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Alexander modules of irreducible C-groups

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Abstract. We give a complete description of the Alexander modules of knotted n-manifolds in the sphere S^{n+2} for $n \geqslant 2$ and the Alexander modules of irreducible Hurwitz curves. This description is applied to investigate the properties of the first homology groups of cyclic coverings of the sphere S^{n+2} and the complex projective plane $\mathbb{CP}^2$ branched respectively along knotted n-manifolds and irreducible Hurwitz (in particular, algebraic) curves.

Introduction

The class $\mathcal{C}$ of C-groups and the subclass $\mathcal{H}$ of Hurwitz C-groups (see the definitions below) play a very important role in the geometry of submanifolds of codimension two. For example, it is well known that knot and link groups (given by their Wirtinger presentations) are C-groups and any C-group G may be realized as the group of a *linked* n-manifold with $n \geqslant 2$, that is, as the fundamental group $\pi_1(S^{n+2} \setminus V)$ of the complement of a closed oriented n-manifold V (without boundary) in the $(n+2)$-dimensional sphere S^{n+2} (see [1]) and conversely. Note that a C-group G is isomorphic to $\pi_1(S^{n+2} \setminus \bigsqcup S^n)$ for some union $\bigsqcup S^n$ of linked n-dimensional spheres with $n \geqslant 3$ if and only if $H_2 G = 0$ (see [2]). Some other results describing the groups $\pi_1(S^{n+2} \setminus \bigsqcup S^n)$ can be found in [3] and [4].

Let $H \subset \mathbb{CP}^2$ be an algebraic (or, more generally, Hurwitz[1]) curve or a pseudo-holomorphic curve with respect to some ω-tamed almost complex structure on $\mathbb{CP}^2$, where ω is the Fubini–Study symplectic form on $\mathbb{CP}^2$, $\deg H = m$. The Zariski–van Kampen presentation of $\pi_1 = \pi_1\bigl(\mathbb{CP}^2 \setminus (H \cup L)\bigr)$ endows π_1 with the structure of a Hurwitz C-group of degree m, where L is the line 'at infinity'. (In other words, L is a fibre of the linear projection $\mathrm{pr}\colon \mathbb{CP}^2 \to \mathbb{CP}^1$ and is in general position with respect to H. If H is a pseudo-holomorphic curve, then the projection pr is determined by a pencil of pseudo-holomorphic lines.) As proved in [7], every Hurwitz C-group G of degree m can be realized as the fundamental group $\pi_1\bigl(\mathbb{CP}^2 \setminus (H \cup L)\bigr)$ for some Hurwitz (resp. pseudo-holomorphic) curve H, $\deg H = 2^n m$, with singularities of the form $w^m - z^m = 0$, where n depends on the Hurwitz C-representation of G. Thus the class $\mathcal{H}$ coincides with the class $\bigl\{\pi_1\bigl(\mathbb{CP}^2 \setminus (H \cup L)\bigr)\bigr\}$ of fundamental groups of complements of 'affine' Hurwitz (resp. 'affine' pseudo-holomorphic)

[1]The definition of Hurwitz curves can be found in [5] or [6].

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curves. It contains the subclass of fundamental groups of complements of plane affine algebraic curves.

By definition, a *C-group* is a group together with a finite presentation

$$G_W = \langle x_1, \dots, x_m \mid x_i = w_{i,j,k}^{-1} x_j w_{i,j,k}, w_{i,j,k} \in W \rangle, \tag{1}$$

where $W = \{w_{i,j,k} \in \mathbb{F}_m \mid 1 \leqslant i, j \leqslant m,\ 1 \leqslant k \leqslant h(i,j)\}$ consists of elements of the free group $\mathbb{F}_m$ freely generated by $x_1, \dots, x_m$ (it is possible that $w_{i_1,j_1,k_1} = w_{i_2,j_2,k_2}$ for $(i_1, j_1, k_1) \neq (i_2, j_2, k_2)$) and $h\colon \{1, \dots, m\}^2 \to \mathbb{Z}$ is some function. Such a presentation is called a *C-presentation* (C means that all the relations are given by conjugations). Let $\varphi_W : \mathbb{F}_m \to G_W$ be the canonical epimorphism. The elements $\varphi_W(x_i) \in G$, $1 \leqslant i \leqslant m$, and their conjugates are called *C-generators* of G. Let $f\colon G_1 \to G_2$ be a homomorphism of C-groups. It is called a *C-homomorphism* if the images of the C-generators of G_1 under f are C-generators of G_2. C-groups are considered up to C-isomorphism. Some properties of C-groups were investigated in [7]–[10].

A C-presentation (1) is called a *Hurwitz C-presentation of degree m* if the word $w_{i,i,1}$ coincides with the product $x_1 \dots x_m$ for every $i = 1, \dots, m$. A C-group G is called a *Hurwitz C-group* (*of degree m*) if there is an $m \in \mathbb{N}$ such that G possesses a Hurwitz C-presentation of degree m. In other words, a C-group G is a Hurwitz C-group of degree m if there are C-generators $x_1, \dots, x_m$ generating G such that the product $x_1 \dots x_m$ belongs to the centre of G. Note that the degree of a Hurwitz C-group G is not canonically defined: it depends on the Hurwitz C-presentation of G. We denote the class of all Hurwitz C-groups by $\mathcal{H}$.

It is easy to show that G/G' is a finitely generated free abelian group for any C-group G. Here $G' = [G, G]$ is the commutator subgroup of G. A C-group G is said to be *irreducible* if $G/G' \simeq \mathbb{Z}$. We say that G consists of *k irreducible components* if $G/G' \simeq \mathbb{Z}^k$. If G is a Hurwitz C-group realized as the fundamental group $\pi_1\big(\mathbb{CP}^2 \setminus (H \cup L)\big)$ of the complement of some Hurwitz curve H, then the number of irreducible components of G is equal to the number of irreducible components of H. Similarly, if a C-group G consisting of k irreducible components is realized as the group of a linked n-manifold V (that is, $G = \pi_1(S^{n+2} \setminus V)$), then the number of connected components of V is equal to k.

The free group $\mathbb{F}_n$ with fixed free generators is a C-group. For any C-group G we have a well-defined canonical C-epimorphism $\nu\colon G \to \mathbb{F}_1$ sending the C-generators of G to the C-generator of $\mathbb{F}_1$. We denote its kernel by N. Note that if G is an irreducible C-group, then N coincides with G'. In what follows we assume that all C-groups under consideration are irreducible.

Let G be an irreducible C-group. The C-epimorphism ν induces the exact sequence of groups

$$1 \to G'/G'' \to G/G'' \xrightarrow{\nu_*} \mathbb{F}_1 \to 1,$$

where $G'' = [G', G']$. The C-generator of $\mathbb{F}_1$ acts on G'/G'' by conjugation: $\tilde{x}^{-1} g \tilde{x}$, where $g \in G'$ and $\tilde{x}$ is one of the C-generators of G. We denote this action by t. The group $A_0(G) = G'/G''$ is abelian and the action t endows $A_0(G)$ with the structure of a Λ-module, where $\Lambda = \mathbb{Z}[t, t^{-1}]$ is the ring of Laurent polynomials with integer coefficients. The Λ-module $A_0(G)$ is called the *Alexander module* of the C-group G. The action t induces an action $h_{\mathbb{C}}$ on $A_{\mathbb{C}} = A_0(G)\otimes\mathbb{C}$, and it is easy to see that its characteristic polynomial $\det(h_{\mathbb{C}} - t\,\mathrm{Id})$ belongs to $\mathbb{Q}[t]$. Let $a \in \mathbb{N}$ be the smallest positive integer such that $a\det(h_{\mathbb{C}} - t\,\mathrm{Id}) \in \mathbb{Z}[t]$. The polynomial $\Delta(t) = a\det(h_{\mathbb{C}} - t\,\mathrm{Id})$ is called the *Alexander polynomial* of the C-group G. If H is an algebraic, or Hurwitz, or pseudo-holomorphic irreducible curve in $\mathbb{CP}^2$ (resp. if $V \subset S^{n+2}$ is a *knotted* (that is, connected smooth oriented without boundary) n-manifold, $n \geqslant 1$) and we put $G = \pi_1\bigl(\mathbb{CP}^2 \setminus (H \cup L)\bigr)$ (resp. $G = \pi_1(S^{n+2} \setminus V)$), then the Alexander module $A_0(G)$ of the group G and its Alexander polynomial $\Delta(t)$ are called the *Alexander module* and *Alexander polynomial* of the curve H (resp. of the knotted manifold V). We note that the Alexander module $A_0(H)$ and the Alexander polynomial $\Delta(t)$ of the curve H are independent of the choice of the generic (pseudo-holomorphic) line L. Some results concerning Alexander modules of knotted spheres are to be found in [11], [12].

Properties of Alexander polynomials of Hurwitz curves were studied in [6] and [13]. In particular, it was proved that if H is an irreducible Hurwitz curve of degree m, then its Alexander polynomial $\Delta(t)$ has the following properties:

(i) $\Delta(t) \in \mathbb{Z}[t]$ and $\deg\Delta(t)$ is even,

(ii) $\Delta(0) = \Delta(1) = 1$,

(iii) $\Delta(t)$ divides the polynomial $(t^m - 1)^{m-2}$.

Moreover, a polynomial $P(t) \in \mathbb{Z}[t]$ is the Alexander polynomial of an irreducible Hurwitz curve if and only if the roots of $P(t)$ are roots of unity and $P(1) = 1$.

Suppose that $G = \pi_1\bigl(\mathbb{CP}^2\setminus(H\cup L)\bigr)$ is the fundamental group of the complement of an irreducible 'affine' Hurwitz curve of degree m (resp. $G = \pi_1(S^{n+2} \setminus V)$ is the group of a knotted n-manifold, $n \geqslant 1$). The homomorphism $\nu\colon G \to \mathbb{F}_1$ determines an infinite unramified cyclic covering $f_\infty\colon X_\infty \to \mathbb{CP}^2 \setminus (H \cup L)$ (resp. $f_\infty\colon X_\infty \to S^{n+2} \setminus V$). We have $H_1(X_\infty, \mathbb{Z}) = G'/G''$ and the action of t on $H_1(X_\infty, \mathbb{Z})$ coincides with the action of a generator h of the covering transformation group of the covering f_∞.

For every $k \in \mathbb{N}$ let $\mathrm{mod}_k\colon \mathbb{F}_1 \to \mu_k = \mathbb{F}_1/\langle t^k\rangle$ be the natural epimorphism to the cyclic group μ_k of order k. The covering f_∞ factors through the cyclic covering $f'_k\colon X'_k \to \mathbb{CP}^2 \setminus (H \cup L)$ (resp. $f'_k\colon X'_k \to S^{n+2} \setminus V$) associated with the epimorphism $\mathrm{mod}_k \circ \nu$, $f_\infty = f'_k \circ g_k$. Since any Hurwitz curve H has only analytic singularities, the covering f'_k can be extended (see [6]) to a map $\widetilde{f}_k\colon \widetilde{X}_k \to X$ branched along H and possibly along L. Here $\widetilde{X}_k$ stands for a closed four-dimensional variety which is locally isomorphic over each singular point of H to the complex-analytic singularity given by $w^k = F(u, v)$, where $F(u, v) = 0$ is a local equation of H at the singular point. Over a neighbourhood of every common point of H and L, the variety $\widetilde{X}_k$ is locally isomorphic to the singularity given locally by $w^k = vu^d$, where d is the smallest non-negative integer such that $m + d$ is divisible by k. If $\tilde{f}_k^{-1}(L) \subset \mathrm{Sing}\,\widetilde{X}_k$, then $\widetilde{X}_k$ can be normalized (as in the algebraic case) and we obtain a covering $\tilde{f}_{k,\mathrm{norm}}\colon \widetilde{X}_{k,\mathrm{norm}} \to \mathbb{CP}^2$, where $\widetilde{X}_{k,\mathrm{norm}}$ is a singular analytic variety at each of its finitely many singular points. The map $\tilde{f}_{k,\mathrm{norm}}$ is branched along H and possibly along the line L 'at infinity' (if k does not divide $\deg H$, then $\tilde{f}_{k,\mathrm{norm}}$ is branched along L). One can resolve the singularities of $\widetilde{X}_{k,\mathrm{norm}}$ and obtain a smooth manifold $\overline{X}_k$, $\dim_{\mathbb{R}} \overline{X}_k = 4$. Let $\sigma\colon \overline{X}_k \to \widetilde{X}_{k,\mathrm{norm}}$ be the resolution of singularities, $E = \sigma^{-1}(\operatorname{Sing} \widetilde{X}_{k,\mathrm{norm}})$ the proper transform of the set of singular points of $\widetilde{X}_{k,\mathrm{norm}}$, and $\bar{f}_k = \widetilde{f}_{k,\mathrm{norm}} \circ \sigma$. The action h induces an action $\bar{h}_k$ on $\overline{X}_k$ and an action $\bar{h}_{k*}$ on $H_1(\overline{X}_k, \mathbb{Z})$.

Similarly, the covering $f'_k \colon X'_k \to S^{n+2} \setminus V$ can be extended to a smooth map f_k: $X_k \to S^{n+2}$ branched along V, where X_k is a smooth compact $(n+2)$-manifold, and the action h induces actions h_k on X_k and h_{k*} on $H_1(X_k, \mathbb{Z})$. The action h_{k*} endows $H_1(X_k, \mathbb{Z})$ with the structure of a Λ-module.

It was shown in [6] that for any Hurwitz curve H the covering space $\overline{X}_k$ can be embedded as a symplectic submanifold in a complex projective rational 3-fold whose symplectic structure is given by an integer Kähler form. It was also proved that the first Betti number $b_1(\overline{X}_k) = \dim_{\mathbb{C}} H_1(\overline{X}_k, \mathbb{C})$ of $\overline{X}_k$ is equal to $r_{k,\neq 1}$, where $r_{k,\neq 1}$ is the number of roots of the Alexander polynomial $\Delta(t)$ of the curve $\overline{H}$ which are kth roots of unity not equal to 1.

Let M be a Noetherian Λ-module. We say that M is $(t-1)$-*invertible* if multiplication by $t-1$ is an automorphism of M. A Λ-module M is said to be t-*unipotent* if there is an $n \in \mathbb{N}$ such that multiplication by t^n is the identity automorphism of M. The *unipotence index* of a t-unipotent module M is the smallest $k \in \mathbb{N}$ such that

$$t^k - 1 \in \operatorname{Ann}(M) = \bigl\{ f(t) \in \Lambda \mid f(t)v = 0 \quad \forall v \in M \bigr\}.$$

Let M be a Noetherian $(t-1)$-invertible Λ-module. The t-unipotent Λ-module $A_n(M) = M/(t^k - 1)M$ is called the *kth derived Alexander module* of M. If M is the Alexander module of some C-group G (resp. knotted n-manifold V or Hurwitz curve H), then $A_k(M)$ is called the *kth derived Alexander module* of G (resp. of V or H) and is denoted by $A_k(G)$ (resp. by $A_k(V)$ or $A_k(H)$).

Here are the main results of this paper.

Theorem 0.1. *A Λ-module M is the Alexander module of a knotted n-manifold for $n \geqslant 2$ if and only if M is a Noetherian $(t-1)$-invertible Λ-module.*

Theorem 0.2. *Suppose that V is a knotted n-manifold, $n \geqslant 1$, and $f_k \colon X_k \to S^{n+2}$ is the cyclic covering branched along V. Then $H_1(X_k, \mathbb{Z})$ is isomorphic as a Λ-module to the kth derived Alexander module $A_k(V)$ of V.*

Similar statements hold for algebraic and, more generally, Hurwitz (resp. pseudo-holomorphic) curves. Namely, we have the following theorems.

Theorem 0.3. *A Λ-module M is the Alexander module of an irreducible Hurwitz (resp. pseudo-holomorphic) curve if and only if M is a Noetherian $(t-1)$-invertible t-unipotent Λ-module. In particular, the Alexander module of an irreducible algebraic plane curve is a Noetherian $(t-1)$-invertible t-unipotent Λ-module.*

The unipotence index of the Alexander module $A_0(H)$ of an irreducible plane algebraic (resp. Hurwitz or pseudo-holomorphic) curve H is a divisor of $\deg H$.

Corollary 0.4. *The Alexander module $A_0(H)$ of any irreducible plane algebraic (or Hurwitz, or pseudo-holomorphic) curve H is finitely generated over $\mathbb{Z}$, that is, $A_0(H)$ is a finitely generated abelian group.*

A finitely generated abelian group G is the Alexander module $A_0(H)$ of some irreducible Hurwitz or pseudo-holomorphic curve H if and only if one can find *a positive integer m and an automorphism $h \in \operatorname{Aut}(G)$ such that $h^m = \operatorname{Id}$ and $h - \operatorname{Id}$ is also an automorphism of G.*

Theorem 0.5. *Suppose that H is an algebraic (or Hurwitz, or pseudo-holomorphic) irreducible curve in $\mathbb{CP}^2$, $\deg H = m$, and $\bar f_k \colon \overline{X}_k \to \mathbb{CP}^2$ is a resolution of singularities of the cyclic covering of degree $\deg \bar f_k = k$ branched along H and possibly along the line L 'at infinity'. Then*

$$H_1(\overline{X}_k \setminus E, \mathbb{Z}) \simeq A_k(H), \qquad H_1(\overline{X}_k, \mathbb{Q}) \simeq A_k(H) \otimes \mathbb{Q},$$

where $E = \sigma^{-1}(\operatorname{Sing} \widetilde{X}_{k,\mathrm{norm}})$ and $A_k(H)$ is the kth derived Alexander module of H.

It should be noticed that the epimorphism $H_1(\overline{X}_k \setminus E, \mathbb{Z}) \simeq A_k(H) \to H_1(\overline{X}_k, \mathbb{Z})$ induced by the embedding $\overline{X}_k \setminus E \hookrightarrow \overline{X}_k$ need not be an isomorphism in the general case of Hurwitz curves (see Example 4.6 below).

Corollary 0.6. *Suppose that H is an algebraic (or Hurwitz, or pseudo-holomorphic) irreducible curve in $\mathbb{CP}^2$, $\deg H = m$, and $\bar f_k \colon \overline{X}_k \to \mathbb{CP}^2$ is a resolution of singularities of the cyclic covering of degree $\deg f_k = k$ branched along H and possibly along the line 'at infinity'. Then the following assertions hold.*

(i) *The first Betti number $b_1(\overline{X}_k)$ of $\overline{X}_k$ is even.*
(ii) *If $k = p^r$ for some prime p, then $H_1(\overline{X}_k, \mathbb{Q}) = 0$.*
(iii) *If k and m are coprime, then $H_1(\overline{X}_k, \mathbb{Z}) = 0$.*
(iv) *$H_1(\overline{X}_2, \mathbb{Z})$ is a finite abelian group of odd order.*

Note also that any C-group G can be realized (see [7]) as $\pi_1\bigl(\Delta^2 \setminus (C \cap \Delta^2)\bigr)$, where $\Delta^2 = \bigl\{|z| < 1\bigr\} \times \bigl\{|w| < 1\bigr\} \subset \mathbb{C}^2$ is a bidisc and $C \subset \mathbb{C}^2$ is a non-singular algebraic curve such that the restriction of $\mathrm{pr}_1 \colon \Delta^2 \to \bigl\{|z| < 1\bigr\}$ to $C \cap \Delta^2$ is a proper map. Therefore the analogues of Theorems 0.1, 0.2 and Corollaries 0.4, 0.6 hold in this case as well.

