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Automorphisms of Galois coverings of generic m-canonical projections

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Abstract. We investigate the automorphism groups of Galois coverings induced by pluricanonical generic coverings of projective spaces. In dimensions one and two, it is shown that such coverings yield sequences of examples where specific actions of the symmetric group S_d on curves and surfaces cannot be deformed together with the action of S_d into manifolds whose automorphism group does not coincide with S_d. As an application, we give new examples of complex and real G-varieties which are diffeomorphic but not deformation equivalent.

Keywords: generic coverings of projective lines and planes, Galois group of a covering, Galois extensions, automorphism group of a projective variety.

> Perhaps the simplest combinatorial entity is the group of the $n!$ permutations of n things. This group has a different constitution for each individual number n. The question is whether there are nevertheless some asymptotic uniformities prevailing for large n or for some distinctive class of large n. Mathematics has still little to tell about such problems.
>
> *H. Weyl* [1]

Introduction

0.1. Terminological conventions. By a *covering* we understand a branched covering, that is, a finite morphism $f\colon X \to Y$ from a normal irreducible projective variety X onto a non-singular projective variety Y, everything being defined over the field $\mathbb{C}$ of complex numbers. With every covering f we associate the branch locus $B \subset Y$, the ramification locus $R \subset f^{-1}B \subset X$, and the unramified part $X \setminus f^{-1}(B) \to Y \setminus B$, which is the maximal unramified subcovering of f. Being unramified, it determines a homomorphism ψ from the fundamental group $\pi_1(Y \setminus B)$ to the symmetric group S_d acting on a set of d elements, where d is the degree of f. This homomorphism ψ (called the *monodromy* of f) is uniquely determined by f up to an inner automorphism of S_d. Conversely, the Grauert–Remmert–Riemann extension theorem implies that the conjugacy class of ψ uniquely determines the

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covering f up to isomorphism. The image $\operatorname{im}\psi = G \subset S_d$ is a transitive subgroup of S_d.

It is well known that for every covering f there is a unique (up to isomorphism) minimal Galois covering $\tilde{f}\colon \widetilde{X} \to Y$ which factors through f: $\tilde{f} = f \circ h$, where $h\colon \widetilde{X} \to X$ is again a Galois covering. The covering $\tilde{f}$ is called the *Galois expansion of* f. The Galois expansion $\tilde{f}$ is characterized by the minimality property: any Galois covering which factors through f can be factored through $\tilde{f}$. Using the Grauert–Remmert–Riemann extension theorem, one can obtain the Galois expansion of f from the non-diagonal component of the fibre product of d copies of the unramified part $X\setminus f^{-1}(B) \to Y\setminus B$ of f. In particular, the Galois group $\operatorname{Gal}(\widetilde{X}/Y)$ of the covering $\tilde{f}\colon \widetilde{X} \to Y$ is naturally identified with $G = \psi(\pi_1(Y \setminus B))$.

In this paper we study actions of finite groups on the Galois expansions of generic coverings of projective spaces. To give a precise definition of a generic covering, we introduce some preliminary definitions concerning actions of symmetric groups.

Let I be a finite set consisting of $|I| = d$ elements and let $I_1 \cup \dots \cup I_k = I$ be a partition of I, $|I_i| = d_i \geqslant 1$, $\sum_{i=1}^k d_i = d$. Such a partition determines a unique (up to conjugation) embedding of $S_{d_1} \times \dots \times S_{d_k}$ into the symmetric group S_d, which is referred to as the *standard embedding.* A representation $S_{d_i} \subset \mathrm{GL}(V_i)$, where V_i is a vector space over $\mathbb{C}$, is called a *standard representation of rank* $d_i - 1$ if there is a basis $e_1, \dots, e_{r_i}$ of V_i with $r_i = \dim V_i \geqslant d_i - 1$ such that the action $\sigma_{(j,j+1)}$ of the transposition $(j, j+1) \in S_{d_i}$ is given by

$$\sigma_{(j,j+1)}(e_l) = \begin{cases} e_l & \text{if } l \neq j, j+1, \\ e_{j+1} & \text{if } l = j, \\ e_j & \text{if } l = j+1 \end{cases}$$

for $j \neq d_i - 1$ and by

$$\sigma_{(d_i-1,d_i)}(e_l) = \begin{cases} e_l & \text{if } l \neq d_i - 1, \\ -\sum_{s=1}^{d_i-1} e_s & \text{if } l = d_i - 1 \end{cases}$$

for $j = d_i - 1$. A set of standard representations of the symmetric groups $S_{d_i} \subset \mathrm{GL}(V_i)$ of ranks $d_i - 1$, $i = 1, \dots, k$, determines a representation $S_{d_1} \times \dots \times S_{d_k} \subset \mathrm{GL}(V)$ with $V = V_1 \oplus \dots \oplus V_k$, which will be referred to as a *standard representation of the product* $S_{d_1} \times \dots \times S_{d_k}$ *of rank* $\sum d_i - k$. It is easy to see that if $S_{d_1} \times \dots \times S_{d_k} \subset \mathrm{GL}(V)$ is a standard representation of rank $\sum d_i - k$, then the codimension in V of the subspace consisting of all vectors that are fixed under the action of $S_{d_1} \times \dots \times S_{d_k}$ is equal to $\sum d_i - k$.

Let the group S_d act on a projective manifold Y. We say that the action of S_d on Y is *generic* if the stabilizer $\mathrm{St}_a \subset S_d$ of each point $a \in Y$ is a standard embedding of a product of symmetric groups and the action induced by St_a on the tangent space T_aY is a standard representation (the product and the representation both depend on a). By this definition, if the action of S_d on Y is standard, then the quotient space Y/S_d is a projective manifold.

A covering $f\colon X \to \mathbb{P}^{\dim X}$ of degree d is said to be *generic* if the Galois group $G = \operatorname{Gal}(\widetilde{X}/\mathbb{P}^{\dim X})$ of the Galois expansion of f is the full symmetric group S_d, the varieties X and $\widetilde{X}$ are smooth and the action of G on $\widetilde{X}$ is generic. This definition implies that, for any generic covering f of degree d, the group $\mathrm{Gal}(\widetilde{X}/\mathbb{P}^{\dim X})$ is the full symmetric group S_d and the subgroup $\mathrm{Gal}(\widetilde{X}/X)$ coincides with $S_{d-1} \subset S_d$.

Suppose that X is non-singular and $\dim X = 1$. In this case, a covering $f\colon X \to \mathbb{P}^1$ branched over $B \subset \mathbb{P}^1$ is generic if and only if $|f^{-1}(b)| = \deg f - 1$ for every $b \in B$. For generic coverings, the stabilizer $\mathrm{St}_{\tilde b} \subset S_d = \mathrm{Gal}(\widetilde{X}/\mathbb{P}^1)$ of every point $\tilde b \in \tilde f^{-1}(b)$, $b \in B$, is generated by a transposition. This determines a standard embedding of $\mathrm{St}_{\tilde b} = S_2$ into S_d.

Suppose that X is non-singular and $\dim X = 2$. In this case, the covering $f\colon X \to \mathbb{P}^2$ is generic if and only if the following conditions hold: f is branched over a cuspidal curve $B \subset \mathbb{P}^2$, we have $|f^{-1}(b)| = \deg f - 1$ for each non-singular point b of B and $|f^{-1}(b)| = \deg f - 2$ if b is a node or a cusp of B. For generic coverings, the stabilizer $\mathrm{St}_{\tilde b} \subset S_d = \mathrm{Gal}(\widetilde{X}/\mathbb{P}^2)$ of every point $\tilde b \in \tilde f^{-1}(b)$, $b \in B$, is generated: by a transposition if b is non-singular point of B, and then $\mathrm{St}_{\tilde b} = S_2$; by two non-commuting transpositions if b is a cusp of B, and then $\mathrm{St}_{\tilde b} = S_3$; by two commuting transpositions if b is a node of B, and then $\mathrm{St}_{\tilde b} = S_2 \times S_2$ (see § 3.1 for a detailed exposition).

Whatever the dimension of the covering, the automorphism group $\mathrm{Aut}(\widetilde{X})$ of the variety $\widetilde{X}$ contains the symmetric group S_d, but the following examples show that one cannot generally expect $\mathrm{Aut}(\widetilde{X})$ and S_d to be equal.

As a first example, we take a generic covering $f_1\colon Y = \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d+1$ and let $\tilde f\colon \widetilde{Y} \to \mathbb{P}^1$ be the Galois expansion $\tilde f = f_1 \circ h_1$ of f_1. Then $\mathrm{Gal}(\widetilde{Y}/\mathbb{P}^1) = S_{d+1}$, $Y = \widetilde{Y}/S_d$ and $h_1\colon \widetilde{Y} \to Y = \mathbb{P}^1$ is a Galois covering with Galois group $\mathrm{Gal}(\widetilde{Y}/Y) = S_d$. The covering $h_1\colon \widetilde{Y} \to Y = \mathbb{P}^1$ may be regarded as the Galois expansion $\widetilde{X} = \widetilde{Y} \to Y = \mathbb{P}^1$ (with Galois group $\mathrm{Gal}(\widetilde{X}/\mathbb{P}^1) = S_d$) of a covering $f\colon X \to Y = \mathbb{P}^1$, $X = \widetilde{X}/S_{d-1}$. Then f is a generic covering of degree d. Hence, if we start from $f\colon X \to \mathbb{P}^1$, we see that its Galois group $\mathrm{Gal}(\widetilde{X}/\mathbb{P}^1) = S_d$ does not coincide with $\mathrm{Aut}(\widetilde{X})$ because the group $\mathrm{Aut}(\widetilde{X}) = \mathrm{Aut}(\widetilde{Y})$ contains at least the group S_{d+1}.

Another example may be obtained as follows. Let $f_1\colon Y = \mathbb{P}^1 \to \mathbb{P}^1$ be a generic covering of degree d branched over some $B_1 \subset \mathbb{P}^1$. Take points $x, y \in \mathbb{P}^1$ outside B_1 and let $f_2\colon Z = \mathbb{P}^1 \to \mathbb{P}^1$ be a cyclic covering of degree p branched at x and y. Consider the fibre product $X = Y \times_{\mathbb{P}^1} Z$ and its projection $f\colon X \to Z = \mathbb{P}^1$ onto the second factor. It is easy to see that f is a generic covering and $\mathrm{Aut}(\widetilde{X})$ contains $\mathrm{Gal}(\widetilde{X}/\mathbb{P}^1) \times \mathbb{Z}/p\mathbb{Z}$.

0.2. Main results. The aim of our research is to give numerical conditions on a generic covering $f\colon X \to \mathbb{P}^{\dim X}$ which ensure that $\mathrm{Aut}(\widetilde{X}) = \mathrm{Gal}(\widetilde{X}/\mathbb{P}^{\dim X})$ and which are preserved under deformations of the Galois expansion.

To state the results obtained, we introduce another auxiliary notion. A covering $f\colon X \to \mathbb{P}^{\dim X}$ is said to be *numerically m-canonical* (*m-canonical* for short) if it is given by $\dim X + 1$ sections (without common zeros) of a line bundle numerically equivalent to the mth power $K_X^{\otimes m}$ of the canonical bundle K_X of X.

It is known that m-canonical coverings exist only if the Kodaira dimension of X coincides with the dimension of X. If $\dim X = 2$, then X must also be minimal and contain no (-2)-curves. If $\dim X = 1$, then the genus of X must be greater than or equal to 2. Conversely, it is well known that any curve of genus $g \geqslant 2$

possesses an m-canonical covering for $m \geqslant 1$ and, as shown in [2], any minimal surface of general type containing no (-2)-curves also possesses an m-canonical covering, at least for $m \geqslant 10$.

Theorem 0.1. *Let X be a curve of genus $g \geqslant 2$ and $\tilde{f}\colon \widetilde{X} \to \mathbb{P}^1$ the Galois expansion of an m-canonical generic covering $f\colon X \to \mathbb{P}^1$. If $m(g-1) \geqslant 500$, then the Galois group $\operatorname{Gal}(\widetilde{X}/\mathbb{P}^1)$ is the full automorphism group of $\widetilde{X}$.*

Theorem 0.2. *Let X be a surface of general type and assume that it possesses an m-canonical generic covering $f\colon X \to \mathbb{P}^2$, $m \geqslant 2$. If $m^2K_X^2 \geqslant 2\cdot 84^2$ and $\tilde{f}\colon \widetilde{X} \to \mathbb{P}^2$ is the Galois expansion of f, then the Galois group $\operatorname{Gal}(\widetilde{X}/\mathbb{P}^2)$ is the full automorphism group of $\widetilde{X}$.*

As a corollary, we see that the G-curves in Theorem 0.1 and G-surfaces in Theorem 0.2 provide infinitely many examples of *saturated* connected components of the moduli space of G-varieties with $G = S_d$. Here a component is said to be *saturated* if, for any G-variety representing a point of this component, G is the full automorphism group of V. (It may be worthwhile to point out other simple examples of saturated components with $G \neq S_d$. These are the components given by the curves and surfaces whose automorphism group has maximal possible order, $84(g-1)$ for curves and 42^2K^2 for surfaces. One might also mention deformation-rigid varieties with non-trivial automorphism groups.)

As another application of the theorems above, we give counterexamples to the Dif = Def problem for complex and real G-varieties. Namely, we construct pairs of complex (resp. real) varieties V_1, V_2 such that the actions of $\operatorname{Aut} V_1$ and $\operatorname{Aut} V_2$ (resp. $\operatorname{Kl} V_1$ and $\operatorname{Kl} V_2$, where $\operatorname{Kl} V$ is the group formed by the regular isomorphisms $X \to X$ and $X \to \overline{X}$) are diffeomorphic but not deformation equivalent. As far as we know, such examples are new, especially in real geometry. (It may be worth noting that the surfaces in our counterexamples [3] to the Dif = Def problem for real structures have diffeomorphic real structures, but the actions of the Kleinian groups on these surfaces are not diffeomorphic.)

0.3. Contents of the paper. The proofs of Theorems 0.1 and 0.2 consist of two parts. First (in § 1) we use the methods of group theory to study minimal expansions of symmetric groups. This restricts the number of possible cases. Then the remaining cases are investigated by geometric methods (in § 2 for Theorem 0.1 and in § 3 for Theorem 0.2). In § 4 we describe the applications mentioned above.

§ 1. Minimal expansions of symmetric groups

1.1. Preliminary definitions. To state the group-theoretic assertions used in the proofs of Theorems 0.1 and 0.2, we introduce some definitions. We say that a group G containing the symmetric group S_d possesses the *minimality property* if there is no proper subgroup G_1 of G that contains S_d and does not coincide with S_d. Such a group G is called a *minimal expansion of S_d*.

An embedding $\alpha\colon S_d \subset S_{d+2}$ is said to be *quasi-standard* if the image $\alpha(\sigma_{i,j})$ of every transposition $\sigma_{i,j} = (i,j) \in S_d$, $1 \leqslant i,j \leqslant d$, is the product $\alpha(\sigma_{i,j}) = (i,j)(d+1,d+2)$ of two transpositions, (i,j) and $(d+1,d+2)$, in S_{d+2}. We note that the image of S_d under a quasi-standard embedding is contained in the alternating subgroup A_{d+2} of S_{d+2}. Such an embedding $\alpha\colon S_d \hookrightarrow A_{d+2}$ is said to be *standard*.

