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On Chisini's conjecture. II

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Abstract. We prove that if $S \subset \mathbb{P}^N$ is a smooth projective surface and $f\colon S \to \mathbb{P}^2$ is a generic linear projection branched over a cuspidal curve $B \subset \mathbb{P}^2$, then S is uniquely determined (up to isomorphism) by B.

Let $B \subset \mathbb{P}^2$ be an irreducible plane algebraic curve over $\mathbb{C}$ with ordinary cusps and nodes as the only singularities. We denote the degree of B by $2d$ and let g be the genus of its desingularization, c the number of cusps and n the number of nodes. The curve B is called the *discriminant curve of a generic covering of the projective plane* if there is a finite morphism $f\colon S \to \mathbb{P}^2$, $\deg f \geqslant 3$, satisfying the following conditions:

(i) S is a non-singular irreducible projective surface,

(ii) f is unramified over $\mathbb{P}^2 \setminus B$,

(iii) $f^*(B) = 2R + C$, where R is a non-singular irreducible reduced curve and C is a reduced curve,

(iv) the morphism $f_{|R} : R \to B$ coincides with the normalization of B.

Such morphisms f are called *generic coverings of the projective plane* $\mathbb{P}^2$.

A generic covering $f\colon S \to \mathbb{P}^2$ is called a *generic projection* if the surface S is embedded in some projective space $\mathbb{P}^N$ and $f = \mathrm{pr}_{|S}$ is the restriction to S of some linear projection $\mathrm{pr}\colon \mathbb{P}^N \to \mathbb{P}^2$.

Chisini's conjecture (see [1]) claims that if $f\colon S \to \mathbb{P}^2$ is a generic covering of the projective plane and $\deg f \geqslant 5$, then f is uniquely determined (up to an isomorphism of S) by its discriminant curve.

It was proved in [2] that Chisini's conjecture holds for the discriminant curve B of a generic covering $f\colon S \to \mathbb{P}^2$ if

$$\deg f > \frac{4(3d+g-1)}{2(3d+g-1)-c}\,. \qquad (1)$$

Furthermore, it was observed in [3] that, by the Bogomolov–Miyaoka–Yau inequality, all possible values of the right-hand side of (1) are less than 12 and, therefore, the conjecture holds for the discriminant curves of generic coverings of degree greater than 11. It was also shown in [3] that if S is a surface of non-general type, then the conjecture holds for the discriminant curves of generic coverings $f\colon S \to \mathbb{P}^2$ with $\deg f \geqslant 8$.

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The purpose of this paper is to prove the following theorem.

Theorem. *Let $f\colon S \to \mathbb{P}^2$ be a generic projection. Then the generic covering f is uniquely determined* (*up to an isomorphism of* S) *by its discriminant curve $B \subset \mathbb{P}^2$ except in the case when $S \simeq \mathbb{P}^2$ is embedded in $\mathbb{P}^5$ by polynomials of degree two* (*the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$*) *and f is the restriction to S of a linear projection* $\mathrm{pr}\colon \mathbb{P}^5 \to \mathbb{P}^2$.

Proof. To prove the theorem, we shall show that inequality (1) fails only for the discriminant curves of two continuous families of generic projections onto the projective plane. Then we shall see that the generic coverings $f\colon S \to \mathbb{P}^2$ of one of these exceptional families are uniquely determined by their discriminant curves, while generic projections of the other are the generic projections of $S \simeq \mathbb{P}^2$ embedded in $\mathbb{P}^5$ by the Veronese embedding.

To do this, we consider a generic projection $f\colon S \to \mathbb{P}^2$, where S is a non-singular surface embedded in $\mathbb{P}^N$. Let $\deg S = m$ be the degree of the embedding $S \subset \mathbb{P}^N$ and let $\mathrm{pr}\colon \mathbb{P}^N \to \mathbb{P}^2$ be a linear projection such that $f = \mathrm{pr}_{|S}$. We have $\deg f = \deg S = m$.

Any linear projection $\mathbb{P}^N \to \mathbb{P}^2$ is determined by its centre $\mathbb{P}^{N-3} \subset \mathbb{P}^N$. Therefore the set of linear projections $\mathbb{P}^N \to \mathbb{P}^2$ is parametrized by the points of the Grassmannian $\mathrm{Gr}(N-3, N)$. Let $u_0 \in \mathrm{Gr}(N-3, N)$ be a point for which the generic covering $f = \mathrm{pr}_{u_0|S}$ is the restriction of the projection $\mathrm{pr} = \mathrm{pr}_{u_0}$. There is a Zariski-open subset U_S of the Grassmannian $\mathrm{Gr}(N-3, N)$ such that for every $u \in U_S$ the restriction f_u of the corresponding linear projection pr_u to S is a generic covering of the projective plane. The set U_S is non-empty since $u_0 \in U_S$ by assumption. For all $u \in U_S$, the discriminant curves B_u of the generic coverings f_u have the same genus g, the same degree $\deg B_u = 2d$ and the same numbers c and n of cusps and nodes. Therefore inequality (1) either holds simultaneously for all the f_u, $u \in U_S$, or for none of them. Thus any point of U_S can be taken for u_0 when verifying inequality (1).