The proofs of Theorems 0.1 and 0.3 are given in § 3. In § 1 we describe the properties of Noetherian $(t-1)$-invertible Λ-modules. § 2 is devoted to the properties of Noetherian t-unipotent Λ-modules. In § 4 we prove Theorems 0.2, 0.5 and give some other corollaries of them.

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§ 1. Properties of $(t-1)$-invertible Λ-modules

1.1. Criteria for $(t-1)$-invertibility. Before describing $(t-1)$-invertible Λ-modules, we recall that the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$ is Noetherian. Each element $f \in \Lambda$ can be written as

$$f = \sum_{n_- \leqslant i \leqslant n_+} a_i t^i \in \mathbb{Z}[t, t^{-1}],$$

where $n_-, n_+, i, a_i \in \mathbb{Z}$. If $n_- \geqslant 0$ for some element $f \in \Lambda$, then $f \in \mathbb{Z}[t]$. Such elements are called *polynomials*.

For every $n \in \mathbb{Z}$, $n \neq 0$, we have a well-defined $\mathbb{Z}$-homomorphism

$$f(t) = \sum a_i t^i \mapsto f(n) = \sum a_i n^i.$$

The image $f(n)$ of an element $f(t)$ is called the *value* of $f(t)$ at n. If $f(t)$ is a polynomial, then its value $f(0) = a_0$ is also well defined.

Lemma 1.1. *A Noetherian Λ-module M is $(t-1)$-invertible if and only if multiplication by $t-1$ is a surjective homomorphism of M.*

Proof. This follows from a more general statement. Namely, every surjective Λ-endomorphism $f\colon M \to M$ of a Noetherian Λ-module M is an isomorphism. Indeed, if $\ker f \neq 0$, then the chain of submodules

$$\ker f \subset \ker f^2 \subset \cdots \subset \ker f^n \subset \cdots$$

is strictly increasing because f is surjective. This contradicts the Noetherian property of M.

Let M be a Noetherian $(t-1)$-invertible Λ-module. We consider an element $v \in M$ and write $M_v = \langle v \rangle$ for the principal submodule of M generated by v. Since M is Noetherian, every principal submodule of M is contained in some maximal principal submodule of M.

Lemma 1.2. *Every maximal principal submodule M_v of a $(t-1)$-invertible module M is $(t-1)$-invertible.*

Proof. Since M is $(t-1)$-invertible, there is an element $v_1 \in M$ such that $v = (t-1)v_1$. Therefore $M_v \subset M_{v_1}$. Since M_v is a maximal principal submodule of M, we have $M_v = M_{v_1}$. Therefore $v_1 \in M_v$ and multiplication by $t-1$ determines a surjective endomorphism of M_v. To complete the proof, we apply Lemma 1.1.

Every principal submodule $M_v \subset M$ is isomorphic to the module $\Lambda/\operatorname{Ann}_v$, where $\operatorname{Ann}_v = \{f \in \Lambda \mid fv = 0\}$ is the *annihilator* of v. The annihilator Ann_v of any element $v \in M$ is an ideal of Λ. We write

$$\operatorname{Ann}(M) = \bigcap_{v \in M} \operatorname{Ann}_v = \bigl\{g(t) \in \Lambda \mid g(t)v = 0 \quad \forall v \in M\bigr\}$$

for the *annihilator* of the module M.

Lemma 1.3. *A principal Λ-module $M = \Lambda/I$ is $(t-1)$-invertible if and only if the ideal I contains a polynomial $f(t)$ such that $f(1) = 1$.*

Proof. Let M be generated by an element $v \in M$. If the ideal $I = \operatorname{Ann}_v$ contains a polynomial $f(t)$ with $f(1) = 1$, then $f(t)$ may be written in the form

$$f(t) = (t-1)g(t) + 1 \tag{2}$$

for some polynomial $g(t)$. Therefore $v = (t-1)v_1$, where $v_1 = -g(t)v$. Thus multiplication by $t-1$ is a surjective endomorphism of M. Hence multiplication by $t-1$ is an automorphism of M by Lemma 1.1.

Conversely, if M is $(t-1)$-invertible, then there is an element $v_1 \in M$ such that $v = (t-1)v_1$. Write $v_1 = h(t)v$ for some $h(t) \in \Lambda$. We have $\bigl(1-(t-1)h(t)\bigr)v = 0$. Therefore $1-(t-1)h(t) \in \operatorname{Ann}_v = I$. There is a positive integer k such that $f(t) = t^k\bigl(1-(t-1)h(t)\bigr) \in I \cap \mathbb{Z}[t]$. It is easy to see that $f(1) = 1$.

As a consequence of Lemma 1.3, we obtain the following lemma.

Lemma 1.4. *Every principal submodule of a principal $(t-1)$-invertible module M is $(t-1)$-invertible.*

Proof. Indeed, suppose that M is generated by an element $v \in M$, and the submodule M_1 is generated by $v_1 = h(t)v$. Then $\mathrm{Ann}_v \subset \mathrm{Ann}_{v_1}$.

Since M is $(t-1)$-invertible, Lemma 1.3 yields a polynomial $f(t) \in \mathrm{Ann}_v$ with $f(1) = 1$. Applying Lemma 1.3 again, we see that M_1 is $(t-1)$-invertible because $f(t) \in \mathrm{Ann}_{v_1}$.

Proposition 1.5. *Every submodule of a Noetherian $(t-1)$-invertible Λ-module M is $(t-1)$-invertible.*

Proof. Let N be a submodule of M. Since M is a Noetherian Λ-module, the submodule N is generated by finitely many elements, say $v_1, \dots, v_n$. Every principal submodule $M_{v_i} \subset N \subset M$ is $(t-1)$-invertible by Lemmas 1.2 and 1.4. It follows that multiplication by $t-1$ is a surjective endomorphism of N because it is surjective on each of the submodules $M_{v_i} \subset N$ and the elements $v_1, \dots, v_n$ generate the module N. To complete the proof, we apply Lemma 1.1.

Proposition 1.6. *Every quotient module of a Noetherian $(t-1)$-invertible Λ-module M is $(t-1)$-invertible.*

Proof. This follows from Lemma 1.1.

Lemma 1.7. *Let $M_1, \dots, M_k$ be Noetherian $(t-1)$-invertible Λ-modules. Then the direct sum $M = \bigoplus_{i=1}^k M_i$ is a Noetherian $(t-1)$-invertible Λ-module.*

Proof. This is obvious.

Corollary 1.8. *Any Noetherian $(t-1)$-invertible Λ-module M is a quotient module of the direct sum $\bigoplus_{j=1}^n \Lambda/I_j$ of principal $(t-1)$-invertible Λ-modules Λ/I_j.*

Proof. Since M is a Noetherian Λ-module, it is generated by finitely many elements, say $v_1, \dots, v_n$. Proposition 1.5 implies that every principal submodule $M_{v_i} \subset M$ is $(t-1)$-invertible. Clearly, there is an epimorphism $\bigoplus_{i=1}^n M_{v_i} \mapsto M$.

Remark 1.9. An abelian group G possesses the structure of a $(t-1)$-invertible Λ-module if and only if it has an automorphism t such that $t-1$ is again an automorphism. If G is finitely generated and such an automorphism $t \in \mathrm{Aut}\, G$ is chosen, then G is a Noetherian Λ-module.

We note that an abelian group may have many $(t-1)$-invertible Λ-module structures. For example, the group $\mathbb{Z}/9\mathbb{Z}$ admits 3 such structures: either $tv = 2v$, or $tv = 5v$, or $tv = 8v$, where v is a generator of $\mathbb{Z}/9\mathbb{Z}$.

Theorem 1.10. *A Noetherian Λ-module M is $(t-1)$-invertible if and only if there is a polynomial $f(t) \in \mathrm{Ann}(M)$ such that $f(1) = 1$.*

Proof. If M is $(t-1)$-invertible, then each of its principal submodules M_v is also $(t-1)$-invertible by Proposition 1.5. Hence, by Lemma 1.3, the annihilator Ann_v

of every $v \in M$ contains a polynomial $f_v(t)$ such that $f_v(1) = 1$. If M is generated by $v_1, \dots, v_n$, then the polynomial $f(t) = f_{v_1}(t) \dots f_{v_n}(t)$ has the desired property.

Let us show that if there is a polynomial $f(t) \in \operatorname{Ann}(M)$ with $f(1) = 1$, then M is a $(t-1)$-invertible module. Indeed, in this case every principal submodule M_v of M is $(t-1)$-invertible by Lemma 1.3. Since multiplication by $t-1$ is an isomorphism of every principal submodule M_v of M, we conclude that this multiplication is an isomorphism of M.

Theorem 1.10 implies that *every Noetherian* $(t-1)$*-invertible module* M *is a torsion* Λ*-module* and, therefore,

$$\dim_{\mathbb{Q}} M \otimes \mathbb{Q} < \infty.$$

The following proposition will be used in the proofs of Theorems 0.1 and 0.3.

Proposition 1.11. *Any Noetherian* $(t-1)$*-invertible* Λ*-module* M *is isomorphic to a quotient module* Λ^n/M_1 *of the free* Λ*-module* Λ^n*, where the submodule* M_1 *is generated by elements* $w_1, \dots, w_n, \dots, w_{n+k}$ *of* Λ^n *with the following properties.*

(i) *For* $i = 1, \dots, n$ *we have* $w_i = \big(0, \dots, 0, f_i(t), 0, \dots, 0\big)$*, where the polynomial* $f_i(t)$ *occupies the* i*th position and satisfies* $f_i(1) = 1$*.*

(ii) *We have* $w_{n+j} = (t-1)\overline{w}_{n+j} = \big((t-1)g_{j,1}(t), \dots, (t-1)g_{j,n}(t)\big)$ *for* $j = 1, \dots, k$*, where the* $g_{j,l}(t)$ *are polynomials.*

(iii) *If the polynomials of the form* $t^m - 1$ *belong to* $\operatorname{Ann}(M)$*, then there is an* $m \in \mathbb{N}$ *such that* $t^m - 1 \in \operatorname{Ann}(M)$ *and the vector* w_{n+i} *equals* $(0, \dots, 0, t^m - 1, 0, \dots, 0)$*, where the polynomial* $t^m - 1$ *occupies the* i*th position for* $i = 1, \dots, n$*.*

Proof. We choose any generators $v_1, \dots, v_n$ of the Noetherian Λ-module M. By Theorem 1.10 there are polynomials $f_i(t) \in \operatorname{Ann}_{v_i}$ such that $f_i(1) = 1$. Clearly, there is an epimorphism

$$h\colon \bigoplus_{i=1}^{n} \Lambda/\big(f_i(t)\big) \to M$$

of Λ-modules such that $h(u_i) = v_i$ for $u_i = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 occupies the ith position. The kernel $N = \ker h$ is a Noetherian Λ-module. Let this kernel be generated by

$$u_{n+1} = \big(g_{1,1}(t), \dots, g_{1,n}(t)\big), \qquad \dots, \qquad u_{n+k} = \big(g_{k,1}(t), \dots, g_{k,n}(t)\big).$$

There is no loss of generality in assuming that all the $g_{i,j}(t)$ are polynomials.

The Λ-module $\bigoplus_{i=1}^{n} \Lambda/\big(f_i(t)\big)$ is $(t-1)$-invertible by Theorem 1.10, and N is also a $(t-1)$-invertible Λ-module by Proposition 1.5. Hence the elements $\bar{u}_{n+1} = (t-1)u_{n+1}, \dots, \bar{u}_{n+k} = (t-1)u_{n+k}$ also generate N.

If there is an $m \in \mathbb{N}$ such that the polynomial $t^m - 1$ belongs to $\operatorname{Ann}(M)$, then the elements $(0, \dots, 0, t^m - 1, 0, \dots, 0)$ belong to N. Therefore we can add the elements $(0, \dots, 0, t^m - 1, 0, \dots, 0)$ to the set of elements $\bar{u}_{n+1}, \dots, \bar{u}_{n+k}$ (that generate the module N) and reorder the resulting set of generators $\bar{u}_{n+1}, \dots, \bar{u}_{n+k}$ (where we put $k := n + k$) in such a way that $\bar{u}_{n+j} = (0, \dots, 0, t^m - 1, 0, \dots, 0)$ for $j = 1, \dots, n$, where $t^m - 1$ occupies the jth position.

To complete the proof, note that the kernel M_1 of the composite $h \circ \nu \colon \Lambda^n \to M$ of h and the natural epimorphism $\nu \colon \Lambda^n \to \bigoplus_{i=1}^n \Lambda/\bigl(f_i(t)\bigr)$ is generated by the elements

$$w_i = \bigl(0, \dots, 0, f_i(t), 0, \dots, 0\bigr), \qquad i = 1, \dots, n,$$

where the polynomial $f_i(t)$ occupies the ith position, along with the elements

$$w_{n+i} = \bigl(f_{i,1}(t), \dots, f_{i,n}(t)\bigr) \in \Lambda^n, \qquad i = 1, \dots, k,$$

where the coordinates $f_{i,j}(t)$ of each w_{n+i} coincide with the coordinates $\bar{g}_{i,j}(t)$ of $\bar{u}_{n+i} = \bigl(\bar{g}_{i,1}(t), \dots, \bar{g}_{i,n}(t)\bigr)$. This proves the proposition.

1.2. $\mathbb{Z}$-torsion submodules of $(t-1)$-invertible Λ-modules. An element v of a Λ-module M is said to be of *finite order* if there is an $m \in \mathbb{Z} \setminus \{0\}$ such that $mv = 0$. A Λ-module M is called a $\mathbb{Z}$-*torsion* module if all the elements of M are of finite order. Given any Λ-module M, we write M_{fin} for the subset of M consisting of all elements of finite order. It is easy to see that M_{fin} is a $\mathbb{Z}$-torsion Λ-module. If M is a Noetherian $(t-1)$-invertible Λ-module, then M_{fin} is also a Noetherian $(t-1)$-invertible Λ-module and Propositions 1.5, 1.6 imply that there is an exact sequence of Λ-modules

$$0 \to M_{\mathrm{fin}} \to M \to M_1 \to 0,$$

where M_1 is a Noetherian $(t-1)$-invertible Λ-module containing no non-trivial elements of finite order.

Let $M = M_{\mathrm{fin}}$ be a Noetherian $(t-1)$-invertible Λ-module. Since M is finitely generated over Λ, there is an integer $d \in \mathbb{N}$ such that $dv = 0$ for all $v \in M$. (We call such a number d an *exponent* of M.) Let $d = p_1^{r_1} \dots p_n^{r_n}$ be the prime factorization. We write $M(p_i)$ for the subset of M consisting of all elements $v \in M$ such that $p_i^{r_i} v = 0$. It is easy to see that $M(p_i)$ is a Λ-submodule of M. We call $M(p_i)$ the p_i-*submodule* of M.

Theorem 1.12. *Let $M = M_{\mathrm{fin}}$ be a Noetherian $(t-1)$-invertible Λ-module with exponent $d = p_1^{r_1} \dots p_n^{r_n}$. Then M is isomorphic to the direct sum of its p_i-submodules*:

$$M = \bigoplus_{i=1}^{n} M(p_i).$$

Proof. The proof coincides with that of the corresponding theorem for abelian groups (see, for example, Theorem 8.1 in [14]).

Since the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$ is Noetherian, every ideal I of Λ is finitely generated. We write $I_{\mathrm{pol}} = I \cap \mathbb{Z}[t]$ for the corresponding ideal of the polynomial ring $\mathbb{Z}[t]$. It is well known that $I = \Lambda I_{\mathrm{pol}}$, that is, every ideal I of Λ is generated by polynomials.

We recall that $\mathbb{Z}[t]$ is a factorial ring. Its units are precisely the units of $\mathbb{Z}$, and its prime elements are either primes of $\mathbb{Z}$ or polynomials $q(t) = \sum a_i t^i$ that are irreducible in $\mathbb{Q}[t]$ and have content 1 (that is, the greatest common divisor of their coefficients a_i is equal to 1). For any non-zero polynomials $q_1(t), q_2(t) \in \mathbb{Z}[t]$, Euclid's algorithm enables us to find polynomials $h_1(t)$, $h_2(t)$, $r(t) \in \mathbb{Z}[t]$ and a constant $d \in \mathbb{Z}$, $d \neq 0$, such that

$$h_1(t)q_1(t) + h_2(t)q_2(t) = dr(t), \tag{3}$$

where $r(t)$ is the greatest common divisor of the polynomials $q_1(t)$ and $q_2(t)$.

Lemma 1.13. *Suppose that M is a Noetherian $(t-1)$-invertible Λ-module and $t^n - 1 \in \mathrm{Ann}(M)$ for some $n = p^r$, where p is a prime. Then M is a $\mathbb{Z}$-torsion module.*

Proof. If the polynomial $t^n - 1 = (t-1)(t^{n-1} + \cdots + t + 1)$ belongs to $\mathrm{Ann}(M)$, then $g_n(t) = t^{n-1} + \cdots + t + 1 \in \mathrm{Ann}(M)$ because M is $(t-1)$-invertible. When $n = p^r$, each factor in the formula

$$g_{p^r}(t) = \prod_{i=1}^{r} \Phi_{p^i}(t) = \prod_{i=1}^{r} \sum_{j=0}^{p-1} t^{jp^{i-1}}$$

is an irreducible element of Λ.

By Theorem 1.10 there is a polynomial $f(t) \in \mathrm{Ann}(M)$ with $f(1) = 1$. If $n = p^r$ for some prime p, then the polynomials $f(t)$ and $g_{p^r}(t)$ have no common irreducible divisors. Indeed, if $g(t)$ is a divisor of $f(t)$, then we must have $g(1) = \pm 1$ since $f(1) = 1$, but $\Phi_{p^i}(1) = p$ for each i. Therefore one can find polynomials $h_1(t)$, $h_2(t)$ and a constant $d \in \mathbb{N}$ such that $h_1(t)f(t) + h_2(t)g_{p^r}(t) = d$. Hence if $g_{p^r}(t) \in \mathrm{Ann}(M)$, then $d \in \mathrm{Ann}(M)$ and, therefore, M is a $\mathbb{Z}$-torsion module.

