $m \geqslant 2m_1 + 1$ (let $m_1 \geqslant m_2$) and the polynomials A, B, and C are sufficiently general, then the surface Y has only ordinary singularities and its double curve is a complete intersection.

Proof. Since the curve D is a complete intersection, the homogeneous ideal $I(D)$ is generated by two elements, $I(D) = (F_1, F_2)$. We write F in the form $F = K_1F_1 + K_2F_2$. It follows from the transversality of intersection of the surfaces $F_1 = 0$ and $F_2 = 0$ along D that the differentials dF_1 and dF_2 are linearly independent at each point of D. In addition, the differential dF is equal to zero at each point in D since D is the double curve of Y. At each point in D we have $dF = K_1dF_1 + K_2dF_2$ and it follows from the linear independence of the differentials dF_1 and dF_2 at these points that K_1 and K_2 belong to the ideal $I(D)$, that is, $K_1 = S_1F_1 + S_2F_2$ and $K_2 = S_3F_1 + S_4F_2$, where the S_i are some homogeneous polynomials. Substituting these expressions for K_1 and K_2 in the expression for F we obtain the desired form of F. The second part of the proposition follows from Proposition 6 and the estimate (33).

6.2. Construction of a degeneration. We prove the following theorem.

Theorem 4. *Surfaces with ordinary singularities the double curves of which are complete intersections have complete degenerations.*

Proof. Let the double curve D of a surface Y be given by the equations $F_1 = 0$, $F_2 = 0$, $\deg F_1 = m_1$, $\deg F_2 = m_2$. Then by Proposition 1 the surface Y is given by an equation of the form

$$AF_1^2 + BF_1F_2 + CF_2^2 = 0.$$

We consider a deformation of this equation killing the first and the third terms and transforming the second term into a product of linear forms. Let a family of surfaces Y_u, $u \in \mathbb{C}$, be given by an equation

$$\begin{aligned}(uB + (1-u)\overline{B})(uF_1 + (1-u)\overline{F}_1)(uF_2 + (1-u)\overline{F}_2)\\ + uA(uF_1 + (1-u)\overline{F}_1)^2 + uC(F_2 + (1-u)\overline{F}_2)^2 = 0,\end{aligned} \tag{35}$$

where $\overline{F}_1 = H_1 \cdots H_{m_1}$, $\overline{F}_2 = H_{m_1+1} \cdots H_{m_1+m_2}$, $\overline{B} = H_{m_1+m_2+1} \cdots H_m$ are products of linear forms such that zeros of the forms are planes $P_1, \dots, P_m$ in general position.

It is easy to see that for $u = 1$ the surface Y_u coincides with Y. It is obvious also that for values u close to 1 the curves D_u given by equations

$$uF_1 + (1-u)\overline{F}_1 = uF_2 + (1-u)\overline{F}_2 = 0,$$

are smooth and they are the double curves of surfaces Y_u with ordinary singularities. For $u = 0$ the degenerated fibre is given by an equation $H_1 \cdots H_m = 0$ and the limit double curve D_0 is given by the equations

$$H_1 \cdots H_{m_1} = H_{m_1+1} \cdots H_{m_1+m_2} = 0. \tag{36}$$

In particular, for $m_1 = m_2 = 2$ the graph of the limit double curve $\mathscr{D} = D_0$ is depicted in Fig. 3.

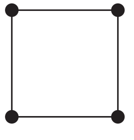


Figure 3

The proof of Theorem 4 is complete.

§ 7. Potentially limit and absolutely nonlimit line arrangements

Until now we have considered complete degenerations of surfaces with ordinary singularities for fixed degree m. The line arrangements $\mathscr{D}\subset\mathscr{L}_m$ which can be obtained in the degenerate fibre are called now *m-limit curves*. (Here $\mathscr{L}_m$ is the double curve of an arrangement of m planes in general position). In § 4 it was shown that there exist line arrangements $\mathscr{D}$ which are not 5-limit. We want to investigate the dependence on m of the property of a curve $\mathscr{D}$ to be m-limit.

If for some m_0 a curve $\mathscr{D}$ is not m_0-limit, but for sufficiently big m and for an embedding $\mathscr{D}\subset\mathscr{L}_m$ the curve $\mathscr{D}$ is m-limit, then $\mathscr{D}$ is called *potentially limit*. If for any embedding $\mathscr{D}\subset\mathscr{L}_m$ a curve $\mathscr{D}$ is not m-limit, then such a line arrangement $\mathscr{D}$ is called *absolutely nonlimit*.