Proposition 1.1. *Let G be a minimal expansion of the symmetric group S_d of index $k=(G:S_d)$. Assume that $k\leqslant cd^n$, where either* (i) *$c=63$ and $n=1$, or* (ii) *$c=(2\cdot 42)^2$ and $n=2$. If $d\geqslant \max(2c,1000)$, then G is one of the following groups.*

1) *$G=S_d\times \mathbb{Z}/p\mathbb{Z}$, where p is a prime and $p\leqslant cd^n$;*

2) *$G=A_d\rtimes D_r$, where $3\leqslant r\leqslant cd^n$, r is odd, D_r is the dihedral group given by the presentation*

$$D_r=\langle \sigma,\tau \mid \sigma^2=\tau^2=(\sigma\tau)^r=1\rangle,$$

and the action (*by conjugation*) *of σ and τ on A_d coincides with that of the transposition $(1,2)\in S_d$ on $A_d\subset S_d$;*

3) *$G=S_{d+1}$ is the symmetric group;*

4) *$G=A_{d+2}$ is the alternating group, the embedding of S_d in A_{d+2} is standard, and such expansions can appear only under assumption* (ii).

The rest of this section is devoted to the proof of Proposition 1.1.

Proof. A priori, one of the following two cases occurs.

Case I. *The group G contains a non-trivial normal subgroup.*

Case II. *The group G is simple.*

Since $d>6$, the group S_d has a unique non-trivial normal subgroup (namely, the alternating group A_d) and, therefore, Case I can be subdivided into the following subcases, where N stands for the non-trivial normal subgroup of G:

Case I_1. $S_d\subset N$.

Case I_2. $N\cap S_d=\{1\}$.

Case I_3. $N\cap S_d=A_d$.

Case I_3 can be further subdivided into two subcases.

Case I_{31}. *A_d is a normal subgroup of G.*

Case I_{32}. *A_d is not a normal subgroup of G.*

1.2. Analysis of Case $\mathbf{I_1}$. It follows from the minimality property that $S_d=N$. Let g_1 be an arbitrary element of $G\setminus S_d$. Then conjugation by g_1 induces an automorphism of S_d. Since $d\geqslant 7$, all automorphisms of S_d are inner. Hence there is a $g_2\in S_d$ such that the element $g=g_1g_2$ commutes with all elements of S_d. Again applying the minimality property, we see that G splits into the direct product of S_d and the cyclic group $\langle g\rangle$ generated by the element g. Moreover, the order of g is a prime number p.

1.3. Analysis of Case $\mathbf{I_{31}}$. By § 1.2, we can assume that S_d is not a normal subgroup of G. Hence there is a $g\in G$ such that the group $S_d'=g^{-1}S_dg$ does not coincide with S_d (but is isomorphic to S_d).

Since A_d is a normal subgroup of G, we have $A_d\subset S_d'\cap S_d$. Furthermore, for any transposition $\sigma\in S_d$, the element $\tau=g^{-1}\sigma g$ (which will be called a transposition in S_d') does not belong to S_d. Thus, it follows from the minimality property that G is generated by elements of A_d and any two transpositions, $\sigma\in S_d$ and $\tau\in S_d'$. Moreover, since conjugation by elements of S_d (resp. S_d') provides the full automorphism group $\mathrm{Aut}(A_d)$ of A_d, we can choose the two generating transpositions $\sigma \in S_d$ and $\tau \in S'_d$ in such a way that $\sigma\tau$ commutes with all elements of A_d. We can also assume that the action (by conjugation) of σ and τ on A_d coincides with the action of the transposition $(1,2) \in S_d$.

Let H denote the subgroup of G generated by σ and τ. Then H is isomorphic to the dihedral group

$$D_r = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^r = 1 \rangle$$

for some $r \in \mathbb{N}$.

It is known that every element $g \in D_r$ either belongs to the cyclic subgroup generated by $\sigma\tau$, or is conjugate to σ or τ. Therefore $A_d \cap H$ is a subgroup of $\langle \sigma\tau \rangle$. Since the element $\sigma\tau$ commutes with all elements of A_d and the centre of A_d is trivial, we conclude that $A_d \cap H = \{1\}$. In addition, A_d is a normal subgroup of G, and G possesses the minimality property, whence

$$G = A_d \rtimes H \simeq A_d \rtimes D_r.$$

Moreover, r is odd since σ and τ are conjugate in G and, therefore, in D_r.

1.4. Analysis of Case $\mathbf{I_2}$. It follows from the minimality property that in this case the group G is isomorphic to the semidirect product $N \rtimes S_d$.

If N is not a simple group, then we can find a minimal non-trivial normal subgroup N_1 of N. First note that N_1 cannot be a normal subgroup of G since G possesses the minimality property. Therefore the set of subgroups of G conjugate to N_1 contains more than one element. Let $\{N_1, \dots, N_s\}$ be the set of subgroups conjugate to N_1 in G, $s \geqslant 2$. Every group N_i, $1 \leqslant i \leqslant s$, is contained in N since N is a normal subgroup of G. Moreover, each of them is a normal subgroup of N since N_1 is normal in N and conjugation by any element of G induces an isomorphism of N. The action of S_d on the set $\{N_1, \dots, N_s\}$ is transitive and this set is an orbit of the action of S_d by conjugation on the set of all subgroups of G.

We claim that $N \simeq N_1 \times \dots \times N_{s_1}$ for some $s_1 \leqslant s$ (possibly after a rearrangement of the N_i).[1] We first note that $N_i \cap N_j = \{1\}$ for $i \neq j$. Indeed, $N_1, \dots, N_s$ are minimal normal subgroups of N and the intersection $N_i \cap N_j$, being an intersection of two normal subgroups, is a normal subgroup. Therefore $[N_1, N_2] \subset N_1 \cap N_2 = \{1\}$. Thus the subgroup N_1N_2, which is generated in N by the elements of N_1 and N_2, is isomorphic to $N_1 \times N_2$ and is again a normal subgroup of N. Using induction, we take some $i < s$ and assume that the subgroup $N_{1,i} = N_1 \cdots N_i$ of N is normal and is isomorphic to $N_1 \times \dots \times N_i$. Then either $N_{1,i} \cap N_{i+1} = \{1\}$, or $N_{i+1} \subset N_{1,i}$. (Here we again use the observation that N_{i+1} is a minimal normal subgroup of N.) If $N_{1,i} \cap N_{i+1} = 1$, then

$$N_{1,i+1} = N_1 \cdots N_{i+1} \simeq N_1 \times \dots \times N_{i+1}$$

since $[N_{1,i}, N_{i+1}] \subset N_{1,i} \cap N_{i+1} = 1$. This inductive procedure stops at some subgroup $N_{1,s_1} \subset N$ which, being normal in N and invariant under the action

[1] *Editor's Note.* The statement that a minimal normal subgroup of a finite group is a direct product of isomorphic simple groups is well known to specialists in group theory (see, for example, [B. Huppert, *Endliche Gruppen* I, Springer-Verlag, Berlin–Heidelberg 1967], Lemma 4.8 a) on p. 21 and Theorem 9.12 a) on p. 51).

of S_d by conjugation, is a normal subgroup of G. Therefore N_{1,s_1} coincides with N. Thus we have $N \simeq N_1 \times \cdots \times N_{s_1}$.

The groups $N_1, \dots, N_s$, which are isomorphic to each other, are simple. Indeed, if N_1 is not simple, then there is a non-trivial normal subgroup $\widetilde{N}_1$ of N_1, so that the group $\widetilde{N}_1 \times \{1\} \times \cdots \times \{1\}$ is a normal subgroup of N. But this contradicts the assumption that N_1 is a minimal normal subgroup of N.

We now claim that $s_1 = s$ if N_1 is a non-abelian simple group. Indeed, arguing by contradiction, suppose that $s_1 < s$. Then $N_{s_1+1} \subset N_1 \times \cdots \times N_{s_1}$, the projection of N_{s_1+1} onto each factor is either an isomorphism or the trivial homomorphism and at least two of them are isomorphisms. Without loss of generality, we can assume that the first two projections are isomorphisms. Thus, if an element $g \in N_{s_1+1}$ is written as a product $g = g_1 g_2 \cdots g_{s_1}$ of elements $g_i \in N_i$, then g_1 is uniquely determined by g_2. On the other hand, N_{s_1+1} is a normal subgroup of N and, therefore, for any $g = g_1 g_2 \cdots g_{s_1} \in N_{s_1+1}$ and any $h \in N_1$, the product $h^{-1} g h = (h^{-1} g_1 h) g_2 \cdots g_{s_1}$ belongs to N_{s_1+1}. This contradiction proves the claim.

If N_1 is a non-abelian simple group, then conjugation by the elements of S_d transitively permutes $N_1, \dots, N_s$ and determines a homomorphism $\psi\colon S_d \to S_s$. Since $d \geqslant 7$, there are only two possibilities: either $s \leqslant 2$ and $A_d \subset \ker\psi$, or ψ is an embedding and, therefore, $s \geqslant d$.

Suppose that N_1 is a non-abelian simple group and $s = 2$. Then conjugation by the elements of A_d determines a homomorphism $\psi\colon A_d \to \operatorname{Aut}(N_1)$. Since the outer automorphism group $\operatorname{Out}(N_1) = \operatorname{Aut}(N_1)/\operatorname{Inn}(N_1)$ is soluble for a simple group N_1 (see [4], Theorem 4.240) and the group A_d is simple, we see that $\psi(A_d) \subset \operatorname{Inn}(N_1)$, and either $\psi(A_d) = 1$ or $\psi\colon A_d \to \operatorname{Inn}(N_1)$ is an embedding in the inner automorphism group. If $\psi(A_d) = 1$, then every element of N_1 commutes with all elements of A_d and, therefore, A_d is a normal subgroup of G since G is generated by the elements of S_d together with a non-identity element $h \in N_1$ and h commutes with all elements of A_d. But this case has already been covered (Case I_{31}). If $\psi\colon A_d \to \operatorname{Inn}(N_1)$ is an embedding, then the order of N_1 is greater than $\frac{1}{2}d!$ and, therefore,

$$k = (G : S_d) \geqslant \left(\frac{1}{2}\, d!\right)^2.$$

But this is impossible because of the assumption that $d \geqslant 1000$ and k does not exceed either $63d$ or $(2 \cdot 42)^2 d^2$.

We now suppose that N_1 is a non-abelian simple group and $s \geqslant d$. Then, since A_5 is the smallest non-abelian simple group,

$$k = (G : S_d) = |N|^s \geqslant |N|^d \geqslant 60^d,$$

which is impossible for the same reason as above.

We finally suppose that N_1 is abelian. Then $N \simeq N_1 \times \cdots \times N_{s_1}$ and again conjugation by the elements of A_d determines a homomorphism $\psi\colon A_d \to \operatorname{Aut}(N_1)$. As above, if $\psi(A_d) = 1$, then A_d is a normal subgroup of G. This case has already been studied.

If $\psi\colon A_d \to \operatorname{Aut}(N_1)$ is an embedding, then Lemma 1.3 (see below) implies that $s_1 \geqslant [\frac{d}{4}]$. Therefore,

$$k = (G : S_d) = |N_1|^{s_1} \geqslant 2^{[\frac{d}{4}]}.$$

This contradicts the assumption that $d \geqslant 1000$ and k does not exceed either $63d$ or $(2 \cdot 42)^2 d^2$.

The rest of this subsection is devoted to a proof of Lemma 1.3, which is based on the following lemma.

Lemma 1.2. *Let F be a finite field of characteristic p and H a subgroup of $\mathrm{PGL}(F,n)$ isomorphic to the alternating group A_{4d_1}, $d_1 \in \mathbb{N}$. Then $n \geqslant d_1$.*

Proof. If $d_1 \leqslant 2$, the statement is obvious. Suppose that the lemma holds for $d_1 \leqslant k$ and put $d_1 = k+1$.

Consider the natural epimorphism $\psi\colon \mathrm{GL}(F,n) \to \mathrm{PGL}(F,n)$. The kernel of ψ consists of scalar matrices $\lambda\,\mathrm{Id}$, $\lambda \in F^*$. Therefore the group $\psi^{-1}(A_{4(k+1)})$ is a central extension of the group $A_{4(k+1)}$.

We write Ξ for the set of all 4-tuples $\{i_1,i_2,i_3,i_4\}$ of pairwise-distinct integers in the range $1 \leqslant i_j \leqslant 4(k+1)$. For every $I \in \Xi$, we denote by x_I the permutation $(i_1,i_2)(i_3,i_4) \in A_{4(k+1)}$. The permutations x_I, $I \in \Xi$, generate the group $A_{4(k+1)}$. Since $4(k+1) > 8$, any two of them are conjugate in $A_{4(k+1)}$. For every pair $I_1, I_2 \in \Xi$ we choose a word w_{I_1,I_2} in the letters x_I such that $x_{I_1} = w_{I_1,I_2}^{-1} x_{I_2} w_{I_1,I_2}$ in $A_{4(k+1)}$. Then we pick elements $\hat{x}_I \in \psi^{-1}(x_I) \subset \psi^{-1}(A_{k+1})$ and put

$$\widetilde{x}_{I_0} = \hat{x}_{I_0}, \qquad I_0 = \{4k+1,\, 4k+2,\, 4k+3,\, 4k+4\},$$
$$\widetilde{x}_I = \widehat{w}_{I,I_0}^{-1} \widetilde{x}_{I_0} \widehat{w}_{I,I_0}, \qquad I \neq I_0,$$

where $\widehat{w}_{I,I_0}$ is obtained from w_{I,I_0} by writing $\hat{x}_I$ instead of x_I.

For every $I \in \Xi$ we have $\widetilde{x}_I = \mu_I \hat{x}_I$, where the $\mu_I \in \ker\psi$ belong to the centre of $\mathrm{GL}(F,n)$. Therefore,

$$\widetilde{x}_I = \widetilde{w}_{I,I_0}^{-1} \widetilde{x}_{I_0} \widetilde{w}_{I,I_0},$$

where $\widetilde{w}_{I,I_0}$ is obtained from w_{I,I_0} by writing $\widetilde{x}_I$ instead of x_I. On the other hand, $x_I^2 = 1$ for every $I \in \Xi$, whence $\widetilde{x}_I^2 = \lambda_I \in \ker\psi$. Since the $\widetilde{x}_I$ are all conjugate and the $\widetilde{x}_I^2 = \lambda_I$ belong to the centre, the λ_I must all be equal. We denote this element by λ.

We regard $\mathrm{GL}(F,n)$ as a subgroup of $\mathrm{GL}(\overline{F},n)$, where $\overline{F}$ is the algebraic closure of the field F. Let $\tilde{A}_{4(k+1)}$ be the subgroup of $\mathrm{GL}(\overline{F},n)$ generated by the $\widetilde{x}_I$, $I \in \Xi$, together with the elements of the centre. It is easy to see that $\tilde{A}_{4(k+1)}$ is a central extension of $A_{4(k+1)}$.