1.3. Principal $(t-1)$-invertible Λ-modules. Let I be a non-zero ideal of the ring Λ. We denote by I_m the subset of I_{pol} consisting of all polynomials $f(t)$ of the smallest degree (let m be this smallest degree). Note that if $f(t) \in I_m \setminus \{0\}$, then $f(0) \neq 0$.

Consider any polynomials $f_1(t), f_2(t) \in I_m$ and write them as $f_i(t) = d_i q_i(t)$, where $d_i \in \mathbb{Z}$ and the polynomials $q_i(t)$ have content 1. We have $q_1(t) = q_2(t)$. Indeed, their greatest common divisor $r(t)$ satisfies $\deg r(t) \leqslant m$ and, moreover, $\deg r(t) = m$ if and only if $q_1(t) = q_2(t)$. On the other hand, (3) implies that $d_2 h_1(t) f_1(t) + d_1 h_2(t) f_2(t) = d_1 d_2 d r(t)$ for some polynomials $h_1(t), h_2(t)$. Therefore $d_1 d_2 d r(t) \in I_{\mathrm{pol}}$ and we must have $\deg r(t) = m$.

Applying Euclid's algorithm for integers, we see that if polynomials $f_i(t) = d_i q(t)$ belong to I_m for $i = 1, 2$, then $d_0 q(t)$ also belongs to I_m, where d_0 is the greatest common divisor of d_1 and d_2. Thus there is a polynomial $f_m(t) = d_m q(t) \in I_m$ such that all polynomials $f(t) \in I_m$ are divisible by $f_m(t)$. The polynomial $f_m(t)$ is uniquely determined up to multiplication by ± 1. It will be called a *leading generator* of the ideal I.

Let I be a non-zero ideal of Λ and $f(t) = d_m q(t)$ a leading generator of I. Then all the polynomials $h(t) \in I$ are divisible by $q(t)$. Indeed, arguing as above, we easily see that if $r(t)$ is the greatest common divisor of $f(t)$ and $h(t)$, then there is a constant d such that $dr(t) \in I$. Since $\deg q(t)$ is the minimal degree of polynomials belonging to I, we must have the equality $r(t) = q(t)$.

The arguments above and Lemma 1.3 prove the following proposition.

Proposition 1.14. *Let $M = M_v$ be a principal $(t-1)$-invertible Λ-module generated by an element v. Then the ideal Ann_v is generated by a finite set of polynomials $f_1(t), \dots, f_k(t)$, where $f_i(t) = d_i q_i(t)$, $d_i \in \mathbb{Z}$, $d_i \neq 0$, and the content of $q_i(t)$ is equal to 1 for all i. Moreover, the polynomials $f_1(t), \dots, f_k(t)$ possess the following properties.*

(i) $\deg f_1 < \deg f_2 \leqslant \cdots \leqslant \deg f_k$.

(ii) $f_i(0) \neq 0$ *for all i.*

(iii) $q_1(1) = 1$.

(iv) $q_1(t) \mid q_i(t)$ *for $i = 2, \dots, k$.*

(v) *If $k > 1$, then $|d_1| > 1$, $d_k = 1$ and $q_k(1) = 1$.*

A set of generators of Ann_v is said to be *good* if it possesses properties (i)–(v) in Proposition 1.14.

Principal $(t-1)$-invertible Λ-modules $M = M_v$ are classified on the basis of properties of their annihilators. We say that M_v is of *finite type* if in a good system $f_1(t), \dots, f_k(t)$ of generators of Ann_v the leading generator $f_1(t) \equiv d_1$ is a constant (that is, $q_1(t) \equiv 1$). A module M_v is said to be of *mixed type* if in a good system $f_1(t), \dots, f_k(t)$ of generators of Ann_v the degree of the leading generator $f_1 = d_1 q_1(t)$ is greater than zero and we have $|d_1| \geqslant 2$. The arguments above imply that if a principal $(t-1)$-invertible Λ-module $M = M_v$ is not of finite or mixed type, then the leading generator $f_1(t)$ of a good system of generators of Ann_v is equal to a polynomial $q_1(t)$ with $q_1(1) = 1$. Therefore Ann_v is a principal ideal generated by $q_1(t)$ because all the polynomials $h(t) \in \mathrm{Ann}_v$ are divisible by $q_1(t)$. Such principal $(t-1)$-invertible Λ-modules are said to be *biprincipal*.

It is easy to see that if $M = M_v$ is a principal Λ-module of finite type and $d_1 \in \mathbb{Z}$ is the leading generator of a good system of generators of Ann_v, then the orders of all elements of M divide d_1. Hence a principal Λ-module M_v is of finite type if and only if it is a $\mathbb{Z}$-torsion module.

If $M = M_v$ is a biprincipal Λ-module, then it has no non-zero elements of finite order. Indeed, let $q(t)$ be a generator of Ann_v. If an element $v_1 = h(t)v$ has order m, then $mh(t) \in \mathrm{Ann}_v$, that is, $mh(t)$ is divisible by $q(t)$. Since t is a unit of the ring Λ, we can assume that $h(t)$ is a polynomial. Since M_v is a biprincipal module, the polynomial $h(t)$ must be divisible by $q(t)$, that is, $v_1 = 0$.

If $M = M_v$ is a Λ-module of mixed type, then there is an exact sequence of Λ-modules

$$0 \to M_1 \to M \to M_2 \to 0,$$

where M_1 is a principal Λ-module of finite type and M_2 is a biprincipal Λ-module. Indeed, let $d_1 q_1(t)$ be the leading generator of Ann_v. Put $v_1 = q_1(t)v$. Then we easily see that the Λ-module $M_1 = M_{v_1} \subset M$ (generated by v_1) is of finite type while the Λ-module $M_2 = M/M_1 \simeq \Lambda/(q_1)$ is biprincipal.

1.4. Finitely $\mathbb{Z}$-generated $(t-1)$-invertible Λ-modules. Every Λ-module M may be regarded as a $\mathbb{Z}$-module, that is, an abelian group.

Proposition 1.15. *A Noetherian Λ-module M is finitely generated over $\mathbb{Z}$ if and only if there is a polynomial*

$$q(t) = \sum_{i=0}^{n} a_i t^i \in \operatorname{Ann}(M)$$

such that $a_n = a_0 = 1$.

Proof. We begin by proving this in the case when $M = M_v$ is a principal Λ-module.

If there is a polynomial $q(t) = \sum_{i=0}^{n} a_i t^i \in \operatorname{Ann}_v$ with $a_n = a_0 = 1$, then we easily see that M is generated over $\mathbb{Z}$ by the elements $v, tv, \dots, t^{n-1}v$.

Suppose that the Λ-module $M = M_v$ is finitely generated over $\mathbb{Z}$ and let $h_1(t)v, \dots, h_m(t)v$ be its generators. Since multiplication by t is an isomorphism of M, we can assume that the $h_i(t)$, $i = 1, \dots, m$, are polynomials with $h_i(0) = 0$. Put $n - 1 = \max\bigl(\deg h_1(t), \dots, \deg h_m(t)\bigr)$. Since $h_1(t)v, \dots, h_m(t)v$ generate M over $\mathbb{Z}$, one can find integers $b_1, \dots, b_m$ and $c_1, \dots, c_m$ such that

$$v = \sum_{i=1}^{m} b_i h_i(t) v \quad \text{and} \quad t^n v = \sum_{i=1}^{m} c_i h_i(t) v.$$

Therefore the polynomials $1 - \sum b_i h_i(t)$ and $t^n - \sum c_i h_i(t)$ belong to Ann_v. Then the polynomial $t^n + 1 - \sum (b_i + c_i) h_i(t)$ has the desired properties.

In the general case, a Noetherian $(t-1)$-invertible Λ-module M is generated by a finite set of elements $v_1, \dots, v_m$. Hence M is finitely generated over $\mathbb{Z}$ if and only if the principal submodules $M_{v_i} \subset M$ are all finitely generated over $\mathbb{Z}$.

If $g(t) \in \operatorname{Ann}(M)$, then $g(t) \in \operatorname{Ann}_{v_i}$ for $i = 1, \dots, m$. In particular, if there is a polynomial $q(t) = \sum_{i=0}^{n} a_i t^i \in \operatorname{Ann}(M)$ with $a_n = a_0 = 1$, then all the M_{v_i} (and therefore M) are finitely generated over $\mathbb{Z}$.

If the principal submodules $M_{v_i} \subset M$ are finitely generated over $\mathbb{Z}$, then there are polynomials $q_i(t) = \sum_{j=0}^{n_i} a_{i,j} t^j \in \operatorname{Ann}_{v_i}$ such that $a_{i,n_i} = a_{i,0} = 1$. Put $n = \sum n_i$. Then we have

$$q(t) = q_1(t) \dots q_n(t) = t^n + 1 + \sum_{j=1}^{n-1} a_j t^j \in \operatorname{Ann}(M)$$

since $q(t) \in \operatorname{Ann}_{v_i}$ for all v_i. The proposition is proved.

Proposition 1.15 implies that many $(t-1)$-invertible biprincipal modules $M = \Lambda/I$ are not finitely generated over $\mathbb{Z}$. More precisely, it is easy to see that *a biprincipal $(t-1)$-invertible module $M = \Lambda/I$ is finitely generated over $\mathbb{Z}$ if and only if the ideal $I = \bigl\langle q(t) \bigr\rangle$ is generated by a polynomial $q(t) = \sum_{i=0}^{n} a_i t^i$ such that $q(1) = 1$ and the coefficients a_0 and a_n are equal to ± 1.*

For example, the $(t-1)$-invertible biprincipal module

$$M_m = \Lambda / \bigl\langle (m+1)t - m \bigr\rangle$$

is never finitely generated over $\mathbb{Z}$ for $m \in \mathbb{N}$.

Theorem 1.16. *Let M be a Noetherian $(t-1)$-invertible $\mathbb{Z}$-torsion module. Then M is finitely generated over $\mathbb{Z}$.*

Proof. By Theorem 1.12, the module M is isomorphic to the direct sum $\oplus M(p_i)$ of finitely many p_i-submodules. Hence it suffices to prove the theorem in the case when M has exponent p^r, where p is a prime. By Corollary 1.8, M is a quotient module of the direct sum $\bigoplus_{j=1}^{n} \Lambda/I_j$ of principal $(t-1)$-invertible Λ-modules Λ/I_j. In our case there is no loss of generality in assuming that each ideal I_j contains the constant p^{r_j} for some $r_j \in \mathbb{N}$. Thus it suffices to prove the theorem in the case when $M = M_v$ is a principal $(t-1)$-invertible Λ-module of exponent p^r. In other words, the ideal $I = \mathrm{Ann}_v$ contains the constant p^r and a polynomial $g(t)$ with $g(1) = 1$.

Suppose that $r = 1$ and $g(t) = \sum a_i t^i$. We put $g_1(t) = \sum_{p|a_i} a_i t^i$ and $\bar{g}(t) = g(t) - g_1(t)$. Then $\bar{g}(t) \in \mathrm{Ann}_v$ because $g(t), g_1(t) \in \mathrm{Ann}_v$. It is easy to see that the numbers $\bar{g}(1)$ and p are coprime since $g(1) = 1$ and $g_1(1) \equiv 0 \pmod{p}$. Moreover, by the construction of $\bar{g}(t)$, every coefficient of $\bar{g}(t)$ is coprime to p. Multiplying $\bar{g}(t)$ by t^{-k}, we can assume that $\bar{g}(0) \neq 0$. Write $\bar{g}(t) = \sum_{i=0}^{m} \bar{a}_i t^i$. Since $\bar{a}_m$ and p are coprime, one can find integers b_1 and c_1 such that $b_1\bar{a}_m + c_1 p = 1$. Similarly, there are integers b_2 and c_2 such that $b_2\bar{a}_0 + c_2 p = 1$. Therefore the polynomial $(b_1 t + b_2)\bar{g}(t) + p(c_1 t^{m+1} + c_2)$ belongs to I and can be written as

$$h(t) = t^{m+1} + 1 + \sum_{i=1}^{m} (b_1\bar{a}_{i-1} + b_2\bar{a}_i)t^i.$$

Thus M_v is finitely generated over $\mathbb{Z}$ by Proposition 1.15.

We now consider the general case of a principal $(t-1)$-invertible Λ-module of exponent p^r. Suppose that any principal $(t-1)$-invertible Λ-module M' of exponent p^{r_1} with $r_1 < r$ is finitely generated over $\mathbb{Z}$. Let $M = M_v$ be a principal $(t-1)$-invertible Λ-module of exponent p^r and M_{v_1} the submodule of M generated by $v_1 = p^{r-1}v$. Then M_{v_1} is of exponent p and the quotient module $M_{\bar{v}} = M_v/M_{v_1}$ is of exponent p^{r-1}. The proof now follows from the exact sequence

$$0 \to M_{v_1} \to M \to M/M_{v_1} \to 0.$$

Corollary 1.17. *Every Noetherian $(t-1)$-invertible $\mathbb{Z}$-torsion module is finite. In other words, it is a finite abelian group.*

Lemma 1.18. *The group $G = \bigoplus_{i=1}^{n} (\mathbb{Z}/2^{r_i}\mathbb{Z})^{m_i}$ possesses no structure of a $(t-1)$-invertible Λ-module if $r_i \neq r_j$ for $i \neq j$ and one of m_i is equal to* 1.

Proof. Suppose that G possesses the structure of a $(t-1)$-invertible Λ-module. Then the subgroup $2^r G$ of G is a Λ-submodule of G for any r and Propositions 1.5 and 1.6 yield that the groups $2^r G$ and $G/2^r G$ are $(t-1)$-invertible Λ-modules. Hence there is no loss of generality in assuming that

$$G = (\mathbb{Z}/2\mathbb{Z}) \oplus \left(\bigoplus_{i=1}^{n} (\mathbb{Z}/2^{r_i}\mathbb{Z})^{m_i} \right),$$

where all $r_i \geqslant 2$ and $m_i \geqslant 2$. We choose generators $v_1, \dots, v_{m+1}$ of G with $m = \sum_{i=1}^{n} m_i$ in such a way that

$$G \simeq (\mathbb{Z}/2\mathbb{Z})v_1 \oplus \Big(\bigoplus_{i=2}^{m+1} (\mathbb{Z}/2^{\overline{r}_i}\mathbb{Z})v_i\Big),$$

where all $\overline{r}_i \geqslant 2$. Consider the $\mathbb{Z}$-submodule $\overline{G}$ of G consisting of all elements $v \in G$ of order at most 4. Clearly, $\overline{G}$ is a Λ-submodule of G and is generated over $\mathbb{Z}$ (and hence over Λ) by $\bar{v}_1 = v_1$ and $\bar{v}_i = 2^{\overline{r}_i - 2} v_i$, $i = 2, \dots, m+1$. It is easy to see that we have an isomorphism of abelian groups

$$\overline{G} \simeq (\mathbb{Z}/2\mathbb{Z})\bar{v}_1 \oplus \Big(\bigoplus_{i=2}^{m+1} (\mathbb{Z}/4\mathbb{Z})\bar{v}_i\Big).$$

By Proposition 1.5, $\overline{G}$ is a $(t-1)$-invertible Λ-module. Multiplication by t is a module automorphism of $\overline{G}$. Write

$$\begin{aligned} t\bar{v}_1 &= a_1\bar{v}_1 + 2\sum_{i=2}^{m+1} b_i\bar{v}_i, \\ t\bar{v}_j &= a_j\bar{v}_1 + \sum_{i=2}^{m+1} c_{j,i}\bar{v}_i, \qquad j = 2, \dots, m+1, \end{aligned} \tag{4}$$

where each coefficient a_j is equal to 0 or 1.

We claim that $a_1 = 1$. Indeed, assume that $a_1 = 0$. Since multiplication by t is an automorphism and $\bar{v}_1, \dots, \bar{v}_{m+1}$ generate $\overline{G}$, we must have an equality $\bar{v}_1 = \sum d_i t\bar{v}_i$, where one of the d_i is odd for some $i \geqslant 2$ as $a_1 = 0$. Furthermore, $\bar{v}_1$ is an element of order 2, whence $2\sum_{i=2}^{m+1} d_i t\bar{v}_i = 0$. On the other hand, $t\bar{v}_2, \dots, t\bar{v}_{m+1}$ are linearly independent over $\mathbb{Z}/4\mathbb{Z}$ since $\bar{v}_2, \dots, \bar{v}_{m+1}$ are linearly independent over $\mathbb{Z}/4\mathbb{Z}$ and multiplication by t is an automorphism. Thus the equality $2\sum_{i=2}^{m+1} d_i t\bar{v}_i = 0$ is impossible if any of the d_i is odd and, therefore, the coefficient a_1 in (4) must be equal to 1.

We claim that $\overline{G}$ cannot be $(t-1)$-invertible. Indeed, we have

$$t\bar{v}_1 = \bar{v}_1 + 2\sum_{i=2}^{m+1} b_i\bar{v}_i.$$

Therefore

$$(t-1)\bar{v}_1 = 2\sum_{i=2}^{m+1} b_i\bar{v}_i.$$

Repeating the argument above with t replaced by $t-1$, we see that multiplication by $t-1$ is not an automorphism of $\overline{G}$ since $(t-1)\bar{v}_1$ is a linear combination of the elements $\bar{v}_2, \dots, \bar{v}_{m+1}$. The lemma is proved.

Theorem 1.19. *Consider an abelian group*

$$G = G_1 \oplus \left(\bigoplus_{i=1}^{n} (\mathbb{Z}/2^{r_i}\mathbb{Z})^{m_i} \right),$$

where $r_i \neq r_j$ for $i \neq j$ and G_1 is a group of odd order. Then G admits the structure of a $(t-1)$-invertible Λ-module if and only if $m_i \geqslant 2$ for all $i = 1, \dots, n$.

Proof. By Theorem 1.12, if $M = M_{\mathrm{fin}}$ is a Noetherian $(t-1)$-invertible Λ-module of exponent $d = p_1^{r_1} \dots p_n^{r_n}$, then M is the direct sum

$$M = \bigoplus_{i=1}^{n} M(p_i)$$

of p_i-submodules that are $(t-1)$-invertible by Proposition 1.5. Each submodule $M(p_i)$ with p_i odd is of odd order and, by Lemma 1.18, the submodule $M(2)$ of M is isomorphic (as an abelian group) to $\bigoplus_{i=1}^{k} (\mathbb{Z}/2^{r_i}\mathbb{Z})^{m_i}$, where $m_i \geqslant 2$ for all $i = 1, \dots, k$.

To prove the converse statement, we make three remarks. First, a finite direct sum of $(t-1)$-invertible Λ-modules is also a $(t-1)$-invertible Λ-module. Second, for every prime $p > 2$, the $(t-1)$-invertible Λ-module $M = \Lambda/I$, where the ideal I is generated by the number p^r and the polynomial $2t-1$, is isomorphic to $\mathbb{Z}/p^r\mathbb{Z}$ as an abelian group. Third, if $n \geqslant 2$, then the $(t-1)$-invertible Λ-module $M = \Lambda/I$, where the ideal I is generated by the number 2^r and the polynomial $t^n - t + 1$, is isomorphic to $(\mathbb{Z}/2^r\mathbb{Z})^n$ as an abelian group.