By Theorem 3 of [4] there is a non-empty Zariski-open subset $V_S \subset \mathrm{Gr}(N-4, N)$ such that for every linear projection $\mathrm{pr}_v\colon \mathbb{P}^N \to \mathbb{P}^3$ with centre at $v \in V_S$, the image $\overline{S} = \mathrm{pr}_v(S)$ of S has only ordinary singular points (that is, singular points which are given locally by one of the following equations: $xy = 0$ (a double curve), $xyz = 0$ (a triple point), $x^2 = y^2 z$ (a pinch)).

Consider the flag manifold $F = F(N-4, N-3, N)$ of linear subspaces $\mathbb{P}^{N-4} \subset \mathbb{P}^{N-3}$ in $\mathbb{P}^N$. We have natural projections $p_1\colon F \to \mathrm{Gr}(N-3, N)$ and $p_2\colon F \to \mathrm{Gr}(N-4, N)$. Clearly, the intersection $W_S = p_1^{-1}(U_S) \cap p_2^{-1}(V_S)$ of two non-empty Zariski-open subsets $p_1^{-1}(U_S)$ and $p_2^{-1}(V_S)$ is a non-empty Zariski-open subset of F. Hence there is no loss of generality in assuming that the generic covering f coincides with the projection f_u, where $u \in U_S$ satisfies $p_1(w) = u$ for some $w \in W_S$. In other words, pr_u can be written as the composite of two projections, $\mathrm{pr}_{p_2(w)}$ and some projection $\overline{\mathrm{pr}}\colon \mathbb{P}^3 \to \mathbb{P}^2$ such that $\overline{S} = \mathrm{pr}_{p_2(w)}(S)$ is a surface in $\mathbb{P}^3$ of degree $\deg \overline{S} = \deg S$ with only ordinary singular points. Let $f_1\colon S \to \overline{S}$ denote the restriction of $\mathrm{pr}_{p_2(w)}$ to S and $f_2\colon \overline{S} \to \mathbb{P}^2$ the restriction of $\overline{\mathrm{pr}}$ to $\overline{S}$. The morphism f_1 is birational. We have $f = f_2 \circ f_1$.

Denote by $D \subset \overline{S}$ the double curve of $\overline{S}$. Write $D = D_1 \cup \cdots \cup D_u$, where D_i $(i = 1, \dots, u)$ are the irreducible components of D. Let g_i and d_i be respectively the genus and degree of the curve D_i. We put $\bar{g} = \sum_{i=1}^{u} g_i$ and $\bar{d} = \sum_{i=1}^{u} d_i$. Denote by t the number of triple points of $\overline{S}$. Note that $0 \leqslant u \leqslant \bar{d}$ and $\bar{g} \geqslant 0$.

We have (see, for example, [5])

$$K_S^2 = m(m-4)^2 - (5m-24)\bar{d} - 4(u-\bar{g}) + 9t, \tag{2}$$

$$e(S) = m^2(m-4) + 6m - (7m-24)\bar{d} - 8(u-\bar{g}) + 15t, \tag{3}$$

where K_S is the canonical class of S and $e(S)$ is its topological Euler characteristic. On the other hand, since $\deg f = \deg S = m$ for a generic projection $f = \mathrm{pr}_{|S}$, we have (see Lemmas 6 and 7 in [2])

$$K_S^2 = 9m - 9d + g - 1, \tag{4}$$

$$e(S) = 3m + 2(g-1) - c. \tag{5}$$

Lemma 1. *We have*

$$2d = m(m-1) - 2\bar{d}, \tag{6}$$

$$\bar{d} \leqslant \frac{(m-1)(m-2)}{2}. \tag{7}$$

Proof. Let L be a generic line in $\mathbb{P}^2$ and $\bar{L} = f_2^{-1}(L)$ its pre-image. Then $\bar{L}$ is an irreducible plane curve of degree m having $\bar{d}$ nodes as its singular points. Therefore the genus $g(\bar{L})$ is equal to $\frac{(m-1)(m-2)}{2} - \bar{d}$, and inequality (7) follows from the inequality $g(\bar{L}) \geqslant 0$.

The covering $f_{2|\bar{L}} \colon \bar{L} \to L$ is a morphism of degree m. It is branched at $2d = (L, B)_{\mathbb{P}^2} = \deg B$ points. It follows that $2g(\bar{L}) - 2 = -2m + 2d$ by Hurwitz' formula. Thus we have

$$-2m + 2d = (m-1)(m-2) - 2\bar{d} - 2,$$

that is, $2d = m(m-1) - 2\bar{d}$. The lemma is proved.