7.1. Examples of potentially limit line arrangements. Let us show that a line arrangement $\mathscr{D}$ of type $\Gamma^{3,0}_{3,0,1}$ is potentially limit. It was shown in Theorem 2 that $\Gamma^{3,0}_{3,0,1}$ is not 5-limit. Let us show that $\Gamma^{3,0}_{3,0,1}$ is m-limit for $m\geqslant 7$. The line arrangement $\mathscr{D}$ consists of three lines $L_1\cup L_2\cup L_3$, which lie in a plane P and are cut out by three planes defined by equations $H_1=0$, $H_2=0$, and $H_3=0$. Such an arrangement $\mathscr{D}$ is limit for surfaces Y the double curves D of which are complete intersections of a smooth cubic and a plane. Indeed, let $F_1(x)=0$ be an equation of a smooth cubic Y_1 and $F_2(x)=0$ an equation of a plane $Y_2=P$. Consider a surface $F(x)=0$, where F is defined by formula (34) and consider a family of surfaces Y_u defined by equation (35). It follows from (36) that the limit double curve $\mathscr{D}$ is defined by equations $H_1H_2H_3=F_2=0$ and it is of type $\Gamma^{3,0}_{3,0,1}$.

Analogously one can show that a line arrangement of type $\Gamma^{(1,0)(2,0)}_{(2)(2,1)}$ also is potentially limit.

7.2. An example of an absolutely nonlimit line arrangement. The following theorem gives an example of absolutely nonlimit line arrangement. Note that this line arrangement is a degeneration of a smooth space curve of degree 4 and genus 1.

Theorem 5. *The line arrangement $\mathscr{D}$ of type $\Gamma^{4,0}_{3,1,1}$ (see Fig. 2) is absolutely nonlimit.*

Proof. Let $\mathscr{D}$ be m-limit for some m, that is, $\mathscr{D}$ is the limit double curve of a complete degeneration of surfaces Y_u, $u\in U$, defined by equations $F_u(x)=0$. We have $\overline{d}=4$, $\tau_3=0$, and $\tau_2=4$. By Proposition 1 the double curve $D\subset Y$ is a smooth irreducible curve in $\mathbb{P}^3$ of degree $\overline{d}=4$ and genus $\overline{g}=1$. As is known, such curves D are complete intersections of two quadrics, and, consequently, a polynomial F defining the surface Y can be written in the form (34):

$$F=AQ_1^2+BQ_1Q_2+CQ_2^2, \tag{37}$$

where $Q_1(x)=0$ and $Q_2(x)=0$ are equations of these quadrics and A, B and C are homogeneous polynomials of degree $m-4$.

Let us show that the family of surfaces $F_u(x)=0$, $u\in U$, can be written in the form

$$F_u=A_uQ_{1,u}^2+B_uQ_{1,u}Q_{2,u}+C_uQ_{2,u}^2 \tag{38}$$

(maybe after a base change). For this let us consider the universal family of surfaces given by equations of the form (37). The base of this family $\mathscr{F}$ is an open subset of the space of coefficients of the forms Q_1, Q_2, A, B, C. Denote by $\mathscr{H}_{m,4,1}$ the space parametrizing the surfaces of degree m the double curve of which is a smooth curve of degree 4 and genus 1. Obviously, we have a rational dominant map $\mathscr{F} \to \mathscr{H}_{m,4,1}$. The family of surfaces $F_u(x) = 0$ defines a map $U \to \mathscr{H}_{m,4,1}$. We can assume that $U \subset \mathscr{H}_{m,4,1}$. If a curve $\widetilde{U} \subset \mathscr{F}$ is mapped to the curve U we get a family of surfaces (38) parametrized by points of $\widetilde{U}$.

A line arrangement $\mathscr{D}$ of type $\Gamma^{4,0}_{3,1,1}$ consists of three lines in a plane and a fourth line not in this plane and intersecting one of these lines. Such curves are degenerations of space elliptic curves of degree 4 (see [14]). But if a curve $\mathscr{D}$ is the limit double curve of the family (38), then the double curves D_u are given by a family of ideals $J_u = (Q_{1,u}, Q_{2,u})$. By [14], in order that a degenerate curve coincide with $\mathscr{D}$ it is necessary that for $u = u_0$ the quadratic forms Q_{1,u_0} and Q_{2,u_0} split into a product of linear forms with one common form: $Q_{1,u_0} = HH_1$ and $Q_{2,u_0} = HH_2$. But then it follows from (38) that the degenerate surface Y_{u_0} contains a multiple plane $H = 0$, and we obtain a contradiction with the definition of a complete degeneration of surfaces with ordinary singularities.

§ 8. Virtual degeneracy

If for a surface $Y \subset \mathbb{P}^3$ of degree m with ordinary singularities there exists a complete degeneration in the sense of the definition given in the introduction, then, for brevity, we call Y *a completely degenerative surface*. In § 5 we gave examples of surfaces which are not completely degenerative.

