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Alexander modules of irreducible $C$-groups

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Abstract. We give a complete description of the Alexander modules of knotted $n$-manifolds in the sphere $S^{n+2}$ for $n \geq 2$ and the Alexander modules of irreducible Hurwitz curves. This description is applied to investigate the properties of the first homology groups of cyclic coverings of the sphere $S^{n+2}$ and the complex projective plane $\mathbb{CP}^2$ branched respectively along knotted $n$-manifolds and irreducible Hurwitz (in particular, algebraic) curves.

Introduction

The class $C$ of $C$-groups and the subclass $H$ of Hurwitz $C$-groups (see the definitions below) play a very important role in the geometry of submanifolds of codimension two. For example, it is well known that knot and link groups (given by their Wirtinger presentations) are $C$-groups and any $C$-group $G$ may be realized as the group of a linked $n$-manifold with $n \geq 2$, that is, as the fundamental group $\pi_1(S^{n+2} \setminus V)$ of the complement of a closed oriented $n$-manifold $V$ (without boundary) in the $(n+2)$-dimensional sphere $S^{n+2}$ (see [1]) and conversely. Note that a $C$-group $G$ is isomorphic to $\pi_1(S^{n+2} \setminus \bigcup S^n)$ for some union $\bigcup S^n$ of linked $n$-dimensional spheres with $n \geq 3$ if and only if $H_2G = 0$ (see [2]). Some other results describing the groups $\pi_1(S^{n+2} \setminus \bigcup S^n)$ can be found in [3] and [4].

Let $H \subset \mathbb{CP}^2$ be an algebraic (or, more generally, Hurwitz\(^1\)) curve or a pseudo-holomorphic curve with respect to some $\omega$-tamed almost complex structure on $\mathbb{CP}^2$, where $\omega$ is the Fubini–Study symplectic form on $\mathbb{CP}^2$, $\deg H = m$. The Zariski–van Kampen presentation of $\pi_1 = \pi_1(\mathbb{CP}^2 \setminus (H \cup L))$ endows $\pi_1$ with the structure of a Hurwitz $C$-group of degree $m$, where $L$ is the line ‘at infinity’. (In other words, $L$ is a fibre of the linear projection $pr: \mathbb{CP}^2 \to \mathbb{CP}^1$ and is in general position with respect to $H$. If $H$ is a pseudo-holomorphic curve, then the projection $pr$ is determined by a pencil of pseudo-holomorphic lines.) As proved in [7], every Hurwitz $C$-group $G$ of degree $m$ can be realized as the fundamental group $\pi_1(\mathbb{CP}^2 \setminus (H \cup L))$ for some Hurwitz (resp. pseudo-holomorphic) curve $H$, $\deg H = 2^m m$, with singularities of the form $w^m - z^m = 0$, where $n$ depends on the Hurwitz $C$-representation of $G$. Thus the class $H$ coincides with the class $\{ \pi_1(\mathbb{CP}^2 \setminus (H \cup L)) \}$ of fundamental groups of complements of ‘affine’ Hurwitz (resp. ‘affine’ pseudo-holomorphic) curves.

\(^1\)The definition of Hurwitz curves can be found in [5] or [6].

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curves. It contains the subclass of fundamental groups of complements of plane affine algebraic curves.

By definition, a C-group is a group together with a finite presentation

$$G_W = \langle x_1, \ldots, x_m \mid x_i = w_{i,j,k}^{-1} x_j w_{i,j,k}, w_{i,j,k} \in W \rangle,$$

where $W = \{ w_{i,j,k} \in \mathbb{F}_m \mid 1 \leq i, j \leq m, 1 \leq k \leq h(i,j) \}$ consists of elements of the free group $\mathbb{F}_m$ freely generated by $x_1, \ldots, x_m$ (it is possible that $w_{i_1,j_1,k_1} = w_{i_2,j_2,k_2}$ for $(i_1, j_1, k_1) \neq (i_2, j_2, k_2)$ and $h : \{1, \ldots, m\}^2 \to \mathbb{Z}$ is some function. Such a presentation is called a C-presentation (C means that all the relations are given by conjugations). Let $\varphi_W : \mathbb{F}_m \to G_W$ be the canonical epimorphism. The elements $\varphi_W(x_i) \in G$, $1 \leq i \leq m$, and their conjugates are called C-generators of $G$. Let $f : G_1 \to G_2$ be a homomorphism of C-groups. It is called a C-homomorphism if the images of the C-generators of $G_1$ under $f$ are C-generators of $G_2$. C-groups are considered up to C-isomorphism. Some properties of C-groups were investigated in [7]–[10].

A C-presentation (1) is called a Hurwitz C-presentation of degree $m$ if the word $w_{i,i,1}$ coincides with the product $x_1 \ldots x_m$ for every $i = 1, \ldots, m$. A C-group $G$ is called a Hurwitz C-group (of degree $m$) if there is an $m \in \mathbb{N}$ such that $G$ possesses a Hurwitz C-presentation of degree $m$. In other words, a C-group $G$ is a Hurwitz C-group of degree $m$ if there are C-generators $x_1, \ldots, x_m$ generating $G$ such that the product $x_1 \ldots x_m$ belongs to the centre of $G$. Note that the degree of a Hurwitz C-group $G$ is not canonically defined: it depends on the Hurwitz C-presentation of $G$. We denote the class of all Hurwitz C-groups by $\mathcal{H}$.

It is easy to show that $G/G'$ is a finitely generated free abelian group for any C-group $G$. Here $G' = [G, G]$ is the commutator subgroup of $G$. A C-group $G$ is said to be irreducible if $G/G' \simeq \mathbb{Z}$. We say that $G$ consists of $k$ irreducible components if $G/G' \simeq \mathbb{Z}^k$. If $G$ is a Hurwitz C-group realized as the fundamental group $\pi_1(\mathbb{C}P^2 \setminus (H \cup L))$ of the complement of some Hurwitz curve $H$, then the number of irreducible components of $G$ is equal to the number of irreducible components of $H$. Similarly, if a C-group $G$ consisting of $k$ irreducible components is realized as the group of a linked $n$-manifold $V$ (that is, $G = \pi_1(S^{n+2} \setminus V)$), then the number of connected components of $V$ is equal to $k$.

The free group $\mathbb{F}_n$ with fixed free generators is a C-group. For any C-group $G$ we have a well-defined canonical C-epimorphism $\nu : G \to \mathbb{F}_1$ sending the C-generators of $G$ to the C-generator of $\mathbb{F}_1$. We denote its kernel by $N$. Note that if $G$ is an irreducible C-group, then $N$ coincides with $G'$. In what follows we assume that all C-groups under consideration are irreducible.

Let $G$ be an irreducible C-group. The C-epimorphism $\nu$ induces the exact sequence of groups

$$1 \to G'/G'' \to G/G'' \overset{\nu}{\longrightarrow} \mathbb{F}_1 \to 1,$$

where $G'' = [G', G']$. The C-generator of $\mathbb{F}_1$ acts on $G'/G''$ by conjugation: $\tilde{x}^{-1} g \tilde{x}$, where $g \in G'$ and $\tilde{x}$ is one of the C-generators of $G$. We denote this action by $t$. The group $A_0(G) = G'/G''$ is abelian and the action $t$ endows $A_0(G)$ with the structure of a $\Lambda$-module, where $\Lambda = \mathbb{Z}[t, t^{-1}]$ is the ring of Laurent polynomials with integer coefficients. The $\Lambda$-module $A_0(G)$ is called the Alexander module of
the $C$-group $G$. The action $t$ induces an action $h_C$ on $A_C = A_0(G) \otimes \mathbb{C}$, and it is easy to see that its characteristic polynomial $\det(h_C - t \text{Id})$ belongs to $\mathbb{Q}[t]$. Let $a \in \mathbb{N}$ be the smallest positive integer such that $a \det(h_C - t \text{Id}) \in \mathbb{Z}[t]$. The polynomial $\Delta(t) = a \det(h_C - t \text{Id})$ is called the Alexander polynomial of the $C$-group $G$. If $H$ is an algebraic, or Hurwitz, or pseudo-holomorphic irreducible curve in $\mathbb{C}P^2$ (resp. if $V \subset S^{n+2}$ is a knotted (that is, connected smooth oriented without boundary) $n$-manifold, $n \geq 1$) and we put $G = \pi_1(\mathbb{C}P^2 \setminus (H \cup L))$ (resp. $G = \pi_1(S^{n+2} \setminus V)$), then the Alexander module $A_0(G)$ of the group $G$ and its Alexander polynomial $\Delta(t)$ are called the Alexander module and Alexander polynomial of the curve $H$ (resp. of the knotted manifold $V$). We note that the Alexander module $A_0(H)$ and the Alexander polynomial $\Delta(t)$ of the curve $H$ are independent of the choice of the generic (pseudo-holomorphic) line $L$. Some results concerning Alexander modules of knotted spheres are to be found in [11], [12].

Properties of Alexander polynomials of Hurwitz curves were studied in [6] and [13]. In particular, it was proved that if $H$ is an irreducible Hurwitz curve of degree $m$, then its Alexander polynomial $\Delta(t)$ has the following properties:

(i) $\Delta(t) \in \mathbb{Z}[t]$ and $\deg \Delta(t)$ is even,

(ii) $\Delta(0) = \Delta(1) = 1$,

(iii) $\Delta(t)$ divides the polynomial $(t^m - 1)^{m-2}$.

Moreover, a polynomial $P(t) \in \mathbb{Z}[t]$ is the Alexander polynomial of an irreducible Hurwitz curve if and only if the roots of $P(t)$ are roots of unity and $P(1) = 1$.

Suppose that $G = \pi_1(\mathbb{C}P^2 \setminus (H \cup L))$ is the fundamental group of the complement of an irreducible ‘affine’ Hurwitz curve of degree $m$ (resp. $G = \pi_1(S^{n+2} \setminus V)$ is the group of a knotted $n$-manifold, $n \geq 1$). The homomorphism $\nu: G \to \mathbb{F}_1$ determines an infinite unramified cyclic covering $f_\infty: X_\infty \to \mathbb{C}P^2 \setminus (H \cup L)$ (resp. $f_\infty: X_\infty \to S^{n+2} \setminus V$). We have $H_1(X_\infty, \mathbb{Z}) = G'/G''$ and the action of $t$ on $H_1(X_\infty, \mathbb{Z})$ coincides with the action of a generator $h$ of the covering transformation group of the covering $f_\infty$.

For every $k \in \mathbb{N}$ let $\text{mod}_k: \mathbb{F}_1 \to \mu_k = \mathbb{F}_1/\langle t^k \rangle$ be the natural epimorphism to the cyclic group $\mu_k$ of order $k$. The covering $f_\infty$ factors through the cyclic covering $f_k': X'_k \to \mathbb{C}P^2 \setminus (H \cup L)$ (resp. $f_k': X'_k \to S^{n+2} \setminus V$) associated with the epimorphism $\text{mod}_k \circ \nu$, $f_\infty = f_k' \circ g_k$. Since any Hurwitz curve $H$ has only analytic singularities, the covering $f_k'$ can be extended (see [6]) to a map $\tilde{f}_k: \tilde{X}_k \to X$ branched along $H$ and possibly along $L$. Here $\tilde{X}_k$ stands for a closed four-dimensional variety which is locally isomorphic over each singular point of $H$ to the complex-analytic singularity given by $w^k = F(u, v)$, where $F(u, v) = 0$ is a local equation of $H$ at the singular point. Over a neighbourhood of every common point of $H$ and $L$, the variety $\tilde{X}_k$ is locally isomorphic to the singularity given locally by $w^k = vu^d$, where $d$ is the smallest non-negative integer such that $m + d$ is divisible by $k$.

If $\tilde{f}_k^{-1}(L) \subset \text{Sing} \tilde{X}_k$, then $\tilde{X}_k$ can be normalized (as in the algebraic case) and we obtain a covering $\tilde{f}_{k,\text{norm}}: \tilde{X}_{k,\text{norm}} \to \mathbb{C}P^2$, where $\tilde{X}_{k,\text{norm}}$ is a singular analytic variety at each of its finitely many singular points. The map $\tilde{f}_{k,\text{norm}}$ is branched along $H$ and possibly along the line $L$ ‘at infinity’ (if $k$ does not divide $\deg H$), then $\tilde{f}_{k,\text{norm}}$ is branched along $L$). One can resolve the singularities of $\tilde{X}_{k,\text{norm}}$ and obtain a smooth manifold $\overline{X}_k$, $\dim_{\mathbb{R}} \overline{X}_k = 4$. Let $\sigma: \overline{X}_k \to \tilde{X}_{k,\text{norm}}$ be the
resolution of singularities, $E = \sigma^{-1}(\text{Sing } \tilde{X}_{k,\text{norm}})$ the proper transform of the set of singular points of $\tilde{X}_{k,\text{norm}}$, and $\tilde{f}_k = f_{k,\text{norm}} \circ \sigma$. The action $h$ induces an action $\tilde{h}_k$ on $\tilde{X}_k$ and an action $\tilde{h}_{k*}$ on $H_1(\tilde{X}_k, \mathbb{Z})$.

Similarly, the covering $f_k': X'_k \to S^{n+2} \setminus V$ can be extended to a smooth map $f_k: X_k \to S^{n+2}$ branched along $V$, where $X_k$ is a smooth compact $(n + 2)$-manifold, and the action $h$ induces actions $h_k$ on $X_k$ and $h_{k*}$ on $H_1(X_k, \mathbb{Z})$. The action $h_{k*}$ endows $H_1(X_k, \mathbb{Z})$ with the structure of a $\Lambda$-module.

It was shown in [6] that for any Hurwitz curve $H$ the covering space $\tilde{X}_k$ can be embedded as a symplectic submanifold in a complex projective rational 3-fold whose symplectic structure is given by an integer Kähler form. It was also proved that the first Betti number $b_1(\tilde{X}_k) = \dim_{\mathbb{C}} H_1(\tilde{X}_k, \mathbb{C})$ of $\tilde{X}_k$ is equal to $r_{k, \neq 1}$, where $r_{k, \neq 1}$ is the number of roots of the Alexander polynomial $\Delta(t)$ of the curve $\overline{H}$ which are $k$th roots of unity not equal to 1.

Let $M$ be a Noetherian $\Lambda$-module. We say that $M$ is $(t-1)$-invertible if multiplication by $t-1$ is an automorphism of $M$. A $\Lambda$-module $M$ is said to be $t$-unipotent if there is an $n \in \mathbb{N}$ such that multiplication by $t^n$ is the identity automorphism of $M$. The unipotence index of a $t$-unipotent module $M$ is the smallest $k \in \mathbb{N}$ such that

$$t^k - 1 \in \text{Ann}(M) = \{ f(t) \in \Lambda \mid f(t)v = 0 \ \forall v \in M \}.$$

Let $M$ be a Noetherian $(t-1)$-invertible $\Lambda$-module. The $t$-unipotent $\Lambda$-module $A_n(M) = M/(t^k - 1)M$ is called the $k$th derived Alexander module of $M$. If $M$ is the Alexander module of some $C$-group $G$ (resp. knotted $n$-manifold $V$ or Hurwitz curve $H$), then $A_k(M)$ is called the $k$th derived Alexander module of $G$ (resp. of $V$ or $H$) and is denoted by $A_k(G)$ (resp. by $A_k(V)$ or $A_k(H)$).

Here are the main results of this paper.

**Theorem 0.1.** A $\Lambda$-module $M$ is the Alexander module of a knotted $n$-manifold for $n \geq 2$ if and only if $M$ is a Noetherian $(t-1)$-invertible $\Lambda$-module.

**Theorem 0.2.** Suppose that $V$ is a knotted $n$-manifold, $n \geq 1$, and $f_k: X_k \to S^{n+2}$ is the cyclic covering branched along $V$. Then $H_1(X_k, \mathbb{Z})$ is isomorphic as a $\Lambda$-module to the $k$th derived Alexander module $A_k(V)$ of $V$.

Similar statements hold for algebraic and, more generally, Hurwitz (resp. pseudo-holomorphic) curves. Namely, we have the following theorems.

**Theorem 0.3.** A $\Lambda$-module $M$ is the Alexander module of an irreducible Hurwitz (resp. pseudo-holomorphic) curve if and only if $M$ is a Noetherian $(t-1)$-invertible $t$-unipotent $\Lambda$-module. In particular, the Alexander module of an irreducible algebraic plane curve is a Noetherian $(t-1)$-invertible $t$-unipotent $\Lambda$-module.

The unipotence index of the Alexander module $A_0(H)$ of an irreducible plane algebraic (resp. Hurwitz or pseudo-holomorphic) curve $H$ is a divisor of $\deg H$.

**Corollary 0.4.** The Alexander module $A_0(H)$ of any irreducible plane algebraic (or Hurwitz, or pseudo-holomorphic) curve $H$ is finitely generated over $\mathbb{Z}$, that is, $A_0(H)$ is a finitely generated abelian group.

A finitely generated abelian group $G$ is the Alexander module $A_0(H)$ of some irreducible Hurwitz or pseudo-holomorphic curve $H$ if and only if one can find
a positive integer \(m\) and an automorphism \(h \in \text{Aut}(G)\) such that \(h^m = \text{Id}\) and \(h - \text{Id}\) is also an automorphism of \(G\).

**Theorem 0.5.** Suppose that \(H\) is an algebraic (or Hurwitz, or pseudo-holomorphic) irreducible curve in \(\mathbb{CP}^2\), \(\deg H = m\), and \(f_k : X_k \to \mathbb{CP}^2\) is a resolution of singularities of the cyclic covering of degree \(\deg f_k = k\) branched along \(H\) and possibly along the line \(L\) ‘at infinity’. Then

\[
H_1(X_k \setminus E, \mathbb{Z}) \simeq A_k(H), \quad H_1(X_k, \mathbb{Q}) \simeq A_k(H) \otimes \mathbb{Q},
\]

where \(E = \sigma^{-1}(\text{Sing } X_{k,\text{norm}})\) and \(A_k(H)\) is the \(k\)th derived Alexander module of \(H\).

It should be noticed that the epimorphism \(H_1(X_k \setminus E, \mathbb{Z}) \simeq A_k(H) \to H_1(X_k, \mathbb{Z})\) induced by the embedding \(X_k \setminus E \hookrightarrow X_k\) need not be an isomorphism in the general case of Hurwitz curves (see Example 4.6 below).

**Corollary 0.6.** Suppose that \(H\) is an algebraic (or Hurwitz, or pseudo-holomorphic) irreducible curve in \(\mathbb{CP}^2\), \(\deg H = m\), and \(f_k : X_k \to \mathbb{CP}^2\) is a resolution of singularities of the cyclic covering of degree \(\deg f_k = k\) branched along \(H\) and possibly along the line ‘at infinity’. Then the following assertions hold.

(i) The first Betti number \(b_1(X_k)\) of \(X_k\) is even.

(ii) If \(k = p^r\) for some prime \(p\), then \(H_1(X_k, \mathbb{Q}) = 0\).

(iii) If \(k\) and \(m\) are coprime, then \(H_1(X_k, \mathbb{Z}) = 0\).

(iv) \(H_1(X_2, \mathbb{Z})\) is a finite abelian group of odd order.