It follows from (2)–(6) that

$$g - 1 = \frac{m(2m^2 - 7m + 5)}{2} - 5(m-3)\bar{d} - 4(u-\bar{g}) + 9t, \tag{8}$$

$$c = m(m-1)(m-2) - 3(m-2)\bar{d} + 3t. \tag{9}$$

Substituting equations (6), (8) and (9) in inequality (1) and performing obvious transformations, we easily see that (1) is equivalent to the inequality

$$(m-2)\bigl[m(m-1)(m-2) - (7m-24)\bar{d} - 8(u-\bar{g})\bigr] + 3(5m-12)t > 0. \tag{10}$$

Therefore, by Theorem 1 of [2], to prove the theorem it suffices to show that if the inequality

$$(m-2)\bigl[m(m-1)(m-2) - (7m-24)\bar{d} - 8(u-\bar{g})\bigr] + 3(5m-12)t \leqslant 0 \tag{11}$$

holds for a surface $\overline{S} \subset \mathbb{P}^3$ with ordinary singular points, then either $f\colon S \to \mathbb{P}^2$ is a projection of the projective plane embedded in $\mathbb{P}^5$ by the Veronese embedding, or f is uniquely determined (up to an isomorphism of S) by its discriminant curve B.

By the main result of [3] we can assume that $m \leqslant 11$.

Lemma 2. *Chisini's conjecture holds for the discriminant curves of generic projections $f\colon S \to \mathbb{P}^2$ if $6 \leqslant \deg S = m \leqslant 11$ and $K_S^2 \leqslant 3e(S)$.*

Proof. It follows from equations (4), (5) and the inequality $K_S^2 \leqslant 3e(S)$ that

$$3c \leqslant 9d + 5(g-1). \tag{12}$$

Assume that the conjecture does not hold for the discriminant curve B of some generic projection $f\colon S \to \mathbb{P}^2$, $\deg f = \deg S = m$. Then the invariants of B do not satisfy (1). Hence they satisfy the inequality

$$\frac{4(3d+g-1)}{2(3d+g-1)-c} \geqslant m$$

or, equivalently,

$$c \geqslant \frac{2(m-2)}{m}(3d+g-1). \tag{13}$$

It follows from inequalities (12) and (13) that

$$6(m-2)\big(3d+(g-1)\big) \leqslant 3mc \leqslant m\big(9d+5(g-1)\big)$$

and hence

$$6(m-2)\big(3d+(g-1)\big) \leqslant m\big(9d+5(g-1)\big),$$

that is,

$$g-1 \geqslant \frac{9(m-4)}{12-m}d \tag{14}$$

(we have $m \leqslant 11$ by hypothesis). Therefore, applying inequality (13), we have

$$c \geqslant \frac{2(m-2)}{m}(3d+g-1) \geqslant \frac{2(m-2)}{m}\left(3d+\frac{9(m-4)}{12-m}d\right),$$

that is,

$$c \geqslant \frac{12(m-2)}{12-m}d. \tag{15}$$

Since $\frac{\deg B(\deg B-3)}{2} = c+n+g-1$ and $n \geqslant 0$, we have

$$d(2d-3) \geqslant c+g-1. \tag{16}$$

Therefore we have

$$d(2d-3) \geqslant c+g-1 \geqslant \frac{12(m-2)}{12-m}d + \frac{9(m-4)}{12-m}d$$

and hence

$$2d-3 \geqslant \frac{12(m-2)}{12-m} + \frac{9(m-4)}{12-m} = \frac{21m-60}{12-m},$$

that is,

$$d \geqslant \frac{3(3m-4)}{12-m}. \tag{17}$$

If $m = 11$, then inequality (17) yields that $d \geqslant 87$. On the other hand, we have $d \leqslant 55$ by Lemma 1, a contradiction.

If $m = 10$, then inequality (17) implies that $d \geqslant 39$. Hence we have $\bar{d} \leqslant 6$ by Lemma 1. On the other hand, inequality (11) implies that

$$8\big(720 - 46\bar{d} - 8(u - \bar{g})\big) + 114t \leqslant 0.$$

Since $t \geqslant 0$, we must have

$$720 - 46\bar{d} - 8(u - \bar{g}) \leqslant 0.$$

Therefore,

$$720 \leqslant 46\bar{d} + 8(u - \bar{g}) \leqslant 54\bar{d}$$

because $u - \bar{g} \leqslant \bar{d}$. Finally, we obtain that $\bar{d} \geqslant \frac{720}{54}$, which contradicts $\bar{d} \leqslant 6$.