§ 2. t-unipotent $\mathbb{Z}[t, t^{-1}]$-modules

2.1. Properties of t-unipotent Λ-modules. The following proposition is a simple corollary of Propositions 1.5 and 1.6.

Proposition 2.1. *Every Λ-submodule M_1 and every quotient module M/M_1 of a Noetherian $(t-1)$-invertible t-unipotent Λ-module M is a $(t-1)$-invertible t-unipotent Λ-module.*

Lemma 2.2. *Let $M_1, \dots, M_n$ be Noetherian $(t-1)$-invertible t-unipotent Λ-modules. Then the direct sum $M = \bigoplus_{i=1}^{n} M_i$ is a Noetherian $(t-1)$-invertible t-unipotent Λ-module.*

Proof. By Lemma 1.7, M is a Noetherian $(t-1)$-invertible Λ-module and there are $k_i \in \mathbb{N}$ such that $t^{k_i} - 1 \in \mathrm{Ann}(M_i)$. Since every polynomial $t^{k_i} - 1$ $(i = 1, \dots, n)$ is a divisor of $t^k - 1$, we easily see that $t^k - 1 \in \mathrm{Ann}(M)$ for $k = k_1 \dots k_n$. The lemma is proved.

Proposition 2.1 and Lemma 2.2 yield the following proposition.

Proposition 2.3. *A Noetherian Λ-module M is $(t-1)$-invertible and t-unipotent if and only if each of its principal submodules M_v is $(t-1)$-invertible and t-unipotent.*

Theorem 2.4. *Any Noetherian $(t-1)$-invertible $\mathbb{Z}$-torsion Λ-module is t-unipotent.*

Proof. Let M be a Noetherian $(t-1)$-invertible $\mathbb{Z}$-torsion Λ-module. By Corollary 1.17, M consists of finitely many elements. Hence the automorphism of M defined by multiplication by t has finite order, say k. Then $t^k v = v$ for all $v \in M$ or, in other words, $t^k - 1 \in \operatorname{Ann}(M)$. The theorem is proved.

The following propositions describe biprincipal $(t-1)$-invertible t-unipotent modules and principal $(t-1)$-invertible t-unipotent modules of mixed type.

Proposition 2.5. *Let $M = \Lambda/I$ be a biprincipal $(t-1)$-invertible t-unipotent Λ-module, where the ideal $I = \langle g(t) \rangle$ is generated by a polynomial $g(t)$. Then the following assertions hold.*

(i) *All the roots of $g(t)$ are roots of unity.*

(ii) *$g(t)$ has no multiple roots.*

(iii) *If ξ is a kth root of unity (that is, $\xi^k = 1$) and $k = p^r$ for some prime p, then ξ is not a root of $g(t)$.*

(iv) $g(1) = \pm 1$.

(v) $\deg g(t)$ *is even.*

Proof. To prove assertions (i) and (ii), note that $t^k - 1 \in I$ for some k because M is a t-unipotent module. Hence $t^k - 1$ is divisible by $g(t)$.

Let us prove (iii)–(v). By Theorem 1.10 there is a polynomial $f(t) \in I$ with $f(1) = 1$. We have $f(t) = h(t)g(t)$ for some polynomial $h(t) \in \mathbb{Z}[t]$ since I is generated by $g(t)$. Thus $g(1) = \pm 1$ (and we may assume that $g(1) = 1$) because we have

$$1 = f(1) = h(1)g(1),$$

where $h(1), g(1) \in \mathbb{Z}$. On the other hand, if a primitive p^rth root ξ of unity is a root of $g(t)$ for some prime p, then $g(t)$ must be divisible by the p^rth cyclotomic polynomial $\Phi_{p^r}(t)$. In other words, there is a polynomial $h(t) \in \mathbb{Z}[t]$ such that $g(t) = \Phi_{p^r}(t)h(t)$. Thus $1 = g(1) = \Phi_{p^r}(1)h(1)$ and we obtain a contradiction because $\Phi_{p^r}(1) = p$.

To complete the proof, we use assertions (iii) and (iv) to conclude that the numbers $\xi = \pm 1$ are not roots of $g(t)$, whence all the roots of $g(t)$ are non-real. Since $g(t) \in \mathbb{Z}[t]$, we see that if ξ is a root of $g(t)$, then so is its complex conjugate $\bar{\xi}$. Therefore $\deg g(t)$ is even (because we have $\bar{\xi} \neq \xi$ for all roots of unity different from ± 1). This proves the proposition.

Proposition 2.6. *Let $M = \Lambda/I$ be a principal $(t-1)$-invertible t-unipotent Λ-module of mixed type and let $f(t) = dg(t)$ be the leading generator of the ideal I, where $d \in \mathbb{N}$ and $g(t)$ is a polynomial of content* 1*. Then $g(t)$ satisfies conditions* (i)–(v) *of Proposition* 2.5.

Proof. Let v be a generator of M and let M_1 be the Λ-submodule of M generated by $v_1 = g(t)v$. We have an exact sequence of Λ-modules

$$0 \to M_1 \to M \to M/M_1 \to 0,$$

where M_1 is a principal module of finite type and $M_2 = M/M_1$ is a biprincipal Λ-module isomorphic to $\Lambda/\langle g(t)\rangle$. By Proposition 2.1, M_2 is a $(t-1)$-invertible t-unipotent module. Hence the proposition follows from Proposition 2.5.

Let M be a Noetherian $(t-1)$-invertible t-unipotent Λ-module. The *unipotence index* of M is the smallest $k \in \mathbb{N}$ such that $t^k - 1 \in \operatorname{Ann}(M)$.

Lemma 2.7. *If M is a Noetherian $(t-1)$-invertible t-unipotent Λ-module of unipotence index k, then $\sum_{i=0}^{k-1} t^i \in \operatorname{Ann}(M)$.*

Proof. We have $t^k - 1 = (t-1)\sum_{i=0}^{k-1} t^i \in \operatorname{Ann}(M)$. Since M is a $(t-1)$-invertible Λ-module, it follows that $\big(\sum_{i=0}^{k-1} t^i\big)v = 0$ for all $v \in M$. The lemma is proved.

Lemma 2.8. *Every Noetherian $(t-1)$-invertible Λ-module M of unipotence index 2 is a finite $\mathbb{Z}$-module of odd order.*

Proof. It follows from Lemma 1.13 and Corollary 1.17 that M is finite. By Lemma 2.7 the polynomial $t+1$ belongs to $\operatorname{Ann}(M)$. Therefore $tv = -v$ for all $v \in M$. In particular, if v is of order 2, then $tv = v$. This is impossible since M is $(t-1)$-invertible. Hence M has no elements of even order. The lemma is proved.

Proposition 2.9. *A cyclic group G of order $n = p_1^{r_1}\dots p_m^{r_m}$ (where $p_1,\dots,p_m$ are primes) possesses the structure of a $(t-1)$-invertible Λ-module of unipotence index k if and only if the polynomial $\sum_{i=0}^{k-1} t^i$ has a root $a_j \neq 1$ in the field $\mathbb{Z}/p_j\mathbb{Z}$ for every $j = 1,\dots,m$.*

Proof. By Theorem 1.12 it suffices to consider only the case when $m = 1$, that is, $n = p^r$ for some prime p.

Suppose that the cyclic group G of order $n = p^r$ possesses the structure of a $(t-1)$-invertible Λ-module of unipotence index k. Then the subgroup $G_p = p^{r-1}G$ consisting of all elements of order p is also a $(t-1)$-invertible Λ-module of unipotence index k. Therefore $\sum_{i=0}^{k-1} t^i \in \operatorname{Ann}(G_p)$. Let $v \in G_p$ be a generator of G_p. Since G_p is a $(t-1)$-invertible module, we have $tv = av$ for some $a \not\equiv 1 \pmod p$. Hence $\sum_{i=0}^{k-1} a^i v = 0$. It follows that $\sum_{i=0}^{k-1} a^i \equiv 0 \pmod p$, that is, the polynomial $\sum_{i=0}^{k-1} t^i$ has a root (different from 1) in the field $\mathbb{Z}/p\mathbb{Z}$.

Conversely, let $a \not\equiv 1 \pmod p$ be a root of the polynomial $\sum_{i=0}^{k-1} t^i$ in the field $\mathbb{Z}/p\mathbb{Z}$ and let v be a generator of the cyclic group G of order p^r. We define an action of t on the $\mathbb{Z}$-module G by putting $t(v) = av$. This endows G with the structure of a $(t-1)$-invertible Λ-module since $a \not\equiv 1 \pmod p$. It is easy to see that $t^k - 1 \in \operatorname{Ann}(G)$. The proposition is proved.

Theorem 2.10. *Every Noetherian $(t-1)$-invertible t-unipotent Λ-module M is finitely generated over $\mathbb{Z}$.*

Proof. The theorem follows from Proposition 1.15 since the polynomial $t^k - 1$ belongs to $\operatorname{Ann}(M)$ for some $k \in \mathbb{Z}$.

Using Theorem 2.10 and the structure theorem for finitely generated $\mathbb{Z}$-modules, we see that for every $(t-1)$-invertible t-unipotent Λ-module M, we have the isomorphism (of $\mathbb{Z}$-modules)

$$M \simeq M_{\mathrm{fin}} \oplus \mathbb{Z}^k, \tag{5}$$

where M_{fin} is the submodule of M consisting of all elements of finite order. The rank k of the free part of M in the decomposition (5) is called the *Betti number* of the Noetherian $(t-1)$-invertible t-unipotent Λ-module M.

Theorem 2.11. *The Betti number of any Noetherian $(t-1)$-invertible t-unipotent Λ-module M is even.*

Proof. By definition, the Betti number of M coincides with the Betti number of the Noetherian $(t-1)$-invertible t-unipotent Λ-module $M_{\mathrm{free}} = M/M_{\mathrm{fin}}$.

The module M_{free} has no non-zero elements of finite order. Hence the annihilator Ann_v of every element $v \in M_{\mathrm{free}}$ is a principal ideal generated by some polynomial $g_v(t)$ that satisfies conditions (i)–(v) of Proposition 2.5.

Suppose that M_{free} is generated by $v_1, \dots, v_m$ over Λ. Then there is a surjective Λ-homomorphism

$$f\colon \Lambda/\big\langle g_{v_1}(t)\big\rangle \oplus \dots \oplus \Lambda/\big\langle g_{v_m}(t)\big\rangle \to M_{\mathrm{free}}.$$

We regard the modules $\widetilde{M} = \oplus\Lambda/\big\langle g_{v_i}(t)\big\rangle$ and M_{free} as free $\mathbb{Z}$-modules and write $h_{\widetilde{M}}$ (resp. $h_{M_{\mathrm{free}}}$) for the automorphism of $\widetilde{M}$ (resp. M_{free}) of multiplication by t. It is easy to see that the characteristic polynomial $\widetilde{\Delta}(t) = \det(h_{\widetilde{M}} - t\,\mathrm{Id})$ coincides with the product $g_{v_1}(t)\dots g_{v_m}(t)$ up to a sign. The characteristic polynomial $\Delta(t) = \det(h_{M_{\mathrm{free}}} - t\,\mathrm{Id})$ divides the polynomial $\widetilde{\Delta}(t)$ since the homomorphism f is surjective and t-equivariant. It follows that all the roots of $\Delta(t)$ are roots of unity different from ± 1 and, therefore, $\deg \Delta(t)$ is even. To complete the proof, we note that the Betti number of M_{free} coincides with $\deg \Delta(t)$.

2.2. Derived Alexander modules. Given any Noetherian $(t-1)$-invertible Λ-module M, we consider an infinite sequence of Noetherian $(t-1)$-invertible t-unipotent Λ-modules

$$A_n(M) = M/(t^n - 1)M, \qquad n \in \mathbb{N}. \tag{6}$$

The module $A_n(M)$ is called the *nth derived Alexander module* of the Λ-module M. We note that $A_1(M) = 0$ since M is $(t-1)$-invertible. It is also clear that $A_n\big(A_n(M)\big) = A_n(M)$.

We easily see that every Λ-homomorphism $f\colon M_1 \to M_2$ of $(t-1)$-invertible modules determines a well-defined sequence of Λ-homomorphisms

$$f_{n*}\colon A_n(M_1) \to A_n(M_2),$$

$n \in \mathbb{N}$. In other words, the map $M \mapsto \big\{A_n(M)\big\}$ is a functor from the category of Noetherian $(t-1)$-invertible Λ-modules to the category of infinite sequences of Noetherian $(t-1)$-invertible t-unipotent Λ-modules.

Proposition 2.12. *If* $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ *is an exact sequence of Noetherian* $(t-1)$*-invertible* Λ*-modules, then*

$$A_n(M_2) \simeq A_n(M)/\operatorname{im} f_{n*}\big(A_n(M_1)\big).$$

If $M = \bigoplus_{i=1}^k M_i$ *is the direct sum of the Noetherian* $(t-1)$*-invertible* Λ*-modules* M_i*, then*

$$A_n(M) \simeq \bigoplus_{i=1}^{k} A_n(M_i).$$

Proof. This is obvious.

Proposition 2.13. *Suppose that* p *is a prime,* $r \in \mathbb{N}$ *and* M *is any Noetherian* $(t-1)$*-invertible* Λ*-module. Then the derived Alexander module* $A_{p^r}(M)$ *is finite.*

Proof. This follows from Lemma 1.13 and Corollary 1.17.

Example 2.14. Consider the module $M_m = \Lambda/\big\langle (m+1)t - m\big\rangle$, where $m \in \mathbb{N}$. Its nth derived Alexander module

$$A_n(M_m) \simeq \mathbb{Z}/\big((m+1)^n - m^n\big)\mathbb{Z}$$

is a cyclic group of order $(m+1)^n - m^n$, and multiplication by t is given by

$$tv = (-1)^{n+1} m\left(\sum_{i=1}^{n-1} (-1)^i \binom{n}{i} (m+1)^{n-i-1}\right) v$$

for all $v \in A_n(M_m)$.

Proof. The module $M_m = \Lambda/\big\langle (m+1)t - m\big\rangle$ is isomorphic to the Λ-submodule $\mathbb{Z}\big[\frac{m}{m+1}, \frac{m+1}{m}\big] \subset \mathbb{Q}$, where we put $t = \frac{m}{m+1}$ and $tv = \frac{m}{m+1} v$ for $v \in \mathbb{Q}$. Therefore we have

$$A_n(M_m) \simeq M_m/(t^n - 1)M_m \simeq \mathbb{Z}\left[\frac{m+1}{m}, \frac{m}{m+1}\right] \Big/ \left\langle \left(\frac{m}{m+1}\right)^n - 1 \right\rangle.$$

We easily see that the module $\mathbb{Z}\big[\frac{m}{m+1}, \frac{m+1}{m}\big]$ is equal to the sum of the $\mathbb{Z}$-submodules $\mathbb{Z}\big[\frac{1}{m+1}\big]$ and $\mathbb{Z}\big[\frac{1}{m}\big] \subset \mathbb{Q}$:

$$\mathbb{Z}\left[\frac{m}{m+1}, \frac{m+1}{m}\right] = \mathbb{Z}\left[\frac{1}{m+1}\right] + \mathbb{Z}\left[\frac{1}{m}\right].$$

Indeed, it is clear that

$$\mathbb{Z}\left[\frac{m}{m+1}, \frac{m+1}{m}\right] \subset \mathbb{Z}\left[\frac{1}{m+1}\right] + \mathbb{Z}\left[\frac{1}{m}\right].$$

We also have

$$\left(\frac{m+1}{m}\right)^n = \frac{\sum_{i=0}^n \binom{n}{i} m^{n-i}}{m^n}$$

and, therefore,

$$\frac{1}{m^n} = \left(\frac{m+1}{m}\right)^n - \sum_{i=0}^{n-1} \binom{n}{i} \frac{1}{m^i}.$$

We similarly have

$$\frac{1}{(m+1)^n} = \sum_{i=0}^{n-1} (-1)^{n+1+i} \binom{n}{i} \frac{1}{(m+1)^i} + (-1)^n \left(\frac{m}{m+1}\right)^n.$$

In particular, $\frac{1}{m} = \frac{m+1}{m} - 1$ and $\frac{1}{m+1} = 1 - \frac{m}{m+1}$. Using induction, we see that $\frac{1}{m^n}, \frac{1}{(m+1)^n} \in \mathbb{Z}\big[\frac{m}{m+1}, \frac{m+1}{m}\big]$ for all n and, therefore,

$$\mathbb{Z}\left[\frac{1}{m+1}\right] + \mathbb{Z}\left[\frac{1}{m}\right] \subset \mathbb{Z}\left[\frac{m}{m+1}, \frac{m+1}{m}\right].$$

Hence,

$$\begin{aligned} A_n(M_m) &\simeq \mathbb{Z}\left[\frac{m+1}{m}, \frac{m}{m+1}\right] \Big/ \left\langle \left(\frac{m}{m+1}\right)^n - 1 \right\rangle \\ &\simeq \mathbb{Z}\left[\frac{m}{m+1}, \frac{m+1}{m}\right] \Big/ \big\langle (m+1)^n - m^n \big\rangle. \end{aligned}$$

We claim that every element $v \in \mathbb{Z}\big[\frac{m}{m+1}, \frac{m+1}{m}\big]$ is equivalent to some element $v_{in} \in \mathbb{Z} \subset \mathbb{Z}\big[\frac{m}{m+1}, \frac{m+1}{m}\big]$ modulo the ideal $I = \big\langle (m+1)^n - m^n \big\rangle$. To see this, it suffices to prove that for every k there are $v_{in,k}, u_{in,k} \in \mathbb{Z}$ such that

$$\frac{1}{m^k} \equiv v_{in,k} \pmod{I}, \qquad \frac{1}{(m+1)^k} \equiv u_{in,k} \pmod{I}.$$

We concentrate on proving the existence of $v_{in,k}$ since the proof of the existence of $u_{in,k}$ is similar. We have

$$\frac{(m+1)^n - m^n}{m^k} = \sum_{i=1}^{n} \binom{n}{i} m^{n-i-k} \equiv 0 \pmod{I}$$

and, therefore,

$$\frac{1}{m^k} \equiv -\sum_{j=k+1-n}^{k-1} \binom{n}{n+j-k} \frac{1}{m^j} \pmod{I}.$$

In particular,

$$\frac{1}{m} \equiv -\sum_{j=0}^{n-2} \binom{n}{n-j-1} m^j \pmod{I}.$$

The existence of $v_{in,k}$ now follows by induction on k.