Note also that any \(C\)-group \(G\) can be realized (see [7]) as \(\pi_1(\Delta^2 \setminus (C \cap \Delta^2))\), where \(\Delta^2 = \{|z| < 1\} \times \{|w| < 1\} \subset \mathbb{C}^2\) is a bidisc and \(C \subset \mathbb{C}^2\) is a non-singular algebraic curve such that the restriction of \(\text{pr}_1 : \Delta^2 \to \{|z| < 1\}\) to \(C \cap \Delta^2\) is a proper map. Therefore the analogues of Theorems 0.1, 0.2 and Corollaries 0.4, 0.6 hold in this case as well.

The proofs of Theorems 0.1 and 0.3 are given in §3. In §1 we describe the properties of Noetherian \((t-1)\)-invertible \(\Lambda\)-modules. §2 is devoted to the properties of Noetherian \(t\)-unipotent \(\Lambda\)-modules. In §4 we prove Theorems 0.2, 0.5 and give some other corollaries of them.

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§1. Properties of \((t-1)\)-invertible \(\Lambda\)-modules

**1.1. Criteria for \((t-1)\)-invertibility.** Before describing \((t-1)\)-invertible \(\Lambda\)-modules, we recall that the ring \(\Lambda = \mathbb{Z}[t, t^{-1}]\) is Noetherian. Each element \(f \in \Lambda\) can be written as

\[
f = \sum_{n_- \leq i \leq n_+} a_i t^i \in \mathbb{Z}[t, t^{-1}],
\]

where \(n_-, n_+, i, a_i \in \mathbb{Z}\). If \(n_- \geq 0\) for some element \(f \in \Lambda\), then \(f \in \mathbb{Z}[t]\). Such elements are called polynomials.

For every \(n \in \mathbb{Z}, n \neq 0\), we have a well-defined \(\mathbb{Z}\)-homomorphism

\[
f(t) = \sum a_i t^i \mapsto f(n) = \sum a_i n^i.
\]
The image $f(n)$ of an element $f(t)$ is called the value of $f(t)$ at $n$. If $f(t)$ is a polynomial, then its value $f(0) = a_0$ is also well defined.

**Lemma 1.1.** A Noetherian $\Lambda$-module $M$ is $(t - 1)$-invertible if and only if multiplication by $t - 1$ is a surjective homomorphism of $M$.

**Proof.** This follows from a more general statement. Namely, every surjective $\Lambda$-endomorphism $f: M \to M$ of a Noetherian $\Lambda$-module $M$ is an isomorphism. Indeed, if $\ker f \neq 0$, then the chain of submodules

$$\ker f \subset \ker f^2 \subset \cdots \subset \ker f^n \subset \cdots$$

is strictly increasing because $f$ is surjective. This contradicts the Noetherian property of $M$.

Let $M$ be a Noetherian $(t - 1)$-invertible $\Lambda$-module. We consider an element $v \in M$ and write $M_v = \langle v \rangle$ for the principal submodule of $M$ generated by $v$. Since $M$ is Noetherian, every principal submodule of $M$ is contained in some maximal principal submodule of $M$.

**Lemma 1.2.** Every maximal principal submodule $M_v$ of a $(t - 1)$-invertible module $M$ is $(t - 1)$-invertible.

**Proof.** Since $M$ is $(t - 1)$-invertible, there is an element $v_1 \in M$ such that $v = (t - 1)v_1$. Therefore $M_v \subset M_{v_1}$. Since $M_v$ is a maximal principal submodule of $M$, we have $M_v = M_{v_1}$. Therefore $v_1 \in M_v$ and multiplication by $t - 1$ determines a surjective endomorphism of $M_v$. To complete the proof, we apply Lemma 1.1.

Every principal submodule $M_v \subset M$ is isomorphic to the module $\Lambda/\text{Ann}_v$, where $\text{Ann}_v = \{ f \in \Lambda \mid fv = 0 \}$ is the annihilator of $v$. The annihilator $\text{Ann}_v$ of any element $v \in M$ is an ideal of $\Lambda$. We write

$$\text{Ann}(M) = \bigcap_{v \in M} \text{Ann}_v = \{ g(t) \in \Lambda \mid g(t)v = 0 \quad \forall v \in M \}$$

for the annihilator of the module $M$.

**Lemma 1.3.** A principal $\Lambda$-module $M = \Lambda/I$ is $(t - 1)$-invertible if and only if the ideal $I$ contains a polynomial $f(t)$ such that $f(1) = 1$.

**Proof.** Let $M$ be generated by an element $v \in M$. If the ideal $I = \text{Ann}_v$ contains a polynomial $f(t)$ with $f(1) = 1$, then $f(t)$ may be written in the form

$$f(t) = (t - 1)g(t) + 1$$

for some polynomial $g(t)$. Therefore $v = (t - 1)v_1$, where $v_1 = -g(t)v$. Thus multiplication by $t - 1$ is a surjective endomorphism of $M$. Hence multiplication by $t - 1$ is an automorphism of $M$ by Lemma 1.1.

Conversely, if $M$ is $(t - 1)$-invertible, then there is an element $v_1 \in M$ such that $v = (t - 1)v_1$. Write $v_1 = h(t)v$ for some $h(t) \in \Lambda$. We have $(1 - (t - 1)h(t))v = 0$. Therefore $1 - (t - 1)h(t) \in \text{Ann}_v = I$. There is a positive integer $k$ such that $f(t) = t^k(1 - (t - 1)h(t)) \in I \cap \mathbb{Z}[t]$. It is easy to see that $f(1) = 1$. 
As a consequence of Lemma 1.3, we obtain the following lemma.

**Lemma 1.4.** Every principal submodule of a principal \((t-1)\)-invertible module \(M\) is \((t-1)\)-invertible.

**Proof.** Indeed, suppose that \(M\) is generated by an element \(v \in M\), and the submodule \(M_1\) is generated by \(v_1 = h(t)v\). Then \(\text{Ann}_v \subset \text{Ann}_{v_1}\).

Since \(M\) is \((t-1)\)-invertible, Lemma 1.3 yields a polynomial \(f(t) \in \text{Ann}_v\) with \(f(1) = 1\). Applying Lemma 1.3 again, we see that \(M_1\) is \((t-1)\)-invertible because \(f(t) \in \text{Ann}_{v_1}\).

**Proposition 1.5.** Every submodule of a Noetherian \((t-1)\)-invertible \(\Lambda\)-module \(M\) is \((t-1)\)-invertible.

**Proof.** Let \(N\) be a submodule of \(M\). Since \(M\) is a Noetherian \(\Lambda\)-module, the submodule \(N\) is generated by finitely many elements, say \(v_1, \ldots, v_n\). Every principal submodule \(M_{v_i} \subset N \subset M\) is \((t-1)\)-invertible by Lemmas 1.2 and 1.4. It follows that multiplication by \(t-1\) is a surjective endomorphism of \(N\) because it is surjective on each of the submodules \(M_{v_i} \subset N\) and the elements \(v_1, \ldots, v_n\) generate the module \(N\). To complete the proof, we apply Lemma 1.1.

**Proposition 1.6.** Every quotient module of a Noetherian \((t-1)\)-invertible \(\Lambda\)-module \(M\) is \((t-1)\)-invertible.

**Proof.** This follows from Lemma 1.1.

**Lemma 1.7.** Let \(M_1, \ldots, M_k\) be Noetherian \((t-1)\)-invertible \(\Lambda\)-modules. Then the direct sum \(M = \bigoplus_{i=1}^k M_i\) is a Noetherian \((t-1)\)-invertible \(\Lambda\)-module.

**Proof.** This is obvious.

**Corollary 1.8.** Any Noetherian \((t-1)\)-invertible \(\Lambda\)-module \(M\) is a quotient module of the direct sum \(\bigoplus_{j=1}^n \Lambda/I_j\) of principal \((t-1)\)-invertible \(\Lambda\)-modules \(\Lambda/I_j\).

**Proof.** Since \(M\) is a Noetherian \(\Lambda\)-module, it is generated by finitely many elements, say \(v_1, \ldots, v_n\). Proposition 1.5 implies that every principal submodule \(M_{v_i} \subset M\) is \((t-1)\)-invertible. Clearly, there is an epimorphism \(\bigoplus_{i=1}^n M_{v_i} \rightarrow M\).

**Remark 1.9.** An abelian group \(G\) possesses the structure of a \((t-1)\)-invertible \(\Lambda\)-module if and only if it has an automorphism \(t\) such that \(t-1\) is again an automorphism. If \(G\) is finitely generated and such an automorphism \(t \in \text{Aut} G\) is chosen, then \(G\) is a Noetherian \(\Lambda\)-module.

We note that an abelian group may have many \((t-1)\)-invertible \(\Lambda\)-module structures. For example, the group \(\mathbb{Z}/9\mathbb{Z}\) admits 3 such structures: either \(tv = 2v\), or \(tv = 5v\), or \(tv = 8v\), where \(v\) is a generator of \(\mathbb{Z}/9\mathbb{Z}\).

**Theorem 1.10.** A Noetherian \(\Lambda\)-module \(M\) is \((t-1)\)-invertible if and only if there is a polynomial \(f(t) \in \text{Ann}(M)\) such that \(f(1) = 1\).

**Proof.** If \(M\) is \((t-1)\)-invertible, then each of its principal submodules \(M_v\) is also \((t-1)\)-invertible by Proposition 1.5. Hence, by Lemma 1.3, the annihilator \(\text{Ann}_v\)
of every \( v \in M \) contains a polynomial \( f_v(t) \) such that \( f_v(1) = 1 \). If \( M \) is generated by \( v_1, \ldots, v_n \), then the polynomial \( f(t) = f_{v_1}(t) \ldots f_{v_n}(t) \) has the desired property.

Let us show that if there is a polynomial \( f(t) \in \text{Ann}(M) \) with \( f(1) = 1 \), then \( M \) is a \((t-1)\)-invertible module. Indeed, in this case every principal submodule \( M_v \) of \( M \) is \((t-1)\)-invertible by Lemma 1.3. Since multiplication by \( t-1 \) is an isomorphism of every principal submodule \( M_v \) of \( M \), we conclude that this multiplication is an isomorphism of \( M \).

Theorem 1.10 implies that every Noetherian \((t-1)\)-invertible module \( M \) is a torsion \( \Lambda \)-module and, therefore,

\[
\dim \mathbb{Q} M \otimes \mathbb{Q} < \infty.
\]

The following proposition will be used in the proofs of Theorems 0.1 and 0.3.

**Proposition 1.11.** Any Noetherian \((t-1)\)-invertible \( \Lambda \)-module \( M \) is isomorphic to a quotient module \( \Lambda^n/M_1 \) of the free \( \Lambda \)-module \( \Lambda^n \), where the submodule \( M_1 \) is generated by elements \( w_1, \ldots, w_n, \ldots, w_{n+k} \) of \( \Lambda^n \) with the following properties.

(i) For \( i = 1, \ldots, n \) we have \( w_i = (0, \ldots, 0, f_i(t), 0, \ldots, 0) \), where the polynomial \( f_i(t) \) occupies the \( i \)th position and satisfies \( f_i(1) = 1 \).

(ii) We have \( w_{n+j} = (t-1)\overline{w}_{n+j} = ((t-1)g_{j,1}(t), \ldots, (t-1)g_{j,n}(t)) \) for \( j = 1, \ldots, k \), where the \( g_{j,i}(t) \) are polynomials.

(iii) If the polynomials of the form \( t^m - 1 \) belong to \( \text{Ann}(M) \), then there is an \( m \in \mathbb{N} \) such that \( t^m - 1 \in \text{Ann}(M) \) and the vector \( w_{n+i} \) equals \((0, \ldots, 0, t^m - 1, 0, \ldots, 0) \), where the polynomial \( t^m - 1 \) occupies the \( i \)th position for \( i = 1, \ldots, n \).

**Proof.** We choose any generators \( v_1, \ldots, v_n \) of the Noetherian \( \Lambda \)-module \( M \). By Theorem 1.10 there are polynomials \( f_i(t) \in \text{Ann}_{v_i} \) such that \( f_i(1) = 1 \). Clearly, there is an epimorphism

\[
h: \bigoplus_{i=1}^n \Lambda/(f_i(t)) \rightarrow M
\]

of \( \Lambda \)-modules such that \( h(u_i) = v_i \) for \( u_i = (0, \ldots, 0, 1, 0, \ldots, 0) \), where \( 1 \) occupies the \( i \)th position. The kernel \( N = \ker h \) is a Noetherian \( \Lambda \)-module. Let this kernel be generated by

\[
u_{n+1} = (g_{1,1}(t), \ldots, g_{1,n}(t)), \ldots, \quad u_{n+k} = (g_{k,1}(t), \ldots, g_{k,n}(t)).
\]

There is no loss of generality in assuming that all the \( g_{i,j}(t) \) are polynomials.

The \( \Lambda \)-module \( \bigoplus_{i=1}^n \Lambda/(f_i(t)) \) is \((t-1)\)-invertible by Theorem 1.10, and \( N \) is also a \((t-1)\)-invertible \( \Lambda \)-module by Proposition 1.5. Hence the elements \( \bar{u}_{n+1} = (t-1)u_{n+1}, \ldots, \bar{u}_{n+k} = (t-1)u_{n+k} \) also generate \( N \).

If there is an \( m \in \mathbb{N} \) such that the polynomial \( t^m - 1 \) belongs to \( \text{Ann}(M) \), then the elements \((0, \ldots, 0, t^m - 1, 0, \ldots, 0) \) belong to \( N \). Therefore we can add the elements \((0, \ldots, 0, t^m - 1, 0, \ldots, 0) \) to the set of elements \( \bar{u}_{n+1}, \ldots, \bar{u}_{n+k} \) (that generate the module \( N \)) and reorder the resulting set of generators \( \bar{u}_{n+1}, \ldots, \bar{u}_{n+k} \) (where we put \( k := n+k \)) in such a way that \( \bar{u}_{n+j} = (0, \ldots, 0, t^m - 1, 0, \ldots, 0) \) for \( j = 1, \ldots, n \), where \( t^m - 1 \) occupies the \( j \)th position.
To complete the proof, note that the kernel $M_1$ of the composite $h \circ \nu : \Lambda^n \to M$ of $h$ and the natural epimorphism $\nu : \Lambda^n \to \bigoplus_{i=1}^n \Lambda/(f_i(t))$ is generated by the elements

$$w_i = (0, \ldots, 0, f_i(t), 0, \ldots, 0), \quad i = 1, \ldots, n,$$

where the polynomial $f_i(t)$ occupies the $i$th position, along with the elements

$$w_{n+i} = (f_{i,1}(t), \ldots, f_{i,n}(t)) \in \Lambda^n, \quad i = 1, \ldots, k,$$

where the coordinates $f_{i,j}(t)$ of each $w_{n+i}$ coincide with the coordinates $\bar{g}_{i,j}(t)$ of $\bar{u}_{n+i} = (\bar{g}_{i,1}(t), \ldots, \bar{g}_{i,n}(t))$. This proves the proposition.

1.2. **$\mathbb{Z}$-torsion submodules of $(t-1)$-invertible $\Lambda$-modules.** An element $v$ of a $\Lambda$-module $M$ is said to be of finite order if there is an $m \in \mathbb{Z} \setminus \{0\}$ such that $mv = 0$. A $\Lambda$-module $M$ is called a $\mathbb{Z}$-torsion module if all the elements of $M$ are of finite order. Given any $\Lambda$-module $M$, we write $M_{\text{fin}}$ for the subset of $M$ consisting of all elements of finite order. It is easy to see that $M_{\text{fin}}$ is a $\mathbb{Z}$-torsion $\Lambda$-module. If $M$ is a Noetherian $(t-1)$-invertible $\Lambda$-module, then $M_{\text{fin}}$ is also a Noetherian $(t-1)$-invertible $\Lambda$-module and Propositions 1.5, 1.6 imply that there is an exact sequence of $\Lambda$-modules

$$0 \to M_{\text{fin}} \to M \to M_1 \to 0,$$

where $M_1$ is a Noetherian $(t-1)$-invertible $\Lambda$-module containing no non-trivial elements of finite order.

Let $M = M_{\text{fin}}$ be a Noetherian $(t-1)$-invertible $\Lambda$-module. Since $M$ is finitely generated over $\Lambda$, there is an integer $d \in \mathbb{N}$ such that $dv = 0$ for all $v \in M$. (We call such a number $d$ an exponent of $M$.) Let $d = p_1^{r_1} \ldots p_n^{r_n}$ be the prime factorization. We write $M(p_i)$ for the subset of $M$ consisting of all elements $v \in M$ such that $p_i^{r_i}v = 0$. It is easy to see that $M(p_i)$ is a $\Lambda$-submodule of $M$. We call $M(p_i)$ the $p_i$-submodule of $M$.

**Theorem 1.12.** Let $M = M_{\text{fin}}$ be a Noetherian $(t-1)$-invertible $\Lambda$-module with exponent $d = p_1^{r_1} \ldots p_n^{r_n}$. Then $M$ is isomorphic to the direct sum of its $p_i$-submodules:

$$M = \bigoplus_{i=1}^n M(p_i).$$

**Proof.** The proof coincides with that of the corresponding theorem for abelian groups (see, for example, Theorem 8.1 in [14]).

Since the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$ is Noetherian, every ideal $I$ of $\Lambda$ is finitely generated. We write $I_{\text{pol}} = I \cap \mathbb{Z}[t]$ for the corresponding ideal of the polynomial ring $\mathbb{Z}[t]$. It is well known that $I = \Lambda I_{\text{pol}}$, that is, every ideal $I$ of $\Lambda$ is generated by polynomials.

We recall that $\mathbb{Z}[t]$ is a factorial ring. Its units are precisely the units of $\mathbb{Z}$, and its prime elements are either primes of $\mathbb{Z}$ or polynomials $q(t) = \sum a_it^i$ that are irreducible in $\mathbb{Q}[t]$ and have content 1 (that is, the greatest common divisor of their coefficients $a_i$ is equal to 1). For any non-zero polynomials $q_1(t), q_2(t) \in \mathbb{Z}[t]$,
Euclid’s algorithm enables us to find polynomials $h_1(t), h_2(t), r(t) \in \mathbb{Z}[t]$ and a constant $d \in \mathbb{Z}$, $d \neq 0$, such that
\begin{equation}
  h_1(t)q_1(t) + h_2(t)q_2(t) = dr(t),
\end{equation}
where $r(t)$ is the greatest common divisor of the polynomials $q_1(t)$ and $q_2(t)$.

**Lemma 1.13.** Suppose that $M$ is a Noetherian $(t - 1)$-invertible $\Lambda$-module and $t^n - 1 \in \text{Ann}(M)$ for some $n = p^r$, where $p$ is a prime. Then $M$ is a $\mathbb{Z}$-torsion module.

**Proof.** If the polynomial $t^n - 1 = (t - 1)(t^{n-1} + \cdots + t + 1)$ belongs to $\text{Ann}(M)$, then $g_n(t) = t^{n-1} + \cdots + t + 1 \in \text{Ann}(M)$ because $M$ is $(t - 1)$-invertible. When $n = p^r$, each factor in the formula
\begin{equation}
  g_{p^r}(t) = \prod_{i=1}^{r} \Phi_{p^i}(t) = \prod_{i=1}^{r} \sum_{j=0}^{p^i-1} t^{jp^i-1}
\end{equation}
is an irreducible element of $\Lambda$.