8.1. Necessary conditions for complete degeneracy. Let us weaken the notion of degeneracy of a surface. Recall that in §§ 1.1 and 1.2 we defined the type of an irreducible surface $Y \subset \mathbb{P}^3$ of degree m with ordinary singularities and the type of a pair $(\mathscr{P}, \mathscr{D})$ as collections of numerical data:

$$\operatorname{type}(Y) = (m, \overline{d}, k, \overline{g}, t), \qquad \operatorname{type}(\mathscr{P}, \mathscr{D}) = (m, \overline{d}, k, \tau_2, \tau_3).$$

By Proposition 1, if Y is a completely degenerative surface, then the types type(Y) and type($\mathscr{P}, \mathscr{D}$) define each other. We call collections of numbers $(m, \overline{d}, k, \overline{g}, t)$ and $(m, \overline{d}, k, \tau_2, \tau_3)$ *corresponding* to each other if $\tau_3 = t$, and the numbers τ_2 and $\overline{g}$ are connected by formula (15): $\tau_2 = \overline{d} + \overline{g} - k$. We call an irreducible surface Y *virtually degenerative* if there exists an irreducible pair $(\mathscr{P}, \mathscr{D})$ the type of which corresponds to the type of the surface Y. Thus, a surface Y can be not degenerative for a trivial reason: there does not exist a pair $(\mathscr{P}, \mathscr{D})$ of the corresponding type.

Analogously, we call a pair $(\mathscr{P}, \mathscr{D})$ *virtually smoothable outside* $\mathscr{D}$ (or we call a line arrangement $\mathscr{D}$ *virtually m-limit*) if there exists an irreducible surface Y of degree m which has a type corresponding to the type of the pair $(\mathscr{P}, \mathscr{D})$. Thus, the term 'virtual' talks about the fulfillment of necessary numerical conditions for the existence of a complete degeneration.

The proof of Proposition 5 does not use essentially the fact of complete degeneracy of Y, but uses only the connection (arising from it) between the type of Y and the type of the limit pair $(\mathscr{P}, \mathscr{D})$ given by formulae (14): $t = \tau_3$, and (15):

$\overline{g} = \tau_2 - \overline{d} + k$. Therefore, in the case of virtual degeneracy an analogue of Proposition 5 holds.

Proposition 8. *Let the type* $(m, \overline{d}, k, \overline{g}, t)$ *of a surface* Y *correspond to the type* $(m, \overline{d}, k, \tau_2, \tau_3)$ *of a pair* $(\mathscr{P}, \mathscr{D})$. *Then the same formulae* (29)–(32) *as in Proposition* 5 *hold for numerical characteristics of a generic projection* $p\colon Y \to \mathbb{P}^2$.

8.2. Examples of virtually m-limit, but not m-limit arrangements. Consider an irreducible pair $(\mathscr{P}, \mathscr{D})$, where $\mathscr{D}$ is an arrangement of type $\Gamma^{4,0}_{3,1,1}$ from Theorem 5. This pair has type$(\mathscr{P}, \mathscr{D}) = (m, 4, 1, 4, 0)$. In Theorem 5 it was proved that $\mathscr{D}$ is nonlimit for all m. But the line arrangement $\mathscr{D}$ is virtually m-limit for $m \geqslant 5$. Indeed, the type of the pair $(\mathscr{P}, \mathscr{D})$ has the corresponding data set $(m, 4, 1, 1, 0)$. Surfaces Y are of type$(Y) = (m, 4, 1, 1, 0)$, that is, surfaces of degree m, the double curve D of which is an elliptic curve of degree 4, exist. Since D is a complete intersection of two quadrics Q_1 and Q_2, we can take a surface Y given by an equation $F = 0$, where the polynomial F is defined by formula (37). Such surfaces are completely degenerative, but the limit double curve $\mathscr{D}$ has a graph not of type $\Gamma^{4,0}_{3,1,1}$, but a graph of type depicted in Fig. 3.