If $m = 9$, then inequalities (14) and (17) yield that $d \geqslant 23$ and $g - 1 \geqslant 15d$. Therefore, by Lemma 1, we have

$$g - 1 \geqslant 15(36 - \bar{d}), \tag{18}$$

$$\bar{d} \leqslant 28 - 23 = 5. \tag{19}$$

It follows from inequality (11) that

$$7\big(504 - 39\bar{d} - 8(u - \bar{g})\big) + 99t \leqslant 0$$

or, equivalently,

$$99t \leqslant 273\bar{d} + 56(u - \bar{g}) - 3528. \tag{20}$$

Equation (8), with $m = 9$, and inequality (18) imply that

$$468 - 30\bar{d} - 4(u - \bar{g}) + 9t \geqslant 15(36 - \bar{d})$$

or, equivalently,

$$9t \geqslant 15\bar{d} + 4(u - \bar{g}) + 72. \tag{21}$$

It follows from inequalities (20) and (21) that

$$273\bar{d} + 56(u - \bar{g}) - 3528 \geqslant 11\big(15\bar{d} + 4(u - \bar{g}) + 72\big),$$

that is, $4320 \leqslant 108\bar{d} + 12(u - \bar{g}) \leqslant 120\bar{d}$ because $\bar{g} \geqslant 0$ and $u \leqslant \bar{d}$. Therefore $\bar{d} \geqslant 36$. But this contradicts inequality (19).

If $m = 8$, then it follows from (14) that $g - 1 \geqslant 9d$. Therefore, by Lemma 1, we have

$$g - 1 \geqslant 9(28 - \bar{d}), \tag{22}$$

where $\bar{d} \leqslant 21$.

It follows from (11) that

$$6\big(336-32\bar{d}-8(u-\bar{g})\big)+84t\leqslant 0$$

or, equivalently,

$$7t\leqslant 16\bar{d}+4(u-\bar{g})-168. \tag{23}$$

Equation (8), with $m=8$, and inequality (22) imply that

$$308-25\bar{d}-4(u-\bar{g})+9t\geqslant 9(28-\bar{d})$$

or, equivalently,

$$9t\geqslant 16\bar{d}+4(u-\bar{g})-56. \tag{24}$$

It follows from inequalities (23) and (24) that

$$7\big(16\bar{d}+4(u-\bar{g})-56\big)\leqslant 9\big(16\bar{d}+4(u-\bar{g})-168\big),$$

that is, $1120\leqslant 32\bar{d}+8(u-\bar{g})\leqslant 40\bar{d}$ because $\bar{g}\geqslant 0$ and $u\leqslant\bar{d}$. Therefore $\bar{d}\geqslant 28$, which contradicts the inequality $\bar{d}\leqslant 21$.

If $m=7$, then it follows from inequality (14) that $g-1\geqslant\frac{27}{5}d$. Hence we have

$$g-1\geqslant\frac{27}{5}(21-\bar{d}) \tag{25}$$

since $d=21-\bar{d}$ and $\bar{d}\leqslant 15$ by Lemma 1.

Inequality (11) can be rewritten as

$$69t\leqslant 125\bar{d}+40(u-\bar{g})-1050. \tag{26}$$

Equation (8), with $m=7$, and inequality (25) imply that

$$189-20\bar{d}-4(u-\bar{g})+9t\geqslant\frac{27}{5}(21-\bar{d})$$

or, equivalently,

$$45t\geqslant 73\bar{d}+20(u-\bar{g})-378. \tag{27}$$

It follows from inequalities (26) and (27) that

$$15\big(125\bar{d}+40(u-\bar{g})-1050\big)\geqslant 23\big(73\bar{d}+20(u-\bar{g})-378\big),$$

that is, $7056\leqslant 196\bar{d}+140(u-\bar{g})\leqslant 336\bar{d}$ because $\bar{g}\geqslant 0$ and $u\leqslant\bar{d}$. Therefore $\bar{d}\geqslant\frac{7056}{336}=21$, which contradicts inequality $\bar{d}\leqslant 15$.

If $m=6$, then (14) yields that $g-1\geqslant 3d$. Hence we have

$$g-1\geqslant 3(15-\bar{d}) \tag{28}$$

because $d=15-\bar{d}$ and, moreover, $\bar{d}\leqslant 10$ by Lemma 1.

Inequality (11) can be rewritten as

$$27t\leqslant 36\bar{d}+16(u-\bar{g})-240. \tag{29}$$

Equation (8), with $m = 6$, and inequality (28) imply that

$$105 - 15\bar{d} - 4(u - \bar{g}) + 9t \geqslant 45 - 3\bar{d}$$

or, equivalently (multiplying by 3),

$$27t \geqslant 36\bar{d} + 12(u - \bar{g}) - 180. \tag{30}$$

It follows from inequalities (29) and (30) that

$$36\bar{d} + 16(u - \bar{g}) - 240 \geqslant 36\bar{d} + 12(u - \bar{g}) - 180,$$

that is, $u - \bar{g} \geqslant 15$. On the other hand, we have $u - \bar{g} \leqslant 10$ since $\bar{g} \geqslant 0$ and $u \leqslant \bar{d} \leqslant 10$, a contradiction. The lemma is proved.