$\tilde{A}_{4(k+1)}$ contains an element μ such that $\mu^2 = \lambda^{-1}$. Put $y_I = \mu\widetilde{x}_I$. Then $y_I^2 = 1$ and the elements y_I are all conjugate: $y_I = v_{I,I_0}^{-1} y_{I_0} v_{I,I_0}$, where v_{I,I_0} is obtained from w_{I,I_0} by writing y_I instead of x_I. We also have $x_I x_{I_0} = x_{I_0} x_I$ for all $I = \{i_1,i_2,i_3,i_4\} \in \Xi$, $1 \leqslant i_j \leqslant 4k$. It follows that $y_I y_{I_0} = \mu_I y_{I_0} y_I$ for some element μ_I of the centre of $\mathrm{GL}(\overline{F},n)$. Since $y_I^2 = \mathrm{Id}$, we have $\mu_I = \pm\mathrm{Id}$. Since the y_I are all conjugate by words depending on y_I, we see that the μ_I must all be equal. Hence the μ_I with $I \in \Xi$ are all equal to either $\mu = \mathrm{Id}$ or $\mu = -\mathrm{Id}$.

We claim that $\mu = \mathrm{Id}$. Indeed, consider the elements $y_{1,2,3,4}$, $y_{1,2,5,6}$ and put $\widetilde{y}_{3,4,5,6} = y_{1,2,3,4} y_{1,2,5,6}$. We have $\widetilde{y}_{3,4,5,6} = \lambda y_{3,4,5,6}$, where λ is a central element since $x_{3,4,5,6} = x_{1,2,3,4} x_{1,2,5,6}$. We also have $y_{I_0} y_{1,2,3,4} = \mu y_{1,2,3,4} y_{I_0}$ and $y_{I_0} y_{1,2,5,6} = \mu y_{1,2,5,6} y_{I_0}$. Hence, on the one hand,

$$y_{I_0} \widetilde{y}_{3,4,5,6} = y_{I_0} \lambda y_{3,4,5,6} = \lambda\mu y_{3,4,5,6} y_{I_0} = \mu \widetilde{y}_{3,4,5,6} y_{I_0},$$

and on the other,

$$y_{I_0}\widetilde{y}_{3,4,5,6} = y_{I_0}y_{1,2,3,4}y_{1,2,5,6} = \mu^2 y_{1,2,3,4}y_{1,2,5,6}y_{I_0} = \mu^2\widetilde{y}_{3,4,5,6}y_{I_0}.$$

Therefore $\mu = \mathrm{Id}$.

We denote by $\bar{A}_{4(k+1)}$ the subgroup of $\mathrm{GL}(\overline{F}, n)$ generated by all y_I, $I \in \Xi$. Its image in $\mathrm{PGL}(\overline{F}, n)$ is obviously equal to $A_{4(k+1)}$. Consider the subgroup $\bar{A}_{4k}$ of $\bar{A}_{4(k+1)}$ generated by the elements y_I, $I = \{i_1, i_2, i_3, i_4\} \in \Xi$ with $1 \leqslant i_j \leqslant 4k$. The elements of $\bar{A}_{4k}$ commute with y_{I_0}, and the image of $\bar{A}_{4k}$ in $\mathrm{PGL}(\overline{F}, n)$ is A_{4k}.

We first assume that the characteristic $p \neq 2$. Then the vector space $V = \overline{F}^n$ splits into a direct sum $E_+ \oplus E_-$ of eigenspaces corresponding to the eigenvalues ± 1 of y_{I_0}. Since y_{I_0} does not belong to the centre, we have $\dim E_+ \geqslant 1$ and $\dim E_- \geqslant 1$. Since the elements of $\bar{A}_{4k}$ commute with y_{I_0}, the eigenspaces $E_\pm$ are invariant under the action of A_{4k}. This action is non-trivial on at least one of these subspaces, say on E_+. Moreover, since A_{4k} is a simple group, this action induces an embedding of A_{4k} into $\mathrm{PGL}(E_+)$. By the induction hypothesis, we have $\dim E_+ \geqslant k$ and, therefore, $\dim V \geqslant k + 1 = d_1$.

We now suppose that $p = 2$. Then the subspace $E = \{v \in V \mid y_{I_0}(v) = v\}$ of V is invariant under the action of A_{4k} and $\dim E < \dim V$. If the action of A_{4k} on E is non-trivial, then $n = \dim V > \dim E \geqslant k$, that is, $n \geqslant k + 1 = d_1$.

To complete the proof, we claim that if the action of A_{4k} on E is trivial, then the induced action of A_{4k} on V/E is non-trivial. For if both actions are trivial, then we can choose a basis of V in such a way that every element $y \in A_{4k}$ is represented in this basis by a matrix of the form

$$y = \begin{pmatrix} \mathrm{Id}_a & A \\ 0 & \mathrm{Id}_b \end{pmatrix},$$

where $a = \dim E$, $b = \dim V - a$, A is an $a \times b$ matrix and 0 is the zero $b \times a$ matrix. But this is impossible because such matrices form an abelian group while the group A_{4k} is non-abelian. Thus we conclude that the action of A_{4k} on V/E is non-trivial and, therefore,

$$n = \dim V > \dim V/E \geqslant k,$$

whence $n \geqslant k + 1 = d_1$. The lemma is proved.

Lemma 1.3. *Let F be a finite field of characteristic p and H a subgroup of $\mathrm{GL}(F, n)$ isomorphic to the alternating group A_{4d_1}, $d_1 \in \mathbb{N}$. Then $n \geqslant d_1$.*

Proof. This follows from Lemma 1.2 since $\mathrm{PGL}(F, n)$ is the quotient group of $\mathrm{GL}(F, n)$ by its centre and the alternating group has trivial centre.

1.5. Analysis of Case $\mathbf{I_{32}}$. Since N is a normal subgroup and $N \cap S_d = A_d$, we see that the subgroup $\langle N, \sigma\rangle$ of G generated by the elements of N together with the transposition $\sigma \in S_d$ is isomorphic to the semidirect product $N \rtimes \langle\sigma\rangle$. The group S_d is contained in $\langle N, \sigma\rangle$ since $A_d \subset N$. Thus the minimality property of G implies that $G = N \rtimes \langle\sigma\rangle$.

We recall that, by assumption, A_d is not a normal subgroup of G.

We first suppose that the group N is not simple. Pick a minimal non-trivial subgroup N_1 of N. Then we have either $N_1 \cap A_d = \{1\}$ or $N_1 \cap A_d = A_d$ since A_d is simple.

If $N_1 \cap A_d = \{1\}$, then the group $N_2 = \sigma^{-1} N_1 \sigma$ is a normal subgroup of N and $N_2 \cap A_d = \{1\}$. If $N_1 = N_2$, then N_1 is a normal subgroup of G, and this case has already been studied (Case I_2). If $N_1 \neq N_2$, then $[N_1, N_2] \subset N_1 \cap N_2 = \{1\}$ and the group $N_1 N_2 \simeq N_1 \times N_2$ is a normal subgroup of G. The case when $N_1 N_2 \cap A_d = \{1\}$ is also contained in Case I_2. Therefore we can assume that $N = N_1 N_2$. Since $N_i \cap A_d = \{1\}$ for $i = 1, 2$, the projections of A_d onto the factors are embeddings. Therefore $|N_i| \geqslant |A_d| = \frac{d!}{2}$. Hence,

$$k = (G : S_d) = (N : A_d) \geqslant \frac{d!}{2},$$

which is impossible because of the assumptions that $d \geqslant 1000$ and k does not exceed either $63d$ or $(2 \cdot 42)^2 d^2$.

If $N_1 \cap A_d = A_d$, then $N_2 \cap A_d = A_d$, where $N_2 = \sigma^{-1} N_1 \sigma$. If $N_1 = N_2$, then N_1 is a normal subgroup of G. This case is contained in Case I_2. If $N_1 \neq N_2$, then $N_1 \cap N_2$ is a normal subgroup of N and we have $A_d \subset N_1 \cap N_2 \subset N_1$. This contradicts the original assumption that N_1 is a minimal non-trivial normal subgroup of N.

Thus it remains to treat the case when N is a simple group and $G = N \rtimes \langle \sigma \rangle$. Clearly, the group N cannot be cyclic.

If N is isomorphic to some alternating group A_{d_1}, then $d_1 - d = n_1 \geqslant 1$ and

$$k = (G : S_d) = (A_{d_1} : A_d) = (d+1) \cdots (d+n_1).$$

By hypothesis, we have $d \geqslant \max(2c, 1000)$ and $k \leqslant cd^n$, where $c = 63$, $n = 1$ (case (i)) or $c = (2 \cdot 42)^2$, $n = 2$ (case (ii)). Therefore $n_1 \leqslant 1$ in case (i) and $n_2 \leqslant 2$ in case (ii).

If $n_1 = 1$, then $G = S_{d+1}$ (and, moreover, the embedding of S_d in $G = S_{d+1}$ is standard).

Before completing the analysis of other simple groups, let us show that it is impossible to have $n_1 = 2$ under assumption (ii).

Lemma 1.4. *Any embedding $\alpha\colon A_d \to A_{d+2}$ is conjugate to the standard one if $d \geqslant 9$.*

Proof. We consider the standard actions of $A_d \subset S_d$ and $A_{d+2} \subset S_{d+2}$ on the sets $I_d = \{1, 2, \ldots, d\}$ and $I_{d+2} = \{1, 2, \ldots, d+2\}$ respectively. If $\tau \in A_d$ is a cyclic permutation of length 3, then its image $\alpha(\tau)$ is a product $\tau_1 \cdots \tau_s$ of pairwise disjoint cyclic permutations, and we have $\tau_i^3 = 1$ for all $i = 1, \ldots, s$. To prove the lemma, it suffices to show that $s = 1$ for every 3-cycle $\tau \in A_d$. There is no loss of generality in assuming that $\tau = (d-2, d-1, d)$.

Under the action of $\alpha(\tau)$, the set I_{d+2} splits into a disjoint union of s orbits $O_{3,i}$, $i = 1, \ldots, s$, of length 3 and $d+2-3s$ orbits $O_{1,i}$, $i = 1, \ldots, d+2-3s$, of length 1. Consider the subgroup A_{d-3} of A_d whose elements leave the points $d-2$, $d-1$, $d \in I_d$ fixed. The elements of A_{d-3} commute with τ. Hence the group $\alpha(A_{d-3})$ acts on the set of all orbits $O_{3,i}$ and on the set of all orbits $O_{1,i}$. This action determines a homomorphism $\beta\colon A_{d-3} \to S_s \times S_{d+2-3s}$. But we have $s < d-3$ for $d > 9$, and if $s > 1$, then $d+2-3s < d-3$. Hence the homomorphism β is trivial for $s > 1$ because A_{d-3} is a simple group and $|A_{d-3}| > |S_s|$, $|A_{d-3}| > |S_{d+2-3s}|$ if $d > 9$. Therefore α induces a homomorphism from A_{d-3} to the direct product of s copies of S_3, and this induced homomorphism must also be trivial. Finally, we see that if $s > 1$, then α cannot be an embedding. The lemma is proved.

By Lemma 1.4, we can assume that $G \simeq A_{d+2} \rtimes \langle\sigma\rangle$ and the embedding $A_d \subset A_{d+2}$ is standard. We claim that in this case $G \simeq S_{d+2}$ and the embedding $S_d \subset G \simeq S_{d+2}$ is also standard, so the expansion $S_d \subset G$ of the symmetric group S_d does not possess the minimality property.

Indeed, consider the natural homomorphism

$$i\colon \operatorname{Inn}(G) \to \operatorname{Aut}(A_{d+2}) \simeq S_{d+2}.$$

Clearly, $i(A_{d+2}) = A_{d+2} \subset S_{d+2}$. To prove that $G \simeq S_{d+2}$, it suffices to show that $i(\sigma)$ is not an inner automorphism of A_{d+2} (we recall that σ is a transposition as an element of S_d). If $i(\sigma) \in \operatorname{Inn}(A_{d+2})$, then there is an element $\tau \in A_{d+2}$ such that the product $\gamma = \sigma\tau$ commutes with all elements of A_{d+2}. In particular, it commutes with τ and, therefore, it commutes with σ. Since $\sigma \notin A_{d+2}$, we have $\gamma = \sigma\tau \neq 1$ and the group $\langle S_d, \gamma\rangle$ (which is generated in G by γ and the elements of S_d) is isomorphic to $S_d \times \langle\gamma\rangle$. But the existence of such a subgroup in $G \simeq A_{d+2} \rtimes \langle\sigma\rangle$ contradicts the minimality property of G.

To prove that the embedding $j\colon S_d \subset G \simeq S_{d+2}$ is standard, we note that $j(\sigma)$ is a product $\sigma_1 \cdots \sigma_s$ of an odd number of pairwise disjoint transpositions $\sigma_i \in S_{d+2}$. We must show that $s = 1$. Assume that $s \geqslant 3$. As in the proof of Lemma 1.4, we consider the standard actions of S_d and S_{d+2} on the sets $I_d = \{1, 2, \dots, d\}$ and $I_{d+2} = \{1, 2, \dots, d+2\}$ respectively. Let $\sigma \in S_d$ be the transposition $(d-1, d)$. Under the action of $j(\sigma)$, the set I_{d+2} splits into the disjoint union of s orbits $O_{2,l}$ ($l = 1, \dots, s$) of length 2 and $d+2-2s$ orbits $O_{1,l}$ ($l = 1, \dots, d+2-2s$) of length 1. Consider the subgroup S_{d-2} of S_d which leaves fixed the elements $d-1, d \in I_d$. The elements of S_{d-2} commute with σ. Hence the group $j(S_{d-2})$ acts on the sets of orbits $O_{2,j}$ and $O_{1,j}$. This action determines a homomorphism $\beta\colon S_{d-2} \to S_s \times S_{d+2-2s}$. But we have $s < d-2$ (recall that $d \geqslant 1000$), and if $s \geqslant 3$, then $d+2-2s < d-3$. Therefore the composite of β and the projection onto each factor has a non-trivial kernel if $s \geqslant 3$. This kernel is either A_{d-2} or the whole group S_{d-2}. Therefore the image of any element of S_{d-2} under the embedding j has order at most 4, which is impossible if $d-2 \geqslant 5$.

The following lemma forbids the appearance of other simple groups N in the product $G = N \rtimes \langle\sigma\rangle$ and thus completes the investigation of Case I_{32}.

Lemma 1.5. *Suppose that G is a simple group different from an alternating group and containing a subgroup H_d isomorphic either to the symmetric group S_d or to the alternating group A_d, $d \geqslant 1000$. Then $(G : H_d) > 84^2 d^2$.*

Proof. The proof uses the classification of finite simple groups (see [4]).

The group G is non-abelian since H_d is non-abelian. Moreover, G cannot be a sporadic simple group since the order of no such group can be divisible by $\frac{d!}{2}$ if $d \geqslant 33$ (the sporadic simple groups possess the following property: either the multiplicity of the prime 11 in their order does not exceed 2, or the order is not divisible by 13) while the order $|G|$ is divisible by $|H_d|$, which is in turn divisible by $\frac{d!}{2}$.

Let G be a simple group of Lie type. Then G is a subgroup of either $\mathrm{GL}(F,n)$ or $\mathrm{PGL}(F,n)$, where F is a finite field. We denote the number of elements of F by q. Since $H_d \subset G$, we have $n \geqslant [\frac{d}{4}]$ by Lemmas 1.2 and 1.3.

Suppose that G is one of the following groups: $A_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$, ${}^2A_n(q^2)$, ${}^2D_n(q^2)$. Then

$$|G| \geqslant q^{r^2},$$

where $r=[\frac{n}{2}]$. Since $[\frac{d}{8}]^2-16 > d\log_2 d$ for $d \geqslant 1000 > 3!$, we have $(G : H_d) > 168^2 d^2$.