8.3. Examples of virtually nonlimit line arrangements. Consider an arrangement of m planes $\mathscr{P} = \mathscr{P}_m$ in general position in $\mathbb{P}^3$. Let $\mathscr{P}_{m-1} \subset \mathscr{P}_m$ be the arrangement of some $m-1$ of these planes and $\mathscr{D} = \mathscr{L}_{m-1}$ the double curve of the surface $\mathscr{P}_{m-1}$. Then

$$\overline{d} = \binom{m-1}{2}, \qquad \tau_0 = \tau_2 = 0, \qquad \tau_1 = \binom{m-1}{2}, \qquad \tau_3 = \binom{m-1}{3}$$

and the pair $(\mathscr{P}, \mathscr{D})$ has type

$$\operatorname{type}(\mathscr{P}, \mathscr{D}) = \left(m, \binom{m-1}{2}, \binom{m-1}{2}, 0, \binom{m-1}{3}\right). \tag{39}$$

Proposition 9. *If* $(\mathscr{P}, \mathscr{D})$ *is an irreducible pair of type* (39) *and* $m \geqslant 5$, *then* $\mathscr{D}$ *is not virtually limit.*

Proof. Assume that $\mathscr{D}$ is virtually limit. Then there exists a surface Y of type

$$\operatorname{type}(Y) = \left(m, \binom{m-1}{2}, \binom{m-1}{2}, 0, \binom{m-1}{3}\right)$$

corresponding to the type of the pair $(\mathscr{P}, \mathscr{D})$.

Consider a generic projection of the surface Y to the plane. Then, by Proposition 8 the branch curve B of the generic projection has the following invariants:

$$\deg B = 2d = 2(m-1), \qquad g-1 = \binom{m-1}{2} - m + 1, \qquad c = 3\binom{m-1}{2}, \qquad n = 0.$$

But then the degree of the curve $\check{B}$ dual to the branch curve B equals

$$\begin{aligned}\deg \check{B} &= \deg B(\deg B - 1) - 3c \\ &= 2(m-1)(2m-3) - 9\binom{m-1}{2} = \left(3 - \frac{1}{2}m\right)(m-1) \leqslant 2\end{aligned}$$

for $m \geqslant 5$, and this is impossible.

8.4. Examples of surfaces completely degenerative only virtually. Consider a surface Y in § 5 which is not completely degenerative. The double curve D of Y is the Hartshorne curve. The curve D is smooth, has degree $\overline{d} = 30$ and genus $\overline{g} = 113$. Then $\operatorname{type}(Y) = (m, 30, 1, 113, 0)$ and, consequently, $\tau_2 = \overline{d} + \overline{g} - k = 142$ and the collection corresponding to the type of Y is $(m, 30, 1, 142, 0)$. Let us show that the surface Y is virtually degenerative.

Proposition 10. *For $m \geqslant 31$ there exist irreducible pairs of type*

$$\operatorname{type}(\mathscr{P}, \mathscr{D}) = (m, 30, 1, 142, 0).$$

Proof. Consider the graph $\Gamma(\mathscr{D})$ depicted in Fig. 4.

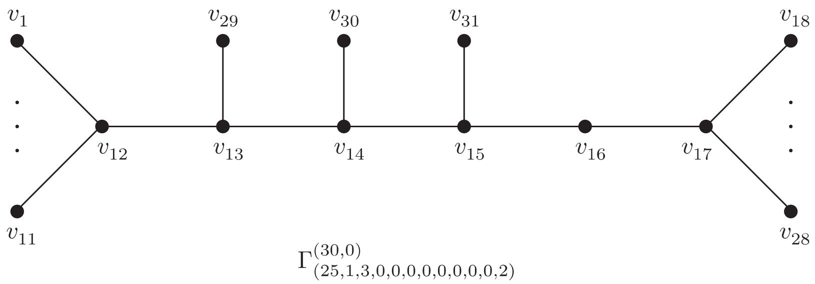


Figure 4

It has 31 vertices, two of which (v_{12} and v_{17}) are of valence 12, three vertices (v_{13}, v_{14}, v_{15}) are of valence 3, the rest are of valence 1. Applying (12) we find $\tau_2 = 142$. Therefore, $\Gamma(\mathscr{D})$ is in fact of the mentioned type. It is easy to see that the pair $(\mathscr{P}, \mathscr{D})$ is irreducible, that is, the graph of the complementary curve $\mathscr{R} = \mathscr{L} \setminus \mathscr{D}$ is connected.

8.5. Examples of virtually nondegenerative surfaces. Below we prove the existence of a surface $Y \subset \mathbb{P}^3$ with ordinary singularities whose double curve D has a unique triple point and consists of three components: two lines L_1 and L_2 and a conic Q. Such a curve D lies on the union of two planes P_1 and P_2 such that $L_1 \cup L_2 \subset P_1$, $Q \subset P_2$, $s = L_1 \cap L_2 \cap Q \in P_1 \cap P_2 = L$, $L_i \neq L$ for $i = 1, 2$ and Q transversally intersects the line L.