It follows from this argument that

$$A_n(M_m) \simeq \mathbb{Z}\left[\frac{m}{m+1}, \frac{m+1}{m}\right] \Big/ \big\langle (m+1)^n - m^n \big\rangle$$

is a cyclic group generated by the image $\bar{1}$ of the element

$$1 \in \mathbb{Z}\Big[\frac{m}{m+1}, \frac{m+1}{m}\Big].$$

We have $\big((m+1)^n - m^n\big)\bar{1} = 0$, whence the order of $A_n(M_m)$ divides $(m+1)^n - m^n$.

We claim that the order of $A_n(M_m)$ is equal to $(m+1)^n - m^n$. Indeed, suppose that $k \in \mathbb{Z}$ satisfies $k\bar{1} = 0$. Then

$$k = \Big(\sum_{i_1 \leqslant i \leqslant i_2} a_i \frac{1}{(m+1)^i} + \sum_{j_1 \leqslant j \leqslant j_2} b_j \frac{1}{m^j}\Big)\big((m+1)^n - m^n\big),$$

where $a_i, b_j \in \mathbb{Z}$. Multiplying both sides by $(m+1)^{i_2}$ and m^{j_2} if $i_2 > 0$ or $j_2 > 0$, we obtain the equality

$$(m+1)^{i_2} m^{j_2} k = C\big((m+1)^n - m^n\big)$$

with some constant $C \in \mathbb{Z}$. It follows that $(m+1)^n - m^n$ divides k because the numbers m, $m+1$ and $(m+1)^n - m^n$ are pairwise coprime.

To calculate the action of t on the cyclic group

$$A_n(M_m) \simeq \mathbb{Z}/\big((m+1)^n - m^n\big)\mathbb{Z},$$

we note that

$$t\bar{1} = \overline{\frac{m}{m+1}} = (-1)^{n+1} m\Big(\sum_{i=1}^{n-1}(-1)^i \binom{n}{i}(m+1)^{n-i-1}\Big)\bar{1}$$

because an argument similar to the previous one yields that

$$\frac{1}{m+1} \equiv (-1)^{n+1}\sum_{i=1}^{n-1}(-1)^i \binom{n}{i}(m+1)^{n-i-1} \pmod{I}.$$

Proposition 2.15. *An abelian group G is isomorphic (as a $\mathbb{Z}$-module) to the derived Alexander module $A_2(M)$ of some Noetherian $(t-1)$-invertible Λ-module M if and only if G is a finite group of odd order.*

Proof. By Lemma 2.8 it suffices to prove that for every finite group G of odd order there is a Noetherian $(t-1)$-invertible Λ-module M with $A_2(M) \simeq G$.

We represent G as a direct sum of cyclic groups,

$$G = \bigoplus_{i=1}^{k} G_i,$$

and let $n_i = 2m_i + 1$ be the order of G_i.

For every i we consider the Λ-module M_{m_i} in Example 2.14. Then $A_2(M_{m_i})$ is a cyclic group of order $(m_i+1)^2 - m_i^2 = 2m_i + 1$. Hence the proposition follows from Proposition 2.12 if we put

$$M = \bigoplus_{i=1}^{k} M_{m_i}.$$

Theorem 2.16. *Let M be a Noetherian $(t-1)$-invertible t-unipotent Λ-module of unipotence index k. Then the sequence $A_1(M),\dots,A_n(M),\dots$ of its derived Alexander modules has period k, that is, $A_n(M)\simeq A_{n+k}(M)$ for all n. If n and k are coprime, then $A_n(M)=0$.*

Proof. Since the unipotence index of M is equal to k, Lemma 2.7 yields that the polynomial $f_k(t)=\sum_{i=0}^{k-1}t^i$ belongs to $\mathrm{Ann}(M)$. Moreover, to obtain $A_n(M)$ from M, we need only take the quotient of M by the relations $f_n(t)v=0$ for all $v\in M$, where $f_n(t)=\sum_{i=0}^{n-1}t^i$. To prove the periodicity of the sequence (6), it remains to note that

$$f_{n+k}(t)=t^nf_k(t)+f_n(t).$$

Suppose that n, k are coprime and polynomials $f_k(t)$, $f_n(t)$ belong to $\mathrm{Ann}(M)$. Applying Euclid's algorithm to $f_k(t)$ and $f_n(t)$, we easily deduce the existence of polynomials $g_k(t)$ and $g_n(t)$ such that

$$f_k(t)g_k(t)+f_n(t)g_n(t)=1$$

since n and k are coprime. Therefore $\mathrm{Ann}(M)=\Lambda$ and thus $A_n(M)=0$.

Example 2.17. The Λ-module $M=\Lambda/\langle t^2-t+1\rangle$ has the following derived Alexander modules:

$$A_{6k\pm1}(M)=0,\qquad A_{6k\pm2}(M)\simeq\mathbb{Z}/3\mathbb{Z},\qquad A_{6k+3}(M)\simeq(\mathbb{Z}/2\mathbb{Z})^2.$$

Multiplication of $\mathbb{Z}/3\mathbb{Z}$ by t coincides with multiplication by 2. Multiplication of $(\mathbb{Z}/2\mathbb{Z})^2$ by t coincides with a cyclic permutation of the non-zero elements of $A_{6k+3}(M)$.

Proof. The unipotence index of M is equal to 6 because t^2-t+1 divides t^6-1. Therefore $A_{6k\pm1}(M)=0$.

To compute $A_{6k+2}(M)$, it suffices to find $A_2(M)$. We have $A_2(M)=\Lambda/\langle t^2-t+1,t+1\rangle$. Since

$$t^2-t+1=(t-2)(t+1)+3,$$

we have $\Lambda/\langle t^2-t+1,t+1\rangle=\Lambda/\langle t+1,3\rangle\simeq\mathbb{Z}/3\mathbb{Z}$.

To compute $A_{6k+3}(M)$, it suffices to find $A_3(M)$. We have $A_3(M)=\Lambda/\langle t^2-t+1,t^2+t+1\rangle$. Since

$$t^2+t+1=t^2-t+1+2t,$$

we have $\Lambda/\langle t^2-t+1,t^2+t+1\rangle=\Lambda/\langle t^2+t+1,2\rangle\simeq(Z/2\mathbb{Z})^2$.

To compute $A_{6k+4}(M)$, it suffices to find $A_4(M)$. We have $A_4(M)=\Lambda/\langle t^2-t+1,t^3+t^2+t+1\rangle$. Since

$$t^3+t^2+t+1=(t+2)(t^2-t+1)+2t-1,$$

we see that the module $\Lambda/\langle t^2-t+1,t^3+t^2+t+1\rangle=\Lambda/\langle t^2-t+1,2t-1\rangle$ is isomorphic to the quotient module $M/(2t-1)M$. Let v be a generator of the biprincipal module M. Using the basis $v_1=v$, $v_2=tv$ of M over $\mathbb{Z}$, we easily see that the module $(2t-1)M$ is generated by the elements $2v_2-v_1$ and $t(2v_2-v_1)=v_2-2v_1$ because $tv_2=v_2-v_1$. Using another basis $e_1=v_1$, $e_2=v_2-2v_1$, we have $2v_2-v_1=2e_2+3e_1$, that is, $(2t-1)M$ is generated by $3e_1$ and e_2 over $\mathbb{Z}$. Therefore $A_4(M)\simeq\mathbb{Z}/3\mathbb{Z}$.

§ 3. Alexander modules of irreducible C-groups

3.1. Proofs of Theorems 0.1 and 0.3. We recall that the class of irreducible C-groups coincides with the class of fundamental groups of knotted n-manifolds V for every $n \geqslant 2$. Knot groups are also C-groups when given by the Wirtinger presentation. Similarly, the class of irreducible Hurwitz C-groups coincides with the class of fundamental groups of complements of irreducible 'affine' Hurwitz (or pseudo-holomorphic) curves. It contains the subclass of fundamental groups of complements of algebraic irreducible plane affine curves. Therefore speaking about the Alexander modules of knotted n-manifolds (resp. irreducible Hurwitz or pseudo-holomorphic curves) is the same as speaking about the Alexander modules of irreducible C-groups (resp. irreducible Hurwitz C-groups). Therefore Theorems 0.1 and 0.3 are equivalent to the following theorems.

Theorem 3.1. *A Λ-module M is the Alexander module of an irreducible C-group if and only if it is Noetherian and $(t-1)$-invertible.*

Theorem 3.2. *A Λ-module M is the Alexander module of an irreducible Hurwitz C-group if and only if it is a Noetherian $(t-1)$-invertible t-unipotent Λ-module. The unipotence index of the Alexander module $A_0(G)$ of an irreducible Hurwitz C-group G of degree m is a divisor of m.*

Proof. Let

$$G = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle \tag{7}$$

be a C-presentation of a C-group G and let $\mathbb{F}_m$ be the free group freely generated by the C-generators $x_1, \dots, x_m$. We write $\frac{\partial}{\partial x_i}$ for the Fox derivative (see [15]). This is an endomorphism of the group ring $\mathbb{Z}[\mathbb{F}_m]$ over $\mathbb{Z}$ of the free group $\mathbb{F}_m$ such that $\frac{\partial}{\partial x_i} \colon \mathbb{Z}[\mathbb{F}_m] \to \mathbb{Z}[\mathbb{F}_m]$ is a $\mathbb{Z}$-linear map with the following properties:

$$\frac{\partial x_j}{\partial x_i} = \delta_{i,j}, \qquad \frac{\partial uv}{\partial x_i} = \frac{\partial u}{\partial x_i} + u\frac{\partial v}{\partial x_i} \tag{8}$$

for all $u, v \in \mathbb{Z}[\mathbb{F}_m]$. The matrix

$$\mathcal{A}(G) = \nu_*\left(\frac{\partial r_i}{\partial x_j}\right) \in \operatorname{Mat}_{n\times m}\bigl(\mathbb{Z}[t, t^{-1}]\bigr)$$

is called the *Alexander matrix* of the C-group G given by presentation (7), where the r_i $(i = 1, \dots, n)$ are the relators of G and the homomorphism $\nu_* \colon \mathbb{Z}[\mathbb{F}_m] \to \mathbb{Z}[\mathbb{F}_1] \simeq \mathbb{Z}[t, t^{-1}]$ is induced by the canonical C-epimorphism $\nu \colon \mathbb{F}_m \to \mathbb{F}_1$.

The following lemma is a generalization (to the case of C-groups) of a well-known assertion concerning the Alexander matrices of Wirtinger presentations of knot groups (see [15]).

Lemma 3.3. *Let $\mathcal{A}(G)$ be the Alexander matrix of a C-group G given by the presentation* (7)*. Then the sum of the columns of $\mathcal{A}(G)$ is equal to zero.*

Proof. Each relator r_i has the form

$$r = wx_jw^{-1}x_l^{-1},$$

where w is a word in the letters $x_1^{\pm 1}, \dots, x_m^{\pm 1}$, and x_j, x_l are letters.

Let us use induction on the length $l(w)$ of the word w to show that

$$\sum_{k=1}^{m} \nu_*\Big(\frac{\partial r}{\partial x_k}\Big) = 0.$$

If $l(w) = 0$ (that is, $r := x_j x_l^{-1}$), then we have

$$\nu_*\Big(\frac{\partial r}{\partial x_k}\Big) = \begin{cases} 1 & \text{if } k = j, \\ -1 & \text{if } k = l, \\ 0 & \text{if } k \neq j \text{ and } k \neq l. \end{cases}$$

In this case we see that $\sum_{k=1}^{m} \nu_*\big(\frac{\partial r}{\partial x_k}\big) = 0$.

Suppose that the equation $\sum_{k=1}^{m} \nu_*\big(\frac{\partial r}{\partial x_k}\big) = 0$ holds for all words $r = wx_j w^{-1} x_l^{-1}$ with $l(w) \leqslant L$. Consider any word $r = wx_j w^{-1} x_l^{-1}$ of length $l(w) = L + 1$. Put $r_1 = w_1 x_j w_1^{-1} x_l^{-1}$, where $w = x_i^{\varepsilon} w_1$, $\varepsilon = \pm 1$, and $l(w_1) = L$. We consider only the case when $i \neq j$, $i \neq l$, $j \neq l$ and $\varepsilon = 1$. The proof that $\sum_{k=1}^{m} \nu_*\big(\frac{\partial r}{\partial x_k}\big) = 0$ in all other cases is similar.

It follows from (8) that

$$\nu_*\Big(\frac{\partial r}{\partial x_k}\Big) = \begin{cases} t\nu_*\big(\frac{\partial r_1}{\partial x_k}\big) & \text{if } k \neq i,\ \ k \neq j,\ \ k \neq l, \\ 1 + t\nu_*\big(\frac{\partial r_1}{\partial x_k}\big) - t & \text{if } k = i, \\ t\nu_*\big(\frac{\partial r_1}{\partial x_k}\big) & \text{if } k = j, \\ t\big(\nu_*\big(\frac{\partial r_1}{\partial x_k}\big) + 1\big) - 1 & \text{if } k = l \end{cases}$$

and it is easy to see that $\sum_{k=1}^{m} \nu_*\big(\frac{\partial r}{\partial x_k}\big) = 0$. The lemma is proved.

For every monomial $a_i t^i \in \mathbb{Z}[t]$ we consider the word

$$w_{a_i t^i}(x_1, x_2) = \big(x_2^i x_1 x_2^{-(i+1)}\big)^{a_i}.$$

For every polynomial $g(t) = \sum_{i=0}^{k} a_i t^i \in \mathbb{Z}[t]$ we put

$$w_{g(t)}(x_1, x_2) = \prod_{i=0}^{k} w_{a_i t^i}(x_1, x_2).$$

We also associate with any polynomial $f(t) = (1 - t)g(t) + 1$ the word

$$r_{f(t)}(x_1, x_2) = w_{g(t)}(x_1, x_2) x_1 w_{g(t)}^{-1}(x_1, x_2) x_2^{-1}. \tag{9}$$

For every vector $u = (1-t)\bar{u} = \big((1-t)g_1(t), \dots, (1-t)g_m(t)\big)$ we consider the word

$$r_u(x_1, \dots, x_{m+1}) = w_u(x_1, \dots, x_{m+1}) x_{m+1} w_u^{-1}(x_1, \dots, x_{m+1}) x_{m+1}^{-1}, \tag{10}$$

where

$$w_u(x_1, \dots, x_{m+1}) = \prod_{i=1}^{m} w_{g_i(t)}(x_i, x_{m+1}).$$

Lemma 3.4. *Take a polynomial $f(t) = (1-t)g(t) + 1$ and a vector*

$$u = \big((1-t)g_1(t), \ldots, (1-t)g_m(t)\big).$$

Then we have

$$\nu_*\left(\frac{\partial r_{f(t)}}{\partial x_1}\right) = f(t), \qquad \nu_*\left(\frac{\partial r_u}{\partial x_i}\right) = (1-t)g_i(t), \qquad i = 1, \ldots, m.$$

Proof. Take $f(t) = (1-t)g(t) + 1$. It follows from (8) that

$$\nu_*\left(\frac{\partial w_{g(t)}(x_1, x_2)}{\partial x_1}\right) = -\nu_*\left(\frac{\partial w^{-1}_{g(t)}(x_1, x_2)}{\partial x_1}\right) = g(t)$$

since we have $w_{g(t)}(x_1, x_2) w^{-1}_{g(t)}(x_1, x_2) = 1$ and

$$\nu_*\big(w_{g(t)}(x_1, x_2)\big) = \nu_*\big(w_{a_i t^i}(x_1, x_2)\big) = 1,$$

$$\nu_*\left(\frac{\partial w_{a_i t^i}(x_1, x_2)}{\partial x_1}\right) = a_i t^i.$$

Therefore,

$$\begin{aligned}\nu_*\left(\frac{\partial r_{f(t)}}{\partial x_1}\right) &= \nu_*\left(\frac{\partial\big(w_{g(t)}(x_1, x_2) x_1 w^{-1}_{g(t)}(x_1, x_2) x_2^{-1}\big)}{\partial x_1}\right)\\ &= g(t) + 1 - t g(t) = f(t).\end{aligned}$$

The proof of the second equation of the lemma is similar.

Proposition 3.5. *The Alexander module $A_0(G)$ of the C-group G with the presentation (7) is isomorphic to the quotient module $\Lambda^{m-1}/M(G)$, where the submodule $M(G)$ of Λ^{m-1} is generated by the rows of the matrix $\bar{\mathcal{A}}$ formed by the first $m-1$ columns of the Alexander matrix $\mathcal{A}(G)$.*

Proof. To describe the Alexander module of a C-group G, we follow [16] (see also [8]). Given a C-group G with C-presentation (7), we consider the following complex K with a single vertex x_0. Its one-dimensional skeleton is a wedge product of oriented circles s_i ($1 \leqslant i \leqslant m$) that are in one-to-one correspondence with the C-generators of G in the presentation (7). Furthermore, $K \setminus \big(\bigcup s_i\big) = \bigsqcup_{i=1}^n \mathring{D}_i$ is a disjoint union of open discs. Each disc D_i corresponds to the relator $r_i = x_{j_{i,1}}^{\varepsilon_{i,1}} \ldots x_{j_{i,k_i}}^{\varepsilon_{i,k_i}}$ in (7), where $\varepsilon_{i,j} = \pm 1$. This disc is glued to the wedge product $\vee s_i$ by identifying the boundary $\partial D_j = D_j \setminus \mathring{D}_i$ and the closed path $s_{j_{i,1}}^{\varepsilon_{i,1}} \ldots s_{j_{i,k_i}}^{\varepsilon_{i,k_i}} \subset \vee s_i$. Clearly, $\pi_1(K, x_0) \simeq G$.

The C-homomorphism $\nu\colon G \to \mathbb{F}_1$ determines an infinite cyclic covering $f\colon \widetilde{K} \to K$ such that $\pi_1(\widetilde{K}) = N$ and $H_1(\widetilde{K}, \mathbb{Z}) = N/N'$, where $N = \ker \nu$. The group $\mathbb{F}_1$ acts on $\widetilde{K}$.

Let $\widetilde{K}_0 = f^{-1}(x_0)$ and let $\widetilde{K}_1$ be the one-dimensional skeleton of the complex $\widetilde{K}$. We consider the following exact sequences of homomorphisms of homology groups with coefficients in $\mathbb{Z}$:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & H_1(\widetilde{K}) & & \\
 & & & & & & \downarrow & & \\
\longrightarrow & H_2(\widetilde{K},\widetilde{K}_1) & \xrightarrow{\alpha} & H_1(\widetilde{K}_1,\widetilde{K}_0) & \xrightarrow{\beta} & & H_1(\widetilde{K},\widetilde{K}_0) & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle\partial} & & \\
 & & & & & & H_0(\widetilde{K}_0) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array} \tag{11}$$

The action of $\mathbb{F}_1$ on $\widetilde{K}$ endows each group in these sequences with the structure of a Λ-module. We fix a vertex $p_0 \in \widetilde{K}_0$. Let $p_i = t^i p_0$ be the result of the action of the element $t^i \in \mathbb{F}_1$ on the point p_0. Then $H_1(\widetilde{K}_1,\widetilde{K}_0)$ is a free Λ-module whose generators $\bar{s}_i$ are edges that join p_0 to p_1 and are mapped by f onto the loops s_i. The result of the action of t^i on the generator $\bar{s}_j$ is an edge that begins at the vertex p_i and is mapped by f onto the loop s_j.