To complete the proof of the lemma, we note that all the other simple groups of Lie type have a non-trivial irreducible linear representation of dimension less than 250. Therefore, by Lemma 1.3, they cannot have a subgroup isomorphic to A_d if $d \geqslant 1000$.

1.6. Analysis of Case II. Lemma 1.5 shows that it only remains to consider the case when $S_d \subset G = A_{d_1}$.

The embedding $S_d \subset A_{d_1}$ induces an embedding $S_d \subset S_{d_1}$. Since any embedding $S_d \subset S_{d+1}$ is standard, we have $d_1 - d = n_1 \geqslant 2$. By hypothesis, we have $d \geqslant \max(2c, 1000)$ and $(A_{d_1} : S_d) = \frac{1}{2}(d+1)\cdots(d+n_1) \leqslant cd^n$, where either (i) $c=63$, $n=1$ or (ii) $c=(2\cdot 42)^2$, $n=2$. Therefore $n_1 \leqslant 2$.

We claim that if $n_1 = 2$, then the embedding $S_d \subset A_{d+2}$ is standard. Indeed, by Lemma 1.4, the embedding $S_d \subset A_{d+2}$ induces a standard embedding $A_d \subset A_{d+2}$. Moreover, the image in $A_{d+2} \subset S_{d+2}$ of a transposition $\sigma \in S_d$ is a product of an even number s of mutually commuting transpositions σ_i in S_{d+2}. To show that the embedding $S_d \subset A_{d+2}$ is standard, it suffices to prove that $s=2$. We omit the proof since it coincides almost verbatim with that of Lemma 1.4.

§ 2. Proof of Theorem 0.1

2.1. Minimal expansions of the Galois groups of generic coverings. We denote by $\overline{g} = g-1$ the arithmetic genus of X, $\overline{g} \geqslant 1$, and by B the branch locus of $f\colon X \to \mathbb{P}^1$ in $\mathbb{P}^1$. Since f is m-canonical, we have

$$d = \deg f = 2m\overline{g}.$$

Applying the Hurwitz formula to f, we get

$$|B| = 2d + 2\overline{g} = 2(2m+1)\overline{g}.$$

The branch locus of $\tilde{f}$ (the Galois expansion of f) coincides with B, and the ramification indices of all ramification points of $\tilde{f}$ are equal to 2. Thus, applying the Hurwitz formula to $\tilde{f}$, we have

$$2\widetilde{g} = -2d! + \frac{1}{2}\, d!\, |B| = d!\,(d + \overline{g} - 2), \tag{2.1}$$

where $\widetilde{g} = g(\widetilde{X}) - 1$ is the arithmetic genus of $\widetilde{X}$.

Assume that $\mathrm{Aut}(\widetilde{X}) \neq \mathrm{Gal}(\widetilde{X}/\mathbb{P}^1)$ and choose a subgroup G of $\mathrm{Aut}(\widetilde{X})$ such that $S_d \subset G$ is a minimal expansion of S_d.

We denote the index of S_d in G by $k = (G : S_d)$. The Hurwitz bound on the orders of the automorphism groups of algebraic curves (see, for example, [5]) implies that $|G| \leqslant 84\widetilde{g}$. Therefore,

$$k \leqslant 42(d + \overline{g} - 2). \tag{2.2}$$

In particular, we have

$$k < 63d. \tag{2.3}$$

Thus it follows from Proposition 1.1 that G is one of the following groups:

1) $G = S_d \times \mathbb{Z}/p\mathbb{Z}$, where $p \geqslant 2$ is a prime;

2) $G = A_d \rtimes D_r$, where $r \geqslant 3$, r is odd, D_r is the dihedral group given by the presentation

$$D_r = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^r = 1 \rangle,$$

and the action (by conjugation) of σ and τ on A_d coincides with that of the transposition $(1, 2) \in S_d$ on $A_d \subset S_d$;

3) $G = S_{d+1}$ is the symmetric group.

2.2. Elimination of the remaining three cases. Consider Case 1). Let γ be a generator of $\mathbb{Z}/p\mathbb{Z}$.

Since the action of γ on $\widetilde{X}$ commutes with the action of any element of S_d, we see that the action of the group $\langle\gamma\rangle$ on $\widetilde{X}$ descends to X and to $\mathbb{P}^1$. We denote the corresponding quotient spaces by $\widetilde{X}_1 = \widetilde{X}/\langle\gamma\rangle$ and $X_1 = X/\langle\gamma\rangle$. Let $\widetilde{r}\colon \widetilde{X} \to \widetilde{X}_1$, $r\colon X \to X_1$, $h_1\colon \widetilde{X}_1 \to X_1$ and $r_P\colon \mathbb{P}^1 \to \mathbb{P}^1/\langle\gamma\rangle \simeq \mathbb{P}^1$ be the corresponding morphisms. We have the following commutative diagram.

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "h"] \arrow[d, "\widetilde{r}"'] & X \arrow[r, "f"] \arrow[d, "r"] & \mathbb{P}^1 \arrow[d, "r_P"] \\
\widetilde{X}_1 \arrow[r, "h_1"'] & X_1 \arrow[r, "f_1"'] & \mathbb{P}^1
\end{tikzcd}$$

The cyclic covering $r_P\colon \mathbb{P}^1 \to \mathbb{P}^1$ is of degree $p \geqslant 2$ and is ramified at two points, say, at $x_1, x_2 \in \mathbb{P}^1$. Therefore the cyclic covering r is ramified at least at $2(d-1)$ points lying in $f^{-1}(x_1) \cup f^{-1}(x_2)$. The ramification indices at these points are equal to p. By the Hurwitz formula, we have

$$2\overline{g} \geqslant 2p(g(X_1) - 1) + 2(d-1)(p-1),$$

whence

$$2\overline{g} \geqslant -2p + 2(2m\overline{g} - 1)(p-1).$$

We finally get the inequality

$$p \leqslant \frac{(2m+1)\overline{g} + 1}{2m\overline{g} - 2} = 1 + \frac{\overline{g} + 3}{2m\overline{g} - 2} < 2,$$

which shows that Case 1) is impossible.

Consider Case 2). By choosing a suitable pair of generators σ, τ of D_r, we have $S_d = A_d \rtimes \langle\sigma\rangle \subset G$ while the group $S'_d = A_d \rtimes \langle\tau\rangle$ is conjugate to S_d and does not coincide with S_d (but is isomorphic to S_d). Moreover, $A_d \subset S'_d \cap S_d$. We denote the corresponding quotient spaces by $X_1 = \widetilde{X}/S'_{d-1}$, $\mathbb{P}^1 = \widetilde{X}/S'_d$ and $X_0 = \widetilde{X}/A_d$. They can be arranged in the following commutative diagram, where the morphisms f_{0i}, $i = 1, 2$, are of degree two and, since f is a generic covering, f_{01} is branched over all points belonging to B.

$$\begin{tikzcd}
 & \widetilde{X} \arrow[dl, "h"'] \arrow[d, "h_0"] \arrow[dr, "h^1"] & \\
X \arrow[d, "f"'] & X_0 \arrow[dl, "f_{01}"] \arrow[dr, "f_{02}"'] & X_1 \arrow[d, "f_1"] \\
\mathbb{P}^1 & & \mathbb{P}^1
\end{tikzcd}$$

The morphisms f_{0i}, $i = 1, 2$, of degree 2 determine an embedding $i\colon X_0 \to \mathbb{P}^1 \times \mathbb{P}^1$ whose image $i(X_0)$ is a curve of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore $i(X_0)$ is an elliptic curve, and the projections of $i(X_0)$ onto each factor are branched at four points. On the other hand, f_{01} is branched at every point of B while $|B| = 2d + 2\overline{g} > 4$. Therefore Case 2) is impossible.

Consider Case 3). First note that the embedding of S_d in $G = S_{d+1}$ is standard.

We consider the quotient space $\widetilde{X}/G$ and the quotient map $\bar{f}\colon \widetilde{X} \to \widetilde{X}/G$. The morphism $\bar{f}$ factors through $\tilde{f}$, whence $\widetilde{X}/G \simeq \mathbb{P}^1$ and $\bar{f}$ is the following composite of morphisms:

$$\widetilde{X} \xrightarrow{\ h\ } X \xrightarrow{\ f\ } \mathbb{P}^1 \xrightarrow{\ r\ } \mathbb{P}^1 ,$$

where r is a morphism of degree $d+1$. Since S_d and S_{d+1} have no common normal subgroup, $\bar{f}$ is the Galois expansion of r.

We denote by $B_1 \subset \mathbb{P}^1$ the branch locus of r and compare the cardinality of B with that of $r(B) \subset B_1$.

The symmetric group S_{d+1} acts as a permutation group on $I = \{1, \dots, d+1\} \subset \mathbb{N}$. We put $H_i = \{\gamma \in S_{d+1} \mid \gamma(i) = i\}$, so that our group S_d is equal to H_{d+1}. The groups H_i are all conjugate. Therefore all the coverings $\tilde{f}_i\colon \widetilde{X} \to \widetilde{X}/H_i \simeq \mathbb{P}^1$ are Galois expansions of generic coverings.

Let $a \in \widetilde{X}$ be a ramification point of $\tilde{f}_{d+1} = \tilde{f}$. The stabilizer $\mathrm{St}_a(\bar{f}) = \{g \in G \mid g(a) = a\}$ is a cyclic group. Its order is equal to the ramification index of $\bar{f}$ at a. Let τ be a generator of $\mathrm{St}_a(\bar{f})$. The intersection $\mathrm{St}_a(\bar{f}) \cap S_d = \mathrm{St}_a(\tilde{f}_{d+1})$ is a group of order 2 generated by a transposition $\sigma \in H_{d+1}$ since $\tilde{f}_{d+1}$ is the Galois expansion of a generic covering. Therefore $\sigma = \tau^k$ and $\tau^{2k} = 1$.

First we claim that k is odd. Indeed, let us write τ as a product of disjoint cycles:

$$\tau = (i_{1,1}, \dots, i_{1,k_1}) \cdots (i_{s,1}, \dots, i_{s,k_s}).$$

We can assume that (up to a rearrangement) $\sigma = (i_{1,1}, \dots, i_{1,k_1})^k$ and $(i_{j,1}, \dots, i_{j,k_j})^k = 1$ for $j = 2, \dots, s$. Then we easily see that $k_1 = 2$, k is odd and the k_j are divisors of k for $j = 2, \dots, s$.

We now claim that $k=1$ and, therefore, $\tau=\sigma$. For each i, the intersection $\mathrm{St}_a(\bar f)\cap H_i=\mathrm{St}_a(\tilde f_i)$ is a group of order at most 2, and if its order is equal to 2, then it is also generated by a transposition $\sigma_i\in H_i$ since $\tilde f_i$ is conjugate to $\tilde f_{d+1}$. On the other hand, the element

$$\sigma\tau=(i_{2,1},\dots,i_{2,k_2})\cdots(i_{s,1},\dots,i_{s,k_s})\in \mathrm{St}_a(\bar f)\cap H_{i_{1,1}}=\mathrm{St}_a(\tilde f_{i_{1,1}})$$

is of odd order. Therefore $\sigma=\tau$.

We now consider the fibre $\bar f^{-1}(\bar f(a))$ containing the point a: it can be identified with the set of right cosets $\{\mathrm{St}_a(\bar f)\gamma\}$ in S_{d+1}. Hence the stabilizer $\mathrm{St}_{(a)\gamma}(\bar f)$ of the point $(a)\gamma$ is generated by the transposition $\gamma^{-1}\sigma\gamma$.

The fibre $\bar f^{-1}(\bar f(a))$ splits into the disjoint union of the orbits under the action of H_{d+1}. Each orbit is a fibre of $\tilde f_{d+1}$. There is no loss of generality in assuming that the group $\mathrm{St}_a(\bar f)$ is generated by $\sigma=(1,2)$. Then we easily see that each of these orbits can be identified with one of the sets $F_i=\{\mathrm{St}_a(\bar f)\gamma\mid\gamma\in\sigma_iH_{d+1}\}$, where $\sigma_i=(i,d+1)$ if $2\leqslant i<d+1$, and $\sigma=(1,2)$ if $i=d+1$. (The transpositions $\sigma_1=(1,d+1)$ and $\sigma_2=(2,d+1)$ determine the same orbit under the action of H_{d+1} because $(1,2)(1,d+1)(1,2)=\mathrm{Id}\cdot(2,d+1)$.)

The points $a_i=(a)\sigma_i$ have the same stabilizer,

$$\mathrm{St}_{a_i}(\bar f)=\langle(1,2)\rangle\subset H_{d+1},$$

if $i>2$. Hence all the points belonging to F_i with $i\geqslant 3$ are ramification points of $\tilde f_{d+1}$ and, therefore, $\tilde f_{d+1}(F_i)\in B$. It is easy to see that the points belonging to F_2 are not ramification points of $\tilde f_{d+1}$. Thus the point $\tilde f_{d+1}(F_2)$ is a ramification point of r.

As a corollary, we obtain that if $\tilde b\in r(B)$, then the fibre $r^{-1}(\tilde b)$ consists of $d-1$ points belonging to B and one point (a ramification point of r) which does not belong to B. Hence the number $|B|=2d+2\overline{g}=2(m+1)\overline{g}$ is divisible by $d-1=2m\overline{g}-1$. Then $2\overline{g}+1$ must be divisible by $2m\overline{g}-1$. But this is possible only when $\overline{g}=1$ and $m=1$ or $m=2$.

§ 3. Proof of Theorem 0.2

3.1. Local behaviour of generic coverings and their Galois expansions. In this subsection we specialize to the case of surfaces the definitions related to generic actions of symmetric groups, compare our definitions with traditional definitions of generic coverings, deduce the local properties of generic coverings from those of such actions, and fix the corresponding notation and notions.

We recall that the Galois expansion $\tilde f\colon\widetilde X\to\mathbb P^2$ of a generic covering $f\colon X\to\mathbb P^2$ of degree d factors through f: $\tilde f=f\circ h$, where $h\colon\widetilde X\to X$ is a Galois covering with Galois group $\mathrm{Gal}(\widetilde X/X)=S_{d-1}\subset S_d=\mathrm{Gal}(\widetilde X/\mathbb P^2)$. The branch locus $B\subset\mathbb P^2$ of the covering f coincides with that of $\tilde f$. We say that f is *generic* if the action of S_d on $\widetilde X$ is generic. The latter means that the stabilizer $\mathrm{St}_a(S_d)$ of any point $a\in\widetilde X$ is a product of symmetric groups (depending on a) embedded in S_d in the standard way and that the actions induced by $\mathrm{St}_a(S_d)$ on the tangent spaces $T_a\widetilde X$ are standard representations of rank $\leqslant 2$ (see the introduction).