By Theorem 1.10 there is a polynomial $f(t) \in \text{Ann}(M)$ with $f(1) = 1$. If $n = p^r$ for some prime $p$, then the polynomials $f(t)$ and $g_{p^r}(t)$ have no common irreducible divisors. Indeed, if $g(t)$ is a divisor of $f(t)$, then we must have $g(1) = \pm 1$ since $f(1) = 1$, but $\Phi_{p^i}(1) = p$ for each $i$. Therefore one can find polynomials $h_1(t)$, $h_2(t)$ and a constant $d \in \mathbb{N}$ such that $h_1(t)f(t) + h_2(t)g_{p^r}(t) = d$. Hence if $g_{p^r}(t) \in \text{Ann}(M)$, then $d \in \text{Ann}(M)$ and, therefore, $M$ is a $\mathbb{Z}$-torsion module.

### 1.3. Principal $(t - 1)$-invertible $\Lambda$-modules.

Let $I$ be a non-zero ideal of the ring $\Lambda$. We denote by $I_m$ the subset of $I_{\text{pol}}$ consisting of all polynomials $f(t)$ of the smallest degree (let $m$ be this smallest degree). Note that if $f(t) \in I_m \setminus \{0\}$, then $f(0) \neq 0$.

Consider any polynomials $f_1(t), f_2(t) \in I_m$ and write them as $f_i(t) = d_i q_i(t)$, where $d_i \in \mathbb{Z}$ and the polynomials $q_i(t)$ have content 1. We have $q_1(t) = q_2(t)$. Indeed, their greatest common divisor $r(t)$ satisfies $\deg r(t) \leq m$ and, moreover, $\deg r(t) = m$ if and only if $q_1(t) = q_2(t)$. On the other hand, (3) implies that $d_2 h_1(t)f_1(t) + d_1 h_2(t)f_2(t) = d_1 d_2 dr(t)$ for some polynomials $h_1(t), h_2(t)$. Therefore $d_1 d_2 dr(t) \in I_{\text{pol}}$ and we must have $\deg r(t) = m$.

Applying Euclid’s algorithm for integers, we see that if polynomials $f_i(t) = d_i q(t)$ belong to $I_m$ for $i = 1, 2$, then $d_0 q(t)$ also belongs to $I_m$, where $d_0$ is the greatest common divisor of $d_1$ and $d_2$. Thus there is a polynomial $f_m(t) = d_m q(t) \in I_m$ such that all polynomials $f(t) \in I_m$ are divisible by $f_m(t)$. The polynomial $f_m(t)$ is uniquely determined up to multiplication by $\pm 1$. It will be called a *leading generator* of the ideal $I$.

Let $I$ be a non-zero ideal of $\Lambda$ and $f(t) = d_m q(t)$ a leading generator of $I$. Then all the polynomials $h(t) \in I$ are divisible by $q(t)$. Indeed, arguing as above, we easily see that if $r(t)$ is the greatest common divisor of $f(t)$ and $h(t)$, then there is a constant $d$ such that $dr(t) \in I$. Since $\deg q(t)$ is the minimal degree of polynomials belonging to $I$, we must have the equality $r(t) = q(t)$. 

The arguments above and Lemma 1.3 prove the following proposition.

**Proposition 1.14.** Let \( M = M_v \) be a principal \((t - 1)\)-invertible \( \Lambda \)-module generated by an element \( v \). Then the ideal \( \text{Ann}_v \) is generated by a finite set of polynomials \( f_1(t), \ldots, f_k(t) \), where \( f_i(t) = d_iq_i(t) \), \( d_i \in \mathbb{Z} \), \( d_i \neq 0 \), and the content of \( q_i(t) \) is equal to 1 for all \( i \). Moreover, the polynomials \( f_1(t), \ldots, f_k(t) \) possess the following properties.

(i) \( \deg f_1 < \deg f_2 \leq \cdots \leq \deg f_k \).
(ii) \( f_i(0) \neq 0 \) for all \( i \).
(iii) \( q_1(1) = 1 \).
(iv) \( q_1(t) \mid q_i(t) \) for \( i = 2, \ldots, k \).
(v) If \( k > 1 \), then \( |d_1| > 1 \), \( d_k = 1 \) and \( q_k(1) = 1 \).

A set of generators of \( \text{Ann}_v \) is said to be **good** if it possesses properties (i)–(v) in Proposition 1.14.

Principal \((t - 1)\)-invertible \( \Lambda \)-modules \( M = M_v \) are classified on the basis of properties of their annihilators. We say that \( M_v \) is of **finite type** if in a good system \( f_1(t), \ldots, f_k(t) \) of generators of \( \text{Ann}_v \) the leading generator \( f_1(t) \equiv d_1 \) is a constant (that is, \( q_1(t) \equiv 1 \)). A module \( M_v \) is said to be of **mixed type** if in a good system \( f_1(t), \ldots, f_k(t) \) of generators of \( \text{Ann}_v \) the degree of the leading generator \( f_1 = d_1q_1(t) \) is greater than zero and we have \( |d_1| \geq 2 \). The arguments above imply that if a principal \((t - 1)\)-invertible \( \Lambda \)-module \( M = M_v \) is not of finite or mixed type, then the leading generator \( f_1(t) \) of a good system of generators of \( \text{Ann}_v \) is equal to a polynomial \( q_1(t) \) with \( q_1(1) = 1 \). Therefore \( \text{Ann}_v \) is a principal ideal generated by \( q_1(t) \) because all the polynomials \( h(t) \in \text{Ann}_v \) are divisible by \( q_1(t) \). Such principal \((t - 1)\)-invertible \( \Lambda \)-modules are said to be **biprincipal**.

It is easy to see that if \( M = M_v \) is a principal \( \Lambda \)-module of finite type and \( d_1 \in \mathbb{Z} \) is the leading generator of a good system of generators of \( \text{Ann}_v \), then the orders of all elements of \( M \) divide \( d_1 \). Hence a principal \( \Lambda \)-module \( M_v \) is of finite type if and only if it is a \( \mathbb{Z} \)-torsion module.

If \( M = M_v \) is a biprincipal \( \Lambda \)-module, then it has no non-zero elements of finite order. Indeed, let \( q(t) \) be a generator of \( \text{Ann}_v \). If an element \( v_1 = h(t)v \) has order \( m \), then \( mh(t) \in \text{Ann}_v \), that is, \( mh(t) \) is divisible by \( q(t) \). Since \( t \) is a unit of the ring \( \Lambda \), we can assume that \( h(t) \) is a polynomial. Since \( M_v \) is a biprincipal module, the polynomial \( h(t) \) must be divisible by \( q(t) \), that is, \( v_1 = 0 \).

If \( M = M_v \) is a \( \Lambda \)-module of mixed type, then there is an exact sequence of \( \Lambda \)-modules

\[ 0 \to M_1 \to M \to M_2 \to 0, \]

where \( M_1 \) is a principal \( \Lambda \)-module of finite type and \( M_2 \) is a biprincipal \( \Lambda \)-module. Indeed, let \( d_1q_1(t) \) be the leading generator of \( \text{Ann}_v \). Put \( v_1 = q_1(t)v \). Then we easily see that the \( \Lambda \)-module \( M_1 = M_{v_1} \subset M \) (generated by \( v_1 \)) is of finite type while the \( \Lambda \)-module \( M_2 = M/M_1 \simeq \Lambda/(q_1) \) is biprincipal.

**1.4. Finitely \( \mathbb{Z} \)-generated \((t - 1)\)-invertible \( \Lambda \)-modules.** Every \( \Lambda \)-module \( M \) may be regarded as a \( \mathbb{Z} \)-module, that is, an abelian group.
Proposition 1.15. A Noetherian $\Lambda$-module $M$ is finitely generated over $\mathbb{Z}$ if and only if there is a polynomial

$$q(t) = \sum_{i=0}^{n} a_i t^i \in \text{Ann}(M)$$

such that $a_n = a_0 = 1$.

Proof. We begin by proving this in the case when $M = M_v$ is a principal $\Lambda$-module.

If there is a polynomial $q(t) = \sum_{i=0}^{n} a_i t^i \in \text{Ann}_v$ with $a_n = a_0 = 1$, then we easily see that $M$ is generated over $\mathbb{Z}$ by the elements $v, tv, \ldots, t^{n-1}v$.

Suppose that the $\Lambda$-module $M = M_v$ is finitely generated over $\mathbb{Z}$ and let $h_1(t)v, \ldots, h_m(t)v$ be its generators. Since multiplication by $t$ is an isomorphism of $M$, we can assume that the $h_i(t)$, $i = 1, \ldots, m$, are polynomials with $h_i(0) = 0$. Put $n - 1 = \max(\deg h_1(t), \ldots, \deg h_m(t))$. Since $h_1(t)v, \ldots, h_m(t)v$ generate $M$ over $\mathbb{Z}$, one can find integers $b_1, \ldots, b_m$ and $c_1, \ldots, c_m$ such that

$$v = \sum_{i=1}^{m} b_i h_i(t)v \quad \text{and} \quad t^n v = \sum_{i=1}^{m} c_i h_i(t)v.$$

Therefore the polynomials $1 - \sum b_i h_i(t)$ and $t^n - \sum c_i h_i(t)$ belong to $\text{Ann}_v$. Then the polynomial $t^n + 1 - \sum (b_i + c_i) h_i(t)$ has the desired properties.

In the general case, a Noetherian $(t-1)$-invertible $\Lambda$-module $M$ is generated by a finite set of elements $v_1, \ldots, v_m$. Hence $M$ is finitely generated over $\mathbb{Z}$ if and only if the principal submodules $M_{v_i} \subset M$ are all finitely generated over $\mathbb{Z}$.

If $g(t) \in \text{Ann}(M)$, then $g(t) \in \text{Ann}_{v_i}$ for $i = 1, \ldots, m$. In particular, if there is a polynomial $q(t) = \sum_{i=0}^{n} a_i t^i \in \text{Ann}(M)$ with $a_n = a_0 = 1$, then all the $M_{v_i}$ (and therefore $M$) are finitely generated over $\mathbb{Z}$.

If the principal submodules $M_{v_i} \subset M$ are finitely generated over $\mathbb{Z}$, then there are polynomials $q_i(t) = \sum_{j=0}^{n_i} a_{i,j} t^j \in \text{Ann}_{v_i}$ such that $a_{i,n_i} = a_{i,0} = 1$. Put $n = \sum n_i$. Then we have

$$q(t) = q_1(t) \ldots q_n(t) = t^n + 1 + \sum_{j=1}^{n-1} a_{j} t^j \in \text{Ann}(M)$$

since $q(t) \in \text{Ann}_{v_i}$ for all $v_i$. The proposition is proved.

Proposition 1.15 implies that many $(t-1)$-invertible biprincipal modules $M = \Lambda/I$ are not finitely generated over $\mathbb{Z}$. More precisely, it is easy to see that a biprincipal $(t-1)$-invertible module $M = \Lambda/I$ is finitely generated over $\mathbb{Z}$ if and only if the ideal $I = \langle q(t) \rangle$ is generated by a polynomial $q(t) = \sum_{i=0}^{n} a_i t^i$ such that $q(1) = 1$ and the coefficients $a_0$ and $a_n$ are equal to $\pm 1$.

For example, the $(t-1)$-invertible biprincipal module

$$M_m = \Lambda/\langle (m+1)t - m \rangle$$

is never finitely generated over $\mathbb{Z}$ for $m \in \mathbb{N}$. 
Theorem 1.16. Let $M$ be a Noetherian $(t-1)$-invertible $\mathbb{Z}$-torsion module. Then $M$ is finitely generated over $\mathbb{Z}$.

Proof. By Theorem 1.12, the module $M$ is isomorphic to the direct sum $\bigoplus M(p_i)$ of finitely many $p_i$-submodules. Hence it suffices to prove the theorem in the case when $M$ has exponent $p^r$, where $p$ is a prime. By Corollary 1.8, $M$ is a quotient module of the direct sum $\bigoplus_{j=1}^n \Lambda/I_j$ of principal $(t-1)$-invertible $\Lambda$-modules $\Lambda/I_j$. In our case there is no loss of generality in assuming that each ideal $I_j$ contains the constant $p^r$ for some $r_j \in \mathbb{N}$. Thus it suffices to prove the theorem in the case when $M = M_v$ is a principal $(t-1)$-invertible $\Lambda$-module of exponent $p^r$. In other words, the ideal $I = \text{Ann}_\Lambda$ contains the constant $p^r$ and a polynomial $g(t)$ with $g(1) = 1$.

Suppose that $r = 1$ and $g(t) = \sum a_i t^i$. We put $g_1(t) = \sum_{p | a_i} a_i t^i$ and $\tilde{g}(t) = g(t) - g_1(t)$. Then $\tilde{g}(t) \in \text{Ann}_\Lambda$ because $g(t), g_1(t) \in \text{Ann}_\Lambda$. It is easy to see that the numbers $g(1)$ and $p$ are coprime since $g(1) = 1$ and $g_1(1) \equiv 0 \pmod{p}$. Moreover, by the construction of $\tilde{g}(t)$, every coefficient of $\tilde{g}(t)$ is coprime to $p$. Multiplying $\tilde{g}(t)$ by $t^{-k}$, we can assume that $\tilde{g}(0) \neq 0$. Write $\tilde{g}(t) = \sum_{i=0}^m \tilde{a}_it^i$. Since $\tilde{a}_m$ and $p$ are coprime, one can find integers $b_1$ and $c_1$ such that $b_1\tilde{a}_m + c_1p = 1$. Similarly, there are integers $b_2$ and $c_2$ such that $b_2\tilde{a}_0 + c_2p = 1$. Therefore the polynomial $(b_1t + b_2)\tilde{g}(t) + p(c_1t^{m+1} + c_2)$ belongs to $I$ and can be written as

$$h(t) = t^{m+1} + 1 + \sum_{i=1}^m (b_1\tilde{a}_{i-1} + b_2\tilde{a}_i)t^i.$$ 

Thus $M_v$ is finitely generated over $\mathbb{Z}$ by Proposition 1.15.

We now consider the general case of a principal $(t-1)$-invertible $\Lambda$-module of exponent $p^r$. Suppose that any principal $(t-1)$-invertible $\Lambda$-module $M'$ of exponent $p^{r_1}$ with $r_1 < r$ is finitely generated over $\mathbb{Z}$. Let $M = M_v$ be a principal $(t-1)$-invertible $\Lambda$-module of exponent $p^r$ and $M_{v_1}$ the submodule of $M$ generated by $v_1 = p^{r-1}v$. Then $M_{v_1}$ is of exponent $p$ and the quotient module $M_v = M/M_{v_1}$ is of exponent $p^{r-1}$. The proof now follows from the exact sequence

$$0 \to M_{v_1} \to M \to M/M_{v_1} \to 0.$$ 

Corollary 1.17. Every Noetherian $(t-1)$-invertible $\mathbb{Z}$-torsion module is finite. In other words, it is a finite abelian group.

Lemma 1.18. The group $G = \bigoplus_{i=1}^n (\mathbb{Z}/2r_i\mathbb{Z})^{m_i}$ possesses no structure of a $(t-1)$-invertible $\Lambda$-module if $r_i \neq r_j$ for $i \neq j$ and one of $m_i$ is equal to 1.

Proof. Suppose that $G$ possesses the structure of a $(t-1)$-invertible $\Lambda$-module. Then the subgroup $2^r G$ of $G$ is a $\Lambda$-submodule of $G$ for any $r$ and Propositions 1.5 and 1.6 yield that the groups $2^r G$ and $G/2^r G$ are $(t-1)$-invertible $\Lambda$-modules. Hence there is no loss of generality in assuming that

$$G = (\mathbb{Z}/2\mathbb{Z}) \oplus \left( \bigoplus_{i=1}^n (\mathbb{Z}/2r_i\mathbb{Z})^{m_i} \right),$$ 

where $\bigoplus_{i=1}^n (\mathbb{Z}/2r_i\mathbb{Z})^{m_i}$ is the direct sum of cyclic groups of order $2r_i$ for each $i$.

Alexander modules of irreducible $C$-groups
where all $r_i \geq 2$ and $m_i \geq 2$. We choose generators $v_1, \ldots, v_{m+1}$ of $G$ with $m = \sum_{i=1}^{n} m_i$ in such a way that

$$G \cong (\mathbb{Z}/2\mathbb{Z})v_1 \oplus \bigoplus_{i=2}^{m+1} (\mathbb{Z}/2^{r_i} \mathbb{Z})v_i,$$

where all $r_i \geq 2$. Consider the $\mathbb{Z}$-submodule $\overline{G}$ of $G$ consisting of all elements $v \in G$ of order at most 4. Clearly, $\overline{G}$ is a $\Lambda$-submodule of $G$ and is generated over $\mathbb{Z}$ (and hence over $\Lambda$) by $\overline{v}_1 = v_1$ and $\overline{v}_i = 2^{r_i-2}v_i$, $i = 2, \ldots, m+1$. It is easy to see that we have an isomorphism of abelian groups

$$\overline{G} \cong (\mathbb{Z}/2\mathbb{Z})\overline{v}_1 \oplus \bigoplus_{i=2}^{m+1} (\mathbb{Z}/4\mathbb{Z})\overline{v}_i.$$

By Proposition 1.5, $\overline{G}$ is a $(t-1)$-invertible $\Lambda$-module. Multiplication by $t$ is a module automorphism of $\overline{G}$. Write

$$t\overline{v}_1 = a_1 \overline{v}_1 + 2 \sum_{i=2}^{m+1} b_i \overline{v}_i,$$

$$t\overline{v}_j = a_j \overline{v}_1 + \sum_{i=2}^{m+1} c_{j,i} \overline{v}_i, \quad j = 2, \ldots, m+1, \quad (4)$$

where each coefficient $a_j$ is equal to 0 or 1.

We claim that $a_1 = 1$. Indeed, assume that $a_1 = 0$. Since multiplication by $t$ is an automorphism and $\overline{v}_1, \ldots, \overline{v}_{m+1}$ generate $\overline{G}$, we must have an equality $\overline{v}_1 = \sum d_i t\overline{v}_i$, where one of the $d_i$ is odd for some $i \geq 2$ as $a_1 = 0$. Furthermore, $\overline{v}_1$ is an element of order 2, whence $2 \sum_{i=2}^{m+1} d_i t\overline{v}_i = 0$. On the other hand, $t\overline{v}_2, \ldots, t\overline{v}_{m+1}$ are linearly independent over $\mathbb{Z}/4\mathbb{Z}$ since $\overline{v}_2, \ldots, \overline{v}_{m+1}$ are linearly independent over $\mathbb{Z}/4\mathbb{Z}$ and multiplication by $t$ is an automorphism. Thus the equality $2 \sum_{i=2}^{m+1} d_i t\overline{v}_i = 0$ is impossible if any of the $d_i$ is odd and, therefore, the coefficient $a_1$ in (4) must be equal to 1.

We claim that $\overline{G}$ cannot be $(t-1)$-invertible. Indeed, we have

$$t\overline{v}_1 = \overline{v}_1 + 2 \sum_{i=2}^{m+1} b_i \overline{v}_i.$$

Therefore

$$(t-1)\overline{v}_1 = 2 \sum_{i=2}^{m+1} b_i \overline{v}_i.$$

Repeating the argument above with $t$ replaced by $t-1$, we see that multiplication by $t-1$ is not an automorphism of $\overline{G}$ since $(t-1)\overline{v}_1$ is a linear combination of the elements $\overline{v}_2, \ldots, \overline{v}_{m+1}$. The lemma is proved.
Theorem 1.19. Consider an abelian group

\[ G = G_1 \oplus \left( \bigoplus_{i=1}^{n} (\mathbb{Z}/2^{r_i} \mathbb{Z})^{m_i} \right), \]

where \( r_i \neq r_j \) for \( i \neq j \) and \( G_1 \) is a group of odd order. Then \( G \) admits the structure of a \((t - 1)\)-invertible \( \Lambda \)-module if and only if \( m_i \geq 2 \) for all \( i = 1, \ldots, n \).