Let us show that a surface Y with ordinary singularities and with double curve D described above cannot be virtually completely degenerated. Indeed, the type of such a surface Y is $\operatorname{type}(Y) = (m, 4, 3, 0, 1)$. If Y is virtually completely degenerated, then an irreducible pair $(\mathscr{P}, \mathscr{D})$ with $\operatorname{type}(\mathscr{P}, \mathscr{D}) = (m, 4, 3, 1, 1)$ must exist, that is, there must exist four double lines $\mathscr{D}$ of a plane arrangement $\mathscr{P}$ having the following invariants: $\tau_2 = \tau_3 = 1$ and, moreover, after removing the triple point the curve $\mathscr{D}$ is decomposed into three ($k = 3$) connected components.

Note that there is a degeneration of the curve D into the union of four lines having one triple point and one double point (to obtain such a degeneration we need to degenerate the conic into a pair of lines lying in P_2 so that one of the lines passes through the point s and the other one does not).

Nevertheless, it is easy to see that there does not exist such a union of double curves of any plane arrangement $\mathscr{P} = \cup P_i$ in general position. Indeed, three lines meeting at the triple point must be pairwise intersections of three planes, say P_1, P_2 and P_3. It follows from the conditions $k = 3$ and $\tau_2 = 1$ that the fourth line must intersect one (and only one) of these lines (without loss of generality we can assume that the fourth line intersects the line $L_{1,2}$). This means that the fourth line is the intersection of the fourth plane P_4 of the arrangement $\mathscr{P}$ with one of the two planes P_1, P_2 and then this line must intersect either $L_{1,3}$ or $L_{2,3}$. But this is impossible since $\tau_2 = 1$.

Let us prove the existence of a surface Y with ordinary singularities, $\deg Y \geqslant 8$, the double curve of which is D. For this let us blow up the point s in $\mathbb{P}^3$ and after that let us consecutively blow up the proper transforms of the curves L_1, L_2, and Q. Let $\sigma\colon \widetilde{\mathbb{P}}^3 \to \mathbb{P}^3$ be a composition of these monoidal transformations, E, E_1, E_2, $E_3 \subset \widetilde{\mathbb{P}}^3$ the proper transforms of the exceptional divisors of each of these blow-ups, and let $\widetilde{P} = \sigma^*(P)$ be the total inverse image of a plane $P \subset \mathbb{P}^3$.

Let us show that a generic member $\widetilde{X}$ of the linear system

$$|m\widetilde{P} - 3E - 2E_1 - 2E_2 - 2E_3|$$

is a smooth surface if $m \geqslant 8$. For this let us choose a system of homogeneous coordinates in $\mathbb{P}^3$ as follows (recall that the systems of homogeneous coordinates in $\mathbb{P}^3$ are uniquely determined by the choice of ordered sets of four planes in general position). We take P_1 and P_2 for the first two coordinate planes. As the third plane we take any plane $P_3 \neq P_2$ passing through s and touching the conic Q at this point. We choose the fourth coordinate plane P_4 such that it is in general position with P_1, P_2 and P_3 and touches the conic Q at some point $s_1 \in Q$, $s_1 \neq s$. In the coordinate system chosen in such a way the lines L_i, $i = 1, 2$, are given by equations

$$z_1 = (c_i z_2 + z_3) = 0, \qquad c_1 \neq c_2,$$

and the curve Q is given by

$$z_2 = (z_3 z_4 - c_3 z_1^2) = 0, \qquad c_3 \neq 0.$$

The triple point s of D has coordinates $(0 : 0 : 0 : 1)$.

Denote by F_i the following homogeneous polynomials:

$$F_1 = z_1, \qquad F_2 = z_3 z_4 + c_1 z_2 z_4 - c_3 z_1^2, \qquad F_3 = z_3 z_4 + c_2 z_2 z_4 - c_3 z_1^2, \qquad F_4 = z_4,$$

and consider a linear system of surfaces $\overline{S}_{a_1,a_2,a_3} \subset \mathbb{P}^3$ given by the homogeneous equation

$$F_1 F_2 F_3 F_4^3 + a_1 F_1^2 F_2^2 F_4^2 + a_2 F_1^2 F_3^2 F_4^2 + a_3 F_2^2 F_3^2 = 0.$$

We also denote by $x_i = z_i / z_4$ inhomogeneous coordinates in the chart

$$V_4 = \mathbb{P}^3 \setminus P_4 \simeq \mathbb{C}^3$$

and put

$$S_{a_1,a_2,a_3} = \overline{S}_{a_1,a_2,a_3} \cap V_4.$$

Claim 1. (i) *For almost all points* (a_1, a_2, a_3) *the proper transforms*

$$\sigma^{-1}(S_{a_1,a_2,a_3}) \subset \sigma^{-1}(V_4)$$

are nonsingular surfaces.