On the other hand, in dimension 2 it is more traditional to define a generic covering $f\colon X \to \mathbb{P}^2$ of degree d as a covering with the following local behaviour. The branch locus B of f is a cuspidal curve. Over a neighbourhood U of a smooth point of B, the pre-image $f^{-1}(U)$ splits into a disjoint union of $d-1$ connected components. In one of them, the covering is two-sheeted and isomorphic to the projection of the surface $x = z^2$ onto the (x, y)-plane in a neighbourhood of the origin, and in the others it is a local isomorphism. Over a neighbourhood of a cusp of B, the pre-image splits into a disjoint union of $d-2$ connected components. In one of them, the covering is a pleat (that is, a three-sheeted covering isomorphic to the projection of the surface $y = z^3 + xz$ onto the (x, y)-plane in a neighbourhood of the origin), and in the others it is a local isomorphism. Over a neighbourhood of a node of B, the pre-image splits into a disjoint union of $d-2$ connected components. The covering is two-sheeted in two of them, and the restriction of the covering to the union of these components is isomorphic to the projection of the union of the surfaces $x = z^2$ and $y = z^2$ onto the (x, y)-plane in a neighbourhood of the origin, while the restriction of the covering to any other component is a local isomorphism. The non-trivial local Galois groups in the corresponding three cases are $\mathbb{Z}/2$, S_3, and $\mathbb{Z}/2 \times \mathbb{Z}/2$. All their non-trivial representations in $\mathrm{GL}(2, \mathbb{C})$ that produce non-singular quotients are standard representations of rank $\leqslant 2$. Hence our definition of a generic covering coincides with the traditional one.

The global behaviour of generic coverings is easy to see from the above local models. In particular, we have $f^*(B) = 2R + C$, where R (the ramification locus of f) is a non-singular curve, C is a reduced curve and C is non-singular over the non-singular points of B. Moreover, the intersection points of R and C lie over the nodes and cusps of B. There are two intersection points over each node (and these intersections are transversal) and one intersection point over each cusp (this intersection is a simple tangency).

Consider the same behaviour from the viewpoint of the action of S_d on $\widetilde{X}$. The properties of this action are used repeatedly in the rest of this section.

For every point $a \in \widetilde{X}$, the action of $\mathrm{St}_a(S_d)$ on a small neighbourhood of a can be linearized. (This can be done by Cartan's linearization procedure [6], which is reproduced below in the proof of Lemma 4.2. It gives a suitable coordinate change that enables us to identify the local action with the action induced on the tangent space $T_a\widetilde{X}$.) We shall treat the possibilities for $\mathrm{St}_a(S_d)$ case by case, analyzing the local behaviour of f, $\tilde{f}$ and h in accordance with the definition of standard representations.

If the group $\mathrm{St}_a(S_d) = S_2$ is generated by a transposition $\sigma \in S_d$, the ramification divisor $\widetilde{R}$ of $\tilde{f}$ coincides in some neighbourhood of a with the set of fixed points of σ, to be denoted by $\widetilde{R}_\sigma$. This set is a smooth curve and, in particular, $\widetilde{R}$ is smooth at a. The image $h(a)$ of a belongs to the ramification locus R of f (equivalently, a does not belong to the ramification locus of h) if and only if $\sigma \notin S_{d-1}$. Moreover, $h(\widetilde{R}_\sigma)$ coincides with R in a neighbourhood of $h(a)$ if $\sigma \notin S_{d-1}$ (otherwise it coincides with the curve C defined above). We see that the curve $h(\widetilde{R}_\sigma)$ is smooth at $h(a)$ in both cases ($\sigma \in S_{d-1}$, $\sigma \notin S_{d-1}$). Moreover, in both cases, the point $\tilde{f}(a)$ belongs to B and B is non-singular at $\tilde{f}(a)$.

If the group $\mathrm{St}_a(S_d) = S_2 \times S_2$ is generated by two commuting transpositions $\sigma_1 \in S_d$ and $\sigma_2 \in S_d$, then a belongs to $\widetilde{R}_{\sigma_1} \cap \widetilde{R}_{\sigma_2}$, and the curves $\widetilde{R}_{\sigma_1}$ and $\widetilde{R}_{\sigma_2}$ are non-singular and meet each other transversally at a. Furthermore, the curves $h(\widetilde{R}_{\sigma_1})$ and $h(\widetilde{R}_{\sigma_2})$ are non-singular and meet each other transversally. If one of the transpositions (say, σ_1) does not belong to S_{d-1}, then $h(\widetilde{R}_{\sigma_1})$ is contained in the divisor R and, moreover, coincides with R in a neighbourhood of $h(a)$. If $\sigma_1 \in S_{d-1}$, then $h(\widetilde{R}_{\sigma_1})$ is not contained in R (but is contained in C). If both σ_1 and σ_2 belong to S_{d-1}, then $h(a)$ is not a ramification point of f. (In this case, $h(a)$ is a node of C and we have $C = h(\widetilde{R}_{\sigma_1}) \cup h(\widetilde{R}_{\sigma_2})$ in a neighbourhood of $h(a)$.) In every case, $\tilde{f}(a)$ is a node of B.

If the group $\mathrm{St}_a(S_d) = S_3$ is generated by two non-commuting transpositions $\sigma_1 \in S_d$ and $\sigma_2 \in S_d$, then a belongs to $\widetilde{R}_{\sigma_1} \cap \widetilde{R}_{\sigma_2} \cap \widetilde{R}_{\sigma_3}$, where $\sigma_3 = \sigma_1\sigma_2\sigma_1$, and the curves $\widetilde{R}_{\sigma_1}$, $\widetilde{R}_{\sigma_2}$ and $\widetilde{R}_{\sigma_3}$ are non-singular and meet each other pairwise transversally at a. If all three transpositions belong to S_{d-1}, then $h(a)$ is not a ramification point of f. Otherwise, one and only one of the transpositions (say, σ_3) belongs to S_{d-1}, and then the curves $h(\widetilde{R}_{\sigma_1}) = h(\widetilde{R}_{\sigma_2})$ and $h(\widetilde{R}_{\sigma_3})$ are non-singular and tangent to each other while the curve $h(\widetilde{R}_{\sigma_1}) = h(\widetilde{R}_{\sigma_2})$ coincides with R (and $h(\widetilde{R}_{\sigma_3})$ coincides with C) in a neighbourhood of $h(a)$. In every case, $\tilde{f}(a)$ is a cusp of B.

3.2. Invariants of m-canonical generic coverings. Let $f\colon X \to \mathbb{P}^2$ be a generic m-canonical covering branched along a cuspidal curve $B \subset \mathbb{P}^2$. Then X is a minimal surface of general type containing no (-2)-curves, and the degree of f is equal to

$$d = \deg f = m^2 K_X^2.$$

By the formula for the canonical divisor of a finite covering, we have $K_X = f^* K_{\mathbb{P}^2} + [R]$. Hence the divisor R is numerically equivalent to $(3m+1)K_X$. Since, in addition, the curve R is non-singular and X contains no (-2)-curves (if f is an m-canonical generic covering, then K_X is ample), we see that R is irreducible. Since the curve B is birationally isomorphic to R, it too is irreducible. Thus we can apply the results of [7]. In particular, we get the following formulae for the degree $\deg B$ and the number c of cusps of B (see [7], proof of Theorem 2):

$$\deg B = m(3m+1)K_X^2, \tag{3.1}$$

$$c = (12m^2 + 9m + 3)K_X^2 - 12p_a, \tag{3.2}$$

where $p_a = p_g - q + 1$ is the arithmetical genus of X. Note that if the degree of the generic covering f exceeds 2, then its branch curve B necessarily has cuspidal singular points, that is, $c > 0$. (Indeed, the image in S_d of the monodromy of f is a transitive subgroup of S_d. Since for generic coverings this image is generated by transpositions, we see that it coincides with the whole of S_d. Therefore the group $\pi_i(\mathbb{P}^2 \setminus B)$ is non-abelian for $d \geqslant 3$. On the other hand, by Zariski's theorem, the group $\pi_i(\mathbb{P}^2 \setminus B)$ is abelian if B is a nodal curve.)

Finally, we apply to $\tilde{f}$ the projection formula for the canonical divisor. This yields that $K_{\widetilde{X}} = \tilde{f}^*(K_{\mathbb{P}^2}) + [\widetilde{R}] = \tilde{f}^*(K_{\mathbb{P}^2} + \frac{1}{2}[B])$ and, therefore,

$$K_{\widetilde{X}}^2 = \frac{1}{4}(\deg B - 6)^2 d! = d!\biggl(\frac{m(3m+1)}{2}\,K_X^2 - 3\biggr)^2. \tag{3.3}$$

3.3. Minimal expansions of the Galois groups of generic coverings. Assume that $\mathrm{Aut}(\widetilde{X}) \neq \mathrm{Gal}(\widetilde{X}/\mathbb{P}^2)$ and choose a subgroup G of $\mathrm{Aut}(\widetilde{X})$ such that $S_d \subset G$ is a minimal expansion of S_d. We write $k = (G : S_d)$ for the index of S_d in G.

The Xiao bound (see [8]) for the order of the automorphism group of a surface of general type states that $|G| \leqslant 42^2 K^2_{\widetilde{X}}$. It follows that

$$k \leqslant 42^2 \left(\frac{m(3m+1)}{2}\, K_X^2 - 3 \right)^2. \tag{3.4}$$

We finally get

$$k < (2 \cdot 42)^2 d^2. \tag{3.5}$$

Proposition 1.1 now yields that G must be one of the groups in the following list.

1) $G = S_d \times \mathbb{Z}/p\mathbb{Z}$, $p \geqslant 2$, where p is a prime.

2) $G = A_d \rtimes D_r$, where $r \geqslant 3$, r is odd, D_r is the dihedral group with the presentation

$$D_r = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^r = 1 \rangle,$$

and the action (by conjugation) of σ and τ on A_d coincides with that of the transposition $(1,2) \in S_d$ on $A_d \subset S_d$.

3) $G = S_{d+1}$ is the symmetric group.

4) $G = A_{d+2}$ is the alternating group and the embedding of S_d in $G = A_{d+2}$ is standard.

3.4. Case 1). Let g be a generator of $\mathbb{Z}/p\mathbb{Z}$. Since the action of g on $\widetilde{X}$ commutes with that of any element of S_d, we see as in the proof of Theorem 0.1 that the action of the group $\langle g \rangle = \mathbb{Z}/p\mathbb{Z}$ descends to X and to $\mathbb{P}^2$. We denote the corresponding quotient spaces by $\widetilde{X}_1 = \widetilde{X}/\langle g \rangle$ and $X_1 = X/\langle g \rangle$, and let $\widetilde{r}\colon \widetilde{X} \to \widetilde{X}_1$, $r\colon X \to X_1$, $h_1\colon \widetilde{X}_1 \to X_1$ and $r_P\colon \mathbb{P}^2 \to \mathbb{P}^2/\langle g \rangle = Y$ be the corresponding morphisms. We have the following commutative diagram.

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "h"] \arrow[d, "\widetilde{r}"'] & X \arrow[r, "f"] \arrow[d, "r"] & \mathbb{P}^2 \arrow[d, "r_P"] \\
\widetilde{X}_1 \arrow[r, "h_1"'] & X_1 \arrow[r, "f_1"'] & Y
\end{tikzcd}$$

The automorphism g of $\mathbb{P}^2$ is determined by a linear map $\mathbb{C}^3 \to \mathbb{C}^3$ of order p. Hence it has either three isolated fixed points, say $x_1, x_2, x_3 \in \mathbb{P}^2$, or just one, say $x \in \mathbb{P}^2$, and a fixed line $E \subset \mathbb{P}^2$. The cyclic covering $r_P\colon \mathbb{P}^2 \to \mathbb{P}^2$ is accordingly of degree p and ramified either at the three points x_1, x_2, x_3 or at the point x and along the line E. We consider these two cases separately.

If r_P is ramified at three isolated points, then r is ramified at least at $3(d-2)$ points lying in $F = f^{-1}(x_1) \cup f^{-1}(x_2) \cup f^{-1}(x_3)$, and the ramification index of each of them is equal to p. The points of F are the only fixed points of the automorphism g acting on X. Therefore, by the Lefschetz fixed point theorem, we have

$$|F| = \sum_{i=0}^{4} (-1)^i \mathrm{tr}_i, \tag{3.6}$$

where tr_i is the trace of the linear transformation g^* acting on $H^i(X, \mathbb{R})$. It follows from (3.6) that

$$3(d-2) \leqslant |F| \leqslant \sum_{i=0}^{4} |\mathrm{tr}_i| \leqslant \sum_{i=0}^{4} b_i(X) = e(X) + 4b_1(X) \tag{3.7}$$

(where e stands for the topological Euler characteristic). On the other hand, Noether's formula $1 - q + p_g = \frac{K_X^2 + e(X)}{12}$ with $2q = b_1$ yields that

$$e(X) + 4b_1(X) = 12 - K_X^2 + 12p_g - 4q \leqslant 12 - K_X^2 + 12p_g. \tag{3.8}$$

Therefore, combining (3.7) and (3.8) with Noether's inequality $2p_g \leqslant K_X^2 + 4$, we get

$$3(m^2 K_X^2 - 2) \leqslant 12 - K_X^2 + 12p_g \leqslant 5K_X^2 + 36. \tag{3.9}$$

Hence,

$$(3m^2 - 5)K_X^2 \leqslant 42.$$

This contradicts the inequality $m^2 K_X^2 \geqslant 2 \cdot 84^2$ if $m \geqslant 2$ since we have $K_X^2 \geqslant 1$ for any surface of general type.

We now assume that r_P is ramified at a point $x \in \mathbb{P}^2$ and along a line $E \subset \mathbb{P}^2$. Then every line $L \subset \mathbb{P}^2$ through x is invariant under the action of g on $\mathbb{P}^2$. Hence every curve $C = f^{-1}(L) \subset X$ is invariant under the action of g on X. Pick a generic line L passing through x. By the Hurwitz formula,

$$2(g(C) - 1) = -2d + \deg B - 2m^2 K_X^2 + m(3m+1)K_X^2 = m(m+1)K_X^2, \tag{3.10}$$

where $g(C)$ is the geometric genus of C.

We consider the restriction $r_{|C} \colon C \to C/\langle g \rangle = Z \subset X_1$ of r to C. The cyclic covering $r_{|C}$ has degree p and is branched at least at $2d - 3 = 2m^2 K_X^2 - 3$ points belonging to $f^{-1}(x) \cup f^{-1}(L \cap E)$. Hence,

$$2(g(C) - 1) \geqslant 2p(g(Z) - 1) + (2m^2 K_X^2 - 3)(p-1). \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$m(m+1)K_X^2 \geqslant -2p + (2m^2 K_X^2 - 3)(p-1)$$

and, therefore,

$$m(3m+1)K_X^2 \geqslant (2m^2 K_X^2 - 5)p.$$

Finally, since $(m^2 - m)K^2 \geqslant 84^2 > 10$, we get

$$p \leqslant \frac{m(3m+1)K_X^2}{2m^2 K_X^2 - 5} < 2,$$

contrary to the fact that p is integer and $p > 1$.

3.5. Case 2). Group-theoretic part. Since r is odd and $r \geqslant 3$, the conjugacy class of σ in D_r consists of r elements $\sigma_1, \dots, \sigma_r$, $\sigma_1 = \sigma$, $\sigma_2 = \tau$. For every i with $1 \leqslant i \leqslant r$, the group $S_{d,i}$ (generated in G by σ_i and the elements of A_d) is isomorphic to $A_d \rtimes \langle \sigma_i \rangle \simeq S_d$. The groups $S_{d,i}$ are conjugate to each other in G. Moreover, the element $\sigma_i \in S_{d,i}$ acts on $A_d \subset S_{d,i}$ as the transposition $(1,2)$. We write $\sigma_{i,(i_1,i_2)}$ for the element of $S_{d,i}$ which is conjugate to σ_i and acts on A_d as the transposition (i_1,i_2). Given two disjoint subsets $J_1 \neq \varnothing$, J_2 of the set $I = \{1, \dots, d\}$, we denote by $S_{J_1,J_2,i}$ the subgroup of $G = A_d \rtimes D_r$ generated by the elements $\sigma_{i,(i_1,i_2)}$, $(i_1,i_2) \in (J_1 \times J_1) \cup (J_2 \times J_2)$.