Proof. By Theorem 1.12, if \( M = M_{\text{fin}} \) is a Noetherian \((t - 1)\)-invertible \( \Lambda \)-module of exponent \( d = p_1^{r_1} \cdots p_n^{r_n} \), then \( M \) is the direct sum

\[ M = \bigoplus_{i=1}^{n} M(p_i) \]

of \( p_i \)-submodules that are \((t - 1)\)-invertible by Proposition 1.5. Each submodule \( M(p_i) \) with \( p_i \) odd is of odd order and, by Lemma 1.18, the submodule \( M(2) \) of \( M \) is isomorphic (as an abelian group) to \( \bigoplus_{i=1}^{k} (\mathbb{Z}/2^{r_i} \mathbb{Z})^{m_i} \), where \( m_i \geq 2 \) for all \( i = 1, \ldots, k \).

To prove the converse statement, we make three remarks. First, a finite direct sum of \((t - 1)\)-invertible \( \Lambda \)-modules is also a \((t - 1)\)-invertible \( \Lambda \)-module. Second, for every prime \( p > 2 \), the \((t - 1)\)-invertible \( \Lambda \)-module \( M = \Lambda/I \), where the ideal \( I \) is generated by the number \( p^r \) and the polynomial \( 2t - 1 \), is isomorphic to \( \mathbb{Z}/p^r \mathbb{Z} \) as an abelian group. Third, if \( n \geq 2 \), then the \((t - 1)\)-invertible \( \Lambda \)-module \( M = \Lambda/I \), where the ideal \( J \) is generated by the number \( 2^r \) and the polynomial \( t^n - t + 1 \), is isomorphic to \( (\mathbb{Z}/2^r \mathbb{Z})^n \) as an abelian group.

§ 2. \( t \)-unipotent \( \mathbb{Z}[t, t^{-1}] \)-modules

2.1. Properties of \( t \)-unipotent \( \Lambda \)-modules. The following proposition is a simple corollary of Propositions 1.5 and 1.6.

Proposition 2.1. Every \( \Lambda \)-submodule \( M_1 \) and every quotient module \( M/M_1 \) of a Noetherian \((t - 1)\)-invertible \( t \)-unipotent \( \Lambda \)-module \( M \) is a \((t - 1)\)-invertible \( t \)-unipotent \( \Lambda \)-module.

Lemma 2.2. Let \( M_1, \ldots, M_n \) be Noetherian \((t - 1)\)-invertible \( t \)-unipotent \( \Lambda \)-modules. Then the direct sum \( M = \bigoplus_{i=1}^{n} M_i \) is a Noetherian \((t - 1)\)-invertible \( t \)-unipotent \( \Lambda \)-module.

Proof. By Lemma 1.7, \( M \) is a Noetherian \((t - 1)\)-invertible \( \Lambda \)-module and there are \( k_i \in \mathbb{N} \) such that \( t^{k_i} - 1 \in \text{Ann}(M_i) \). Since every polynomial \( t^{k_i} - 1 \) \((i = 1, \ldots, n)\) is a divisor of \( t^k - 1 \), we easily see that \( t^k - 1 \in \text{Ann}(M) \) for \( k = k_1 \cdots k_n \). The lemma is proved.

Proposition 2.1 and Lemma 2.2 yield the following proposition.

Proposition 2.3. A Noetherian \( \Lambda \)-module \( M \) is \((t - 1)\)-invertible and \( t \)-unipotent if and only if each of its principal submodules \( M_v \) is \((t - 1)\)-invertible and \( t \)-unipotent.
Theorem 2.4. Any Noetherian \((t-1)\)-invertible \(\mathbb{Z}\)-torsion \(\Lambda\)-module is \(t\)-unipotent.

Proof. Let \(M\) be a Noetherian \((t-1)\)-invertible \(\mathbb{Z}\)-torsion \(\Lambda\)-module. By Corollary 1.17, \(M\) consists of finitely many elements. Hence the automorphism of \(M\) defined by multiplication by \(t\) has finite order, say \(k\). Then \(t^kv = v\) for all \(v \in M\) or, in other words, \(t^k - 1 \in \text{Ann}(M)\). The theorem is proved.

The following propositions describe biprincipal \((t-1)\)-invertible \(t\)-unipotent modules and principal \((t-1)\)-invertible \(t\)-unipotent modules of mixed type.

Proposition 2.5. Let \(M = \Lambda/I\) be a biprincipal \((t-1)\)-invertible \(t\)-unipotent \(\Lambda\)-module, where the ideal \(I = \langle g(t) \rangle\) is generated by a polynomial \(g(t)\). Then the following assertions hold.

(i) All the roots of \(g(t)\) are roots of unity.
(ii) \(g(t)\) has no multiple roots.
(iii) If \(\xi\) is a \(k\)th root of unity (that is, \(\xi^k = 1\)) and \(k = p^r\) for some prime \(p\), then \(\xi\) is not a root of \(g(t)\).
(iv) \(g(1) = \pm 1\).
(v) \(\deg g(t)\) is even.

Proof. To prove assertions (i) and (ii), note that \(t^k - 1 \in I\) for some \(k\) because \(M\) is a \(t\)-unipotent module. Hence \(t^k - 1\) is divisible by \(g(t)\).

Let us prove (iii)–(v). By Theorem 1.10 there is a polynomial \(f(t) \in I\) with \(f(1) = 1\). We have \(f(t) = h(t)g(t)\) for some polynomial \(h(t) \in \mathbb{Z}[t]\) since \(I\) is generated by \(g(t)\). Thus \(g(1) = \pm 1\) (and we may assume that \(g(1) = 1\)) because we have

\[1 = f(1) = h(1)g(1),\]

where \(h(1), g(1) \in \mathbb{Z}\). On the other hand, if a primitive \(p^r\)th root \(\xi\) of unity is a root of \(g(t)\) for some prime \(p\), then \(g(t)\) must be divisible by the \(p^r\)th cyclotomic polynomial \(\Phi_{p^r}(t)\). In other words, there is a polynomial \(h(t) \in \mathbb{Z}[t]\) such that \(g(t) = \Phi_{p^r}(t)h(t)\). Thus \(1 = g(1) = \Phi_{p^r}(1)h(1)\) and we obtain a contradiction because \(\Phi_{p^r}(1) = p\).

To complete the proof, we use assertions (iii) and (iv) to conclude that the numbers \(\xi = \pm 1\) are not roots of \(g(t)\), whence all the roots of \(g(t)\) are non-real. Since \(g(t) \in \mathbb{Z}[t]\), we see that if \(\xi\) is a root of \(g(t)\), then so is its complex conjugate \(\bar{\xi}\). Therefore \(\deg g(t)\) is even (because we have \(\bar{\xi} \neq \xi\) for all roots of unity different from \(\pm 1\)). This proves the proposition.

Proposition 2.6. Let \(M = \Lambda/I\) be a principal \((t-1)\)-invertible \(t\)-unipotent \(\Lambda\)-module of mixed type and let \(f(t) = dg(t)\) be the leading generator of the ideal \(I\), where \(d \in \mathbb{N}\) and \(g(t)\) is a polynomial of content 1. Then \(g(t)\) satisfies conditions (i)–(v) of Proposition 2.5.

Proof. Let \(v\) be a generator of \(M\) and let \(M_1\) be the \(\Lambda\)-submodule of \(M\) generated by \(v_1 = g(t)v\). We have an exact sequence of \(\Lambda\)-modules

\[0 \to M_1 \to M \to M/M_1 \to 0,\]
where $M_1$ is a principal module of finite type and $M_2 = M/M_1$ is a biprincipal $\Lambda$-module isomorphic to $\Lambda/(g(t))$. By Proposition 2.1, $M_2$ is a $(t - 1)$-invertible $t$-unipotent module. Hence the proposition follows from Proposition 2.5.

Let $M$ be a Noetherian $(t - 1)$-invertible $t$-unipotent $\Lambda$-module. The unipotence index of $M$ is the smallest $k \in \mathbb{N}$ such that $t^k - 1 \in \text{Ann}(M)$.

**Lemma 2.7.** If $M$ is a Noetherian $(t - 1)$-invertible $t$-unipotent $\Lambda$-module of unipotence index $k$, then $\sum_{i=0}^{k-1} t^i \in \text{Ann}(M)$.

**Proof.** We have $t^k - 1 = (t - 1) \sum_{i=0}^{k-1} t^i \in \text{Ann}(M)$. Since $M$ is a $(t - 1)$-invertible $\Lambda$-module, it follows that $(\sum_{i=0}^{k-1} t^i)v = 0$ for all $v \in M$. The lemma is proved.

**Lemma 2.8.** Every Noetherian $(t - 1)$-invertible $\Lambda$-module $M$ of unipotence index 2 is a finite $\mathbb{Z}$-module of odd order.

**Proof.** It follows from Lemma 1.13 and Corollary 1.17 that $M$ is finite. By Lemma 2.7 the polynomial $t + 1$ belongs to $\text{Ann}(M)$. Therefore $tv = -v$ for all $v \in M$. In particular, if $v$ is of order 2, then $tv = v$. This is impossible since $M$ is $(t - 1)$-invertible. Hence $M$ has no elements of even order. The lemma is proved.

**Proposition 2.9.** A cyclic group $G$ of order $n = p_1^{r_1} \ldots p_m^{r_m}$ (where $p_1, \ldots, p_m$ are primes) possesses the structure of a $(t - 1)$-invertible $\Lambda$-module of unipotence index $k$ if and only if the polynomial $\sum_{i=0}^{k-1} t^i$ has a root $a_j \neq 1$ in the field $\mathbb{Z}/p_j\mathbb{Z}$ for every $j = 1, \ldots, m$.

**Proof.** By Theorem 1.12 it suffices to consider only the case when $m = 1$, that is, $n = p^r$ for some prime $p$.

Suppose that the cyclic group $G$ of order $n = p^r$ possesses the structure of a $(t - 1)$-invertible $\Lambda$-module of unipotence index $k$. Then the subgroup $G_p = p^{r-1}G$ consisting of all elements of order $p$ is also a $(t - 1)$-invertible $\Lambda$-module of unipotence index $k$. Therefore $\sum_{i=0}^{k-1} t^i \in \text{Ann}(G_p)$. Let $v \in G_p$ be a generator of $G_p$. Since $G_p$ is a $(t - 1)$-invertible module, we have $tv = av$ for some $a \neq 1 (\text{mod } p)$. Hence $\sum_{i=0}^{k-1} a^i v = 0$. It follows that $\sum_{i=0}^{k-1} a^i \equiv 0 (\text{mod } p)$, that is, the polynomial $\sum_{i=0}^{k-1} t^i$ has a root (different from 1) in the field $\mathbb{Z}/p\mathbb{Z}$.

Conversely, let $a \neq 1 (\text{mod } p)$ be a root of the polynomial $\sum_{i=0}^{k-1} t^i$ in the field $\mathbb{Z}/p\mathbb{Z}$ and let $v$ be a generator of the cyclic group $G$ of order $p^r$. We define an action of $t$ on the $\mathbb{Z}$-module $G$ by putting $t(v) = av$. This endows $G$ with the structure of a $(t - 1)$-invertible $\Lambda$-module since $a \neq 1 (\text{mod } p)$. It is easy to see that $t^k - 1 \in \text{Ann}(G)$. The proposition is proved.

**Theorem 2.10.** Every Noetherian $(t - 1)$-invertible $t$-unipotent $\Lambda$-module $M$ is finitely generated over $\mathbb{Z}$.

**Proof.** The theorem follows from Proposition 1.15 since the polynomial $t^k - 1$ belongs to $\text{Ann}(M)$ for some $k \in \mathbb{Z}$. 
Using Theorem 2.10 and the structure theorem for finitely generated \( \mathbb{Z} \)-modules, we see that for every \((t - 1)\)-invertible \( t \)-unipotent \( \Lambda \)-module \( M \), we have the isomorphism (of \( \mathbb{Z} \)-modules)

\[
M \simeq M_{\text{fin}} \oplus \mathbb{Z}^k,
\]

where \( M_{\text{fin}} \) is the submodule of \( M \) consisting of all elements of finite order. The rank \( k \) of the free part of \( M \) in the decomposition (5) is called the Betti number of the Noetherian \((t - 1)\)-invertible \( t \)-unipotent \( \Lambda \)-module \( M \).

**Theorem 2.11.** The Betti number of any Noetherian \((t - 1)\)-invertible \( t \)-unipotent \( \Lambda \)-module \( M \) is even.

**Proof.** By definition, the Betti number of \( M \) coincides with the Betti number of the Noetherian \((t - 1)\)-invertible \( t \)-unipotent \( \Lambda \)-module \( M_{\text{free}} = M/M_{\text{fin}} \).

The module \( M_{\text{free}} \) has no non-zero elements of finite order. Hence the annihilator \( \text{Ann}_v \) of every element \( v \in M_{\text{free}} \) is a principal ideal generated by some polynomial \( g_v(t) \) that satisfies conditions (i)–(v) of Proposition 2.5.

Suppose that \( M_{\text{free}} \) is generated by \( v_1, \ldots, v_m \) over \( \Lambda \). Then there is a surjective \( \Lambda \)-homomorphism \( f: \Lambda/\langle g_{v_1}(t) \rangle \oplus \cdots \oplus \Lambda/\langle g_{v_m}(t) \rangle \to M_{\text{free}} \).

We regard the modules \( \widetilde{M} = \oplus \Lambda/\langle g_{v_i}(t) \rangle \) and \( M_{\text{free}} \) as free \( \mathbb{Z} \)-modules and write \( h_{\widetilde{M}} \) (resp. \( h_{M_{\text{free}}} \)) for the automorphism of \( \widetilde{M} \) (resp. \( M_{\text{free}} \)) of multiplication by \( t \).

It is easy to see that the characteristic polynomial \( \Delta(t) = \det(h_{\widetilde{M}} - t \text{Id}) \) coincides with the product \( g_{v_1}(t) \ldots g_{v_m}(t) \) up to a sign. The characteristic polynomial \( \Delta(t) = \det(h_{M_{\text{free}}} - t \text{Id}) \) divides the polynomial \( \widetilde{\Delta}(t) \) since the homomorphism \( f \) is surjective and \( t \)-equivariant. It follows that all the roots of \( \Delta(t) \) are roots of unity different from \( \pm 1 \) and, therefore, \( \deg \Delta(t) \) is even. To complete the proof, we note that the Betti number of \( M_{\text{free}} \) coincides with \( \deg \Delta(t) \).

### 2.2. Derived Alexander modules.

Given any Noetherian \((t - 1)\)-invertible \( \Lambda \)-module \( M \), we consider an infinite sequence of Noetherian \((t - 1)\)-invertible \( t \)-unipotent \( \Lambda \)-modules

\[
A_n(M) = M/(t^n - 1)M, \quad n \in \mathbb{N}.
\]

The module \( A_n(M) \) is called the \( n \)th derived Alexander module of the \( \Lambda \)-module \( M \). We note that \( A_1(M) = 0 \) since \( M \) is \((t - 1)\)-invertible. It is also clear that \( A_n(A_n(M)) = A_n(M) \).

We easily see that every \( \Lambda \)-homomorphism \( f: M_1 \to M_2 \) of \((t - 1)\)-invertible modules determines a well-defined sequence of \( \Lambda \)-homomorphisms

\[
f_{n*} : A_n(M_1) \to A_n(M_2),
\]

\( n \in \mathbb{N} \). In other words, the map \( M \mapsto \{A_n(M)\} \) is a functor from the category of Noetherian \((t - 1)\)-invertible \( \Lambda \)-modules to the category of infinite sequences of Noetherian \((t - 1)\)-invertible \( t \)-unipotent \( \Lambda \)-modules.
Proposition 2.12. If $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ is an exact sequence of Noetherian $(t - 1)$-invertible $\Lambda$-modules, then

$$A_n(M_2) \simeq A_n(M) / \text{im } f_n^* (A_n(M_1)).$$

If $M = \bigoplus_{i=1}^k M_i$ is the direct sum of the Noetherian $(t - 1)$-invertible $\Lambda$-modules $M_i$, then

$$A_n(M) \simeq \bigoplus_{i=1}^k A_n(M_i).$$

Proof. This is obvious.

Proposition 2.13. Suppose that $p$ is a prime, $r \in \mathbb{N}$ and $M$ is any Noetherian $(t - 1)$-invertible $\Lambda$-module. Then the derived Alexander module $A_{p^r}(M)$ is finite.

Proof. This follows from Lemma 1.13 and Corollary 1.17.

Example 2.14. Consider the module $M_m = \Lambda / \langle (m + 1)t - m \rangle$, where $m \in \mathbb{N}$. Its $n$th derived Alexander module

$$A_n(M_m) \simeq \mathbb{Z}/((m + 1)^n - m^n) \mathbb{Z}$$

is a cyclic group of order $(m + 1)^n - m^n$, and multiplication by $t$ is given by

$$tv = (-1)^{n+1}m \left( \sum_{i=1}^{n-1} (-1)^i \binom{n}{i} (m + 1)^{n-i-1} \right) v$$

for all $v \in A_n(M_m)$.