According to Theorem 2 of [3], if $\deg f \geqslant 8$ and S is a surface of non-general type, then Chisini's conjecture holds for the discriminant curve B of any generic covering $f\colon S \to \mathbb{P}^2$. It is well known (see the classification of algebraic surfaces) that if the Bogomolov–Miyaoka–Yau inequality does not hold for an algebraic surface S, then S is an irregular ruled surface and we have $K_S^2 \leqslant 2e(S)$ and $K_S^2 \leqslant -2$. Therefore, by Lemma 2, to prove the theorem, it suffices to consider only the following cases: $3 \leqslant m \leqslant 7$ and, when $m = 6$ or 7, $K_S^2 \leqslant 2e(S)$ and $K_S^2 \leqslant -2$.

We again assume that the invariants of the surface $\overline{S}$ satisfy (11).

Case $m = 3$. In this case (11) takes the form

$$6 + 3\bar{d} - 8(u - \bar{g}) + 9t \leqslant 0.$$

It follows from (7) that $\bar{d} \leqslant 1$. Hence there are two possibilities: either $\bar{d} = 0$ and, therefore, $u = \bar{g} = t = 0$, or $\bar{d} = 1$ and, therefore, $u = 1$, $\bar{g} = t = 0$ because D is a line in $\mathbb{P}^3$ in this case. In both cases, it is easy to see that inequality (11) does not hold.

Case $m = 4$. In this case (11) takes the form

$$2\big(24 - 4\bar{d} - 8(u - \bar{g})\big) + 24t \leqslant 0.$$

It follows from (7) that $\bar{d} \leqslant 3$ and we have three possibilities: $\bar{d} \leqslant 2$ (and hence $u \leqslant \bar{d} \leqslant 2$, $\bar{g} = t = 0$) or $\bar{d} = 3$, $u = 3$, $\bar{g} = 0$, $t = 1$, or $\bar{d} = 3$, $u = 1$, $\bar{g} = 1$ or $\bar{g} = 0$, $t = 0$. It is easy to see that inequality (11) holds only in the following two cases: $u = \bar{d} = 2$, $\bar{g} = t = 0$ and $u = \bar{d} = 3$, $\bar{g} = 0$, $t = 1$. These exceptional cases will be investigated at the end of the proof of the theorem.

Case $m = 5$. Inequality (11) takes the form

$$3\big(60 - 11\bar{d} - 8(u - \bar{g})\big) + 39t \leqslant 0$$

or, equivalently,

$$60 + 2t \leqslant 11(\bar{d} - t) + 8(u - \bar{g}). \tag{31}$$

By Theorem 11 of [2], Chisini's conjecture holds for all cuspidal curves B of genus $g \leqslant 3$. Therefore, by inequality (8), we have

$$g - 1 = 50 - 10\bar{d} - 4(u - \bar{g}) + 9t \geqslant 3$$

or, equivalently,

$$47 - t \geqslant 10(\bar{d} - t) + 4(u - \bar{g}). \tag{32}$$

By Lemma 1 we have $u \leqslant \bar{d} \leqslant 6$. Therefore $u - \bar{g} \leqslant 6$ and we get the following corollary of inequality (31):

$$12 + 2t \leqslant 11(\bar{d} - t),$$

that is,

$$\bar{d} - t \geqslant 2. \tag{33}$$

Similarly, since $\bar{d} - t \leqslant 6$, inequality (31) implies that

$$-6 + 2t \leqslant 8(u - \bar{g})$$

and, therefore, $u - \bar{g} \geqslant 0$. Applying inequality (32), we have $47 - t \geqslant 10(\bar{d} - t)$, that is, $\bar{d} - t \leqslant 4$. Therefore inequality (31) yields that $16 + 2t \leqslant 8(u - \bar{g})$, that is, $u - \bar{g} \geqslant 2$. Then we have $39 - t \geqslant 10(\bar{d} - t)$ by inequality (32) and, therefore,

$$\bar{d} - t \leqslant 3. \tag{34}$$

It now follows from inequality (31) that

$$27 + 2t \leqslant 8(u - \bar{g}),$$

that is, $u - \bar{g} \geqslant 4$. Hence $u \geqslant 4$ and, therefore, $\bar{d} \geqslant 4$.