(ii) *For* $j = 1, 2, 3$ *and almost all* (a_1, a_2, a_3) *the intersections*

$$\sigma^{-1}(S_{a_1,a_2,a_3}) \cap E_j$$

are nonsingular 2*-sections of the ruled surfaces* E_j*, and the intersections*

$$\sigma^{-1}(s) \cap \sigma^{-1}(S_{a_1,a_2,a_3}) \cap E_j$$

consist of two points.

(iii) *The base locus of the linear system* $\sigma^{-1}(S_{a_1,a_2,a_3})$ *consists of three rational curves lying in* E*, their images after blowing down of the divisors* E_1*,* E_2 *and* E_3 *are lines in the exceptional divisor* E' *of the blow-up of* s*. In some analytic neighborhood of* E'*, after the blow-down the image of a generic surface of this linear system is decomposed into three irreducible nonsingular components each of which intersects transversally* E' *along one of three lines in* $E' \simeq \mathbb{P}^2$*, which is the image of the base locus of the linear system.*

Proof. In the coordinates x_1, x_2, x_3 the linear system of the surfaces S_{a_1,a_2,a_3} is given by

$$x_1(x_3 + c_1x_2 - c_3x_1^2)(x_3 + c_2x_2 - c_3x_1^2) + a_1x_1^2(x_3 + c_1x_2 - c_3x_1^2)^2 \\ + a_2x_1^2(x_3 + c_2x_2 - c_3x_1^2)^2 + a_3(x_3 + c_1x_2 - c_3x_1^2)^2(x_3 + c_2x_2 - c_3x_1^2)^2 = 0.$$

Let us take new coordinates in V_4:

$$y_1 = x_1, \qquad y_2 = x_3 + c_1x_2 - c_3x_1^2, \qquad y_3 = x_3 + c_2x_2 - c_3x_1^2.$$

In these coordinates the linear system of the surfaces S_{a_1,a_2,a_3} is given by

$$y_1y_2y_3 + a_1y_1^2y_2^2 + a_2y_1^2y_3^2 + a_3y_2^2y_3^2 = 0,$$

that is, for almost all points (a_1, a_2, a_3) the surfaces $S_{a_1,a_2,a_3} \cap V_4$ are affine parts of the images of Veronese surfaces under a generic projection into $\mathbb{P}^3$ for which the claim is well known.

To show that a generic member $\widetilde{X}$ of the linear system $|m\widetilde{P} - 3E - 2E_1 - 2E_2 - 2E_3|$ is a nonsingular surface for $m \geqslant 8$, let us note that the surfaces

$$\widetilde{S}_{a_1,a_2,a_3} = \sigma^{-1}(\overline{S}_{a_1,a_2,a_3})$$

belong to the linear system $|8\widetilde{P} - 3E - 2E_1 - 2E_2 - 2E_3|$. It follows from Claim 1 that for any point $p \subset \mathbb{P}^3$, $p \neq s$, after replacing the coordinate plane P_4 by another plane not passing through p, the base locus of the linear system $|m\widetilde{P} - 3E - 2E_1 - 2E_2 - 2E_3|$ does not meet the proper transform $\sigma^{-1}(p)$. Consequently, for any $m \geqslant 8$ the linear system $|m\widetilde{P} - 3E - 2E_1 - 2E_2 - 2E_3|$ has the same base locus as for $m = 8$, since the linear system $|(m-8)\widetilde{P}|$ has no base points. Finally, it follows from Bertini's theorem that the generic member $\widetilde{X}$ of $|m\widetilde{P} - 3E - 2E_1 - 2E_2 - 2E_3|$ satisfies the same properties (i)–(iii) in Claim 1 as the generic member $\sigma^{-1}(S_{a_1,a_2,a_3})$ has. From this it follows that the image $Y = \sigma(\widetilde{X})$ of the generic member $\widetilde{X}$ is a surface with ordinary singularities the double curve of which is D.

§ 9. Concluding remarks

In this section, we formulate some open questions relating to the existence problem of complete degenerations of surfaces with ordinary singular points.

9.1. Absolute and relative complete nondegeneracy. Let X be a smooth projective surface and $g : X \to \mathbb{P}^3$ some 'immersion' (that is, $Y = g(X)$ is a surface with ordinary singularities and the morphism $g\colon X \to Y$ is a normalization of Y). In §§ 5.1 and 8.5 we gave examples of surfaces Y which can not be completely degenerate. The reason for the impossibility to be completely degenerate can originate from the possibility that the 'immersion' g is 'bad', while for some other 'immersion' of X its image, nevertheless, can be completely degenerated. The second possible case is when for all 'immersions' of a surface X in $\mathbb{P}^3$ it is impossible to completely degenerate its image Y, in other words, the reason for the impossibility of complete degenerations lies in the topology of X. In the first case we say that X is *relatively completely degenerative*, and in the second case X is *absolutely completely nondegenerative*. We say also that X is *absolutely completely degenerative* if for any 'immersion' g its image $g(X) = Y$ is completely degenerative.