The free Λ-module $H_2(\widetilde{K},\widetilde{K}_1)$ is generated by the discs $\overline{D}_i$ $(i = 1,\dots,n)$ that correspond to the relators $r_i = x_{j_{i,1}}^{\varepsilon_{i,1}} \dots x_{j_{i,k_i}}^{\varepsilon_{i,k_i}}$. Each disc $\overline{D}_i$ is glued to the one-dimensional skeleton along the product of paths

$$t^{\delta(\varepsilon_{i,1})}\bar{s}_{j_{i,1}}^{\varepsilon_{i,1}}, t^{\delta(\varepsilon_{i,2})+\varepsilon_{i,1}}\bar{s}_{j_{i,2}}^{\varepsilon_{i,2}}, \dots, t^{\delta(\varepsilon_{i,k_i})+\sum_{l=1}^{k_i-1}\varepsilon_{i,l}}\bar{s}_{j_{i,k_i}}^{\varepsilon_{i,k_i}},$$

where $\delta(1) = 0$ and $\delta(-1) = -1$. It is easy to verify that the coordinates of the elements $\alpha(\overline{D}_i) \in H_1(\widetilde{K}_1,\widetilde{K}_0)$ in the basis $\bar{s}_1,\dots,\bar{s}_m$ coincide with the rows $\mathcal{A}_i$ of the Alexander matrix $\mathcal{A}(G)$ of the C-group G with the presentation (7).

It follows from the vertical exact sequence in (11) that $\partial\big(\beta(\bar{s}_i)\big) = (t-1)p_0$ for each generator $\bar{s}_i$ of the module $H_1(\widetilde{K}_1,\widetilde{K}_0)$. We choose a new basis of this module by putting $e_i = \bar{s}_i - \bar{s}_m$, $i = 1,\dots,m-1$, $e_m = \bar{s}_m$. Then $\beta(e_i) \in \ker\partial$ for $i = 1,\dots,m-1$ and $\ker\partial$ is generated by $\beta(e_1),\dots,\beta(e_{m-1})$. Hence we may identify the module $H_1(\widetilde{K})$ with $\beta\big(H_1'(\widetilde{K}_1,\widetilde{K}_0)\big)$, where $H_1'(\widetilde{K}_1,\widetilde{K}_0)$ is the free submodule generated by the elements $e_1,\dots,e_{m-1}$ of the free Λ-module $H_1(\widetilde{K}_1,\widetilde{K}_0)$.

The matrix formed by the coordinates of the elements $\alpha(\overline{D}_i)$ in the basis $e_1,\dots,e_m$ coincides with the matrix $\tilde{\mathcal{A}}(G)$ obtained from $\mathcal{A}(G)$ by replacing the last column by a column of zeros. Hence $H_1(\widetilde{K})$ is isomorphic to the quotient module of the free Λ-module $H_1'(\widetilde{K}_1,\widetilde{K}_0) \simeq \bigoplus_{i=1}^{m-1}\Lambda e_i$ by the submodule $M(G)$ generated by the rows of the matrix $\bar{\mathcal{A}}(G)$ formed by the first $m-1$ columns of $\mathcal{A}(G)$. This proves the proposition.

We use Proposition 1.11 to prove that every Noetherian $(t-1)$-invertible (resp. t-unipotent) Λ-module M is the Alexander module of some irreducible (resp. Hurwitz) C-group. By Proposition 1.11 every Noetherian $(t-1)$-invertible Λ-module M is isomorphic to a quotient module Λ^m/M_1 of the free Λ-module Λ^m. Here the submodule M_1 is generated by elements $u_1,\dots,u_m,\dots,u_{m+k}$ of Λ^m such that

(i) we have $u_i=\big(0,\dots,0,f_i(t),0,\dots,0\big)$ for $i=1,\dots,m$, where the polynomial $f_i(t)$ occupies the ith position and $f_i(1)=1$,

(ii) we have $u_{m+j}=(1-t)\bar u_{m+j}=\big((1-t)g_{j,1}(t),\dots,(1-t)g_{j,m}(t)\big)$ for $j=1,\dots,k$, where the $g_{j,l}(t)$ are polynomials.

If M is a t-unipotent Λ-module of unipotence index n, then we can also assume that

(iii) we have $u_{m+k+i}=(0,\dots,0,t^n-1,0,\dots,0)$ for $i=1,\dots,m$, where the polynomial t^n-1 occupies the ith position.

We write each polynomial $f_i(t)$ as $f_i(t)=(1-t)g_i(t)+1$ and consider the C-group

$$G=\langle x_1,\dots,x_{m+1}\mid r_1,\dots,r_{m+k}\rangle$$

with relators $r_i:=r_{f_i(t)}(x_i,x_{m+1})$ for $i=1,\dots,m$ and $r_{m+j}:=r_u(x_1,\dots,x_{m+1})$ for $j=1,\dots,k$, where the words $r_{f(t)}$ and r_u are defined by (9) and (10). We put $r_{m+k+i}:=x_{m+1}^n x_i x_{m+1}^{-n}x_i^{-1}$ if

$$u_{m+k+i}=(0,\dots,0,t^n-1,0,\dots,0)\in M_1$$

for $i=1,\dots,m$. Let $\overline{G}$ be the group defined by the presentation

$$\overline{G}=\langle x_1,\dots,x_{m+1}\mid r_1,\dots,r_{2m+k}\rangle.$$

Lemma 3.4 implies that the matrix $\tilde{\mathcal{A}}(G)$ (resp. $\tilde{\mathcal{A}}(\overline{G}\,)$) formed by the first m columns of the Alexander matrix $\mathcal{A}(G)$ (resp. $\mathcal{A}(\overline{G}\,)$) coincides with the matrix $\mathcal{U}$ (resp. $\overline{\mathcal{U}}$) formed by the rows $u_1,\dots,u_{m+k}$ (resp. $u_1,\dots,u_{2m+k}$). Therefore, by Proposition 3.5, the Alexander module $A_0(G)$ (resp. $A_0(\overline{G}\,)$) coincides with $M=\Lambda^m/M_1$, where M_1 is generated by the rows $u_1,\dots,u_{m+k}$ (resp. $u_1,\dots,u_{2m+k}$).

We note that G (resp. $\overline{G}$) is an irreducible C-group since all the C-generators $x_1,\dots,x_m$ are conjugate to x_{m+1}. Moreover, $\overline{G}$ is a Hurwitz C-group. Indeed, the relators r_{m+k+j} $(j=1,\dots,m)$ imply that x_{m+1}^n belongs to the centre of $\overline{G}$. Since all the x_i are conjugate to x_{m+1}, we have $x_i^n=x_{m+1}^n$ for all $i=1,\dots,m$. Hence the product $x_1^n\dots x_{m+1}^n$ also belongs to the centre of $\overline{G}$, and $\overline{G}$ possesses a Hurwitz presentation

$$\begin{aligned}\overline{G}=\big\langle y_1,\dots,y_{n(m+1)}\,|\,&\tilde r_1,\dots,\tilde r_{2m+k},\\ &y_{in+1}y_{in+j}^{-1},\ i=0,1,\dots,m,\ j=2,\dots,n,\\ &\big[y_i,\big(y_1\dots y_{n(m+1)}\big)\big],\ i=1,\dots,n(m+1)\big\rangle,\end{aligned}$$

where the relators $\tilde r_i=\tilde r_i(y_1,\dots,y_{n(m+1)})$ are obtained from the relators $r_i=r_i(x_1,\dots,x_{m+1})$ by substituting y_{jn+1} for x_j, $j=1,\dots,m+1$.

The following lemmas complete the proofs of Theorems 0.1 and 0.3.

Lemma 3.6 [17]. *The Alexander module $A_0(G) = G'/G''$ of an irreducible C-group G is a Noetherian $(t-1)$-invertible Λ-module.*

Proof. Since G is an irreducible C-group, its commutator subgroup G' coincides with the kernel of the C-epimorphism $\nu\colon G \to \mathbb{F}_1$. By the Reidemeister–Schreier method, if C-generators $x_1, \dots, x_m$ generate G, then the elements $a_{i,n} = x_m^n x_i x_m^{-(n+1)}$ $(i=1,\dots,m-1,\ n\in\mathbb{Z})$ generate G'. Hence the module $A_0(G)=G'/G''$ is generated by the images $\bar{a}_{i,n}$ of the elements $a_{i,n}$ under the natural epimorphism $G' \to G'/G''$. The action of t on $A_0(G)$ is defined by the conjugation $a \mapsto x_m a x_m^{-1}$ for $a \in G'$. Therefore $t\bar{a}_{i,n} = \bar{a}_{i,n+1}$. Thus the module $A_0(G)$ is generated over Λ by $\bar{a}_{1,0}, \dots, \bar{a}_{m-1,0}$ and is therefore a Noetherian Λ-module.

To show that $A_0(G)$ is a $(t-1)$-invertible Λ-module, we note that every element $g \in G$ may be written as $g = x_m^k a$, where $a \in G'$ and $k = \nu(g)$. Hence the group G' is generated by the elements $[x_m^n a, x_m^k b]$, where $a, b \in G'$. Therefore $A_0(G)$ is generated by their images $\overline{[x_m^n a, x_m^k b]}$. It is easy to see that

$$\begin{aligned}[x_m^n a, x_m^k b] = [x_m^n, a]\bigl(a x_m^{n+k}[b, a^{-1}]x_m^{-(n+k)}a^{-1}\bigr)[a, x_m^{n+k}]&\\ \cdot\bigl(x_m^{n+k}[b, x_m^{-n}]x_m^{-(n+k)}\bigr).&\end{aligned} \tag{12}$$

It follows from (12) that

$$\begin{aligned}\overline{[x_m^n a, x_m^k b]} &= (t^n - 1)\bar{a} + (1 - t^{n+k})\bar{a} + t^{n+k}(1 - t^{-n})\bar{b}\\ &= t^n(1 - t^k)\bar{a} + t^k(t^n - 1)\bar{b} = (t-1)\Biggl(\sum_{i=0}^{n-1} t^{i+k}\bar{b} - \sum_{i=0}^{k-1} t^{i+n}\bar{a}\Biggr)\end{aligned} \tag{13}$$

since $a x_m^{n+k}[b, a^{-1}]x_m^{-(n+k)}a^{-1} \in G''$. We now easily see that multiplication by $t-1$ is an epimorphism of $A_0(G)$ onto itself since the elements of the form $\overline{[x_m^n a, x_m^k b]}$ generate $A_0(G)$ over $\mathbb{Z}$. To complete the proof, we apply Lemma 1.1.

Lemma 3.7 [13]. *The Alexander module of a Hurwitz irreducible C-group of degree m is a Noetherian $(t-1)$-invertible t-unipotent Λ-module of unipotence index d, where d is a divisor of m.*

Proof. If G is a Hurwitz group of degree m, then it is generated by C-generators $x_1, \dots, x_m$ such that the product $x_1 \dots x_m$ belongs to the centre of G. By Lemma 3.6, the Alexander module $A_0(G) = G'/G''$ is a Noetherian $(t-1)$-invertible Λ-module. Multiplication of the module $A_0(G)$ by t is induced by the conjugation $a \mapsto x_m a x_m^{-1}$ for $a \in G'$. Since $\nu(x_m^m) = \nu(x_1 \dots x_m)$, there is an element $a_0 \in G'$ such that $x_m^m = a_0 x_1 \dots x_m$. Hence conjugation by x_m^m is an inner automorphism of G'. Therefore the induced automorphism t^m of G'/G'' is the identity.

3.2. Alexander modules of C-products of C-groups. Let G_1, G_2 be irreducible C-groups and let $x \in G_1$ (resp. $y \in G_2$) be one of the C-generators of G_1 (resp. G_2). We consider the amalgamated product $G_1 *_{\{x=y\}} G_2$. If G_1 and G_2 are given by C-presentations

$$\begin{aligned}G_1 &= \langle x_1, \dots, x_n \mid \mathcal{R}_1\rangle,\\ G_2 &= \langle y_1, \dots, y_m \mid \mathcal{R}_2\rangle,\end{aligned} \tag{14}$$

where $x = x_n$ and $y = y_m$, then the group $G_1 *_{\{x=y\}} G_2$ is given by the C-presentation

$$\langle x_1, \dots, x_{n-1}, y_1, \dots, y_{m-1}, z \mid \widetilde{\mathcal{R}}_1 \cup \overline{\mathcal{R}}_2 \rangle, \tag{15}$$

where each relator $\tilde{r}_i \in \widetilde{\mathcal{R}}_1$ (resp. $\bar{r}_i \in \overline{\mathcal{R}}_2$) is obtained from $r_i \in \mathcal{R}_1$ (resp. $r_i \in \mathcal{R}_2$) by substituting the letter z for the letter x_n (resp. y_m).

If $x' \in G_1$ and $y' \in G_2$ are other C-generators of these groups, then there are inner C-isomorphisms $f_i \colon G_i \to G_i$ such that $f_1(x') = x$ and $f_2(y') = y$ because all the C-generators of an irreducible C-group are conjugate to each other. Hence there is a C-isomorphism

$$f_1 * f_2 \colon G_1 *_{\{x'=y'\}} G_2 \to G_1 *_{\{x=y\}} G_2.$$

In other words, the group $G_1 *_{\{x=y\}} G_2$ is independent (up to C-isomorphism) of the choice of the C-generators x and y. We denote this group by $G_1 *_C G_2$ and call it the *C-product* of the irreducible C-groups G_1 and G_2.

Proposition 3.8. *The Alexander module of the C-product $G = G_1 *_C G_2$ of irreducible C-groups G_1 and G_2 is isomorphic to the direct sum of the Alexander modules of G_1 and G_2:*

$$A_0(G) = A_0(G_1) \oplus A_0(G_2).$$

Proof. This follows easily from Proposition 3.5. Indeed, if G_1 and G_2 are given by the presentations (14), then Proposition 3.5 yields that the Alexander module $A_0(G)$ of the irreducible C-group $G = G_1 *_C G_2$ (given by the presentation (15)) is isomorphic to the quotient module $\Lambda^{n+m-1}/M(G)$, where the submodule $M(G)$ of Λ^{n+m-1} is generated by the rows of the matrix

$$\bar{\mathcal{A}} = \begin{pmatrix} \bar{\mathcal{A}}_1 & 0 \\ 0 & \bar{\mathcal{A}}_2 \end{pmatrix}.$$

Here $\bar{\mathcal{A}}_1$ (resp. $\bar{\mathcal{A}}_2$) is the matrix formed by the first $n-1$ (resp. $m-1$) columns of the matrix $\mathcal{A}(G_1)$ (resp. $\mathcal{A}(G_2)$). It is now easy to see that $A_0(G) = A_0(G_1) \oplus A_0(G_2)$. The proposition is proved.

Let

$$G = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle \tag{16}$$

be a C-presentation of some C-group G. The number $d_P = m - n$ is called the *C-deficiency* of the presentation (16). The number $d_G = \max d_P$, where the maximum is taken over all C-presentations of G, is called the *C-deficiency* of the C-group G. Clearly, the C-deficiency satisfies $d_G \leqslant k$ if the C-group consists of k irreducible components. In particular, if G is an irreducible C-group, then $d_G \leqslant 1$.

Lemma 3.9. *Let $G = G_1 *_C G_2$ be the C-product of the irreducible C-groups G_1 and G_2. Then*

$$d_G \geqslant d_{G_1} + d_{G_2} - 1.$$

In particular, if $d_{G_1} = d_{G_2} = 1$, then $d_G = 1$.

Proof. This follows from formula (15).

3.3. Presentation graphs of C-groups. We associate a *presentation graph* Γ_P with each C-presentation (16). The vertices of Γ_P are labelled by the generators appearing in (16). (In particular, there is a one-to-one correspondence between the vertices of Γ_P and these generators.) The edges of Γ_P are in one-to-one correspondence with the relators r_j appearing in (16). If $r_j := w_j^{-1}(x_1, \dots, x_m)x_{i_1}w_j(x_1, \dots, x_m)x_{i_2}^{-1}$, then the corresponding edge connects the vertices x_{i_1} and x_{i_2}.

Clearly, the C-deficiency satisfies

$$d_P = \dim H_0(\Gamma_P, \mathbb{R}) - \dim H_1(\Gamma_P, \mathbb{R}).$$

Therefore *the C-deficiency d_G of an irreducible C-group G is equal to 1 if and only if G possesses a C-presentation whose graph Γ_P is a tree.*

A C-presentation

$$G = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle \tag{17}$$

is said to be *simple* if each relator r_j in (17) is of the form

$$r_j := x_{i_3}^{-1}x_{i_1}x_{i_3}x_{i_2}^{-1}$$

for some $i_1, i_2, i_3 \in \{1, \dots, m\}$ (that is, the relator r_j is given by $x_{i_2} = x_{i_3}^{-1}x_{i_1}x_{i_3}$).

Remark 3.10. If the presentations (14) of irreducible C-groups G_1 and G_2 are simple, then so is the presentation (15) of their product $G = G_1 *_C G_2$, and the graph Γ_P of the presentation (15) is the wedge product $\Gamma_P = \Gamma_{P_1} \vee_{z=x_n=y_m} \Gamma_{P_2}$ of the graphs Γ_{P_1} and Γ_{P_2} of the presentations (14). In particular, if Γ_{P_1} and Γ_{P_2} are trees, then Γ_P is a tree.

Lemma 3.11. *Every C-group G possesses a simple C-presentation of C-deficiency $d_P = d_G$.*

Proof. Let G be given by a C-presentation of C-deficiency $d_P = d_G$ and let $r := w^{-1}x_iwx_j^{-1}$ be one of its relators (so that $w^{-1}x_iw = x_j$), where $w = x_{i_1}^{\varepsilon_1} \dots x_{i_k}^{\varepsilon_k}$ is a word in the group $\mathbb{F}_m$ and $\varepsilon_l = \pm 1$. Then we can add $k-1$ new generators $x_{m+1}, \dots, x_{m+k-1}$ and replace r by the k simple relations

$$\begin{aligned}
x_{m+1} &= x_{i_1}^{-\varepsilon_1}x_ix_{i_1}^{\varepsilon_1}, \\
x_{m+2} &= x_{i_2}^{-\varepsilon_2}x_{m+1}x_{i_2}^{\varepsilon_2}, \\
&\dots\dots\dots\dots\dots\dots \\
x_{m+k-1} &= x_{i_{k-1}}^{-\varepsilon_{k-1}}x_{m+k-2}x_{i_{k-1}}^{\varepsilon_{k-1}}, \\
x_j &= x_{i_k}^{-\varepsilon_k}x_{m+k-1}x_{i_k}^{\varepsilon_k}.
\end{aligned}$$

Clearly, we get a new C-presentation which has the same C-deficiency and defines the same C-group G.