By inequalities (33) and (34) we have

$$2 \leqslant \bar{d} - t \leqslant 3.$$

Consider the case when $\bar{d} - t = 3$. It follows from (32) that

$$17 - t \geqslant 4(u - \bar{g}). \tag{35}$$

Hence $u - \bar{g} \leqslant 4$. It follows that $u - \bar{g} = 4$, $u = 4$ and $\bar{g} = 0$ because the genera of irreducible components of a curve of degree $\bar{d} \leqslant 6$ having more than four irreducible components must be equal to zero. Moreover, it follows from inequality (35) that $t \leqslant 1$. Therefore $t = 1$ and $\bar{d} = 4$ since $\bar{d} - t = 3$ and $\bar{d} \geqslant 4$. In this case, formulae (8), (9) and Lemma 1 yield that the curve B must have the following invariants:

$$\deg B = 2d = 12, \qquad g = 4, \qquad c = 27, \qquad n = (2d-1)(d-1) - g - c = 24.$$

But this is impossible since, in this case, the degree of the dual curve $\check{B}$ equals $2d(2d-1) - 3c - 2n = 3$ by Plücker's formula and, therefore, the degree of B cannot exceed

$$\deg \check{B}(\deg \check{B} - 1) = 3 \cdot 2 = 6.$$

Consider the case when $\bar{d} - t = 2$. It follows from (31) that

$$38 + 2t \leqslant 8(u - \bar{g}). \tag{36}$$

Hence $u - \bar{g} \geqslant 5$. It follows that $u \geqslant 5$ and $\bar{g} = 0$. Now (36) yields that $u = 6$ because the equation $\bar{d} - t = 2$ and the inequalities $6 \geqslant \bar{d} \geqslant u \geqslant 5$ imply that $t \geqslant 3$ and, therefore, $38 + 2t \geqslant 44$. Thus we have only one possibility:

$$u = 6, \qquad \bar{g} = 0, \qquad \bar{d} = 6, \qquad t = 4.$$

But these values of u, $\bar{g}$, $\bar{d}$ and t do not satisfy inequality (32).

Case $m = 6$ *and* $K_S^2 \leqslant 2e(S)$. Applying formulae (4) and (5), we obtain

$$2c \leqslant 9d + 3(g-1) - 18. \tag{37}$$

Inequality (13) may be written as

$$3c \geqslant 4(3d + g - 1). \tag{38}$$

It follows from inequalities (38) and (37) that

$$24d + 8(g-1) \leqslant 6c \leqslant 27d + 9(g-1) - 54,$$

that is, $3d + g - 1 \geqslant 54$. Since $d = 15 - \bar{d}$, we have

$$g - 1 \geqslant 54 - 3(15 - \bar{d}) = 9 + 3\bar{d}. \tag{39}$$

By assumption, the invariants of $\overline{S}$ satisfy inequality (11) for $m = 6$. Hence they satisfy inequality (29).

Equation (8), with $m = 6$, and inequality (39) imply that

$$105 - 15\bar{d} - 4(u - \bar{g}) + 9t \geqslant 9 + 3\bar{d}$$

or, equivalently,

$$27t \geqslant 72\bar{d} + 12(u - \bar{g}) - 288.$$

By inequality (29) we have $36\bar{d} + 16(u - \bar{g}) - 240 \geqslant 27t$. Hence,

$$36\bar{d} + 16(u - \bar{g}) - 240 \geqslant 72\bar{d} + 12(u - \bar{g}) - 288,$$

that is, $12 \geqslant 9\bar{d} - (u - \bar{g})$. But $u - \bar{g} \leqslant \bar{d}$. Hence $9\bar{d} - (u - \bar{g}) \geqslant 8\bar{d}$ and, therefore, $3 \geqslant 2\bar{d}$. Since $\bar{d}$ is an integer, we must have $\bar{d} \leqslant 1$.

On the other hand, inequality (29) implies that

$$240 \leqslant 9\bar{d} + 27(\bar{d} - t) + 16(u - \bar{g}) \leqslant 52\bar{d}$$

because $\bar{d} - t \leqslant \bar{d}$ and $u - \bar{g} \leqslant \bar{d}$. Therefore we get $\bar{d} \geqslant 5$, a contradiction.

Case $m = 7$ *and* $K_S^2 \leqslant 2e(S)$, $K_S^2 \leqslant -2$. Applying formulae (4), (5), we see from the inequality $K_S^2 \leqslant 2e(S)$ that

$$2c \leqslant 9d + 3(g-1) - 21. \tag{40}$$

We have $K_S^2 \leqslant -2$. Therefore it follows from (2) that

$$K_S^2 = 7 \cdot 9 - 25\bar{d} - 4(u - \bar{g}) + 9t \leqslant -2.$$

Hence,

$$65 \leqslant 65 + 9t \leqslant 25\bar{d} + 4(u - \bar{g}) \leqslant 29\bar{d}$$

because $t \geqslant 0$ and $u - \bar{g} \leqslant \bar{d}$. Thus we have

$$\bar{d} \geqslant 3. \tag{41}$$

Inequality (13) may be written as

$$7c \geqslant 10(3d + g - 1). \tag{42}$$

It follows from (42) and (40) that

$$60d + 20(g-1) \leqslant 14c \leqslant 63d + 21(g-1) - 147,$$

that is,

$$3d + g - 1 \geqslant 147. \tag{43}$$

Since $d = 21 - \bar{d}$, we have

$$g - 1 \geqslant 147 - 3(21 - \bar{d}) = 84 + 3\bar{d}. \tag{44}$$

Therefore inequality (41) implies that

$$g - 1 \geqslant 93. \tag{45}$$

It follows from inequalities (42) and (43) that $c \geqslant 210$. Using inequality (16), we get