Problem 1. (i) Do there exist absolutely completely nondegenerative surfaces X?
(ii) Do there exist absolutely completely degenerative surfaces X?

In the case of a negative answer to any of these problems the normalizations X of completely nondegenerative and completely degenerative surfaces Y described in the article would give examples of relatively completely degenerative surfaces.

9.2. Problem of adjacency. We denote by $\mathscr{H}_{\mathrm{type}(Y)} \subset \mathbb{P}^{\binom{m+3}{3}-1}$ the quasi-projective variety parametrizing the surfaces with ordinary singularities of the same type as the type of a surface Y, $\deg Y = m$. Let $\Pi_m \subset \mathbb{P}^{\binom{m+3}{3}-1}$ be the variety parametrizing arrangements of m planes in $\mathbb{P}^3$ in general position, $\dim \Pi_m = 3m$. It follows from the complete degeneracy of Y that Π_m and the closure of $\mathscr{H}_{\mathrm{type}(Y)}$ have nonempty intersection.

Problem 2. Let Y be a completely degenerative surface with ordinary singularities. Is it true that Π_m lies in the closure of $\mathscr{H}_{\mathrm{type}}(Y)$?

This is a part of the following more general problem: to describe the natural stratification (according to the types of double curves) of the variety $\mathscr{H}_m$ of surfaces in $\mathbb{P}^3$ with ordinary singularities and the adjacencies of these strata.

9.3. Complete degeneracy uniqueness problem. We say that pairs $(\mathscr{P}, \mathscr{D}_1)$ and $(\mathscr{P}, \mathscr{D}_2)$ are *deformation equivalent* if these pairs are fibres of flat families $\mathscr{D}_u \subset \mathscr{P}_u \subset \mathbb{P}^3$, $u \in U$, of plane arrangements in general position and the configurations of double lines contained in the plane arrangements. It is obvious that pairs $(\mathscr{P}_1, \mathscr{D}_1)$ and $(\mathscr{P}_2, \mathscr{D}_2)$ are deformation equivalent if and only if $\deg \mathscr{P}_1 = \deg \mathscr{P}_2$ and the graphs $\Gamma(\mathscr{D}_1)$ and $\Gamma(\mathscr{D}_2)$ are isomorphic.

Let Y be a surface with ordinary singularities. We say that Y has *a unique complete degeneration* if it is completely degenerative and any two of its complete degenerations are deformation equivalent.

Claim 2. *Any surface Y with ordinary singularities whose double curve is of degree at most 4 possesses not more than a unique complete degeneration.*