3.4. Alexander modules of C-groups possessing C-presentations whose graphs are trees. Lemma 3.11 shows that an irreducible C-group G possesses a simple C-presentation whose graph is a tree if and only if the C-deficiency d_G is equal to 1.

Proposition 3.12. *If $M = \bigoplus_{i=1}^{m} M_i$ is the direct sum of the biprincipal $(t-1)$-invertible Λ-modules $M_i = \Lambda/\langle f_i(t)\rangle$, then there is an irreducible C-group G such that the C-deficiency d_G is equal to 1 and $A_0(G) \simeq M$.*

Proof. Consider the C-group given by the presentation

$$G = \langle x_1, x_2 \mid w x_1 w^{-1} x_2^{-1} \rangle, \tag{18}$$

where $w = w(x_1, x_2)$ is a word in letters x_1, x_2 and their inverses. Note that the C-deficiency of G is equal to 1. Applying Proposition 3.5, we see that the Alexander module $A_0(G)$ of an irreducible C-group G given by the presentation (18) is a biprincipal $(t-1)$-invertible Λ-module.

Conversely, it was shown in the proof of Theorem 3.1 that every biprincipal $(t-1)$-invertible Λ-module $M = \Lambda/\langle f(t)\rangle$ is the Alexander module of some irreducible C-group given by (18). To complete the proof, we apply Proposition 3.8 and Remark 3.10.

Corollary 3.13. *Let $M = \bigoplus_{i=1}^{m} M_i$ be the direct sum of the biprincipal $(t-1)$-invertible Λ-modules $M_i = \Lambda/\langle f_i(t)\rangle$. Then for every $n \geqslant 2$ there is a knotted sphere $S^n \subset S^{n+2}$ such that*

$$A_0\big(\pi_1(S^{n+2} \setminus S^n)\big) \simeq M.$$

In particular, a polynomial $f(t) \in \mathbb{Z}[t]$ is the Alexander polynomial $\Delta(t)$ of some knotted sphere $S^n \subset S^{n+2}$ with $n \geqslant 2$ if and only if $f(1) = \pm 1$. Moreover, the Jordan blocks of the Jordan canonical form of the matrix of the automorphism $h_{\mathbb{C}}$ acting on $A_0(S^n) \otimes \mathbb{C}$ can be of arbitrary size.

Proof. Let G be an irreducible C-group given by a simple presentation whose graph is a tree. As shown in [1], for every $n \geqslant 2$ there is a knotted sphere $S^n \subset S^{n+2}$ such that $\pi_1(S^{n+2} \setminus S^n) \simeq G$. This proves the corollary.

Proposition 3.14. *Let G be an irreducible C-group of C-deficiency $d_G = 1$. Then its Alexander module $A_0(G)$ has no non-zero $\mathbb{Z}$-torsion elements.*

Proof. Let G be given by a C-presentation

$$G = \langle x_1, \dots, x_m \mid r_1, \dots, r_{m-1} \rangle. \tag{19}$$

By Proposition 3.5 the Alexander module $A_0(G)$ is isomorphic to the quotient module $\Lambda^{m-1}/M(G)$, where the submodule $M(G)$ of Λ^{m-1} is generated by the rows of the matrix $\bar{\mathcal{A}}$ formed by the first $m-1$ columns of the Alexander matrix $\mathcal{A}(G)$ of the group G given by (19). The size of the matrix $\bar{\mathcal{A}}$ is $(m-1) \times (m-1)$.

Lemma 3.15. *The determinant $\Delta(t) = \det \bar{\mathcal{A}}$ satisfies $\Delta(1) = \pm 1$.*

Proof. The proof coincides with that of the corresponding statement for knot groups (see, for example, [15]).

We denote the rows of the matrix $\bar{\mathcal{A}}$ by $\mathcal{A}_j$, $j = 1, \dots, m-1$. The module $A_0(G)$ has a non-zero $\mathbb{Z}$-torsion element if and only if there is a vector

$u = \big(f_1(t), \dots, f_{m-1}(t)\big)$ such that $u \notin M(G)$ and $ku \in M(G)$ for some $k \in \mathbb{N}$. Assume that there is such a vector u. Then there are elements $g_j(t) \in \Lambda$ such that $ku = \sum g_j(t)\mathcal{A}_j$ and at least one coefficient of one of these elements $g_j(t)$ is not divisible by k.

There is no loss of generality in assuming that all the elements $f_i(t)$ and $g_j(t)$ belong to $\mathbb{Z}[t]$. By Cramér's rule we have

$$g_j(t) = \frac{\Delta_j(t)}{\Delta(t)},$$

where $\Delta_j(t)$ is the determinant of the matrix obtained from $\bar{\mathcal{A}}$ by substituting the row ku for the row $\mathcal{A}_j$. Hence all the coefficients of all the polynomials $\frac{\Delta_j(t)}{\Delta(t)}$ are divisible by k, a contradiction.

Remark 3.16. Let G be an irreducible C-group given by a presentation of C-deficiency $d_P = d_G = 1$ and let $\bar{\mathcal{A}}$ be the matrix obtained from the Alexander matrix $\mathcal{A}$ by removing its last column. Then the determinant $\Delta(t) = \det \bar{\mathcal{A}}$ coincides with the Alexander polynomial $\Delta_G(t)$ of the group G.

3.5. Finitely $\mathbb{Z}$-generated Alexander modules of irreducible C-groups.

Theorem 3.17. *The Alexander module $A_0(G)$ of an irreducible C-group G is finitely generated over $\mathbb{Z}$ if and only if the leading coefficient a_n and the constant coefficient a_0 of the Alexander polynomial $\Delta_G(t) = \sum_{i=0}^{n} a_i t^i$ of G are equal to ± 1.*

Proof. By Theorem 3.1, $A_0(G)$ is a Noetherian $(t-1)$-invertible Λ-module. Let $A_0(G)_{\mathrm{fin}}$ be the $\mathbb{Z}$-torsion submodule of the Alexander module $A_0(G)$. By Theorem 1.16, the module $A_0(G)_{\mathrm{fin}}$ is finitely generated over $\mathbb{Z}$.

Consider the quotient module $M = A_0(G)/A_0(G)_{\mathrm{fin}}$. It is $\mathbb{Z}$-torsion free. Hence there is a natural embedding $M \hookrightarrow M_{\mathbb{Q}} = M \otimes \mathbb{Q}$. We have $\dim_{\mathbb{Q}} M_{\mathbb{Q}} < \infty$ since M is a Noetherian Λ-torsion module.

Let $h_{\mathbb{Q}}$ be the automorphism of $M_{\mathbb{Q}}$ induced by multiplication by t. By definition, we have $\Delta_G(t) = a \det(h_{\mathbb{Q}} - t\,\mathrm{Id})$, where $a \in \mathbb{N}$ is the smallest positive integer such that $a \det(h_{\mathbb{Q}} - t\,\mathrm{Id}) \in \mathbb{Z}[t]$.

If the Alexander module $A_0(G)$ is finitely generated over $\mathbb{Z}$, then M is a free finitely generated $\mathbb{Z}$-module. Let h be the automorphism of M induced by multiplication by t. We have $\det h = \pm 1$ and

$$\det(h - t\,\mathrm{Id}) = \det(h_{\mathbb{Q}} - t\,\mathrm{Id}) \in \mathbb{Z}[t].$$

Therefore $\Delta_G(t) = \det(h - t\,\mathrm{Id})$, the leading coefficient satisfies $a_n = (-1)^n$, where $n = \operatorname{rk} M$, and we have $a_0 = \det h = \pm 1$.

Now suppose that the leading coefficient a_n and the constant coefficient a_0 of the Alexander polynomial $\Delta_G(t)$ of G are equal to ± 1. By the Cayley–Hamilton theorem, $\Delta_G(t) \in \operatorname{Ann}(M_{\mathbb{Q}})$. Therefore $\Delta_G(t) \in \operatorname{Ann}(M)$ and the module M is finitely generated over $\mathbb{Z}$ by Proposition 1.15. The theorem is proved.

Remark 3.18. Let G be a C-group given by a C-presentation $G = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$ and let $\mathcal{A}(G)$ be its Alexander matrix. Then the Alexander polynomial $\Delta_G(t)$ coincides (up to multiplication by $\pm t^k$) with the greatest common divisor of the $(m-1)$-rowed minors of $\mathcal{A}(G)$.

3.6. Alexander modules of some irreducible C-groups. At the end of this section we compute the Alexander modules for some irreducible C-groups.

Example 3.19. The Alexander module $A_0(\mathrm{Br}_{m+1})$ of the braid group Br_{m+1} is trivial if $m \geqslant 4$ (or $m = 1$) and is isomorphic to $\Lambda/\langle t^2 - t + 1\rangle$ for $m = 2, 3$.

This statement is well known, but we give a proof for completeness.

Proof. The braid group Br_{m+1} is given by the presentation

$$\mathrm{Br}_{m+1} = \bigl\langle x_1, \dots, x_m \bigm| [x_i, x_j] \quad \text{for } |i-j| \geqslant 2, \\ x_i x_{i+1} x_i x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1} \quad \text{for } i = 1, \dots, m-1 \bigr\rangle.$$

We note that this is a C-presentation of an irreducible C-group.

By Proposition 3.5, to calculate $A_0(\mathrm{Br}_{m+1})$ we must find the matrix $\bar{\mathcal{A}}(\mathrm{Br}_{m+1})$.

The relations $[x_m, x_i]$ $(i = 1, \dots, m-2)$ yield the rows

$$\bigl(0, \dots, 0, t-1, 0, \dots, 0\bigr), \tag{20}$$

where $t-1$ occupies the ith position for $i = 1, \dots, m-2$. If $m \geqslant 4$, then the relator $[x_{m-1}, x_1]$ yields the row

$$(t-1, 0, \dots, 0, 1-t). \tag{21}$$

If $m \geqslant 4$, then the rows (20) and the row (21) generate the submodule $(t-1)\Lambda^{m-1}$ of Λ^{m-1}. On the other hand, these rows belong to the module $M(\mathrm{Br}_{m+1})$. It follows that $A_0(\mathrm{Br}_{m+1}) = 0$ since $A_0(\mathrm{Br}_{m+1}) \simeq \Lambda^{m-1}/M(\mathrm{Br}_{m+1})$ is a Noetherian $(t-1)$-invertible Λ-module and $(t-1)\Lambda^{m-1} \subset M(\mathrm{Br}_{m+1})$.

If $m = 2$, the presentation of Br_3 contains only one relator

$$r := x_1 x_2 x_1 x_2^{-1} x_1^{-1} x_2^{-1}.$$

We have $\nu_*\bigl(\frac{\partial r}{\partial x_1}\bigr) = 1 + t^2 - t$ and, therefore, $A_0(\mathrm{Br}_3) \simeq \Lambda/\langle t^2 - t + 1\rangle$.

If $m = 3$, the presentation of Br_4 contains only three relators:

$$\begin{aligned} r_1 &:= x_1 x_2 x_1 x_2^{-1} x_1^{-1} x_2^{-1}, \\ r_2 &:= x_2 x_3 x_2 x_3^{-1} x_2^{-1} x_3^{-1}, \\ r_3 &:= x_1 x_3 x_1^{-1} x_3^{-1}. \end{aligned}$$

We have

$$\nu_*\left(\frac{\partial r_1}{\partial x_1}\right) = -\nu_*\left(\frac{\partial r_1}{\partial x_2}\right) = \nu_*\left(\frac{\partial r_2}{\partial x_2}\right) = t^2 - t + 1, \\ \nu_*\left(\frac{\partial r_3}{\partial x_1}\right) = 1 - t.$$

Therefore the module $M(\mathrm{Br}_3) \subset \Lambda^2$ is generated by the vectors

$$v_1 = \bigl(t^2 - t + 1, -(t^2 - t + 1)\bigr), \qquad v_2 = (0, t^2 - t + 1), \qquad v_3 = (1 - t, 0).$$

Hence $A_0(\mathrm{Br}_3) \simeq \Lambda/\langle t^2 - t + 1\rangle$.

Example 3.20. The Alexander module of the C-group

$$G_m = \langle x_1, x_2 \mid (x_1^{-1}x_2)^m x_1 (x_1^{-1}x_2)^{-m} x_2^{-1} \rangle, \qquad m \in \mathbb{N},$$

is isomorphic to $A_0(G) \simeq \Lambda/\langle (m+1)t - m \rangle$.

These irreducible C-groups are interesting because they are non-Hopfian for $m \geqslant 2$ and, therefore, they are not residually finite. The group G_m is isomorphic to the Baumslag–Solitar group $\langle a, x_1 \mid x_1^{-1} a^m x_1 a^{-(m+1)} \rangle$ (see [18]) if we put $x_2 = x_1 a$. We also note that, by Corollary 3.13, each of these groups can be realized as $\pi_1(S^4 \setminus S^2)$ for some knotted sphere $S^2 \subset S^4$.

Proof. Straightforward calculations show that

$$\nu_*\left(\frac{\partial r}{\partial x_1}\right) = -mt^{-1} + m + 1,$$

where $r := (x_1^{-1}x_2)^m x_1 (x_1^{-1}x_2)^{-m} x_2^{-1}$. Therefore the Alexander module $A_0(G)$ is isomorphic to $\Lambda/\langle (m+1)t - m \rangle$.

§ 4. The first homology groups of cyclic coverings

4.1. Proofs of Theorems 0.2 and 0.5. We prove Theorems 0.2 and 0.5 simultaneously.

In the notation of the introduction, let X be either the sphere S^{n+2} (case I) or the projective plane $\mathbb{CP}^2$ (case II), and let X' be either the complement of a knotted n-manifold V in S^{n+2} or the complement of the union of an irreducible Hurwitz curve H and a line L 'at infinity' in $\mathbb{CP}^2$. We recall that the fundamental group $G = \pi_1(X')$ is an irreducible C-group.

Consider the infinite cyclic covering $f = f_\infty \colon X_\infty \to X'$ corresponding to the C-epimorphism $\nu \colon G \to \mathbb{F}_1$ with $\ker \nu = G'$. Let $h \in \operatorname{Deck}(X_\infty/X') \simeq \mathbb{F}_1$ be the covering transformation corresponding to a C-generator $x \in \mathbb{F}_1$. We regard X' as the quotient space $X' = X_\infty/\mathbb{F}_1$. In such a situation Milnor [19] considered an exact sequence of chain complexes

$$0 \to C_.(X_\infty) \xrightarrow{h-\mathrm{id}} C_.(X_\infty) \xrightarrow{f_*} C_.(X') \to 0,$$

which gives an exact sequence of homology groups with integer coefficients:

$$\cdots \to H_1(X_\infty) \xrightarrow{t-\mathrm{id}} H_1(X_\infty) \xrightarrow{f_*} H_1(X') \xrightarrow{\partial} H_0(X_\infty) \to 0, \tag{22}$$

where $t = h_*$.

The action h_* endows each group $H_i(X_\infty)$ with the structure of a Λ-module, and $H_1(X_\infty) \simeq G'/G''$ is the Alexander module of the C-group G. If we regard the $H_i(X')$ as Λ-modules with the trivial action of t, then (22) becomes an exact sequence of Λ-modules. We also note that the action of $t \in \Lambda$ on $H_0(X_\infty) \simeq \mathbb{Z}$ is trivial, that is, t is the identity automorphism of $H_0(X_\infty)$.

Let $\langle h^k \rangle \subset \mathbb{F}_1$ be the infinite cyclic group generated by the element h^k. Then we can regard the manifold X'_k as a quotient manifold: $X'_k = X_\infty/\langle h^k \rangle$. Moreover, we have $X' = X'_k/\mu_k$, where $\mu_k = \mathbb{F}_1/\langle h^k\rangle$ is the cyclic group of order k. We write h_k for the automorphism of X'_k induced by h. Then h_k is a generator of the covering transformation group $\operatorname{Deck}(X'_k/X') = \mu_k$ acting on X'_k.

In case I it is easy to see that the manifold X'_k can be embedded in a compact smooth manifold X_k with the following properties.

(i) The action of h_k on X'_k and the map $f'_k\colon X'_k \to X'$ extend to give an action (again denoted by h_k) on X_k and a smooth map

$$f_k\colon X_k \to X \simeq X_k/\{h_k\}.$$

(ii) The set of fixed points of h_k coincides with $f_k^{-1}(V) = \overline{V}$. The restriction $f_{k|\overline{V}}\colon \overline{V} \to V$ of f_k to $\overline{V}$ is a smooth isomorphism.

In case II, the covering f'_k can be extended to a map $\tilde{f}_{k\,\mathrm{norm}}\colon \widetilde{X}_{k,\mathrm{norm}} \to X$ which is branched along H and possibly along L, where the variety $\widetilde{X}_{k,\mathrm{norm}}$ is a singular analytic variety near its finitely many singular points. Let $\sigma\colon \overline{X}_k \to \widetilde{X}_{k,\mathrm{norm}}$ be a resolution of these singularities, $E = \sigma^{-1}(\operatorname{Sing}\widetilde{X}_{k,\mathrm{norm}})$ the proper transform of the set of singular points of $\widetilde{X}_{k,\mathrm{norm}}$, and $\bar{f}_k = \tilde{f}_{k,\mathrm{norm}} \circ \sigma$. We denote the proper transforms of H and L by $R = \tilde{f}^{-1}_{k,\mathrm{norm}}(H)$ and $R_\infty = \tilde{f}^{-1}_{k,\mathrm{norm}}(L)$. The restriction of $\tilde{f}_{k,\mathrm{norm}}$ to R is one-to-one. The restriction of $\tilde{f}_{k,\mathrm{norm}}$ to R_∞ is a k_0-sheeted cyclic covering, where $k_0 = \mathrm{GCD}(k, m)$ and the ramification index of $\tilde{f}_{k,\mathrm{norm}}$ along R_∞ is equal to $k_\infty = \frac{k}{k_0}$. As in the algebraic case, we easily see that R_∞ is irreducible. We denote the proper transform of R by $\overline{R} = \sigma^{-1}(R)$. Note that k_0 divides m. If we put $m_0 = \frac{m}{k_0}$, then $m_0 \in \mathbb{N}$.

We write $X_k = \overline{X}_k \setminus E$ for the non-singular part of $\widetilde{X}_{k,\mathrm{norm}}$. We have embeddings $i_k\colon X'_k \hookrightarrow X_k$ and $j_k\colon X_k \hookrightarrow \overline{X}_k$.