Let $\mathrm{St}_a \subset \mathrm{Aut}\, \widetilde{X}$ be the stabilizer of a point $a \in \widetilde{X}$. For any subgroup H of $\mathrm{Aut}\, \widetilde{X}$ we put $\mathrm{St}_a(H) = H \cap \mathrm{St}_a$. For every point $a \in \widetilde{X}$, the action induced by $\mathrm{St}_a(S_d)$ on the tangent space $T_a\widetilde{X}$ is a standard representation of rank $\leqslant 2$ and the group $\mathrm{St}_a(S_d)$ is either trivial or isomorphic to $S_{J_1,J_2,1}$, where either $2 \leqslant |J_1| \leqslant 3$ and $J_2 = \varnothing$, or $|J_1| = |J_2| = 2$. Since the groups $S_{d,i}$ are all conjugate, we see that the groups $\mathrm{St}_a(S_{d,i})$ possess the same properties for every i and every $a \in \widetilde{X}$. Hence the intersection $\mathrm{St}_a(S_{d,i}) \cap A_d$ is generated by the cyclic permutation $(i_1,i_2,i_3) \in A_d$ if $J_1 = \{i_1,i_2,i_3\}$ and $J_2 = \varnothing$, and by the product of two transpositions if $J_1 = \{i_1,i_2\}$ and $J_2 = \{i_3,i_4\}$. The group $\mathrm{St}_a(S_{d,i}) \cap A_d$ is trivial in all the remaining cases (when $|J_1| = 2$ and $J_2 = \varnothing$ or $\mathrm{St}_a(S_{d,i}) = \{1\}$). It follows that if $\mathrm{St}_a(S_{d,1}) = S_{J_1,J_2,1}$ with $|J_1| = 3$ and $J_2 = \varnothing$, then the group $\mathrm{St}_a(S_{d,i})$ coincides with $S_{J_1',J_2',i}$ for every i, where $J_1' = J_1$ and $J_2' = \varnothing$. Similarly, if $\mathrm{St}_a(S_{d,1}) = S_{J_1,J_2,1}$, where $J_1 = \{i_1,i_2\}$ and $J_2 = \{i_3,i_4\}$, then the group $\mathrm{St}_a(S_{d,i})$ coincides with $S_{J_1',J_2',i}$ for every i, where $J_1' = J_1$ and $J_2' = J_2$. If $\mathrm{St}_a(S_{d,1}) = S_{J_1,J_2,1}$, where $J_1 = \{i_1,i_2\}$ and $J_2 = \varnothing$, then for each i either $\mathrm{St}_a(S_{d,i})$ is trivial or $\mathrm{St}_a(S_{d,i})$ coincides with $S_{J_1',J_2',i}$, where $|J_1'| = 2$ and $J_2' = \varnothing$.

Let us investigate in more detail the case when $\mathrm{St}_a(S_{d,i})$ coincides with $S_{J_1,J_2,i}$ for every i, where $|J_1| = 3$ and $J_2 = \varnothing$. Suppose that $J_1 = \{i_1,i_2,i_3\}$ and denote the cyclic permutation $(i_1,i_2,i_3) \in A_d$ by y. Let F_3 be the subgroup of $\mathrm{St}_a(G)$ generated by $x_1 = \sigma_{1,(i_1,i_2)}$ (conjugate to σ), $x_2 = \sigma_{2,(i_1,i_2)}$ (conjugate to τ) and y. It is easy to see that F_3 has the presentation

$$F_3 = \big\langle x_1, x_2, y \mid x_1^2 = x_2^2 = (x_1x_2)^r = y^3 = [y, x_1x_2] = [y, x_2x_1] = 1, \\ x_1^{-1}yx_1 = y^{-1}, x_2^{-1}yx_2 = y^{-1} \big\rangle \tag{3.12}$$

(recall that $r \geqslant 3$). The group F_3 is non-abelian and has a maximal normal subgroup N_3 generated by y and $z = x_1x_2$. This subgroup is isomorphic to the direct product $\langle y \rangle \times \langle z \rangle$ of cyclic groups of orders 3 and r respectively. On the other hand, well-known properties of finite subgroups of $\mathrm{GL}(2,\mathbb{C})$ (see, for example, [9]) imply that the quotient of F_3 by its centre is either a cyclic group, a dihedral group, or one of the groups A_4, A_5, S_5. However, the centre of the group $F_3 = (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$, is trivial since r is odd. It follows that r cannot be divisible by 3 (we recall that the branch curve B has at least one cuspidal singular point; see § 3.2) and that F_3 must be isomorphic to the dihedral group $D_{r'}$, $r' = 3r$.

Again using the classification of the conjugacy classes of finite subgroups of $\mathrm{GL}(2,\mathbb{C})$, we see that the action of $F_3 \simeq D_{3r}$ near a is isomorphic to the unique 2-dimensional linear representation of D_{3r}. In particular, the set of fixed points of the automorphism $\sigma_{i,(i_1,i_2)}$, $i_1, i_2 \in J_1$, in a neighbourhood of a is a smooth curve, to be denoted by $\widetilde{R}_{i,(i_1,i_2)}$. Any two curves $\widetilde{R}_{i,(i_1,i_2)}$ and $\widetilde{R}_{i',(i'_1,i'_2)}$ with $(i,(i_1,i_2)) \neq (i',(i'_1,i'_2))$ are distinct and meet transversally at a.

We now examine the case when $\mathrm{St}_a(S_{d,i}) = S_{J_1,J_2,i}$, where $J_1 = \{i_1,i_2\}$ and $J_2 = \{i_3,i_4\}$. Let $F_{2,2}$ be the subgroup of $\mathrm{St}_a(G)$ generated by $x_1 = \sigma_{1,(i_1,i_2)}$ (conjugate to σ), $x_2 = \sigma_{2,(i_1,i_2)}$ (conjugate to τ) and $y = (i_1,i_2)(i_3,i_4) \in A_d$. It is easy to see that $F_{2,2}$ has the presentation

$$F_{2,2} = \left\langle x_1, x_2, y \mid x_1^2 = x_2^2 = (x_1x_2)^r = y^2 = [y,x_1] = [y,x_2] = 1 \right\rangle \tag{3.13}$$

(we recall that r is odd and $r \geqslant 3$). The group $F_{2,2}$ has a maximal normal subgroup $N_{2,2}$ generated by y and $z = x_1x_2$. This subgroup is isomorphic to the direct product $\langle y\rangle \times \langle z\rangle$ of cyclic groups of orders 2 and r respectively. Therefore $F_{2,2}$ is isomorphic to the dihedral group D_{2r}. The classification of the conjugacy classes of finite subgroups of $\mathrm{GL}(2,\mathbb{C})$ implies that the action of $F_{2,2} \simeq D_{2r}$ near a is isomorphic to the unique 2-dimensional representation of D_{2r}. In particular, for all $i_1, i_2, i'_1, i'_2 \in J_1 \sqcup J_2$ with $(i,(i_1,i_2)) \neq (i',(i'_1,i'_2))$, we see that the curves $\widetilde{R}_{i,(i_1,i_2)}$ and $\widetilde{R}_{i',(i'_1,i'_2)}$ are distinct and meet transversally at a.

Finally, we consider the case when $\mathrm{St}_a(S_{d,1}) = S_{J_1,J_2,1}$ and $\mathrm{St}_a(S_{d,2}) = S_{J'_1,J'_2,2}$, where $J_1 = \{j_1,j_2\}$, $J_2 = \varnothing$, $J'_1 = \{j_3,j_4\}$, $J'_2 = \varnothing$. We claim that in this case $J_1 = J'_1$.

Indeed, if $|J_1 \cap J'_1| = 1$, we can assume that $j_2 = j_3$ and, therefore, $\sigma_{2,(j_3,j_4)} = \eta^{-1}\sigma_{2,(j_1,j_2)}\eta$, where $\eta = (j_4,j_2,j_1) \in A_d$. We have $\eta^3 = 1$ and

$$\begin{aligned}(\sigma_{1,(j_1,j_2)}\sigma_{2,(j_3,j_4)})^r &= (\sigma_{1,(j_1,j_2)}\eta\sigma_{2,(j_1,j_2)}\eta^{-1})^r = (\sigma_{1,(j_1,j_2)}\sigma_{2,(j_1,j_2)}\eta)^r \\ &= (\sigma_{1,(j_1,j_2)}\sigma_{2,(j_1,j_2)})^r\eta^r = \eta^r = \eta^{\pm 1}\end{aligned}$$

since r is not divisible by 3. Hence $\eta \in \mathrm{St}_a(G)$, contrary to the assumption that $\mathrm{St}_a(S_{d,1}) = S_{J_1,J_2,1}$.

If $J_1 \cap J'_1 = \varnothing$, then $\sigma_{2,(j_3,j_4)} = \eta^{-1}\sigma_{2,(j_1,j_2)}\eta$, where $\eta = (j_1,j_3)(j_2,j_4) \in A_d$. We put $\eta_1 = (j_1,j_2)(j_3,j_4) \in A_d$. Then $\eta^2 = \eta_1^2 = 1$ and

$$\begin{aligned}(\sigma_{1,(j_1,j_2)}\sigma_{2,(j_3,j_4)})^r &= (\sigma_{1,(j_1,j_2)}\eta\sigma_{2,(j_1,j_2)}\eta^{-1})^r \\ &= (\sigma_{1,(j_1,j_2)}\sigma_{2,(j_1,j_2)}\sigma^{-1}_{2,(j_1,j_2)}\eta\sigma_{2,(j_1,j_2)}\eta^{-1})^r \\ &= (\sigma_{1,(j_1,j_2)}\sigma_{2,(j_1,j_2)}\eta_1)^r = \eta_1.\end{aligned}$$

Hence $\eta_1 \in \mathrm{St}_a(G)$, contrary to the assumption that $\mathrm{St}_a(S_{d,1}) = S_{J_1,J_2,1}$.

3.6. Case 2). Geometric part. We have $A_d = S_{d,i} \cap S_{d,j}$ for $i \neq j$. Denote the corresponding quotient spaces by $X_i = \widetilde{X}/S_{d-1,i}$, $\mathbb{P}^2 = \widetilde{X}/S_{d,i}$ and $X_0 = \widetilde{X}/A_d$. They can be arranged in a commutative diagram (a fragment of which is shown below), where f_{0i} is a morphism of degree 2 for $i = 1, \dots, r$.

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "h_i"] \arrow[d, "h_0"] \arrow[dr, "h_j"] & \\ X_i \arrow[d, "f_i"] & X_0 \arrow[dl, "f_{0i}"] \arrow[dr, "f_{0j}"] & X_j \arrow[d, "f_j"] \\ \mathbb{P}^2 & & \mathbb{P}^2 \end{tikzcd}$$

Since for every i, the morphism f_{0i} is conjugate to the covering $f_{01}\colon X_0 \to \mathbb{P}^2$ branched along $B_1 = B$, we see that f_{0i} is a covering of $\mathbb{P}^2$ branched over the points of a cuspidal curve $B_i \subset \mathbb{P}^2$ having the same degree and the same number of cusps and nodes as B. We denote the ramification curve of the covering f_{0i} by $R_{i,0} \subset X_0$.

The group D_r acts on X_0. The image of $\sigma_{i,(j_1,j_2)} \in S_{d,i}$ (see § 3.5) under the natural epimorphism from G to D_r coincides with σ_i. Therefore the set of fixed points of σ_i coincides with $R_{i,0}$.

The surface X_0 is a normal projective variety. The set of its singular points coincides with $f_{01}^{-1}(\operatorname{Sing} B_1)$: we have a singular point of type A_2 over every cusp and a singular point of type A_1 over every node. Hence all the points of $f_{01}^{-1}(\operatorname{Sing} B_1)$ belong to $R_{i,0}$ for every i, $1 \leqslant i \leqslant r$. On the other hand, the observations at the end of § 3.5 imply that no two curves $R_{i,0}$ and $R_{j,0}$ with $i \neq j$ can meet at a non-singular point of X_0.

Let $\nu\colon Z \to X_0$ be the minimal resolution of singularities. Its exceptional divisor is given by

$$\overline{E} = \nu^{-1}(\operatorname{Sing} X_0) = \bigcup_{k=1}^{c} (\overline{E}_{1,s_k} \cup \overline{E}_{2,s_k}) \cup \bigcup_{l=2c+1}^{2c+n} \overline{E}_{s_l},$$

where $\overline{E}_{1,s_k}, \overline{E}_{2,s_k}$ $(1 \leqslant k \leqslant c)$ are the irreducible components of $\overline{E}$ contracted to the cusp s_k of X_0, and $\overline{E}_{s_l}$ $(l = 2c+1, \dots, 2c+n)$ is the irreducible component of $\overline{E}$ contracted to the node s_l of X_0.

Since f_{0i} is a double covering branched along a cuspidal curve B_{i0}, the above minimal resolution of singularities fits into the following commutative diagram:

$$\begin{tikzcd} Z \arrow[r, "\nu"] \arrow[d, "\bar f_{0i}"'] & X_0 \arrow[d, "f_{0i}"] \\ \overline{\mathbb{P}}^2 \arrow[r, "\nu_i"'] & \mathbb{P}^2 \end{tikzcd}$$

where ν_i blows up each singular point of B_i once and $\bar f_{0i}$ is a two-sheeted covering of $\overline{\mathbb{P}}^2$ branched over the proper transform $\overline{B}_i \subset \overline{\mathbb{P}}^2$ of B_i.

Here is a more explicit description that enables us to count the intersection numbers.

Let s be a cusp of B_i and $E \subset \overline{\mathbb{P}}^2$ the exceptional curve $\nu_i^{-1}(s) = \mathbb{P}^1$ of ν_i lying over s. Then the curve $\overline{B}_i \subset \overline{\mathbb{P}}^2$ meets E at one point, is non-singular at this point and has a simple tangency to E there. The lift $\overline{R}_{i,0} = \bar f_{0i}^{-1}(\overline{B}_i)$ is the ramification curve of $\bar f_{0i}$. It is non-singular and coincides with the proper transform of $R_{i,0}$. The set $\bar f_{0i}^{-1}(E)$ splits into a union $\overline{E}_{1,s} \cup \overline{E}_{2,s} \subset Z$ of two smooth curves intersecting transversally, so that

$$(\overline{E}_{1,s}^2)_Z = (\overline{E}_{2,s}^2)_Z = -2,$$
$$(\overline{E}_{1,s}, \overline{E}_{2,s})_Z = (\overline{E}_{1,s}, \overline{R}_{i,0})_Z = (\overline{E}_{2,s}, \overline{R}_{i,0})_Z = 1.$$

Let s be a node of B_i. Then the curve $\overline{B}_i \subset \overline{\mathbb{P}}^2$ meets the exceptional curve $E = \nu_i^{-1}(s)$ at two points, is non-singular at these points and intersects E transversally.