Proof. The module $M_m = \Lambda / \langle (m + 1)t - m \rangle$ is isomorphic to the $\Lambda$-submodule $\mathbb{Z}\left[\frac{m}{m+1}, \frac{m+1}{m}\right] \subset \mathbb{Q}$, where we put $t = \frac{m}{m+1}$ and $tv = \frac{m}{m+1}v$ for $v \in \mathbb{Q}$. Therefore we have

$$A_n(M_m) \simeq M_m / (t^n - 1)M_m \simeq \mathbb{Z}\left[\frac{m+1}{m}, \frac{m}{m+1}\right] / \langle \left( \frac{m}{m+1} \right)^n - 1 \rangle.$$

We easily see that the module $\mathbb{Z}\left[\frac{m}{m+1}, \frac{m+1}{m}\right]$ is equal to the sum of the $\mathbb{Z}$-submodules $\mathbb{Z}\left[\frac{1}{m+1}\right]$ and $\mathbb{Z}\left[\frac{1}{m}\right] \subset \mathbb{Q}$:

$$\mathbb{Z}\left[\frac{m}{m+1}, \frac{m+1}{m}\right] = \mathbb{Z}\left[\frac{1}{m+1}\right] + \mathbb{Z}\left[\frac{1}{m}\right].$$

Indeed, it is clear that

$$\mathbb{Z}\left[\frac{m}{m+1}, \frac{m+1}{m}\right] \subset \mathbb{Z}\left[\frac{1}{m+1}\right] + \mathbb{Z}\left[\frac{1}{m}\right].$$

We also have

$$\left( \frac{m+1}{m} \right)^n = \sum_{i=0}^{n} \binom{n}{i} m^{n-i}.$$
and, therefore,

\[ \frac{1}{m^n} = \left( \frac{m+1}{m} \right)^n - \sum_{i=0}^{n-1} \binom{n}{i} \frac{1}{m^i}. \]

We similarly have

\[ \frac{1}{(m+1)^n} = \sum_{i=0}^{n-1} (-1)^{n+i+1} \binom{n}{i} \frac{1}{(m+1)^i} + (-1)^n \left( \frac{m}{m+1} \right)^n. \]

In particular, \( \frac{1}{m} = \frac{m+1}{m} - 1 \) and \( \frac{1}{m+1} = 1 - \frac{m}{m+1} \). Using induction, we see that \( \frac{1}{m^n}, \frac{1}{(m+1)^n} \in \mathbb{Z}[\frac{m}{m+1}, \frac{m+1}{m}] \) for all \( n \) and, therefore,

\[ \mathbb{Z}\left[ \frac{1}{m+1} \right] + \mathbb{Z}\left[ \frac{1}{m} \right] \subset \mathbb{Z}\left[ \frac{m}{m+1}, \frac{m+1}{m} \right]. \]

Hence,

\[ A_n(M_m) \simeq \mathbb{Z}\left[ \frac{m+1}{m}, \frac{m}{m+1} \right] / \langle (m+1)^n - 1 \rangle \]

\[ \simeq \mathbb{Z}\left[ \frac{m}{m+1}, \frac{m+1}{m} \right] / \langle (m+1)^n - m^n \rangle. \]

We claim that every element \( v \in \mathbb{Z}[\frac{m}{m+1}, \frac{m+1}{m}] \) is equivalent to some element \( v_{in} \in \mathbb{Z} \subset \mathbb{Z}[\frac{m}{m+1}, \frac{m+1}{m}] \) modulo the ideal \( I = \langle (m+1)^n - m^n \rangle \). To see this, it suffices to prove that for every \( k \) there are \( v_{in,k}, u_{in,k} \in \mathbb{Z} \) such that

\[ \frac{1}{m^k} \equiv v_{in,k} \pmod{I}, \quad \frac{1}{(m+1)^k} \equiv u_{in,k} \pmod{I}. \]

We concentrate on proving the existence of \( v_{in,k} \) since the proof of the existence of \( u_{in,k} \) is similar. We have

\[ \frac{(m+1)^n - m^n}{m^k} = \sum_{i=1}^{n} \binom{n}{i} m^{n-i-k} \equiv 0 \pmod{I} \]

and, therefore,

\[ \frac{1}{m^k} \equiv - \sum_{j=k+1-n}^{n-1} \binom{n}{n+j-k} \frac{1}{m^j} \pmod{I}. \]

In particular,

\[ \frac{1}{m} \equiv - \sum_{j=0}^{n-2} \binom{n}{n-j-1} m^j \pmod{I}. \]

The existence of \( v_{in,k} \) now follows by induction on \( k \).

It follows from this argument that

\[ A_n(M_m) \simeq \mathbb{Z}\left[ \frac{m}{m+1}, \frac{m+1}{m} \right] / \langle (m+1)^n - m^n \rangle \]
is a cyclic group generated by the image $\bar{1}$ of the element

$$1 \in \mathbb{Z} \left[ \frac{m}{m+1}, \frac{m+1}{m} \right].$$

We have $((m+1)^n - m^n)\bar{1} = 0$, whence the order of $A_n(M_m)$ divides $(m+1)^n - m^n$.

We claim that the order of $A_n(M_m)$ is equal to $(m+1)^n - m^n$. Indeed, suppose that $k \in \mathbb{Z}$ satisfies $k\bar{1} = 0$. Then

$$k = \left( \sum_{i_1 \leq i \leq i_2} a_i \frac{1}{(m+1)^i} + \sum_{j_1 \leq j \leq j_2} b_j \frac{1}{m^j} \right) ((m+1)^n - m^n),$$

where $a_i, b_j \in \mathbb{Z}$. Multiplying both sides by $(m+1)^{i_2}$ and $m^{j_2}$ if $i_2 > 0$ or $j_2 > 0$, we obtain the equality

$$(m+1)^{i_2}m^{j_2}k = C((m+1)^n - m^n)$$

with some constant $C \in \mathbb{Z}$. It follows that $(m+1)^n - m^n$ divides $k$ because the numbers $m$, $m+1$ and $(m+1)^n - m^n$ are pairwise coprime.

To calculate the action of $t$ on the cyclic group

$$A_n(M_m) \simeq \mathbb{Z}/((m+1)^n - m^n)\mathbb{Z},$$

we note that

$$t\bar{1} = \frac{m}{m+1} = (-1)^{n+1}m \left( \sum_{i=1}^{n-1} (-1)^i \binom{n}{i} (m+1)^{n-i-1} \right) \bar{1}$$

because an argument similar to the previous one yields that

$$\frac{1}{m+1} \equiv (-1)^{n+1} \sum_{i=1}^{n-1} (-1)^i \binom{n}{i} (m+1)^{n-i-1} \pmod{I}.$$

**Proposition 2.15.** An abelian group $G$ is isomorphic (as a $\mathbb{Z}$-module) to the derived Alexander module $A_2(M)$ of some Noetherian $(t-1)$-invertible $\Lambda$-module $M$ if and only if $G$ is a finite group of odd order.

**Proof.** By Lemma 2.8 it suffices to prove that for every finite group $G$ of odd order there is a Noetherian $(t-1)$-invertible $\Lambda$-module $M$ with $A_2(M) \simeq G$.

We represent $G$ as a direct sum of cyclic groups,

$$G = \bigoplus_{i=1}^{k} G_i,$$

and let $n_i = 2m_i + 1$ be the order of $G_i$.

For every $i$ we consider the $\Lambda$-module $M_{m_i}$ in Example 2.14. Then $A_2(M_{m_i})$ is a cyclic group of order $(m_i + 1)^2 - m_i^2 = 2m_i + 1$. Hence the proposition follows from Proposition 2.12 if we put

$$M = \bigoplus_{i=1}^{k} M_{m_i}.$$
Theorem 2.16. Let $M$ be a Noetherian $(t - 1)$-invertible $t$-unipotent $\Lambda$-module of unipotence index $k$. Then the sequence $A_1(M), \ldots, A_n(M), \ldots$ of its derived Alexander modules has period $k$, that is, $A_n(M) \simeq A_{n+k}(M)$ for all $n$. If $n$ and $k$ are coprime, then $A_n(M) = 0$.

Proof. Since the unipotence index of $M$ is equal to $k$, Lemma 2.7 yields that the polynomial $f_k(t) = \sum_{i=0}^{k-1} t^i$ belongs to $\text{Ann}(M)$. Moreover, to obtain $A_n(M)$ from $M$, we need only take the quotient of $M$ by the relations $f_n(t)v = 0$ for all $v \in M$, where $f_n(t) = \sum_{i=0}^{n-1} t^i$. To prove the periodicity of the sequence (6), it remains to note that

$$f_{n+k}(t) = t^n f_k(t) + f_n(t).$$

Suppose that $n$, $k$ are coprime and polynomials $f_k(t)$, $f_n(t)$ belong to $\text{Ann}(M)$. Applying Euclid’s algorithm to $f_k(t)$ and $f_n(t)$, we easily deduce the existence of polynomials $g_k(t)$ and $g_n(t)$ such that

$$f_k(t)g_k(t) + f_n(t)g_n(t) = 1$$

since $n$ and $k$ are coprime. Therefore $\text{Ann}(M) = \Lambda$ and thus $A_n(M) = 0$.

Example 2.17. The $\Lambda$-module $M = \Lambda/\langle t^2 - t + 1 \rangle$ has the following derived Alexander modules:

$$A_{6k \pm 1}(M) = 0, \quad A_{6k \pm 2}(M) \simeq \mathbb{Z}/3\mathbb{Z}, \quad A_{6k + 3}(M) \simeq (\mathbb{Z}/2\mathbb{Z})^2.$$  

Multiplication of $\mathbb{Z}/3\mathbb{Z}$ by $t$ coincides with multiplication by 2. Multiplication of $(\mathbb{Z}/2\mathbb{Z})^2$ by $t$ coincides with a cyclic permutation of the non-zero elements of $A_{6k+3}(M)$.

Proof. The unipotence index of $M$ is equal to 6 because $t^2 - t + 1$ divides $t^6 - 1$. Therefore $A_{6k \pm 1}(M) = 0$.

To compute $A_{6k+2}(M)$, it suffices to find $A_2(M)$. We have $A_2(M) = \Lambda/\langle t^2 - t + 1, t + 1 \rangle$. Since

$$t^2 - t + 1 = (t - 2)(t + 1) + 3,$$

we have $\Lambda/\langle t^2 - t + 1, t + 1 \rangle = \Lambda/\langle t + 1, 3 \rangle \simeq \mathbb{Z}/3\mathbb{Z}$.

To compute $A_{6k+3}(M)$, it suffices to find $A_3(M)$. We have $A_3(M) = \Lambda/\langle t^2 - t + 1, t^2 + t + 1 \rangle$. Since

$$t^2 + t + 1 = t^2 - t + 1 + 2t,$$

we have $\Lambda/\langle t^2 - t + 1, t^2 + t + 1 \rangle = \Lambda/\langle t^2 + t + 1, 2 \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$.

To compute $A_{6k+4}(M)$, it suffices to find $A_4(M)$. We have $A_4(M) = \Lambda/\langle t^2 - t + 1, t^3 + t^2 + t + 1 \rangle$. Since

$$t^3 + t^2 + t + 1 = (t + 2)(t^2 - t + 1) + 2t - 1,$$

we see that the module $\Lambda/\langle t^2 - t + 1, t^3 + t^2 + t + 1 \rangle$ is isomorphic to the quotient module $M/(2t - 1)M$. Let $v$ be a generator of the biprincipal module $M$. Using the basis $v_1 = v, v_2 = tv$ of $M$ over $\mathbb{Z}$, we easily see that the module $(2t - 1)M$ is generated by the elements $2v_2 - v_1$ and $t(2v_2 - v_1) = v_2 - 2v_1$ because $tv_2 = v_2 - v_1$. Using another basis $e_1 = v_1, e_2 = v_2 - 2v_1$, we have $2v_2 - v_1 = 2e_2 + 3e_1$, that is, $(2t - 1)M$ is generated by $3e_1$ and $e_2$ over $\mathbb{Z}$. Therefore $A_4(M) \simeq \mathbb{Z}/3\mathbb{Z}$. 
§ 3. Alexander modules of irreducible $C$-groups

3.1. Proofs of Theorems 0.1 and 0.3. We recall that the class of irreducible $C$-groups coincides with the class of fundamental groups of knotted $n$-manifolds $V$ for every $n \geq 2$. Knot groups are also $C$-groups when given by the Wirtinger presentation. Similarly, the class of irreducible Hurwitz $C$-groups coincides with the class of fundamental groups of complements of irreducible ‘affine’ Hurwitz (or pseudo-holomorphic) curves. It contains the subclass of fundamental groups of complements of algebraic irreducible plane affine curves. Therefore speaking about the Alexander modules of knotted $n$-manifolds (resp. irreducible Hurwitz or pseudo-holomorphic curves) is the same as speaking about the Alexander modules of irreducible $C$-groups (resp. irreducible Hurwitz $C$-groups). Therefore Theorems 0.1 and 0.3 are equivalent to the following theorems.

Theorem 3.1. A $\Lambda$-module $M$ is the Alexander module of an irreducible $C$-group if and only if it is Noetherian and $(t-1)$-invertible.

Theorem 3.2. A $\Lambda$-module $M$ is the Alexander module of an irreducible Hurwitz $C$-group if and only if it is a Noetherian $(t-1)$-invertible $t$-unipotent $\Lambda$-module.

The unipotence index of the Alexander module $A_0(G)$ of an irreducible Hurwitz $C$-group $G$ of degree $m$ is a divisor of $m$.

Proof. Let 

\[ G = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle \quad (7) \]

be a $C$-presentation of a $C$-group $G$ and let $F_m$ be the free group freely generated by the $C$-generators $x_1, \ldots, x_m$. We write $\frac{\partial}{\partial x_i}$ for the Fox derivative (see [15]). This is an endomorphism of the group ring $Z[F_m]$ over $Z$ of the free group $F_m$ such that 

\[ \frac{\partial}{\partial x_i} : Z[F_m] \to Z[F_m] \]

is a $Z$-linear map with the following properties:

\[ \frac{\partial x_j}{\partial x_i} = \delta_{i,j}, \quad \frac{\partial uv}{\partial x_i} = \frac{\partial u}{\partial x_i} + u \frac{\partial v}{\partial x_i} \quad (8) \]

for all $u, v \in Z[F_m]$. The matrix

\[ A(G) = \nu_* \left( \frac{\partial r_i}{\partial x_j} \right) \in \text{Mat}_{n \times m}(Z[t, t^{-1}]) \]

is called the Alexander matrix of the $C$-group $G$ given by presentation (7), where the $r_i$ ($i = 1, \ldots, n$) are the relators of $G$ and the homomorphism $\nu_* : Z[F_m] \to Z[F_1] \simeq Z[t, t^{-1}]$ is induced by the canonical $C$-epimorphism $\nu : F_m \to F_1$.

The following lemma is a generalization (to the case of $C$-groups) of a well-known assertion concerning the Alexander matrices of Wirtinger presentations of knot groups (see [15]).

Lemma 3.3. Let $A(G)$ be the Alexander matrix of a $C$-group $G$ given by the presentation (7). Then the sum of the columns of $A(G)$ is equal to zero.

Proof. Each relator $r_i$ has the form

\[ r = wx_jw^{-1}x_i^{-1}, \]

where $w$ is a word in the letters $x_1^{\pm 1}, \ldots, x_m^{\pm 1}$, and $x_j, x_l$ are letters.
Let us use induction on the length \( l(w) \) of the word \( w \) to show that
\[
\sum_{k=1}^{m} \nu_{*} \left( \frac{\partial r}{\partial x_{k}} \right) = 0.
\]

If \( l(w) = 0 \) (that is, \( r := x_{j}x_{l}^{-1} \)), then we have
\[
\nu_{*} \left( \frac{\partial r}{\partial x_{k}} \right) = \begin{cases} 
1 & \text{if } k = j, \\
-1 & \text{if } k = l, \\
0 & \text{if } k \neq j \text{ and } k \neq l.
\end{cases}
\]

In this case we see that \( \sum_{k=1}^{m} \nu_{*} \left( \frac{\partial r}{\partial x_{k}} \right) = 0. \)

Suppose that the equation \( \sum_{k=1}^{m} \nu_{*} \left( \frac{\partial r}{\partial x_{k}} \right) = 0 \) holds for all words \( r = wx_{j}w^{-1}x_{l}^{-1} \) with \( l(w) \leq L \). Consider any word \( r = wx_{j}w^{-1}x_{l}^{-1} \) of length \( l(w) = L + 1 \). Put \( r_{1} = w_{1}x_{j}w_{1}^{-1}x_{l}^{-1} \), where \( w = x_{j}^{\varepsilon}w_{1}, \varepsilon = \pm 1, \) and \( l(w_{1}) = L \). We consider only the case when \( i \neq j, i \neq l, j \neq l \) and \( \varepsilon = 1 \). The proof that \( \sum_{k=1}^{m} \nu_{*} \left( \frac{\partial r}{\partial x_{k}} \right) = 0 \) in all other cases is similar.

It follows from (8) that
\[
\nu_{*} \left( \frac{\partial r}{\partial x_{k}} \right) = \begin{cases} 
t \nu_{*} \left( \frac{\partial r_{1}}{\partial x_{k}} \right) & \text{if } k \neq i, \ k \neq j, \ k \neq l, \\
1 + t \nu_{*} \left( \frac{\partial r_{1}}{\partial x_{k}} \right) - t & \text{if } k = i, \\
t \nu_{*} \left( \frac{\partial r_{1}}{\partial x_{k}} \right) & \text{if } k = j, \\
t (\nu_{*} \left( \frac{\partial r_{1}}{\partial x_{k}} \right) + 1) - 1 & \text{if } k = l
\end{cases}
\]
and it is easy to see that \( \sum_{k=1}^{m} \nu_{*} \left( \frac{\partial r}{\partial x_{k}} \right) = 0. \) The lemma is proved.

For every monomial \( a_{i}t^{i} \in \mathbb{Z}[t] \) we consider the word
\[
w_{a_{i}t^{i}}(x_{1}, x_{2}) = (x_{2}x_{1}x_{2}^{-(i+1)})^{a_{i}}.
\]
For every polynomial \( g(t) = \sum_{i=0}^{k} a_{i}t^{i} \in \mathbb{Z}[t] \) we put
\[
w_{g(t)}(x_{1}, x_{2}) = \prod_{i=0}^{k} w_{a_{i}t^{i}}(x_{1}, x_{2}).
\]
We also associate with any polynomial \( f(t) = (1 - t)g(t) + 1 \) the word
\[
r_{f(t)}(x_{1}, x_{2}) = w_{g(t)}(x_{1}, x_{2})x_{1}w_{g(t)}^{-1}(x_{1}, x_{2})x_{2}^{-1}.
\]
For every vector \( u = (1 - t)\bar{u} = ((1 - t)g_{1}(t), \ldots, (1 - t)g_{m}(t)) \) we consider the word
\[
r_{u}(x_{1}, \ldots, x_{m+1}) = w_{u}(x_{1}, \ldots, x_{m+1})x_{m+1}w_{u}^{-1}(x_{1}, \ldots, x_{m+1})x_{m+1}^{-1},
\]
where
\[
w_{u}(x_{1}, \ldots, x_{m+1}) = \prod_{i=1}^{m} w_{g_{i}(t)}(x_{i}, x_{m+1}).
\]
Lemma 3.4. Take a polynomial \( f(t) = (1 - t)g(t) + 1 \) and a vector 

\[
    u = ((1 - t)g_1(t), \ldots, (1 - t)g_m(t)).
\]

Then we have

\[
    \nu_* \left( \frac{\partial r_f(t)}{\partial x_1} \right) = f(t), \quad \nu_* \left( \frac{\partial r_u}{\partial x_i} \right) = (1 - t)g_i(t), \quad i = 1, \ldots, m.
\]

Proof. Take \( f(t) = (1 - t)g(t) + 1 \). It follows from (8) that

\[
    \nu_* \left( \frac{\partial w_g(t)(x_1, x_2)}{\partial x_1} \right) = -\nu_* \left( \frac{\partial w_g^{-1}(t)(x_1, x_2)}{\partial x_1} \right) = g(t)
\]

since we have \( w_g(t)(x_1, x_2)w_g^{-1}(t)(x_1, x_2) = 1 \) and

\[
    \nu_* \left( w_g(t)(x_1, x_2) \right) = \nu_* \left( w_{a, t^i}(x_1, x_2) \right) = 1,
\]

\[
    \nu_* \left( \frac{\partial w_{a, t^i}(x_1, x_2)}{\partial x_1} \right) = a_it^i.
\]

Therefore,

\[
    \nu_* \left( \frac{\partial r_f(t)}{\partial x_1} \right) = \nu_* \left( \frac{\partial \left( w_g(t)(x_1, x_2)x_1w_g^{-1}(t)(x_1, x_2)x_2^{-1} \right)}{\partial x_1} \right)
\]

\[
    = g(t) + 1 - tg(t) = f(t).
\]

The proof of the second equation of the lemma is similar.

Proposition 3.5. The Alexander module \( A_0(G) \) of the \( C \)-group \( G \) with the presentation (7) is isomorphic to the quotient module \( \Lambda^{m-1}/M(G) \), where the submodule \( M(G) \) of \( \Lambda^{m-1} \) is generated by the rows of the matrix \( \hat{A} \) formed by the first \( m - 1 \) columns of the Alexander matrix \( A(G) \).