$$d(2d-3) \geqslant c + g - 1 \geqslant 210 + 93 = 303,$$

whence $d \geqslant \frac{3+\sqrt{2433}}{4} > 13$, that is,

$$d \geqslant 14 \tag{46}$$

because d is an integer. Therefore,

$$\bar{d} = 21 - d \leqslant 7. \tag{47}$$

By assumption, the invariants of $\overline{S}$ must satisfy inequality (11) for $m = 7$. Hence they satisfy inequality (26). It follows from (26) that

$$210 - 25\bar{d} - 8(u - \bar{g}) \leqslant 0$$

since $t \geqslant 0$. Then $210 \leqslant 25\bar{d} + 8(u - \bar{g}) \leqslant 33\bar{d}$ because $u - \bar{g} \leqslant \bar{d}$. Therefore $\bar{d} \geqslant \frac{210}{33} = 6 + \frac{4}{11}$, that is, $\bar{d} \geqslant 7$ since $\bar{d}$ is an integer. Applying inequality (47), we must have $\bar{d} = 7$.

Equation (8), with $m = 7$, and inequality (44) imply that

$$181 - 20\bar{d} - 4(u - \bar{g}) + 9t \geqslant 84 + 3\bar{d}.$$

Therefore,

$$9t \geqslant 64 + 4(u - \bar{g}) \tag{48}$$

since $\bar{d} = 7$, and by (26) we have

$$125\bar{d} + 40(u - \bar{g}) - 1050 \geqslant 69t,$$

that is,

$$40(u - \bar{g}) - 175 \geqslant 69t. \tag{49}$$

Combining inequalities (48) and (49), we get

$$3\big(40(u-\bar{g})-175\big)\geqslant 23\big(64+4(u-\bar{g})\big),$$

that is, $28(u-\bar{g})\geqslant 3\cdot 175+23\cdot 64=1997$. On the other hand, $u-\bar{g}\leqslant \bar{d}=7$, a contradiction.

Let us return to the remaining two cases, when $m=4$ and either $u=\bar{d}=2$, $\bar{g}=t=0$, or $u=\bar{d}=3$, $\bar{g}=0$, $t=1$.

First take the case when $m=4$ and $u=\bar{d}=2$, $\bar{g}=t=0$. By formulae (8), (9) and (6) we have $d=4$, $g=1$ and $c=12$. Hence the number n of nodes of B is equal to $d(2d-3)-c-g+1=8$.

Suppose that there is another generic covering $f_2\colon S_2\to\mathbb{P}^2$ with the same discriminant curve B and f_2 is not equivalent to the generic projection f. By Theorem 1 of [2], we have

$$\deg f_2\leqslant\frac{4(3d+g-1)}{2(3d+g-1)-c}=4.$$

Since S_2 is a non-singular surface and the discriminant curve B of f_2 has nodes, $\deg f_2$ cannot be equal to 3. Hence $\deg f_2=4$.

We put $S_1=S$, $R_1=R$, $C_1=C$ and $f_2^*(B)=2R_2+C_2$, where R_2 is the ramification locus of f_2.

Consider the fibre product

$$S_1\times_{\mathbb{P}^2}S_2=\big\{(x,y)\in S_1\times S_2\mid f_1(x)=f_2(y)\big\}$$

and let $X=\widetilde{S_1\times_{\mathbb{P}^2}S_2}$ be the normalization of $S_1\times_{\mathbb{P}^2}S_2$. We denote the corresponding natural morphisms by $g_1\colon X\to S_1$, $g_2\colon X\to S_2$ and $f_{1,2}\colon X\to\mathbb{P}^2$. We have $\deg g_1=\deg f_2=4$, $\deg g_2=\deg f_1=4$ and $\deg f_{1,2}=\deg g_1\cdot\deg f_1=16$. By Propositions 2 and 3 of [2], X is an irreducible non-singular surface.

Let $\widetilde{R}\subset X$ be the curve $g_1^{-1}(R_1)\cap g_2^{-1}(R_2)$, $\widetilde{C}=g_1^{-1}(C_1)\cap g_2^{-1}(C_2)$, $\widetilde{C}_1=g_1^{-1}(R_1)\cap g_2^{-1}(C_2)$ and $\widetilde{C}_2=g_1^{-1}(C_1)\cap g_2^{-1}(R_2)$.