The lift $f_{0i}^{-1}(E) = \overline{E}_s \subset Z$ is the exceptional curve of ν and meets the proper transform $\overline{R}_{i,0} = f_{0i}^{-1}(\overline{B}_i)$ transversally at two non-singular points of $\overline{R}_{i,0}$. Hence we have

$$(\overline{E}_s^2)_Z = -2, \qquad (\overline{E}_s, \overline{R}_{i,0})_Z = 2.$$

We claim that the intersection number $(\overline{R}_{i,0}, \overline{R}_{j,0})_Z$ is independent of i and j if $i \neq j$. Indeed, consider the commutative diagram

$$\begin{tikzcd} \overline{X} \arrow[r, "\overline{\mu}"] \arrow[d, "\bar{h}_0"'] & \widetilde{X} \arrow[d, "h_0"] \\ \overline{Z} \arrow[r, "\mu"'] & X_0 \end{tikzcd}$$

where $\overline{X} = \widetilde{X} \times_{X_0} \overline{Z}$ is the fibre product of $\widetilde{X}$ and $\overline{Z}$ over X_0 while $\mu\colon \overline{Z} \to X_0$ is the composite of ν and the blow ups of all the intersection points of the (-2)-curves $\overline{E}_{1,s_k}$ and $\overline{E}_{2,s_k}$ $(k = 1, \ldots, c)$ lying on Z (these curves are those components of the divisor $\overline{E}$ that are contracted by ν to the cusps of X_0). We denote by $\overline{E}_{1,2,s_k} \subset \overline{Z}$ the exceptional curve lying over the point $\overline{E}_{1,s_k} \cap \overline{E}_{2,s_k}$. To simplify the notation, we again use the symbols $\overline{R}_{i,0}$, $\overline{E}_{1,s_k}$, $\overline{E}_{2,s_k}$ and $\overline{E}_{s_l}$ to denote the strict transforms in $\overline{Z}$ of the curves $\overline{R}_{i,0}$, $\overline{E}_{1,s_k}$, $\overline{E}_{2,s_k}$ $(k = 1, \ldots, c)$ and $\overline{E}_{s_l}$ $(l = 2c+1, \ldots, 2c+n)$ lying in Z.

It is easy to see that $\bar{h}_0$ is a Galois covering branched along the curves $\overline{E}_{\cdot, s_k}$ and $\overline{E}_{s_l}$. The ramification indices over the curves $\overline{E}_{\cdot, s_k}$ are equal to 3 (see the local calculations in [2], § 2), and the ramification indices over the curves $\overline{E}_{s_l}$ are equal to 2. The morphism $\overline{\mu}$ blows up once each of the points lying over the nodes of X_0 and performs three blow ups at each of the points lying over the cusps of X_0. Therefore the strict transforms $\overline{\mu}^{-1}(\widetilde{R}_{i,(j_1,j_2)})$ are pairwise disjoint for $1 \leqslant i \leqslant r$, $1 \leqslant j_1, j_2 \leqslant d$. But $\bigcup_{j_1,j_2} \overline{\mu}^{-1}(\widetilde{R}_{i,(j_1,j_2)}) = \bar{h}_0^{-1}(\overline{R}_{i,0})$. Hence, blowing all the curves $\overline{E}_{1,2,s_k}$ down, we see that

$$(\overline{R}_{i,0}, \overline{R}_{j,0})_Z = c$$

for $i \neq j$ and these intersection numbers are independent of i and j. We also note that the intersection numbers of the curves $\overline{R}_{j,0}$ and any irreducible component of $\overline{E}$ are independent of j.

The action of D_r on X_0 lifts to an action on Z. The curve $\overline{R}_{i,0} \subset Z$ (resp. $R_{i,0} \subset X_0$) is the set of fixed points of $\sigma_i \in D_r$. Since $\sigma_i^{-1}\sigma_j\sigma_i \neq \sigma_j$ for $j \neq i$, we have $\sigma_i(\overline{R}_{j,0}) \neq \overline{R}_{j,0}$ (resp. $\sigma_i(R_{j,0}) \neq R_{j,0}$) for $j \neq i$. In particular, $R_{3,0} = \sigma_1(R_{2,0}) \neq R_{2,0}$ and, therefore, $R_{2,0} + R_{3,0} = f_{01}^{-1}(D)$ for some curve $D \subset \mathbb{P}^2$.

Since D_r acts transitively on the set of curves $\overline{R}_{i,0}$ (resp. on the set of curves $R_{i,0}$), we have $(\overline{R}_{1,0}^2)_Z = (\overline{R}_{2,0}^2)_Z = (\overline{R}_{3,0}^2)_Z$. It was also shown above that

$$(\overline{R}_{1,0}, \overline{R}_{2,0})_Z = (\overline{R}_{1,0}, \overline{R}_{3,0})_Z = (\overline{R}_{2,0}, \overline{R}_{3,0})_Z.$$

Denote by L the subspace of the Néron–Severi group $\mathrm{NS}(Z) \otimes \mathbb{Q}$ orthogonal (with respect to the intersection form) to the subspace V_E generated by $\overline{E}_{1,s_k}$,

$\overline{E}_{2,s_k}$, $k = 1,\dots,c$, and $\overline{E}_{s_l}$, $l = 2c+1,\dots,2c+n$. The intersection form is negative definite on V_E. Therefore, by the Hodge index theorem, the intersection form on L has signature $(1, \dim L - 1)$.

Let us calculate for later use the intersection numbers of some divisors in L. We project the Néron–Severi classes of the divisors $\overline{R}_{i,0}$ to L, denote these projections by $(\overline{R}_{i,0})_L$ and denote their intersections in L by $(\overline{R}_{i,0}\cdot\overline{R}_{j,0})_L$ (these numbers are equal to the corresponding $\mathbb{Q}$-intersection numbers on the $\mathbb{Q}$-variety X_0).

Observe that $f^*_{i,0}B_i = 2R_{i,0}$ and $\nu^*R_{i,0} = \overline{R}_{i,0} \mod L$. Thus,

$$(\overline{R}^2_{i,0})_L = (\overline{R}^2_{j,0})_L = \frac{1}{2}(\deg B)^2 > 0$$

for all i, j. Write $(\overline{R}_{2,0})_L = \lambda(\overline{R}_{1,0})_L + T$, where $T \in L$ is orthogonal to $(\overline{R}_{1,0})_L$. We have

$$(\overline{R}_{2,0}\cdot\overline{R}_{1,0})_L = \lambda(\overline{R}^2_{1,0})_L = (\overline{R}_{3,0}\cdot\overline{R}_{1,0})_L = (\overline{R}_{2,0}\cdot\overline{R}_{3,0})_L$$

since the intersection numbers of the curves $\overline{R}_{j,0}$ and any irreducible component of $\overline{E}$ are independent of j, and the intersection numbers $(\overline{R}_{i,0},\overline{R}_{j,0})_Z$ are also independent of i and j provided that $i \neq j$.

Next, the divisor $(\overline{R}_{2,0}+\overline{R}_{3,0})_L$ coincides with $\nu^*(f^*_{01}(D))$. Therefore $(\overline{R}_{2,0}+\overline{R}_{3,0})_L$ is proportional to $(\overline{R}_{1,0})_L = \frac{1}{2}\nu^*(f^*_{01}(B_1))$. It follows that $(\overline{R}_{3,0})_L = \lambda(\overline{R}_{1,0})_L - T$, where $\lambda > 0$. We have $(\overline{R}^2_{2,0})_L = \lambda^2(\overline{R}^2_{1,0})_L + T^2 = (\overline{R}^2_{1,0})_L$ and, therefore,

$$T^2 = (1-\lambda^2)(\overline{R}^2_{1,0})_L \leqslant 0.$$

Hence $\lambda \geqslant 1$. Moreover, $\lambda = 1$ if and only if $T^2 = 0$, that is, if and only if $T = 0 \in L$.

Since $(\overline{R}_{2,0}\cdot\overline{R}_{3,0})_L = (\overline{R}_{2,0}\cdot\overline{R}_{1,0})_L$, we have

$$\lambda^2(\overline{R}^2_{1,0})_L - T^2 = \lambda(\overline{R}^2_{1,0})_L, \qquad T^2 = (\lambda^2-\lambda)(\overline{R}^2_{1,0})_L \leqslant 0,$$

whence $\lambda \leqslant 1$. Combining this with the previous observations, we get $\lambda = 1$ and $T = 0$. It follows that $(\overline{R}_{i,0})_L = (\overline{R}_{j,0})_L$ for all i, j. This enables us to conclude that $\deg D = \deg B$ since $2(\deg D)^2 = ((\overline{R}_{2,0}+\overline{R}_{3,0})^2)_L = 4(\overline{R}^2_{2,0})_L = 2(\deg B)^2$.

Therefore we have

$$(\overline{R}^2_{1,0})_Z = \frac{1}{2}(\overline{R}_{1,0},\overline{R}_{2,0}+\overline{R}_{3,0})_Z = c.$$

Since, in addition,

$$(\overline{R}^2_{1,0})_Z = \frac{1}{2}\Big(\nu^*_1(B_1) - 2{\sum}' E_{s_k} - 2{\sum}'' E_{s_l}\Big)^2 = \frac{1}{2}(\deg B)^2 - 2c - 2n,$$

where the sum $\sum'$ (resp. $\sum''$) is taken over all cusps (resp. all nodes) of the curve B_1, we get

$$(\deg B)^2 - 4n = 6c. \tag{3.14}$$

On the other hand, we know (see the proof of Lemma 3 in [7]) that

$$(\deg B)^2 - 2n \leqslant 6c$$

for any generic covering of degree $d \geqslant 3$. It is also shown in [10] that $n > 0$ if $d \geqslant 6$. The contradiction between these estimates and (3.14) shows that Case 2) is impossible.

3.7. Case 3). The symmetric group $G = S_{d+1}$ acts on the set $I = \{1, \dots, d+1\} \subset \mathbb{N}$ as a permutation group. We denote by H_i the subgroup $\{\gamma \in S_{d+1} \mid \gamma(i) = i\}$ of S_{d+1}, so that S_d coincides with H_{d+1}.

As in the proof of Theorem 0.1, we consider the quotient space $\widetilde{X}/G = Y$ and the quotient map $\bar{f}\colon \widetilde{X} \to Y$. The surface Y is a normal projective variety. The morphism $\bar{f}$ factors through $\tilde{f}_i$. Hence $\bar{f}$ is the following composite of morphisms:

$$\widetilde{X} \xrightarrow{h} X \xrightarrow{f} \mathbb{P}^2 \xrightarrow{r} Y,$$

where r is a finite morphism of degree $d+1$. Since S_d and S_{d+1} have no common normal subgroups, we see that $\bar{f}$ is the Galois expansion of r.

Let $\overline{B} \subset Y$ be the branch locus of r. We have $r(B) = B_1 \subset \overline{B}$. The pre-image $r^{-1}(B_1)$ is the union of B and some curve $B' \subset \mathbb{P}^2$.

Since Y is a normal projective surface, we can find a non-singular projective curve $L \subset Y \setminus \operatorname{Sing}(Y)$ which transversally intersects $\overline{B}$. We put $E = r^{-1}(L)$, $F = f^{-1}(E)$ and $\widetilde{F} = \tilde{f}^{-1}(E)$. Then $f_{|F}\colon F \to E$ is a generic covering branched over $B \cap E$, $\tilde{f}_{|\widetilde{F}}\colon \widetilde{F} \to E$ is the Galois expansion of the generic covering $f_{|F}\colon F \to E$ with Galois group $\operatorname{Gal}(\widetilde{F}/E) = S_d$, and $\bar{f}_{|\widetilde{F}}\colon \widetilde{F} \to L$ is the Galois expansion of the covering $r_{|E}\colon E \to L$ with Galois group $\operatorname{Gal}(\widetilde{F}/L) = S_{d+1}$.

Consider the image $b_1 = r(b)$ of a point $b \in B \cap E$. As in the proof of Theorem 0.1, we easily see that $r^{-1}(b_1)$ consists of $d-1$ points belonging to B and one point (the ramification point of $r_{|E}$) belonging to B'. In other words, the covering $r_{|B'}\colon B' \to B_1$ is of degree 1, the covering $r_{|B}\colon B \to B_1$ is of degree $d-1$, and $r^*(B_1) = B + 2B'$. In particular, we have

$$\begin{aligned} \deg B' \cdot \deg E &= (B', E)_{\mathbb{P}^2} = (B_1, L)_Y, \\ \deg B \cdot \deg E &= (B, E)_{\mathbb{P}^2} = (d-1)(B_1, L)_Y. \end{aligned}$$

Therefore,

$$\deg B = (d-1) \deg B'.$$

Since $d = m^2 K_X^2$ and $\deg B = m(3m+1)K_X^2$, it follows that $mK_X^2 + 3$ is divisible by $m^2 K_X^2 - 1$. This contradicts the assumption that $m^2 K_X^2 \geqslant 2 \cdot 84^2$.

3.8. Case 4). We denote the standard embedding $S_d \to A_{d+2}$ by α. For every transposition $\sigma \in S_d$, the set $\widetilde{X}_\sigma = \widetilde{X}_{\alpha(\sigma)} \subset \widetilde{X}$ of fixed points of σ is a non-singular curve. Hence, for every element $\tau \in A_{d+2}$ which is conjugate to $\alpha(\sigma)$, the set $\widetilde{X}_\tau$ of fixed points of τ is also a non-singular curve.

It is shown in [10] that if $d > 6$, then the branch curve $B \subset \mathbb{P}^2$ has at least one node. Therefore, for every product $\eta = \sigma_1\sigma_2$ of two commuting transpositions $\sigma_1, \sigma_2 \in S_d$, the set $\widetilde{X}_\eta$ of fixed points of η is finite and non-empty. It follows that the set $\widetilde{X}_{\eta'}$ is finite and non-empty for any element η' which is conjugate to $\alpha(\eta)$

in A_{d+2}. On the other hand, if $\sigma_1 = (j_1, j_2)$ and $\sigma_2 = (j_3, j_4)$, then the elements $\alpha(\sigma_1) = (j_1, j_2)((d+1), (d+2))$, $\alpha(\sigma_2) = (j_3, j_4)((d+1), (d+2))$ and $\alpha(\eta) = (j_1, j_2)(j_3, j_4)$ are conjugate to each other in A_{d+2}, a contradiction.

§ 4. Applications

4.1. Deformation stability. In this subsection we prove a kind of deformation stability of the examples given by Theorems 0.1 and 0.2. To state the corresponding assertions, we need to introduce some notions. Namely, a *G-manifold* is understood to be a non-singular projective manifold equipped with a regular action of the group G. A smooth *G-family* (or *G-deformation*) of G-manifolds is a proper smooth morphism (that is, a proper submersion) $p\colon \mathscr{X} \to B$, where $\mathscr{X}$ and B are smooth quasi-projective varieties and $\mathscr{X}$ is equipped with a regular action of G preserving every fibre of p (preservation of fibres means that $p \circ G = p$).

Proposition 4.1. *If one of the fibres of a smooth G-family is the Galois expansion of a generic covering of* $\mathbb{P}^n$, *then the whole family consists of Galois expansions of generic coverings of* $\mathbb{P}^n$.

To prove this we shall need the following lemma.

Lemma 4.2. *Let G be a finite group and $p\colon \mathscr{X} \to B$ a smooth G-deformation of G-manifolds. Assume that there is an element $g \in G$ whose fixed-point set $\mathscr{X}^g = \{x \in X \mid g(x) = x\}$ is non-empty. Then the following assertions hold.*

(i) *$\mathscr{X}^g$ is a smooth closed submanifold of $\mathscr{X}$.*

(ii) *The restriction of p to $\mathscr{X}^g$ is a smooth proper surjective morphism.*

(iii) *The intersection of $\mathscr{X}^g$ and every fibre X_t, $t \in B$, of p is transversal.*

Proof. It is known that the action of any subgroup H of G can be linearized at any point $x \in \mathscr{X}$ which is fixed by H. In other words, one can find local analytic coordinates in a neighbourhood of x such that the action of H is linear in these coordinates.