Proof. To describe the Alexander module of a \( C \)-group \( G \), we follow [16] (see also [8]). Given a \( C \)-group \( G \) with \( C \)-presentation (7), we consider the following complex \( K \) with a single vertex \( x_0 \). Its one-dimensional skeleton is a wedge product of oriented circles \( s_i \) \((1 \leq i \leq m)\) that are in one-to-one correspondence with the \( C \)-generators of \( G \) in the presentation (7). Furthermore, \( K \setminus (\bigcup_{i=1}^n s_i) = \bigcup_{i=1}^n \hat{D}_i \) is a disjoint union of open discs. Each disc \( D_i \) corresponds to the relator \( r_i = \bar{x}_{j_i,1}^{\varepsilon_{i,1}} \cdots \bar{x}_{j_i,k_i}^{\varepsilon_{i,k_i}} \) in (7), where \( \varepsilon_{i,j} = \pm 1 \). This disc is glued to the wedge product \( \vee s_i \) by identifying the boundary \( \partial D_j = D_j \setminus \hat{D}_i \) and the closed path \( s_{j_i,1}^{\varepsilon_{i,1}} \cdots s_{j_i,k_i}^{\varepsilon_{i,k_i}} \subset \vee s_i \). Clearly, \( \pi_1(K, x_0) \simeq G \).

The \( C \)-homomorphism \( \nu: G \to F_1 \) determines an infinite cyclic covering \( f: \tilde{K} \to K \) such that \( \pi_1(\tilde{K}) = N \) and \( H_1(\tilde{K}, \mathbb{Z}) = N/N' \), where \( N = \text{ker } \nu \). The group \( F_1 \) acts on \( \tilde{K} \).
Let $\widetilde{K}_0 = f^{-1}(x_0)$ and let $\widetilde{K}_1$ be the one-dimensional skeleton of the complex $\widetilde{K}$. We consider the following exact sequences of homomorphisms of homology groups with coefficients in $\mathbb{Z}$:

\[
\begin{array}{cccc}
0 & \rightarrow & H_1(\widetilde{K}) & \rightarrow & H_1(\widetilde{K}_1, \widetilde{K}_0) & \rightarrow & H_1(\widetilde{K}, \widetilde{K}_0) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & H_0(\widetilde{K}_0) & \rightarrow & 0
\end{array}
\]

(11)

The action of $F_1$ on $\widetilde{K}$ endows each group in these sequences with the structure of a $\Lambda$-module. We fix a vertex $p_0 \in \widetilde{K}_0$. Let $p_i = t^ip_0$ be the result of the action of the element $t^i \in F_1$ on the point $p_0$. Then $H_1(\widetilde{K}_1, \widetilde{K}_0)$ is a free $\Lambda$-module whose generators $\bar{s}_i$ are edges that join $p_0$ to $p_1$ and are mapped by $f$ onto the loops $s_i$. The result of the action of $t^i$ on the generator $\bar{s}_j$ is an edge that begins at the vertex $p_i$ and is mapped by $f$ onto the loop $s_j$.

The free $\Lambda$-module $H_2(\widetilde{K}, \widetilde{K}_1)$ is generated by the discs $\bar{D}_i$ ($i = 1, \ldots, n$) that correspond to the relators $r_i = x_{j_{i,1}}^{\varepsilon_{i,1}} \ldots x_{j_{i,k_i}}^{\varepsilon_{i,k_i}}$. Each disc $\bar{D}_i$ is glued to the one-dimensional skeleton along the product of paths

\[
t^{\delta(\varepsilon_{i,1})} s_{j_{i,1}}^{\varepsilon_{i,1}}, t^{\delta(\varepsilon_{i,2})+\varepsilon_{i,1}} s_{j_{i,2}}^{\varepsilon_{i,2}}, \ldots, t^{\delta(\varepsilon_{i,k_i})+\sum_{l=1}^{k_i-1} \varepsilon_{i,l}} s_{j_{i,k_i}}^{\varepsilon_{i,k_i}},
\]

where $\delta(1) = 0$ and $\delta(-1) = -1$. It is easy to verify that the coordinates of the elements $\alpha(\bar{D}_i) \in H_1(\widetilde{K}_1, \widetilde{K}_0)$ in the basis $\bar{s}_1, \ldots, \bar{s}_m$ coincide with the rows $A_i$ of the Alexander matrix $A(G)$ of the $C$-group $G$ with the presentation (7).

It follows from the vertical exact sequence in (11) that $\partial(\beta(\bar{s}_i)) = (t-1)p_0$ for each generator $\bar{s}_i$ of the module $H_1(\widetilde{K}_1, \widetilde{K}_0)$. We choose a new basis of this module by putting $e_i = \bar{s}_i - \bar{s}_m$, $i = 1, \ldots, m-1$, $e_m = \bar{s}_m$. Then $\beta(e_i) \in \ker \partial$ for $i = 1, \ldots, m-1$ and $\ker \partial$ is generated by $\beta(e_1), \ldots, \beta(e_{m-1})$. Hence we may identify the module $H_1(\widetilde{K})$ with $\beta(H'_1(\widetilde{K}_1, \widetilde{K}_0))$, where $H'_1(\widetilde{K}_1, \widetilde{K}_0)$ is the free submodule generated by the elements $e_1, \ldots, e_{m-1}$ of the free $\Lambda$-module $H_1(\widetilde{K}_1, \widetilde{K}_0)$.

The matrix formed by the coordinates of the elements $\alpha(\bar{D}_i)$ in the basis $e_1, \ldots, e_m$ coincides with the matrix $\bar{A}(G)$ obtained from $A(G)$ by replacing the last column by a column of zeros. Hence $H_1(\widetilde{K})$ is isomorphic to the quotient module of the free $\Lambda$-module $H'_1(\widetilde{K}_1, \widetilde{K}_0) \cong \bigoplus_{i=1}^{m-1} \Lambda e_i$ by the submodule $M(G)$ generated by the rows of the matrix $\bar{A}(G)$ formed by the first $m-1$ columns of $A(G)$. This proves the proposition.
We use Proposition 1.11 to prove that every Noetherian \((t-1)\)-invertible (resp. \(t\)-unipotent) \(\Lambda\)-module \(M\) is the Alexander module of some irreducible (resp. Hurwitz) \(C\)-group. By Proposition 1.11 every Noetherian \((t-1)\)-invertible \(\Lambda\)-module \(M\) is isomorphic to a quotient module \(\Lambda^m/M_1\) of the free \(\Lambda\)-module \(\Lambda^m\). Here the submodule \(M_1\) is generated by elements \(u_1, \ldots, u_m, \ldots, u_{m+k}\) of \(\Lambda^m\) such that

(i) we have \(u_i = (0, \ldots, 0, f_i(t), 0, \ldots, 0)\) for \(i = 1, \ldots, m\), where the polynomial \(f_i(t)\) occupies the \(i\)th position and \(f_i(1) = 1\),

(ii) we have \(u_{m+j} = (1-t)\bar{u}_{m+j} = ((1-t)g_{j,1}(t), \ldots, (1-t)g_{j,m}(t))\) for \(j = 1, \ldots, k\), where the \(g_{j,i}(t)\) are polynomials.

If \(M\) is a \(t\)-unipotent \(\Lambda\)-module of unipotence index \(n\), then we can also assume that

(iii) we have \(u_{m+k+i} = (0, \ldots, 0, t^n - 1, 0, \ldots, 0)\) for \(i = 1, \ldots, m\), where the polynomial \(t^n - 1\) occupies the \(i\)th position.

We write each polynomial \(f_i(t)\) as \(f_i(t) = (1-t)g_i(t) + 1\) and consider the \(C\)-group

\[G = \langle x_1, \ldots, x_{m+1} \mid r_1, \ldots, r_{m+k} \rangle\]

with relators \(r_i := r_{f_i(t)}(x_i, x_{m+1})\) for \(i = 1, \ldots, m\) and \(r_{m+j} := r_u(x_1, \ldots, x_{m+1})\) for \(j = 1, \ldots, k\), where the words \(r_{f(t)}\) and \(r_u\) are defined by (9) and (10). We put

\[r_{m+k+i} := x_{m+1}^n x_i x_{m+1}^{n-1} x_i^{-1}\]

\[u_{m+k+i} = (0, \ldots, 0, t^n - 1, 0, \ldots, 0) \in M_1\]

for \(i = 1, \ldots, m\). Let \(\overline{G}\) be the group defined by the presentation

\[\overline{G} = \langle x_1, \ldots, x_{m+1} \mid r_1, \ldots, r_{2m+k} \rangle\]

Lemma 3.4 implies that the matrix \(\tilde{A}(G)\) (resp. \(\tilde{A}(\overline{G})\)) formed by the first \(m\) columns of the Alexander matrix \(A(G)\) (resp. \(A(\overline{G})\)) coincides with the matrix \(U\) (resp. \(\overline{U}\)) formed by the rows \(u_1, \ldots, u_{m+k}\) (resp. \(u_1, \ldots, u_{2m+k}\)). Therefore, by Proposition 3.5, the Alexander module \(A_0(G)\) (resp. \(A_0(\overline{G})\)) coincides with \(M = \Lambda^m/M_1\), where \(M_1\) is generated by the rows \(u_1, \ldots, u_{m+k}\) (resp. \(u_1, \ldots, u_{2m+k}\)).

We note that \(G\) (resp. \(\overline{G}\)) is an irreducible \(C\)-group since all the \(C\)-generators \(x_1, \ldots, x_m\) are conjugate to \(x_{m+1}\). Moreover, \(\overline{G}\) is a Hurwitz \(C\)-group. Indeed, the relators \(r_{m+k+j}\) \((j = 1, \ldots, m)\) imply that \(x_{m+1}^n\) belongs to the centre of \(\overline{G}\). Since all the \(x_i\) are conjugate to \(x_{m+1}\), we have \(x_i^n = x_{m+1}^n\) for all \(i = 1, \ldots, m\). Hence the product \(x_1^n \ldots x_{m+1}^n\) also belongs to the centre of \(\overline{G}\), and \(\overline{G}\) possesses a Hurwitz presentation

\[\overline{G} = \langle y_1, \ldots, y_{n(m+1)} | \tilde{r}_1, \ldots, \tilde{r}_{2m+k},
\]

\[y_{in+j} y_{in+j}^{-1}, i = 0, 1, \ldots, m, j = 2, \ldots, n,
\]

\[\left[ y_i, (y_1 \ldots y_{n(m+1)}) \right], i = 1, \ldots, n(m+1), \]

where the relators \(\tilde{r}_i = \tilde{r}_i(y_1, \ldots, y_{n(m+1)})\) are obtained from the relators \(r_i = r_i(x_1, \ldots, x_{m+1})\) by substituting \(y_{jm+1}\) for \(x_j, j = 1, \ldots, m+1\).
The following lemmas complete the proofs of Theorems 0.1 and 0.3.

**Lemma 3.6** [17]. The Alexander module $A_0(G) = G'/G''$ of an irreducible $C$-group $G$ is a Noetherian $(t-1)$-invertible $\Lambda$-module.

**Proof.** Since $G$ is an irreducible $C$-group, its commutator subgroup $G'$ coincides with the kernel of the $C$-epimorphism $\nu: G \to F_1$. By the Reidemeister–Schreier method, if $C$-generators $x_1, \ldots, x_m$ generate $G$, then the elements $a_{i,n} = x_m^nx_{i,n}x_m^{-(n+1)}$ ($i = 1, \ldots, m-1, n \in \mathbb{Z}$) generate $G'$. Hence the module $A_0(G) = G'/G''$ is generated by the images $\bar{a}_{i,n}$ of the elements $a_{i,n}$ under the natural epimorphism $G' \to G'/G''$. The action of $t$ on $A_0(G)$ is defined by the conjugation $a \mapsto x_max_m^{-1}$ for $a \in G'$. Therefore $t\bar{a}_{i,n} = \bar{a}_{i,n+1}$. Thus the module $A_0(G)$ is generated over $\Lambda$ by $\bar{a}_{1,0}, \ldots, \bar{a}_{m-1,0}$ and is therefore a Noetherian $\Lambda$-module.

To show that $A_0(G)$ is a $(t-1)$-invertible $\Lambda$-module, we note that every element $g \in G$ may be written as $g = x_m^ka, \text{ where } a \in G'$ and $k = \nu(g)$. Hence the group $G'$ is generated by the elements $[x_m^ka, x_m^kb]$, where $a, b \in G'$. Therefore $A_0(G)$ is generated by their images $[x_m^ka, x_m^kb]$. It is easy to see that

$$[x_m^ka, x_m^kb] = [x_m^ka](ax_m^{n+k}[b, a^{-1}]x_m^{-(n+k)}a^{-1})[a, x_m^{n+k}] 
\cdot (x_m^{n+k}[b, x_m^ka]x_m^{-(n+k)}). \tag{12}$$

It follows from (12) that

$$[x_m^ka, x_m^kb] = (t^n - 1)\bar{a} + (1 - t^{n+k})\bar{a} + t^{n+k}(1 - t^{-n})\bar{b} 
= t^n(1 - t^k)\bar{a} + t^k(t^n - 1)\bar{b} = (t - 1)\left(\sum_{i=0}^{n-1} t^{i+k}\bar{b} - \sum_{i=0}^{k-1} t^{i+n}\bar{a}\right). \tag{13}$$

since $ax_m^{n+k}[b, a^{-1}]x_m^{-(n+k)}a^{-1} \in G''$. We now easily see that multiplication by $t - 1$ is an epimorphism of $A_0(G)$ onto itself since the elements of the form $[x_m^ka, x_m^kb]$ generate $A_0(G)$ over $\mathbb{Z}$. To complete the proof, we apply Lemma 1.1.

**Lemma 3.7** [13]. The Alexander module of a Hurwitz irreducible $C$-group of degree $m$ is a Noetherian $(t-1)$-invertible $\text{unipotent} \Lambda$-module of unipotence index $d$, where $d$ is a divisor of $m$.

**Proof.** If $G$ is a Hurwitz group of degree $m$, then it is generated by $C$-generators $x_1, \ldots, x_m$ such that the product $x_1 \ldots x_m$ belongs to the centre of $G$. By Lemma 3.6, the Alexander module $A_0(G) = G'/G''$ is a Noetherian $(t-1)$-invertible $\Lambda$-module. Multiplication of the module $A_0(G)$ by $t$ is induced by the conjugation $a \mapsto x_max_m^{-1}$ for $a \in G'$. Since $\nu(x_m^a) = \nu(x_1 \ldots x_m)$, there is an element $a_0 \in G'$ such that $x_m^m = a_0x_1 \ldots x_m$. Hence conjugation by $x_m^m$ is an inner automorphism of $G'$. Therefore the induced automorphism $t^m$ of $G'/G''$ is the identity.

3.2. Alexander modules of $C$-products of $C$-groups. Let $G_1, G_2$ be irreducible $C$-groups and let $x \in G_1$ (resp. $y \in G_2$) be one of the $C$-generators of $G_1$ (resp. $G_2$). We consider the amalgamated product $G_1 *_{\{x=y\}} G_2$. If $G_1$ and $G_2$ are given by $C$-presentations

$$G_1 = \langle x_1, \ldots, x_n \mid R_1 \rangle,$n
$$G_2 = \langle y_1, \ldots, y_m \mid R_2 \rangle, \tag{14}$$

$$G_1 *_{\{x=y\}} G_2 = \langle x_1, \ldots, x_n, y_1, \ldots, y_m \mid R_1 \cup R_2 \rangle.$$
where \( x = x_n \) and \( y = y_m \), then the group \( G_1 \ast_{\{x=y\}} G_2 \) is given by the \( C \)-presentation
\[
\langle x_1, \ldots, x_{n-1}, y_1, \ldots, y_{m-1}, z \mid \tilde{R}_1 \cup \tilde{R}_2 \rangle,
\]
where each relator \( \tilde{r}_i \in \tilde{R}_1 \) (resp. \( \tilde{r}_i \in \tilde{R}_2 \)) is obtained from \( r_i \in R_1 \) (resp. \( r_i \in R_2 \)) by substituting the letter \( z \) for the letter \( x_n \) (resp. \( y_m \)).

If \( x' \in G_1 \) and \( y' \in G_2 \) are other \( C \)-generators of these groups, then there are inner \( C \)-isomorphisms \( f_i: G_i \to G_i \) such that \( f_1(x') = x \) and \( f_2(y') = y \) because all the \( C \)-generators of an irreducible \( C \)-group are conjugate to each other. Hence there is a \( C \)-isomorphism
\[
f_1 \ast f_2: G_1 \ast_{\{x'=y'\}} G_2 \to G_1 \ast_{\{x=y\}} G_2.
\]
In other words, the group \( G_1 \ast_{\{x=y\}} G_2 \) is independent (up to \( C \)-isomorphism) of the choice of the \( C \)-generators \( x \) and \( y \). We denote this group by \( G_1 \ast_C G_2 \) and call it the \( C \)-product of the irreducible \( C \)-groups \( G_1 \) and \( G_2 \).

**Proposition 3.8.** The Alexander module of the \( C \)-product \( G = G_1 \ast_C G_2 \) of irreducible \( C \)-groups \( G_1 \) and \( G_2 \) is isomorphic to the direct sum of the Alexander modules of \( G_1 \) and \( G_2 \):
\[
A_0(G) = A_0(G_1) \oplus A_0(G_2).
\]

**Proof.** This follows easily from Proposition 3.5. Indeed, if \( G_1 \) and \( G_2 \) are given by the presentations (14), then Proposition 3.5 yields that the Alexander module \( A_0(G) \) of the irreducible \( C \)-group \( G = G_1 \ast_C G_2 \) (given by the presentation (15)) is isomorphic to the quotient module \( \Lambda^{n+m-1}/M(G) \), where the submodule \( M(G) \) of \( \Lambda^{n+m-1} \) is generated by the rows of the matrix
\[
\bar{A} = \begin{pmatrix}
\bar{A}_1 & 0 \\
0 & \bar{A}_2
\end{pmatrix}.
\]
Here \( \bar{A}_1 \) (resp. \( \bar{A}_2 \)) is the matrix formed by the first \( n-1 \) (resp. \( m-1 \)) columns of the matrix \( A(G_1) \) (resp. \( A(G_2) \)). It is now easy to see that \( A_0(G) = A_0(G_1) \oplus A_0(G_2) \). The proposition is proved.

Let
\[
G = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle
\]
be a \( C \)-presentation of some \( C \)-group \( G \). The number \( d_P = m - n \) is called the \( C \)-deficiency of the presentation (16). The number \( d_G = \max d_P \), where the maximum is taken over all \( C \)-presentations of \( G \), is called the \( C \)-deficiency of the \( C \)-group \( G \). Clearly, the \( C \)-deficiency satisfies \( d_G \leq k \) if the \( C \)-group consists of \( k \) irreducible components. In particular, if \( G \) is an irreducible \( C \)-group, then \( d_G \leq 1 \).

**Lemma 3.9.** Let \( G = G_1 \ast_C G_2 \) be the \( C \)-product of the irreducible \( C \)-groups \( G_1 \) and \( G_2 \). Then
\[
d_G \geq d_{G_1} + d_{G_2} - 1.
\]
In particular, if \( d_{G_1} = d_{G_2} = 1 \), then \( d_G = 1 \).

**Proof.** This follows from formula (15).
3.3. Presentation graphs of $C$-groups. We associate a presentation graph $\Gamma_P$ with each $C$-presentation (16). The vertices of $\Gamma_P$ are labelled by the generators appearing in (16). (In particular, there is a one-to-one correspondence between the vertices of $\Gamma_P$ and these generators.) The edges of $\Gamma_P$ are in one-to-one correspondence with the relators $r_j$ appearing in (16). If $r_j := w_j^{-1}(x_1, \ldots, x_m) x_i w_j(x_1, \ldots, x_m) x_i^{-1}$, then the corresponding edge connects the vertices $x_{i_1}$ and $x_{i_2}$.