In cases I and II, the action of h_k on X_k endows the group $H_1(X_k, \mathbb{Z})$ (resp. $H_1(X'_k, \mathbb{Z})$) with the structure of a Λ-module such that the homomorphism

$$i_{k*}\colon H_1(X'_k, \mathbb{Z}) \to H_1(X_k, \mathbb{Z}),$$

induced by the embedding $i\colon X'_k \hookrightarrow X_k$, is a Λ-homomorphism. Clearly, the homomorphism i_{k*} is epimorphic.

In case I let $S \subset X_k$ be a germ of a smooth surface that is transversal to $\overline{V}$ at the point $p \in \overline{V}$, and let $\bar{\gamma} \subset S$ be a small circle centred at p. Since $\overline{V}$ is a smooth connected submanifold of codimension 2 in X_k, we see that $\ker i_{k*}$ is generated by the homology class $[\bar{\gamma}] \in H_1(X'_k, \mathbb{Z})$ that contains the cycle $\bar{\gamma}$.

It is clear that $t\bigl([\bar{\gamma}]\bigr) = [\bar{\gamma}]$, where $t = h_{k*}$. Moreover,

$$f_{k*}\bigl([\bar{\gamma}]\bigr) = \pm k[\gamma] \in H_1(X', \mathbb{Z}) \simeq \mathbb{Z},$$

where $[\gamma]$ is the generator of $H_1(X', \mathbb{Z})$ represented by a simple loop γ around V lying in a surface transversal to V.

In case II let $S \subset X_k$ be a germ of a smooth surface that is transversal to R at the point $p \in R$, and let $\bar{\gamma} \subset S$ be a small circle centred at p. Clearly, the homology class $[\bar{\gamma}] \in H_1(X'_k, \mathbb{Z})$ is invariant under multiplication by t and we have $f_{k*}\bigl([\bar{\gamma}]\bigr) = k[\gamma]$, where $[\gamma]$ is the generator of $H_1\bigl(\mathbb{CP}^2 \setminus (H \cup L), \mathbb{Z}\bigr) \simeq \mathbb{Z}$.

We similarly let $L_1 \subset \mathbb{CP}^2$ be a complex line transversal to L at the point $q \in L \setminus H$, and let γ_∞ be a simple small loop around q lying in L_1. Then $f_k^{-1}(\gamma_\infty)$ splits into a disjoint union of k_0 simple loops $\bar{\gamma}_{\infty,i}$, $i = 1, \dots, k_0$. Since R_∞ is irreducible, any two loops $\bar{\gamma}_{\infty,i}$ and $\bar{\gamma}_{\infty,j}$ belong to the same homology class in $H_1(X'_k, \mathbb{Z})$ (to be denoted by $[\bar{\gamma}_\infty]$). It is easy to see that $t(\bar{\gamma}_{\infty,i}) = \bar{\gamma}_{\infty,i+1}$. Therefore the homology class $[\bar{\gamma}_\infty] \in H_1(X'_n, \mathbb{Z})$ is invariant under multiplication by t. We also note that $f_{k*}\big([\bar{\gamma}_\infty]\big) = k_\infty m[\gamma] = km_0[\gamma]$ because $[\gamma_\infty] = m[\gamma]$.

Lemma 4.1. *The Λ-module $H_1(X'_k, \mathbb{Z})$ is isomorphic to*

$$A_k(G) \oplus H_1(X'_k)_1 \simeq A_k(G) \oplus \mathbb{Z},$$

where $A_k(G)$ is the kth derived Alexander module of the C-group G and

$$H_1(X'_k)_1 = \big\{h \in H_1(X'_k, \mathbb{Z}) \mid (t-1)h = 0\big\}.$$

Proof. To calculate the groups $H_1(X'_k, \mathbb{Z})$, we use the exact sequence

$$\cdots \to H_1(X_\infty, \mathbb{Z}) \xrightarrow{t^k - \mathrm{id}} H_1(X_\infty, \mathbb{Z}) \xrightarrow{g_{k*}} H_1(X'_k, \mathbb{Z}) \xrightarrow{\partial} H_0(X_\infty, \mathbb{Z}) \to 0 \tag{23}$$

for the infinite cyclic covering $g_k = g_{\infty,k} \colon X_\infty \to X'_k$. This sequence is constructed similarly to (22).

Using (23), we get the short exact sequence

$$0 \to H_1(X_\infty)/(t^k - 1)H_1(X_\infty) \xrightarrow{g_{k*}} H_1(X'_k) \xrightarrow{\partial} H_0(X_\infty) \to 0, \tag{24}$$

which is a sequence of Λ-modules.

We introduce the notation

$$M_1 = \ker \partial = \operatorname{im} g_{k*} \simeq H_1(X_\infty)/(t^k - 1)H_1(X_\infty)$$

and $M_2 = H_1(X'_k)_1$.

We have $H_0(X_\infty, \mathbb{Z}) \simeq \mathbb{Z}$. Choose a generator $u \in H_0(X_\infty, \mathbb{Z})$ and let $v_1 \in H_1(X'_k, \mathbb{Z})$ be an element with $\partial(v_1) = u$. Then we have $(t-1)v_1 \in \ker \partial$ because $H_0(X_\infty, \mathbb{Z})$ is a trivial Λ-module and ∂ is a Λ-homomorphism. We fix such an element v_1.

By Theorems 0.1 and 0.3, $H_1(X_\infty, \mathbb{Z}) = A_0(G)$ is a Noetherian $(t-1)$-invertible Λ-module. Therefore, by Proposition 1.6,

$$M_1 \simeq H_1(X_\infty)/(t^k - 1)H_1(X_\infty) = A_k(G)$$

is also a Noetherian $(t-1)$-invertible Λ-module and Theorem 1.10 yields a polynomial $g_1(t) \in \operatorname{Ann}(M_1)$ with $g_1(1) = 1$. We fix such a polynomial $g_1(t)$.

Consider the element $\bar{v}_1 = g_1(t)v_1$. Since $\partial(\bar{v}_1) = g_1(1)u = u$, we have

$$(t-1)\bar{v}_1 = (t-1)g_1(t)v_1 = g_1(t)(t-1)v_1 = 0$$

because $(t-1)v_1 \in M_1$. It follows that $\bar{v}_1 \in M_2$.

We note that $M_1 \cap M_2 = 0$ since M_1 is $(t-1)$-invertible. Therefore ∂ maps M_2 isomorphically onto $H_0(X_\infty, \mathbb{Z})$, that is, the exact sequence (24) splits and we have $H_1(X'_k, \mathbb{Z}) \simeq M_1 \oplus M_2$. The lemma is proved.

Lemma 4.2. *The homomorphism $f_{k*}\colon H_1(X'_k,\mathbb{Z}) \longrightarrow H_0(X',\mathbb{Z})$ has the following properties*:

(i) $\ker f_{k*} = A_k(G) \subset H_1(X'_k,\mathbb{Z})$,

(ii) $\operatorname{im} f_{k*} = k\mathbb{Z} \subset \mathbb{Z} \simeq H_1(X',\mathbb{Z})$ *and the restriction of f_{k*} to $H_1(X'_k)_1$ is an isomorphism of $H_1(X'_k)_1$ onto its image.*

Proof. The group $H_1(X',\mathbb{Z})$ is isomorphic to $G/G' \simeq \mathbb{Z}$. Similarly, the group $H_1(X'_k,\mathbb{Z})$ is isomorphic to G_k/G'_k, where $G_k = \ker \nu_k$,

$$\nu_k = \operatorname{mod}_k \circ \nu\colon G \to \mu_k = \mathbb{Z}/\langle h^k\rangle,$$

and $f_{k*}\colon H_1(X'_k,\mathbb{Z}) \to H_1(X',\mathbb{Z})$ coincides with the homomorphism

$$i_{k*}\colon G_k/G'_k \to G/G'$$

induced by the embedding $i_k\colon G_k \hookrightarrow G$.

Let the C-group G be given by a C-presentation (7). To describe $\ker i_{k*}$ and $\operatorname{im} i_{k*}$, we again consider the two-dimensional complex K described in § 3.1. The complex K has a single vertex x_0. Its one-dimensional skeleton is a wedge product of oriented circles s_j $(1 \leqslant j \leqslant m)$ corresponding to the generators of G appearing in (7), and $K \setminus (\bigcup s_i) = \bigsqcup_{j=1}^{l} \overset{\circ}{D}_j$ is a disjoint union of open discs. For each $j = 1,\dots,l$, $\overset{\circ}{D}_j$ corresponds to the relator r_j appearing in (7). Here l is the number of relators r_i in the presentation (7).

The embedding $i_k\colon G_k \hookrightarrow G$ of groups determines an unramified covering $f_k\colon K_k \to K$, where K_k is a two-dimensional complex with k vertices $p_1,\dots,p_k$, $f_k(p_j) = x_0$. The proper transform $f^{-1}(s_j) = \bigsqcup_{s=1}^{k} \bar{s}_{j,s}$ of each edge s_j is a disjoint union of k edges $\bar{s}_{j,s}$, $1 \leqslant s \leqslant k$. The proper transform $f^{-1}(\overset{\circ}{D}_j) = \bigsqcup_{s=1}^{k} \overset{\circ}{D}_{j,s}$ of each disc $\overset{\circ}{D}_j$ is a disjoint union of k open discs $\overset{\circ}{D}_{j,s}$, $1 \leqslant s \leqslant k$.

Let h_k be a generator of the covering transformation group $\operatorname{Deck}(K_k/K) = \mu_k$ acting on K_k. The homeomorphism h_k induces an action h_{k*} on the chain complex $C.(K_k)$ and an action t on the groups $H_i(K_k,\mathbb{Z})$ endowing each of these groups with the structure of a Λ-module. This structure on $H_1(K_k,\mathbb{Z})$ coincides with the structure on $H_1(X'_k,\mathbb{Z})$ defined above if we identify the groups $H_1(K_k,\mathbb{Z})$ and $H_1(X'_k,\mathbb{Z})$ by means of the isomorphisms $H_1(K_k,\mathbb{Z}) \simeq G_k/G'_k$ and $H_1(X'_k,\mathbb{Z}) \simeq G_k/G'_k$.

Consider the sequence of chain complexes

$$C.(K_k) \xrightarrow{h_{k*}-\mathrm{id}} C.(K_k) \xrightarrow{f_{k*}} C.(K) \to 0.$$

We easily see that $\operatorname{im}(h_{k*} - \mathrm{id}) = \ker f_{k*}$ and

$$\ker(h_{k*} - \mathrm{id}) = \biggl(\sum_{j=0}^{k-1} h_{k*}^{j}\biggr) C.(K_k).$$

The proof of the lemma now follows from the exact sequence

$$\begin{aligned} \cdots \to H_1\bigl(C.\bigl(K_k/\ker(h_{k*}-\mathrm{id})\bigr)\bigr) &\xrightarrow{t^k-1} H_1(K_k) \xrightarrow{f_{k*}} H_1(K) \\ \xrightarrow{\partial} H_0\bigl(C.\bigl(K_k/\ker(h_{k*}-\mathrm{id})\bigr)\bigr) &\xrightarrow{t^k-1} H_0(K_k) \xrightarrow{f_{k*}} H_0(K) \to 0 \end{aligned} \tag{25}$$

since

$$\operatorname{im}\big[H_1\big(C.\big(K_k/\ker(h_{k*}-\mathrm{id})\big)\big) \xrightarrow{t^k-1} H_1(K_k)\big] = A_k(G),$$
$$H_1(K) \simeq \mathbb{Z},$$
$$H_0\big(C.\big(K_k/\ker(h_{k*}-\mathrm{id})\big)\big) \simeq \mathbb{Z}/k\mathbb{Z},$$
$$H_0(K_k) \overset{f_{k*}}{\simeq} H_0(K) \simeq \mathbb{Z}$$

are Λ-modules with the trivial action of t and the exact sequence (25) is a sequence of Λ-homomorphisms of Λ-modules. The lemma is proved.

Theorem 0.2 follows from Lemmas 4.1 and 4.2 because $\ker i_{k*}$ is generated by $[\bar\gamma] \in H_1(X')_1$ and we have $f_{k*}\big([\bar\gamma]\big) = k[\gamma]$.

In case II we similarly get $\ker i_{k*} = H_1(X'_k)_1$. Indeed, $\ker i_{k*}$ is generated by the elements $\bar\gamma$ and $\bar\gamma_\infty \in H_1(X'_k)_1 \simeq \mathbb{Z}$ and we have $f_{k*}\big([\bar\gamma]\big) = k[\gamma]$. Hence the group $H_1(X'_k)_1$ is generated by the element $[\bar\gamma]$.

As a corollary, we see that the restriction of i_{k*} to the submodule $A_k(G)$ of $H_1(X'_k,\mathbb{Z})$ is an isomorphism between $A_k(G)$ and $H_1(X_k,\mathbb{Z})$. Therefore the following lemma yields Theorem 0.5.

Lemma 4.3 [6]. *The homomorphism $j_{k*}\colon H_1(X_k,\mathbb{Q}) \to H_1(\overline{X}_k,\mathbb{Q})$ is an isomorphism.*

4.2. Corollaries of Theorems 0.2 and 0.5.

Corollary 4.4. *Let V be a knotted n-manifold with $n \geqslant 1$ and let $f_k\colon X_k \to S^{n+2}$ be the cyclic covering of degree k branched along V. Then the following assertions hold.*

(i) *The first Betti number $b_1(X_k)$ of X_k is even.*

(ii) *If $k = p^r$, where p is a prime, then the group $H_1(X_k,\mathbb{Z})$ is finite.*

(iii) *A finitely generated abelian group G can be realized as $H_1(X_k,\mathbb{Z})$ for some knotted n-manifold V with $n \geqslant 2$ if and only if there is an automorphism $h \in \mathrm{Aut}(G)$ such that $h^k = \mathrm{Id}$ and $h - \mathrm{Id}$ is again an automorphism of G. In particular, $H_1(X_2,\mathbb{Z})$ is a finite abelian group of odd order, and every finite abelian group G of odd order can be realized as $H_1(X_2,\mathbb{Z})$ for some knotted n-sphere for $n \geqslant 2$.*

Proof. This follows from Theorems 0.1, 0.2, 2.11, Proposition 2.13, Corollary 3.13 and Examples 2.14, 3.20.

Corollary 0.4 follows from Theorems 0.3 and 2.10.

Corollary 0.6 follows immediately from Lemma 4.3 and Corollary 4.5 since the homomorphism $j_{k*}\colon H_1(X_k,\mathbb{Z}) \to H_1(\overline{X}_k,\mathbb{Z})$ is an isomorphism and we have $H_1(\overline{X}_k,\mathbb{Q}) \simeq A_k(H) \otimes \mathbb{Q}$.

Corollary 4.5. *Suppose that H is an algebraic (resp. Hurwitz or pseudo-holomorphic) irreducible curve in $\mathbb{CP}^2$, $\deg H = m$ and $\bar f_k\colon \overline{X}_k \to \mathbb{CP}^2$ is a resolution of singularities of the cyclic covering of degree k branched along H and possibly along the line L 'at infinity'. Put $X_k = \overline{X}_k \setminus E$, where E is the proper transform of the set of singular points of the cyclic covering. Then the following assertions hold.*

(i) *The sequence of groups* $H_1(X_1,\mathbb{Z}),\dots,H_1(X_k,\mathbb{Z}),\dots$ *has period* m, *that is*, $H_1(X_k,\mathbb{Z})\simeq H_1(X_{k+m},\mathbb{Z})$.

(ii) *The first Betti number* $b_1(\overline{X}_k)$ *is equal to the number* $r_{k,\neq 1}$ *of roots of the Alexander polynomial* $\Delta(t)$ *of the curve* H *which are* k*th roots of unity not equal to* 1. *In particular*, $b_1(\overline{X}_k)$ *is even.*

(iii) *If* $k=p^r$, *where* p *is a prime, then the groups* $H_1(X_k,\mathbb{Z})$ *and* $H_1(\overline{X}_k,\mathbb{Z})$ *are finite.*

(iv) *If* k *and* m *are coprime, then* $H_1(\overline{X}_k,\mathbb{Z})=0$.

(v) *A finitely generated abelian group* G *can be realized as* $H_1(X_k,\mathbb{Z})$ *for some Hurwitz* (*resp. pseudo-holomorphic*) *curve* H *if and only if there is an automorphism* $h\in\operatorname{Aut}(G)$ *such that* $h^d=\mathrm{Id}$ *for some divisor* d *of* k *and* $h-\mathrm{Id}$ *is again an automorphism of* G. *Moreover, if the group* G *is realized as* $H_1(X_k,\mathbb{Z})$ *for some curve* H, *then* d *divides* $\deg H$. *In particular*, $H_1(\overline{X}_2,\mathbb{Z})$ *is a finite abelian group of odd order, and every finite abelian group* G *of odd order can be realized as* $H_1(X_2,\mathbb{Z})$ *for some Hurwitz* (*resp. pseudo-holomorphic*) *curve* H *of even degree.*

Proof. This follows from Theorems 0.3, 0.5, 2.11, 2.16 and Propositions 2.13, 2.15.

We note that there are plane algebraic curves H such that the homomorphisms $j_{k*}\colon H_1(X_k,\mathbb{Z})\to H_1(\overline{X}_k,\mathbb{Z})$ are not isomorphisms.

Example 4.6. Let $H\subset\mathbb{CP}^2$ be the curve of degree 6 given by the equation

$$Q^3(z_0,z_1,z_2)+C^2(z_0,z_1,z_2)=0,$$

where Q and C are homogeneous forms of degrees $\deg Q=2$, $\deg C=3$ such that the conic $Q=0$ and the cubic $C=0$ meet each other transversally at 6 points. Then we have $A_2(H)\simeq\mathbb{Z}/3\mathbb{Z}$ but $H_1(\overline{X}_2,\mathbb{Z})=0$.

Proof. The curve H has six singular points (cusps) lying on the conic $Q=0$. It is known ([20], see also [21]) that $\pi_1\bigl(\mathbb{CP}^2\setminus(H\cup L)\bigr)\simeq\mathrm{Br}_3$ as a C-group. Therefore $A_2(H)\simeq\mathbb{Z}/3\mathbb{Z}$ (see Examples 2.17 and 3.19). It is also well known that the minimal resolution of singularities of the two-sheeted covering of $\mathbb{CP}^2$ branched along H is a K3-surface, which is simply connected.

We note that the sequence of homology groups $H_1(X_k,\mathbb{Z})$, $k\in\mathbb{N}$, need not be periodic in the case of knotted n-manifolds $V\subset S^{n+2}$. For example, let $S^2\subset S^4$ be a knotted sphere with $\pi_1(S^4\setminus S^2)\simeq G_m$, where G_m is the group studied in Example 3.20. (Corollary 3.13 shows that this group can be realized as the group of a knotted sphere.) Then $H_1(X_k,\mathbb{Z})$ is a cyclic group of order $(m+1)^k-m^k$ (see Example 2.14).

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