Recall Cartan's linearization procedure for actions of finite groups (see [6]). We start from any system of local coordinates $z_1, \dots, z_n$ taking the value 0 at the chosen point x which is fixed by H. For every $h \in H$ we denote by h' the linear part of the Taylor expansion (with respect to $z_1, \dots, z_n$) of the automorphism h at x. Then the change of coordinates defined by the map

$$\sigma = \frac{1}{|H|} \sum_{g \in H} (g')^{-1} g$$

makes the action of H linear. Namely, it conjugates h and h' for every $h \in H$ since $\sigma \circ h = h' \circ \sigma$. Indeed, we have

$$\sigma \circ h = \frac{1}{|H|} \sum_{g \in H} (g')^{-1} g \circ h = \frac{1}{|H|} \sum_{g \in H} h'(g' \circ h')^{-1} g \circ h = \frac{1}{|H|} h' \sum_{e \in H} (e')^{-1} e = h' \circ \sigma.$$

This change of coordinates is tangent to the identity. Moreover, it acts as the identity on every linear (with respect to $z_1, \dots, z_n$) subspace on which H already acts linearly. Therefore, to prove (i), it suffices to linearize the action of g (whereupon the set $\mathscr{X}^g$ becomes linear in the new coordinates), and to prove (ii) and (iii), it suffices to take any system of local coordinates at $t \in B$ and include its lift into a system of local coordinates $z_1, \dots, z_n$. (This guarantees the surjectivity of the map $T_x(\mathscr{X}^g) \to T_{p(x)}B$ at the level of tangent spaces. Since the morphism $p\colon \mathscr{X} \to B$ is proper and the submanifold $\mathscr{X}^g$ is closed in $\mathscr{X}$, we see that the morphism $p\colon \mathscr{X}^g \to B$ is proper and surjective.)

Proof of Proposition 4.1. We give a proof only for $n \leqslant 2$ since the proof in the general case is similar.

Let X_o, $o \in B$ be a fibre of p which is the Galois expansion of a generic covering $X_o \to \mathbb{P}^n$. The ramification locus of this covering is a union of smooth manifolds $R_{o(i,j)}$, $1 \leqslant i < j \leqslant d$, of codimension one. These manifolds are the fixed-point sets of the transpositions $(i,j) \in S_d$. By Lemma 4.2, the fixed-point sets $\mathscr{X}^{(i,j)} \subset \mathscr{X}$ of the same transpositions acting on $\mathscr{X}$ are also smooth manifolds of codimension one, and the intersection $R_{t(i,j)} = \mathscr{X}^{(i,j)} \cap X_t$ is transversal for every $t \in B$ and all transpositions $(i,j) \in S_d$. Moreover, if $\mathscr{X}^g \neq \varnothing$ for some element $g \in S_d$, $g \neq 1$, then $\mathscr{X}^g \cap X_o \neq \varnothing$ by Lemma 4.2. Since the action of S_d on X_o is generic, it follows that g is a transposition in the case $n = 1$, and g is either a transposition, a product of two commuting transpositions or a cyclic permutation of length three in the case $n = 2$.

Suppose that $n = 2$ and g is a cyclic permutation (j_1, j_2, j_3) (the other cases can be treated in a similar way). We claim that the actions of $S_{\{j_1,j_2,j_3\}}$ on $\mathscr{X}$ and on each fibre X_t, $t \in B$, are generic. Indeed, there is no loss of generality in assuming that $\dim B = 1$. Then by Lemma 4.2, $\mathscr{X}^g$ is a smooth curve and $\mathscr{X}^{(j_1,j_2)}$ and $\mathscr{X}^{(j_1,j_3)}$ are smooth surfaces, and they all meet every fibre transversally. Since $\mathscr{X}^g \cap X_o = R_{o(j_1,j_2)} \cap R_{o(j_1,j_3)}$, another application of Lemma 4.2 yields that $\mathscr{X}^{(j_1,j_2)} \cap \mathscr{X}^{(j_1,j_3)} = \mathscr{X}^g$ and $X_t^g = R_{t(j_1,j_2)} \cap R_{t(j_1,j_3)}$ for all $t \in B$. Therefore $\mathscr{X}^g$ (resp. X_t^g) coincides with the fixed-point set under the action of $S_{\{j_1,j_2,j_3\}}$ on $\mathscr{X}$ (resp. on X_t).

As a result, we see that the actions of S_d on $\mathscr{X}$ and on every X_t, $t \in B$, are generic. Hence the quotient space $\mathscr{X}/S_d$ is a smooth manifold and the induced morphism $p_1\colon \mathscr{X}/S_d \to B$ is smooth and proper. Thus it remains to note that $X_t/S_d = X_o/S_d = \mathbb{P}^n$ for every $t \in B$ because a projective manifold M is isomorphic to $\mathbb{P}^n$ if there is a C^∞-diffeomorphism $M \to \mathbb{P}^n$ which maps the canonical class to the canonical class. (For $n=1$, this was known to Riemann. For $n=2$, one can use the Enriques–Kodaira classification of algebraic surfaces; see [11]. It is worth mentioning a related result of Siu [12]: in any dimension, a compact complex manifold is isomorphic to $\mathbb{P}^n$ if it is deformation equivalent to $\mathbb{P}^n$.)

Corollary 4.3. *G-varieties like those in Theorems* 0.1 *and* 0.2 *form connected components in the moduli spaces of, respectively, G-curves and G-surfaces of general type. These components are saturated* (*see* § 0.2). *In dimension one, G-curves like those in Theorem* 0.1 *form proper subvarieties in the moduli space of curves of general type.*

Proof. The first assertion follows from Proposition 4.1 while the second follows from the first and Theorems 0.1 and 0.2. The third assertion follows from the first along with the observation that if a birational transformation of a one-parameter deformation family of curves of genus $g \geqslant 2$ preserves every fibre and is regular everywhere except for finitely many fibres, then it extends to a transformation which is regular everywhere.

4.2. Examples of Dif $\neq$ Def complex G-manifolds. Here we consider regular actions of finite groups on complex surfaces and construct diffeomorphic actions which are not deformation equivalent. The idea is to pick surfaces that are diffeomorphic but not deformation equivalent and apply Theorem 0.2.

Let X be a rigid non-real minimal surface of general type (that is, a minimal surface of general type which is stable under deformations and not isomorphic to its complex conjugate $\overline{X}$). Such surfaces can be found in [13]. We denote by $Y_1 = \widetilde{X}$ the Galois expansion of a generic m-canonical covering $X \to \mathbb{P}^2$, and by Y_2 its conjugate, $Y_2 = \overline{Y}_1$.

Proposition 4.4. *Suppose that Y_1 and Y_2 are as above and m satisfies the hypotheses of Theorem 0.2. Then the actions of $S_d = \operatorname{Aut} Y_1 = \operatorname{Aut} Y_2$ on Y_1 and Y_2 are diffeomorphic, but Y_1 and Y_2 are not S_d-deformation equivalent.*

Proof. According to Theorem 0.2, $\operatorname{Aut} Y_1 = \operatorname{Aut} Y_2 = S_d$, where d is the degree of the m-canonical covering $X \to \mathbb{P}^2$. The actions of S_d on Y_1 and Y_2 are tautologically diffeomorphic since $Y_2 = \overline{Y}_1$.

Assume that $Y_1 = \widetilde{X}$ and $Y_2 = \overline{Y}_1$ are S_d-deformation equivalent. Let $p\colon \mathscr{X} \to B$ be a smooth S_d-deformation connecting them (the same treatment applies in the case when there is a chain of deformations). By Proposition 4.1, the covering $X_t \to \mathbb{P}^n$ is generic for every $t \in B$. Hence, $\mathscr{X}/S_{d-1} \to B$ is a deformation family connecting $X = Y_1/S_{d-1}$ with $\overline{X} = Y_2/S_{d-1}$, which is a contradiction.

4.3. Examples of Dif $\neq$ Def real G-manifolds. Here we extend the category of G-manifolds. Namely, we consider finite subgroups of Klein extensions of the automorphism group. We recall that the Klein group $\mathrm{Kl}(X)$ of a complex variety X is, by definition, the group consisting of biregular isomorphisms $X \to X$ and $X \to \overline{X}$ (it is sometimes called the *group of dyanalytic automorphisms*). If X is a real manifold and c is a real structure on X, then there is an exact sequence

$$1 \to \langle c \rangle = \mathbb{Z}/2 \to \mathrm{Kl}(X) \to \operatorname{Aut} X \to 1.$$

We consider the real Campedelli surfaces (X_1, c_1) and (X_2, c_2) constructed in [3], § 2. As shown in [3], these surfaces are not real deformation equivalent, but their real structures $c_1\colon X_1 \to X_1$ and $c_2\colon X_2 \to X_2$ are diffeomorphic.

The Campedelli surfaces are minimal surfaces of general type. Thus we can consider m-canonical generic coverings $X_1 \to \mathbb{P}^2$ and $X_2 \to \mathbb{P}^2$. Moreover, we can choose these coverings to be real. This means that they are equivariant with respect to the usual (complex-conjugation) real structure on $\mathbb{P}^2$ and the real structures c_1, c_2 on X_1, X_2. We denote the Galois expansions of such coverings by $\widetilde{X}_1 \to \mathbb{P}^2$ and $\widetilde{X}_2 \to \mathbb{P}^2$. The surfaces $\widetilde{X}_1$ and $\widetilde{X}_2$ are real with real structures lifted from $\mathbb{P}^2$.

Proposition 4.5. *The Klein groups $\mathrm{Kl}(\widetilde{X}_1)$ and $\mathrm{Kl}(\widetilde{X}_2)$ are isomorphic and their actions are diffeomorphic, but there is no equivariant deformation connecting $(\widetilde{X}_1, \mathrm{Kl}(\widetilde{X}_1))$ with $(\widetilde{X}_2, \mathrm{Kl}(\widetilde{X}_2))$.*

Proof. As in the proof of Proposition 4.4, the non-existence of an equivariant deformation follows immediately from Proposition 4.1.

To prove the other two assertions, it suffices to construct a real diffeomorphism $\widetilde{X}_1 \to \widetilde{X}_2$ commuting with the actions of the Galois groups. To do this, it suffices to construct real diffeomorphisms $X_1 \to X_2$, $\mathbb{P}^2 \to \mathbb{P}^2$ that commute with the initial m-canonical generic coverings $X_1 \to \mathbb{P}^2$ and $X_2 \to \mathbb{P}^2$.

We recall some details of the construction of the surfaces X_1, X_2 in [3]. The construction starts from a real one-parameter family of Campedelli line arrangements $\mathscr{L}(t)$, $t \in T = \{|t| \leqslant 1, t \in \mathbb{C}\}$, consisting of seven lines $L_1(t), \dots, L_7(t)$ labelled by the non-zero elements $\alpha \in (\mathbb{Z}/2\mathbb{Z})^3$. These lines are real for real values of t, and the family performs a triangular transformation at $t = 0$ (see the definition of a triangular transformation in [3]). Consider the Galois covering $Y \to \mathbb{P}^2 \times T$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^3$, branched over $\sum_{i=1}^{7} \mathscr{L}_i$, $\mathscr{L}_i = \{(p,t) \in \mathbb{P}^2 \times T \colon p \in L_i(t)\}$ and defined by the chosen labelling of the lines. The fibres Y_t of the projection of Y to T are non-singular Campedelli surfaces for generic t and, in particular, for all $t \neq 0$ sufficiently close to 0. The surfaces X_1 and X_2 we are interested in are given by Y_t with, respectively, positive and negative t close to 0. The fibre Y_0 has two singular points. Each of them is a so-called $T(-4)$-singularity. These points are non-real and complex-conjugate to each other. For every non-singular Campedelli surface Y_t, $t \neq 0$, and every i, $1 \leqslant i \leqslant 7$, the pullback $L_i^*(t) \subset Y_t$ of the line $L_i(t) \subset \mathbb{P}^2$ represents the bi-canonical class, $[L_i^*(t)] = 2K_{Y_t}$.

Since $Y \to \mathbb{P}^2 \times T$ is a finite morphism, the divisors $E_m = m[\mathscr{L}_i^*]$ are relatively very ample for any sufficiently large m (and any i); see, for example, [14]. Pick such an integer m and consider the real (with respect to T) embedding of Y in $\mathbb{P}^N \times T$ defined by the linear system $|E_m|$ (to construct such an embedding, one must twist E_m by the pullback of a very ample divisor on T). Since $[L_i^*(t)] = 2K_{Y_t}$, this embedding determines a $(2m)$-canonical embedding of the Campedelli surfaces Y_t in $\mathbb{P}^N$. Theorem 0.1 of [2] implies that if $m \geqslant 5$, then the linear projection $Y_t \to \mathbb{P}^2$ from a generic subspace $\mathbb{P}^{N-3}$ is a generic covering for all but finitely many values of $t \in T$ and, in particular, for all $t \neq 0$ close to 0. The projection $Y \to \mathbb{P}^2 \times T$ is real for real subspaces $\mathbb{P}^{N-3}$. If the real subspace $\mathbb{P}^{N-3}$ is sufficiently generic, then the two singular points of Y_0 are projected to distinct complex-conjugate points. We denote these singular points by y, $\bar{y}$ and their projections by b, $\bar{b}$. Literally repeating the argument in [2], one can show that the projection $Y_0 \to \mathbb{P}^2$ is everywhere generic. (The projection is said to be generic at the singular points if the fibre of the projection passing through the singular point $y \in Y_0$ (resp. $\bar{y} \in Y_0$) is in general position with respect to the tangent cone $C_b Y_0$ (resp. $C_{\bar{b}} Y_0$).)

We restrict our attention to small values of t. The coverings $Y_t \to \mathbb{P}^2$ are generic for $t \neq 0$, and their branch curves B_t are cuspidal. For $t = 0$, the branch curve B_0 is cuspidal everywhere except for two distinct complex-conjugate points b and $\bar{b}$. We cut out Milnor's small complex-conjugate balls $V(b)$ and $V(\bar{b})$ centred at these points. Using a family of Morse–Lefschetz diffeomorphisms, we complete the isotopy $B_{te^{i\varphi}} \setminus (V(b) \cup V(\bar{b}))$ by an isotopy inside $V(b)$ and then complete it by the complex-conjugate isotopy inside $V(\bar{b})$. The resulting isotopy provides an equivariant diffeomorphism between the Galois coverings branched along B_t and B_{-t} respectively. The Morse–Lefschetz diffeomorphisms may be regarded as transformations acting as the identity on the complement of $V(b) \cup V(\bar{b})$. Hence the epimorphism $\pi_1(\mathbb{P}^2 \setminus B_t) \to S_d$ which defines the Galois coverings is not changed and, therefore, the diffeomorphism constructed between the Galois coverings acts from $\widetilde{X}_1$ to $\widetilde{X}_2$ and is equivariant with respect to the Galois action. It is also equivariant with respect to the real structure. To complete the proof, it remains to note that by Theorem 0.2, the full automorphism groups $\mathrm{Aut}(\widetilde{X}_1)$ and $\mathrm{Aut}(\widetilde{X}_2)$ coincide with the Galois group.

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