Clearly, the $C$-deficiency satisfies

$$d_P = \dim H_0(\Gamma_P, \mathbb{R}) - \dim H_1(\Gamma_P, \mathbb{R}).$$

Therefore the $C$-deficiency $d_G$ of an irreducible $C$-group $G$ is equal to 1 if and only if $G$ possesses a $C$-presentation whose graph $\Gamma_P$ is a tree.

A $C$-presentation

$$G = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle$$

is said to be simple if each relator $r_j$ in (17) is of the form

$$r_j := x_{i_3}^{-1} x_{i_1} x_{i_3} x_{i_2}^{-1}$$

for some $i_1, i_2, i_3 \in \{1, \ldots, m\}$ (that is, the relator $r_j$ is given by $x_{i_2} = x_{i_3}^{-1} x_{i_1} x_{i_3}$).

Remark 3.10. If the presentations (14) of irreducible $C$-groups $G_1$ and $G_2$ are simple, then so is the presentation (15) of their product $G = G_1 \ast_C G_2$, and the graph $\Gamma_P$ of the presentation (15) is the wedge product $\Gamma_P = \Gamma_{P_1} \vee_{z=x_n=y_m} \Gamma_{P_2}$ of the graphs $\Gamma_{P_1}$ and $\Gamma_{P_2}$ of the presentations (14). In particular, if $\Gamma_{P_1}$ and $\Gamma_{P_2}$ are trees, then $\Gamma_P$ is a tree.

Lemma 3.11. Every $C$-group $G$ possesses a simple $C$-presentation of $C$-deficiency $d_P = d_G$.

Proof. Let $G$ be given by a $C$-presentation of $C$-deficiency $d_P = d_G$ and let $r := w^{-1} x_i w x_j^{-1}$ be one of its relators (so that $w^{-1} x_i w = x_j$), where $w = x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$ is a word in the group $\mathbb{F}_m$ and $\varepsilon_l = \pm 1$. Then we can add $k - 1$ new generators $x_{m+1}, \ldots, x_{m+k-1}$ and replace $r$ by the $k$ simple relations

\[
\begin{align*}
x_{m+1} &= x_{i_1}^{-\varepsilon_1} x_{i_1}^{\varepsilon_1}, \\
x_{m+2} &= x_{i_2}^{-\varepsilon_2} x_{m+1} x_{i_2}^{\varepsilon_2}, \\
&\quad \cdots \\
x_{m+k-1} &= x_{i_{k-1}}^{-\varepsilon_{k-1}} x_{m+k-2} x_{i_{k-1}}^{\varepsilon_{k-1}}, \\
x_j &= x_{i_k}^{-\varepsilon_k} x_{m+k-1} x_{i_k}^{\varepsilon_k}.
\end{align*}
\]

Clearly, we get a new $C$-presentation which has the same $C$-deficiency and defines the same $C$-group $G$.

3.4. Alexander modules of $C$-groups possessing $C$-presentations whose graphs are trees. Lemma 3.11 shows that an irreducible $C$-group $G$ possesses a simple $C$-presentation whose graph is a tree if and only if the $C$-deficiency $d_G$ is equal to 1.
Proposition 3.12. If $M = \bigoplus_{i=1}^{m} M_i$ is the direct sum of the biprincipal $(t-1)$-invertible $\Lambda$-modules $M_i = \Lambda/\langle f_i(t) \rangle$, then there is an irreducible $C$-group $G$ such that the $C$-deficiency $d_G$ is equal to 1 and $A_0(G) \simeq M$.

Proof. Consider the $C$-group given by the presentation
\[ G = \langle x_1, x_2 \mid wx_1w^{-1}x_2^{-1} \rangle, \tag{18} \]
where $w = w(x_1, x_2)$ is a word in letters $x_1$, $x_2$ and their inverses. Note that the $C$-deficiency of $G$ is equal to 1. Applying Proposition 3.5, we see that the Alexander module $A_0(G)$ of an irreducible $C$-group $G$ given by the presentation (18) is a biprincipal $(t-1)$-invertible $\Lambda$-module.

Conversely, it was shown in the proof of Theorem 3.1 that every biprincipal $(t-1)$-invertible $\Lambda$-module $M = \Lambda/\langle f(t) \rangle$ is the Alexander module of some irreducible $C$-group given by (18). To complete the proof, we apply Proposition 3.8 and Remark 3.10.

Corollary 3.13. Let $M = \bigoplus_{i=1}^{m} M_i$ be the direct sum of the biprincipal $(t-1)$-invertible $\Lambda$-modules $M_i = \Lambda/\langle f_i(t) \rangle$. Then for every $n \geq 2$ there is a knotted sphere $S^n \subset S^{n+2}$ such that
\[ A_0(\pi_1(S^{n+2} \setminus S^n)) \simeq M. \]

In particular, a polynomial $f(t) \in \mathbb{Z}[t]$ is the Alexander polynomial $\Delta(t)$ of some knotted sphere $S^n \subset S^{n+2}$ with $n \geq 2$ if and only if $f(1) = \pm 1$. Moreover, the Jordan blocks of the Jordan canonical form of the matrix of the automorphism $h_{\mathbb{C}}$ acting on $A_0(S^n) \otimes \mathbb{C}$ can be of arbitrary size.

Proof. Let $G$ be an irreducible $C$-group given by a simple presentation whose graph is a tree. As shown in [1], for every $n \geq 2$ there is a knotted sphere $S^n \subset S^{n+2}$ such that $\pi_1(S^{n+2} \setminus S^n) \simeq G$. This proves the corollary.

Proposition 3.14. Let $G$ be an irreducible $C$-group of $C$-deficiency $d_G = 1$. Then its Alexander module $A_0(G)$ has no non-zero $\mathbb{Z}$-torsion elements.

Proof. Let $G$ be given by a $C$-presentation
\[ G = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_{m-1} \rangle. \tag{19} \]

By Proposition 3.5 the Alexander module $A_0(G)$ is isomorphic to the quotient module $\Lambda^{m-1}/M(G)$, where the submodule $M(G)$ of $\Lambda^{m-1}$ is generated by the rows of the matrix $\hat{A}$ formed by the first $m-1$ columns of the Alexander matrix $\mathcal{A}(G)$ of the group $G$ given by (19). The size of the matrix $\hat{A}$ is $(m-1) \times (m-1)$.

Lemma 3.15. The determinant $\Delta(t) = \det \hat{A}$ satisfies $\Delta(1) = \pm 1$.

Proof. The proof coincides with that of the corresponding statement for knot groups (see, for example, [15]).

We denote the rows of the matrix $\hat{A}$ by $A_j$, $j = 1, \ldots, m-1$. The module $A_0(G)$ has a non-zero $\mathbb{Z}$-torsion element if and only if there is a vector
Let \( u = (f_1(t), \ldots, f_{m-1}(t)) \) such that \( u \not\in M(G) \) and \( ku \in M(G) \) for some \( k \in \mathbb{N} \). Assume that there is such a vector \( u \). Then there are elements \( g_j(t) \in \Lambda \) such that 

\[
ku = \sum g_j(t) \mathcal{A}_j
\]

and at least one coefficient of one of these elements \( g_j(t) \) is not divisible by \( k \).

There is no loss of generality in assuming that all the elements \( f_i(t) \) and \( g_j(t) \) belong to \( \mathbb{Z}[t] \). By Cramér’s rule we have

\[
g_j(t) = \frac{\Delta_j(t)}{\Delta(t)},
\]

where \( \Delta_j(t) \) is the determinant of the matrix obtained from \( \bar{A} \) by substituting the row \( ku \) for the row \( \mathcal{A}_j \). Hence all the coefficients of all the polynomials \( \frac{\Delta_j(t)}{\Delta(t)} \) are divisible by \( k \), a contradiction.

**Remark 3.16.** Let \( G \) be an irreducible \( C \)-group given by a presentation of \( C \)-deficiency \( d_P = d_G = 1 \) and let \( \bar{A} \) be the matrix obtained from the Alexander matrix \( A \) by removing its last column. Then the determinant \( \Delta(t) = \det \bar{A} \) coincides with the Alexander polynomial \( \Delta_G(t) \) of the group \( G \).

### 3.5. Finitely \( \mathbb{Z} \)-generated Alexander modules of irreducible \( C \)-groups

**Theorem 3.17.** The Alexander module \( A_0(G) \) of an irreducible \( C \)-group \( G \) is finitely generated over \( \mathbb{Z} \) if and only if the leading coefficient \( a_n \) and the constant coefficient \( a_0 \) of the Alexander polynomial \( \Delta_G(t) = \sum_{i=0}^{n} a_i t^i \) of \( G \) are equal to \( \pm 1 \).

**Proof.** By Theorem 3.1, \( A_0(G) \) is a Noetherian \( (t-1) \)-invertible \( \Lambda \)-module. Let \( A_0(G)_{\text{fin}} \) be the \( \mathbb{Z} \)-torsion submodule of the Alexander module \( A_0(G) \). By Theorem 1.16, the module \( A_0(G)_{\text{fin}} \) is finitely generated over \( \mathbb{Z} \).

Consider the quotient module \( M = A_0(G)/A_0(G)_{\text{fin}} \). It is \( \mathbb{Z} \)-torsion free. Hence there is a natural embedding \( M \hookrightarrow M_\mathbb{Q} = M \otimes \mathbb{Q} \). We have \( \dim_\mathbb{Q} M_\mathbb{Q} < \infty \) since \( M \) is a Noetherian \( \Lambda \)-torsion module.

Let \( h_\mathbb{Q} \) be the automorphism of \( M_\mathbb{Q} \) induced by multiplication by \( t \). By definition, we have \( \Delta_G(t) = a \det(h_\mathbb{Q} - t \text{ Id}) \), where \( a \in \mathbb{N} \) is the smallest positive integer such that \( a \det(h_\mathbb{Q} - t \text{ Id}) \in \mathbb{Z}[t] \).

If the Alexander module \( A_0(G) \) is finitely generated over \( \mathbb{Z} \), then \( M \) is a free finitely generated \( \mathbb{Z} \)-module. Let \( h \) be the automorphism of \( M \) induced by multiplication by \( t \). We have \( \det h = \pm 1 \) and

\[
\det(h - t \text{ Id}) = \det(h_\mathbb{Q} - t \text{ Id}) \in \mathbb{Z}[t].
\]

Therefore \( \Delta_G(t) = \det(h - t \text{ Id}) \), the leading coefficient satisfies \( a_n = (-1)^n \), where \( n = \text{rk} M \), and we have \( a_0 = \det h = \pm 1 \).

Now suppose that the leading coefficient \( a_n \) and the constant coefficient \( a_0 \) of the Alexander polynomial \( \Delta_G(t) \) of \( G \) are equal to \( \pm 1 \). By the Cayley–Hamilton theorem, \( \Delta_G(t) \in \text{Ann}(M_\mathbb{Q}) \). Therefore \( \Delta_G(t) \in \text{Ann}(M) \) and the module \( M \) is finitely generated over \( \mathbb{Z} \) by Proposition 1.15. The theorem is proved.

**Remark 3.18.** Let \( G \) be a \( C \)-group given by a \( C \)-presentation \( G = \langle x_1, \ldots, x_m \mid r_1, \ldots, r_n \rangle \) and let \( A(G) \) be its Alexander matrix. Then the Alexander polynomial \( \Delta_G(t) \) coincides (up to multiplication by \( \pm t^k \)) with the greatest common divisor of the \( (m-1) \)-rowed minors of \( A(G) \).
3.6. Alexander modules of some irreducible C-groups. At the end of this section we compute the Alexander modules for some irreducible C-groups.

Example 3.19. The Alexander module $A_0(Br_{m+1})$ of the braid group $Br_{m+1}$ is trivial if $m \geq 4$ (or $m = 1$) and is isomorphic to $\Lambda/(t^2 - t + 1)$ for $m = 2, 3$.

This statement is well known, but we give a proof for completeness.

Proof. The braid group $Br_{m+1}$ is given by the presentation

$$Br_{m+1} = \langle x_1, \ldots, x_m \mid [x_i, x_j] \text{ for } |i - j| \geq 2, x_i x_{i+1} x_i x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1} \text{ for } i = 1, \ldots, m - 1 \rangle.$$  

We note that this is a $C$-presentation of an irreducible $C$-group.

By Proposition 3.5, to calculate $A_0(Br_{m+1})$ we must find the matrix $\bar{A}(Br_{m+1})$.

The relations $[x_m, x_i]$ $(i = 1, \ldots, m - 2)$ yield the rows

$$\begin{bmatrix} 0, \ldots, 0, t - 1, 0, \ldots, 0 \end{bmatrix}, \tag{20}$$

where $t - 1$ occupies the $i$th position for $i = 1, \ldots, m - 2$. If $m \geq 4$, then the relator $[x_{m-1}, x_1]$ yields the row

$$\begin{bmatrix} t - 1, 0, \ldots, 0, 1 - t \end{bmatrix}. \tag{21}$$

If $m \geq 4$, then the rows (20) and the row (21) generate the submodule $(t - 1)\Lambda^{m-1}$ of $\Lambda^{m-1}$. On the other hand, these rows belong to the module $M(Br_{m+1})$. It follows that $A_0(Br_{m+1}) = 0$ since $A_0(Br_{m+1}) \simeq \Lambda^{m-1}/M(Br_{m+1})$ is a Noetherian $(t - 1)$-invertible $\Lambda$-module and $(t - 1)\Lambda^{m-1} \subset M(Br_{m+1})$.

If $m = 2$, the presentation of $Br_3$ contains only one relator

$$r := x_1 x_2 x_1 x_2^{-1} x_1^{-1} x_2^{-1}.$$  

We have $\nu_* \left( \frac{\partial r}{\partial x_1} \right) = 1 + t^2 - t$ and, therefore, $A_0(Br_3) \simeq \Lambda/(t^2 - t + 1)$.

If $m = 3$, the presentation of $Br_4$ contains only three relators:

$$r_1 := x_1 x_2 x_1 x_2^{-1} x_1^{-1} x_2^{-1},$$  

$$r_2 := x_2 x_3 x_2 x_3^{-1} x_2^{-1} x_3^{-1},$$  

$$r_3 := x_1 x_3 x_1^{-1} x_3^{-1}.$$  

We have

$$\nu_* \left( \frac{\partial r_1}{\partial x_1} \right) = -\nu_* \left( \frac{\partial r_1}{\partial x_2} \right) = \nu_* \left( \frac{\partial r_2}{\partial x_2} \right) = t^2 - t + 1,$$

$$\nu_* \left( \frac{\partial r_3}{\partial x_1} \right) = 1 - t.$$  

Therefore the module $M(Br_3) \subset \Lambda^2$ is generated by the vectors

$$v_1 = (t^2 - t + 1, -(t^2 - t + 1)), \quad v_2 = (0, t^2 - t + 1), \quad v_3 = (1 - t, 0).$$

Hence $A_0(Br_3) \simeq \Lambda/(t^2 - t + 1)$. 


Example 3.20. The Alexander module of the $C$-group

$$G_m = \langle x_1, x_2 \mid (x_1^{-1} x_2)^m x_1 (x_1^{-1} x_2)^{-m} x_2^{-1} \rangle, \quad m \in \mathbb{N},$$

is isomorphic to $A_0(G) \simeq \Lambda/\langle (m+1)t-m \rangle$.

These irreducible $C$-groups are interesting because they are non-Hopfian for $m \geq 2$ and, therefore, they are not residually finite. The group $G_m$ is isomorphic to the Baumslag–Solitar group $\langle a, x_1 \mid x_1^{-1} a^m x_1 a^{-(m+1)} \rangle$ (see [18]) if we put $x_2 = x_1 a$. We also note that, by Corollary 3.13, each of these groups can be realized as $\pi_1(S^4 \setminus S^2)$ for some knotted sphere $S^2 \subset S^4$.

Proof. Straightforward calculations show that

$$\nu_* \left( \frac{\partial r}{\partial x_1} \right) = -mt^{-1} + m + 1,$$

where $r := (x_1^{-1} x_2)^m x_1 (x_1^{-1} x_2)^{-m} x_2^{-1}$. Therefore the Alexander module $A_0(G)$ is isomorphic to $\Lambda/\langle (m+1)t-m \rangle$.

§ 4. The first homology groups of cyclic coverings

4.1. Proofs of Theorems 0.2 and 0.5. We prove Theorems 0.2 and 0.5 simultaneously.

In the notation of the introduction, let $X$ be either the sphere $S^{n+2}$ (case I) or the projective plane $\mathbb{CP}^2$ (case II), and let $X'$ be either the complement of a knotted $n$-manifold $V$ in $S^{n+2}$ or the complement of the union of an irreducible Hurwitz curve $H$ and a line $L$ ‘at infinity’ in $\mathbb{CP}^2$. We recall that the fundamental group $G = \pi_1(X')$ is an irreducible $C$-group.

Consider the infinite cyclic covering $f = f_\infty: X_\infty \to X'$ corresponding to the $C$-epimorphism $\nu: G \to \mathbb{F}_1$ with $\ker \nu = G'$. Let $h \in \text{Deck}(X_\infty/X') \simeq \mathbb{F}_1$ be the covering transformation corresponding to a $C$-generator $x \in \mathbb{F}_1$. We regard $X'$ as the quotient space $X' = X_\infty/\mathbb{F}_1$. In such a situation Milnor [19] considered an exact sequence of chain complexes

$$0 \to C.(X_\infty) \xrightarrow{h-\text{id}} C.(X_\infty) \xrightarrow{f_*} C.(X') \to 0,$$

which gives an exact sequence of homology groups with integer coefficients:

$$\cdots \to H_1(X_\infty) \xrightarrow{t-\text{id}} H_1(X_\infty) \xrightarrow{f_*} H_1(X') \xrightarrow{\partial} H_0(X_\infty) \to 0, \quad (22)$$

where $t = h_*$. The action $h_*$ endows each group $H_i(X_\infty)$ with the structure of a $\Lambda$-module, and $H_1(X_\infty) \simeq G'/G''$ is the Alexander module of the $C$-group $G$. If we regard the $H_1(X')$ as $\Lambda$-modules with the trivial action of $t$, then (22) becomes an exact sequence of $\Lambda$-modules. We also note that the action of $t \in \Lambda$ on $H_0(X_\infty) \simeq \mathbb{Z}$ is trivial, that is, $t$ is the identity automorphism of $H_0(X_\infty)$.

Let $(h^k) \subset \mathbb{F}_1$ be the infinite cyclic group generated by the element $h^k$. Then we can regard the manifold $X'_k$ as a quotient manifold: $X'_k = X_\infty/\langle h^k \rangle$. Moreover,
we have $X' = X'_k/\mu_k$, where $\mu_k = \mathbb{F}_1/\langle h^k \rangle$ is the cyclic group of order $k$. We write \( h_k \) for the automorphism of $X'_k$ induced by $h$. Then $h_k$ is a generator of the covering transformation group $\text{Deck}(X'_k/X') = \mu_k$ acting on $X'_k$.

In case I it is easy to see that the manifold $X'_k$ can be embedded in a compact smooth manifold $X_k$ with the following properties.