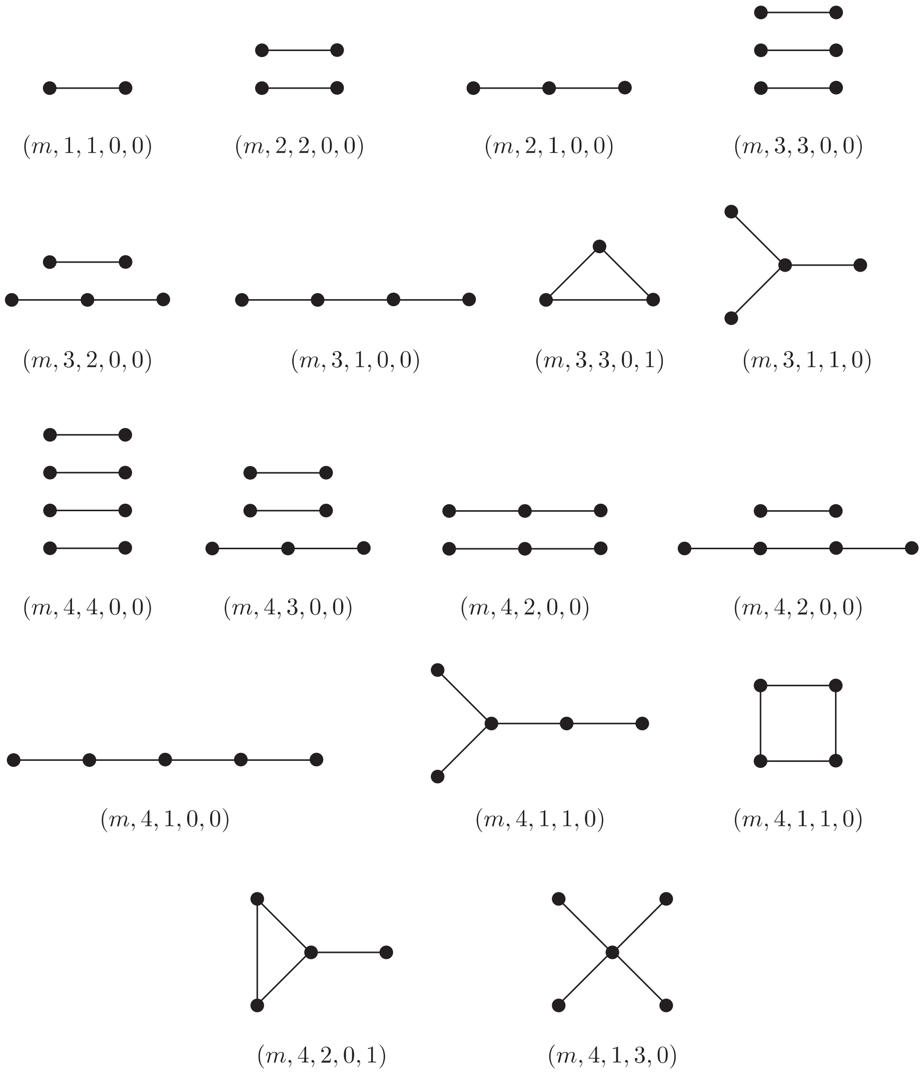


Figure 5

Proof. All possible realizable graphs with the number of edges not more than four are depicted in Fig. 5. The type $(m, \overline{d}, k, \overline{g}, t)$ of a surface Y in which the plane arrangement $\mathscr{P}_m$ in general position can be smoothed outside the configuration of corresponding double curve $\mathscr{D}$ is written under each graph.

One can see from this list of graphs that a surface Y can have more than one complete degeneration only in two cases, when type$(Y) = (m, 4, 2, 0, 0)$ or type$(Y) = (m, 4, 1, 1, 0)$. But in the first case graphs $\Gamma(\mathscr{D})$ have different types $\Gamma^{(2,0)^2}_{(2,1)^2}$ and $\Gamma^{(1,0)(3,0)}_{(2)(2,2)}$, and if a plane arrangement $\mathscr{P}$ is smoothed outside $\mathscr{D}$ with graph $\Gamma^{(2,0)^2}_{(2,1)^2}$, then the double curve D of Y consists of two irreducible components of degree two; while if the smoothing takes place outside $\mathscr{D}$ with graph $\Gamma^{(1,0)(3,0)}_{(2)(2,2)}$, then the irreducible components of D have degrees one and three, that is, in this case the regenerated surfaces Y have different (extended) types. In the second case the graphs $\Gamma(\mathscr{D})$ have the same type $\Gamma^{(4,0)}_{(3,1,1)}$, but the plane arrangements $\mathscr{P}$ cannot be smoothed outside one of configurations of double curves corresponding to this type (see Theorem 5). The proof is complete.

Problem 3. Does any surface Y possess not more than a single complete degeneration?

9.4. Smoothings in the symplectic case. Above we gave many examples of pairs $(\mathscr{P},\mathscr{D})$ which cannot be smoothed outside $\mathscr{D}$. In some cases the obstructions to smoothing were purely topological (for example, the negativity of degree of the dual curve of the branch curve $B \subset \mathbb{P}^2$ of a generic projection to the plane of a smoothed surface Y); in the other cases the obstructions, possibly, have an algebraic-geometric nature (for example, the pairs $(\mathscr{P},\mathscr{D})$ with the curve $\mathscr{D}$ whose graph is depicted in Fig. 4).

To understand better the nature of these obstructions, it is useful to generalize the problem of complete degenerations of algebraic surfaces with ordinary singularities to the case of symplectic varieties. Namely, we say that a compact real four-dimensional subvariety M of $\mathbb{CP}^3$ is a *symplectic variety with ordinary singularities* if for each point $p \in M$ there is a neighbourhood V of p such that

- either the variety $V \cap M$ is decomposed into n smooth components ($n \leqslant 3$) which are symplectic submanifolds of $\mathbb{CP}^3$ with respect to the Fubini-Studi symplectic form and meet transversally along smooth symplectic surfaces (‘double curves’ of M; in the case $n = 3$ the point p is a triple point of M),
- or $V \cap M$ is a complex analytic variety given in some complex analytic coordinates in V by equation $x^2 - yz^2 = 0$ (and in this case the point p is called a *pinch* of M).

The definition of complete degeneration of varieties with ordinary singularities can also be generalized for symplectic varieties.

Problem 4. Do there exist pairs $(\mathscr{P},\mathscr{D})$ such that the plane arrangement $\mathscr{P}$ cannot be smoothed outside $\mathscr{D}$ in the context of algebraic geometry, but this arrangement can be smoothed outside $\mathscr{D}$ in the symplectic context?

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