(i) The action of $h_k$ on $X'_k$ and the map $f'_k: X'_k \to X'$ extend to give an action (again denoted by $h_k$) on $X_k$ and a smooth map

$$f_k: X_k \to X \cong X_k/\langle h_k \rangle.$$ 

(ii) The set of fixed points of $h_k$ coincides with $f_k^{-1}(V) = \bar{V}$. The restriction $f_k|_V: \bar{V} \to V$ of $f_k$ to $\bar{V}$ is a smooth isomorphism.

In case II, the covering $f'_k$ can be extended to a map $\tilde{f}_{k,\text{norm}}: \tilde{X}_{k,\text{norm}} \to X$ which is branched along $H$ and possibly along $L$, where the variety $\tilde{X}_{k,\text{norm}}$ is a singular analytic variety near its finitely many singular points. Let $\sigma: \tilde{X}_k \to \tilde{X}_{k,\text{norm}}$ be a resolution of these singularities, $E = \sigma^{-1}(\text{Sing} \tilde{X}_{k,\text{norm}})$ the proper transform of the set of singular points of $\tilde{X}_{k,\text{norm}}$, and $\tilde{f}_k = \tilde{f}_{k,\text{norm}} \circ \sigma$. We denote the proper transforms of $H$ and $L$ by $R = \tilde{f}_{k,\text{norm}}^{-1}(H)$ and $R_\infty = \tilde{f}_{k,\text{norm}}^{-1}(L)$. The restriction of $\tilde{f}_{k,\text{norm}}$ to $R$ is one-to-one. The restriction of $\tilde{f}_{k,\text{norm}}$ to $R_\infty$ is a $k_0$-sheeted cyclic covering, where $k_0 = \text{GCD}(k, m)$ and the ramification index of $\tilde{f}_{k,\text{norm}}$ along $R_\infty$ is equal to $\kappa_\infty = \frac{k}{k_0}$. As in the algebraic case, we easily see that $R_\infty$ is irreducible.

We denote the proper transform of $R$ by $\bar{R} = \sigma^{-1}(R)$. Note that $k_0$ divides $m$. If we put $m_0 = \frac{m}{k_0}$, then $m_0 \in \mathbb{N}$.

We write $X_k = \tilde{X}_k \setminus E$ for the non-singular part of $\tilde{X}_{k,\text{norm}}$. We have embeddings $i_k: X'_k \hookrightarrow X_k$ and $j_k: X_k \hookrightarrow \tilde{X}_k$.

In cases I and II, the action of $h_k$ on $X_k$ endows the group $H_1(X_k, \mathbb{Z})$ (resp. $H_1(X'_k, \mathbb{Z})$) with the structure of a $\Lambda$-module such that the homomorphism

$$i_k*: H_1(X'_k, \mathbb{Z}) \to H_1(X_k, \mathbb{Z}),$$

induced by the embedding $i: X'_k \hookrightarrow X_k$, is a $\Lambda$-homomorphism. Clearly, the homomorphism $i_k*$ is epimorphic.

In case I let $S \subset X_k$ be a germ of a smooth surface that is transversal to $\bar{V}$ at the point $p \in \bar{V}$, and let $\gamma \subset S$ be a small circle centred at $p$. Since $\bar{V}$ is a smooth connected submanifold of codimension 2 in $X_k$, we see that $\ker i_k*$ is generated by the homology class $[\gamma] \in H_1(X'_k, \mathbb{Z})$ that contains the cycle $\gamma$.

It is clear that $t([\gamma]) = [\gamma]$, where $t = h_{k,*}$. Moreover,

$$f_{k,*}([\gamma]) = \pm k[\gamma] \in H_1(X', \mathbb{Z}) \cong \mathbb{Z},$$

where $[\gamma]$ is the generator of $H_1(X', \mathbb{Z})$ represented by a simple loop $\gamma$ around $V$ lying in a surface transversal to $\bar{V}$.

In case II let $S \subset X_k$ be a germ of a smooth surface that is transversal to $R$ at the point $p \in R$, and let $\gamma \subset S$ be a small circle centred at $p$. Clearly, the homology class $[\gamma] \in H_1(X'_k, \mathbb{Z})$ is invariant under multiplication by $t$ and we have $f_{k,*}([\gamma]) = k[\gamma]$, where $[\gamma]$ is the generator of $H_1(\mathbb{CP}^2 \setminus (H \cup L), \mathbb{Z}) \cong \mathbb{Z}$. 


We similarly let $L_1 \subset \mathbb{CP}^2$ be a complex line transversal to $L$ at the point $q \in L \setminus H$, and let $\gamma_\infty$ be a simple small loop around $q$ lying in $L_1$. Then $f_{k-1}^{-1}(\gamma_\infty)$ splits into a disjoint union of $k_0$ simple loops $\gamma_\infty,i$, $i = 1, \ldots, k_0$. Since $R_\infty$ is irreducible, any two loops $\gamma_\infty,i$ and $\gamma_\infty,j$ belong to the same homology class in $H_1(X', \mathbb{Z})$ (to be denoted by $[\gamma_\infty]$). It is easy to see that $t(\gamma_\infty,i) = \gamma_\infty,i+1$. Therefore the homology class $[\gamma_\infty] \in H_1(X', \mathbb{Z})$ is invariant under multiplication by $t$. We also note that $f_{k+1}([\gamma_\infty]) = k_0m[\gamma] = km_0[\gamma]$ because $[\gamma_\infty] = m[\gamma]$.

**Lemma 4.1.** The $\Lambda$-module $H_1(X'_k, \mathbb{Z})$ is isomorphic to

$$A_k(G) \oplus H_1(X'_k)_1 \simeq A_k(G) \oplus \mathbb{Z},$$

where $A_k(G)$ is the $k$th derived Alexander module of the $G$-group $G$ and

$$H_1(X'_k)_1 = \{ h \in H_1(X'_k, \mathbb{Z}) \mid (t-1)h = 0 \}.$$

**Proof.** To calculate the groups $H_1(X'_k, \mathbb{Z})$, we use the exact sequence

$$\cdots \to H_1(X_\infty, \mathbb{Z}) \xrightarrow{t^k \cdot \text{id}} H_1(X_\infty, \mathbb{Z}) \xrightarrow{g_k} H_1(X'_k, \mathbb{Z}) \xrightarrow{\partial} H_0(X_\infty, \mathbb{Z}) \to 0 \quad (23)$$

for the infinite cyclic covering $g_k = g_{\infty,k} : X_\infty \to X'_k$. This sequence is constructed similarly to (22).

Using (23), we get the short exact sequence

$$0 \to H_1(X_\infty)/(t^k-1)H_1(X_\infty) \xrightarrow{g_k} H_1(X'_k) \xrightarrow{\partial} H_0(X_\infty) \to 0,$$

which is a sequence of $\Lambda$-modules.

We introduce the notation

$$M_1 = \ker \partial = \text{im } g_{k+1} \simeq H_1(X_\infty)/(t^k-1)H_1(X_\infty)$$

and

$$M_2 = H_1(X'_k)_1.$$  

We have $H_0(X_\infty, \mathbb{Z}) \simeq \mathbb{Z}$. Choose a generator $u \in H_0(X_\infty, \mathbb{Z})$ and let $v_1 \in H_1(X'_k, \mathbb{Z})$ be an element with $\partial(v_1) = u$. Then we have $(t-1)v_1 \in \ker \partial$ because $H_0(X_\infty, \mathbb{Z})$ is a trivial $\Lambda$-module and $\partial$ is a $\Lambda$-homomorphism. We fix such an element $v_1$.

By Theorems 0.1 and 0.3, $H_1(X_\infty, \mathbb{Z}) = A_0(G)$ is a Noetherian $(t-1)$-invertible $\Lambda$-module. Therefore, by Proposition 1.6,

$$M_1 \simeq H_1(X_\infty)/(t^k-1)H_1(X_\infty) = A_k(G)$$

is also a Noetherian $(t-1)$-invertible $\Lambda$-module and Theorem 1.10 yields a polynomial $g_1(t) \in \text{Ann}(M_1)$ with $g_1(1) = 1$. We fix such a polynomial $g_1(t)$.

Consider the element $\bar{v}_1 = g_1(t)v_1$. Since $\partial(\bar{v}_1) = g_1(1)u = u$, we have

$$(t-1)\bar{v}_1 = (t-1)g_1(t)v_1 = g_1(t)(t-1)v_1 = 0$$

because $(t-1)v_1 \in M_1$. It follows that $\bar{v}_1 \in M_2$.

We note that $M_1 \cap M_2 = 0$ since $M_1$ is $(t-1)$-invertible. Therefore $\partial$ maps $M_2$ isomorphically onto $H_0(X_\infty, \mathbb{Z})$, that is, the exact sequence (24) splits and we have $H_1(X'_k, \mathbb{Z}) \simeq M_1 \oplus M_2$. The lemma is proved.
Lemma 4.2. The homomorphism $f_{k*} : H_1(X'_k, \mathbb{Z}) \to H_0(X', \mathbb{Z})$ has the following properties:

(i) $\ker f_{k*} = A_k(G) \subset H_1(X'_k, \mathbb{Z})$,

(ii) $\text{im } f_{k*} = k\mathbb{Z} \subset \mathbb{Z} \simeq H_1(X', \mathbb{Z})$ and the restriction of $f_{k*}$ to $H_1(X'_k)_1$ is an isomorphism of $H_1(X'_k)_1$ onto its image.

Proof. The group $H_1(X', \mathbb{Z})$ is isomorphic to $G/G' \simeq \mathbb{Z}$. Similarly, the group $H_1(X'_k, \mathbb{Z})$ is isomorphic to $G_k/G'_k$, where $G_k = \ker \nu_k$,

$$\nu_k = \text{mod}_k \circ \nu : G \to \mu_k = \mathbb{Z}/\langle h^k \rangle,$$

and $f_{k*} : H_1(X'_k, \mathbb{Z}) \to H_1(X', \mathbb{Z})$ coincides with the homomorphism

$$i_{k*} : G_k/G'_k \to G/G'$$

induced by the embedding $i_k : G_k \hookrightarrow G$.

Let the $C$-group $G$ be given by a $C$-presentation (7). To describe $\ker i_{k*}$ and $\text{im } i_{k*}$, we again consider the two-dimensional complex $K$ described in §3.1. The complex $K$ has a single vertex $x_0$. Its one-dimensional skeleton is a wedge product of oriented circles $s_j$ ($1 \leq j \leq m$) corresponding to the generators of $G$ appearing in (7), and $K \setminus (\bigcup s_i) = \bigcup_{j=1}^l D_j$ is a disjoint union of open discs. For each $j = 1, \ldots, l$, $\overline{D}_j$ corresponds to the relator $r_j$ appearing in (7). Here $l$ is the number of relators $r_i$ in the presentation (7).

The embedding $i_k : G_k \hookrightarrow G$ of groups determines an unramified covering $f_k : K_k \to K$, where $K_k$ is a two-dimensional complex with $k$ vertices $p_1, \ldots, p_k$, $f_k(p_j) = x_0$. The proper transform $f^{-1}(s_j) = \bigcup_{s=1}^k \overline{s}_{j,s}$ of each edge $s_j$ is a disjoint union of $k$ edges $\overline{s}_{j,s}$, $1 \leq s \leq k$. The proper transform $f^{-1}(\overline{D}_j) = \bigcup_{s=1}^k \overline{D}_{j,s}$ of each disc $\overline{D}_j$ is a disjoint union of $k$ open discs $\overline{D}_{j,s}, 1 \leq s \leq k$.

Let $h_k$ be a generator of the covering transformation group $\text{Deck}(K_k/K) = \mu_k$ acting on $K_k$. The homeomorphism $h_k$ induces an action $h_{k*}$ on the chain complex $C.(K_k)$ and an action $t$ on the groups $H_1(K_k, \mathbb{Z})$ endowing each of these groups with the structure of a $\Lambda$-module. This structure on $H_1(K_k, \mathbb{Z})$ coincides with the structure on $H_1(X'_k, \mathbb{Z})$ defined above if we identify the groups $H_1(K_k, \mathbb{Z})$ and $H_1(X'_k, \mathbb{Z})$ by means of the isomorphisms $H_1(K_k, \mathbb{Z}) \simeq G_k/G'_k$ and $H_1(X'_k, \mathbb{Z}) \simeq G_k/G'_k$.

Consider the sequence of chain complexes

$$C.(K_k) \xrightarrow{h_{k*} - \text{id}} C.(K_k) \xrightarrow{f_{k*}} C.(K) \to 0.$$

We easily see that $\text{im } (h_{k*} - \text{id}) = \ker f_{k*}$ and

$$\ker(h_{k*} - \text{id}) = \left( \sum_{j=0}^{k-1} h_{k*}^j \right) C.(K_k).$$

The proof of the lemma now follows from the exact sequence

$$\cdots \to H_1(C.(K_k/\ker(h_{k*} - \text{id}))) \xrightarrow{t^{k-1}} H_1(K_k) \xrightarrow{f_{k*}} H_1(K) \xrightarrow{\text{id}} H_0(C.(K_k/\ker(h_{k*} - \text{id}))) \xrightarrow{t^{k-1}} H_0(K_k) \xrightarrow{f_{k*}} H_0(K) \to 0 \quad (25)$$
This follows from Theorems 0.2 and 0.3. Corollary 4.5 follows from Theorems 2.14 and 2.11. Following lemma yields Theorem 4.5.

Lemma 4.3 [6]. The homomorphism \( j_{k*} : H_1(X_k, \mathbb{Q}) \to H_1(\overline{X}_k, \mathbb{Q}) \) is an isomorphism.

4.2. Corollaries of Theorems 0.2 and 0.5.

Corollary 4.4. Let \( V \) be a knotted \( n \)-manifold with \( n \geq 1 \) and let \( f_k : X_k \to S^{n+2} \) be the cyclic covering of degree \( k \) branched along \( V \). Then the following assertions hold.

(i) The first Betti number \( b_1(X_k) \) of \( X_k \) is even.

(ii) If \( k = p^r \), where \( p \) is a prime, then the group \( H_1(X_k, \mathbb{Z}) \) is finite.

(iii) A finitely generated abelian group \( G \) can be realized as \( H_1(X_k, \mathbb{Z}) \) for some knotted \( n \)-manifold \( V \) with \( n \geq 2 \) if and only if there is an automorphism \( h \in \text{Aut}(G) \) such that \( h^k = \text{Id} \) and \( h - \text{Id} \) is again an automorphism of \( G \). In particular, \( H_1(X_2, \mathbb{Z}) \) is a finite abelian group of odd order, and every finite abelian group \( G \) of odd order can be realized as \( H_1(X_2, \mathbb{Z}) \) for some knotted \( n \)-sphere for \( n \geq 2 \).

Proof. This follows from Theorems 0.1, 0.2, 2.11, Proposition 2.13, Corollary 3.13 and Examples 2.14, 3.20.

Corollary 0.4 follows from Theorems 0.3 and 2.10.

Corollary 0.6 follows immediately from Lemma 4.3 and Corollary 4.5 since the homomorphism \( j_{k*} : H_1(X_k, \mathbb{Z}) \to H_1(\overline{X}_k, \mathbb{Z}) \) is an isomorphism and we have \( H_1(\overline{X}_k, \mathbb{Q}) \simeq A_k(H) \otimes \mathbb{Q} \).

Corollary 4.5. Suppose that \( H \) is an algebraic (resp. Hurwitz or pseudo-holomorphic) irreducible curve in \( \mathbb{C}P^2 \), \( \deg H = m \) and \( f_k : \overline{X}_k \to \mathbb{C}P^2 \) is a resolution of singularities of the cyclic covering of degree \( k \) branched along \( H \) and possibly along the line \( L \) ‘at infinity’. Put \( X_k = \overline{X}_k \setminus E \), where \( E \) is the proper transform of the set of singular points of the cyclic covering. Then the following assertions hold.
(i) The sequence of groups $H_1(X_1, \mathbb{Z}), \ldots, H_1(X_k, \mathbb{Z}), \ldots$ has period $m$, that is, $H_1(X_k, \mathbb{Z}) \simeq H_1(X_{k+m}, \mathbb{Z})$.

(ii) The first Betti number $b_1(\mathbb{X}_k)$ is equal to the number $r_k, \neq 1$ of roots of the Alexander polynomial $\Delta(t)$ of the curve $H$ which are $k$th roots of unity not equal to 1. In particular, $b_1(\mathbb{X}_k)$ is even.

(iii) If $k = p^r$, where $p$ is a prime, then the groups $H_1(X_k, \mathbb{Z})$ and $H_1(\mathbb{X}_k, \mathbb{Z})$ are finite.

(iv) If $k$ and $m$ are coprime, then $H_1(\mathbb{X}_k, \mathbb{Z}) = 0$.

(v) A finitely generated abelian group $G$ can be realized as $H_1(X_k, \mathbb{Z})$ for some Hurwitz (resp. pseudo-holomorphic) curve $H$ if and only if there is an automorphism $h \in \text{Aut}(G)$ such that $h^d = \text{Id}$ for some divisor $d$ of $k$ and $h - \text{Id}$ is again an automorphism of $G$. Moreover, if the group $G$ is realized as $H_1(X_k, \mathbb{Z})$ for some curve $H$, then $d$ divides $\deg H$. In particular, $H_1(\mathbb{X}_2, \mathbb{Z})$ is a finite abelian group of odd order, and every finite abelian group $G$ of odd order can be realized as $H_1(X_2, \mathbb{Z})$ for some Hurwitz (resp. pseudo-holomorphic) curve $H$ of even degree.

**Proof.** This follows from Theorems 0.3, 0.5, 2.11, 2.16 and Propositions 2.13, 2.15.

We note that there are plane algebraic curves $H$ such that the homomorphisms $j_{k*}: H_1(X_k, \mathbb{Z}) \to H_1(\mathbb{X}_k, \mathbb{Z})$ are not isomorphisms.

**Example 4.6.** Let $H \subset \mathbb{CP}^2$ be the curve of degree 6 given by the equation

$$Q^3(z_0, z_1, z_2) + C^2(z_0, z_1, z_2) = 0,$$

where $Q$ and $C$ are homogeneous forms of degrees $\deg Q = 2$, $\deg C = 3$ such that the conic $Q = 0$ and the cubic $C = 0$ meet each other transversally at 6 points. Then we have $A_2(H) \simeq \mathbb{Z}/3\mathbb{Z}$ but $H_1(\mathbb{X}_2, \mathbb{Z}) = 0$.

**Proof.** The curve $H$ has six singular points (cusps) lying on the conic $Q = 0$. It is known (\cite{20}, see also \cite{21}) that $\pi_1(\mathbb{CP}^2 \setminus (H \cup L)) \simeq \text{Br}_3$ as a $C$-group. Therefore $A_2(H) \simeq \mathbb{Z}/3\mathbb{Z}$ (see Examples 2.17 and 3.19). It is also well known that the minimal resolution of singularities of the two-sheeted covering of $\mathbb{CP}^2$ branched along $H$ is a K3-surface, which is simply connected.

We note that the sequence of homology groups $H_1(X_k, \mathbb{Z}), k \in \mathbb{N}$, need not be periodic in the case of knotted $n$-manifolds $V \subset S^{n+2}$. For example, let $S^2 \subset S^4$ be a knotted sphere with $\pi_1(S^4 \setminus S^2) \simeq G_m$, where $G_m$ is the group studied in Example 3.20. (Corollary 3.13 shows that this group can be realized as the group of a knotted sphere.) Then $H_1(X_k, \mathbb{Z})$ is a cyclic group of order $(m + 1)^k - m^k$ (see Example 2.14).

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