By Proposition 4 of [2] we have

$$\begin{aligned}\widetilde{R}^2&=2(3d+g-1)-c=12,\\ \widetilde{C}_1^2&=(\deg f_1-2)(3d+g-1)-c=12,\\ \widetilde{C}_2^2&=(\deg f_2-2)(3d+g-1)-c=12,\\ (\widetilde{R},\widetilde{C}_i)&=c=12,\qquad i=1,2.\end{aligned}$$

Applying the arguments used in the proof of Proposition 4 in [2], one can easily see that the intersection number

$$(\widetilde{C}_1,\widetilde{C}_2)=c+2n=28.$$

Hence the determinant

$$\begin{vmatrix}\widetilde{R}^2 & (\widetilde{R},\widetilde{C}_1)\\ (\widetilde{C}_1,\widetilde{R}) & \widetilde{C}_1^2\end{vmatrix}=0$$

and, therefore, the Hodge index theorem yields that the classes $[\widetilde{C}_1]$ and $[\widetilde{R}]$ of the curves $\widetilde{C}_1$ and $\widetilde{R}$ are linearly dependent in the Néron–Severi group $\mathrm{NS}(X)$ of all divisors classes on X modulo numerical equivalence. Since $\widetilde{R}^2 = \widetilde{C}_1^2$, we have $[\widetilde{R}] = [\widetilde{C}_1]$ in $\mathrm{NS}(X)$. Applying the same arguments, we see that $[\widetilde{R}] = [\widetilde{C}_2]$ and, therefore, $[\widetilde{C}_2] = [\widetilde{C}_1]$ in $\mathrm{NS}(X)$. Hence the intersection number $(\widetilde{C}_2, \widetilde{C}_1)$ must be equal to $\widetilde{C}_1^2 = 12$. On the other hand, $(\widetilde{C}_2, \widetilde{C}_1) = 28$, a contradiction.

To complete the proof of the theorem, we note that the last case (when $m = 4$, $u = \bar{d} = 3$, $\bar{g} = 0$, $t = 1$) corresponds to a generic projection $f\colon S \to \mathbb{P}^2$, where the surface $S \simeq \mathbb{P}^2$ is embedded in $\mathbb{P}^5$ by polynomials of degree two (the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$) and f is the restriction to S of a linear projection $\mathrm{pr}\colon \mathbb{P}^5 \to \mathbb{P}^2$ (see, for example, [5]). In this case $B \subset \mathbb{P}^2$ is the dual curve of a smooth cubic, $\deg B = 6$, $c = 9$ and the curve B is the discriminant curve of four inequivalent generic coverings of $\mathbb{P}^2$ (see [1], [6]). Three of these have degree 4 and the other has degree 3.

Corollary 1. *Let S_i be non-singular surfaces, $i = 1, 2$, and let $S_i \subset \mathbb{P}^{N_i}$ be embeddings given by complete linear systems of divisors on S_i. Suppose that neither of these embeddings coincides with the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. Let $f_i = \mathrm{pr}_{i|S_i}\colon S_i \to \mathbb{P}^2$ be two generic coverings ramified over the same cuspidal curve B, where $\mathrm{pr}_i\colon \mathbb{P}^{N_i} \to \mathbb{P}^2$ are linear projections. Then $N_1 = N_2 = N$ and there is a linear transformation $h\colon \mathbb{P}^N \to \mathbb{P}^N$ such that $h(S_1) = S_2$ and $f_1 = f_2 \circ h$.*

Proof. Let $\bar{L}_i = f_i^{-1}(L) \subset S_i$, $i = 1, 2$, be proper transforms of a line L in $\mathbb{P}^2$. By the theorem, there is an isomorphism $h\colon S_1 \to S_2$ such that $f_1 = h \circ f_2$. Hence $h(\bar{L}_1) = \bar{L}_2$ and, therefore, $h^*\big(\mathcal{O}_{S_2}(\bar{L}_2)\big) = \mathcal{O}_{S_1}(\bar{L}_1)$. It follows that

$$N_1 = \dim H^0\big(S_1, \mathcal{O}_{S_1}(\bar{L}_1)\big) = \dim H^0\big(S_2, \mathcal{O}_{S_2}(\bar{L}_2)\big) = N_2$$

and the isomorphism h can be defined by the linear transformation $\mathbb{P}^{N_1} \to \mathbb{P}^{N_2}$ induced by

$$h^*\colon H^0\big(S_2, \mathcal{O}_{S_2}(\bar{L}_2)\big) \to H^0\big(S_1, \mathcal{O}_{S_1}(\bar{L}_1)\big).$$

We also note that if $f\colon S \to \mathbb{P}^2$ is a generic covering with $\deg f = 4$ branched over a cuspidal curve $B \subset \mathbb{P}^2$ with $\deg B = 6$ and $c = 9$, then equations (4) and (5) imply that $K_S^2 = 9$ and $e(S) = 3$. For any line L in $\mathbb{P}^2$, the genus of $f^{-1}(L)$ is equal to $\frac{-2\deg f + (L,B)}{2} + 1 = 0$ by Hurwitz' formula. Therefore $S \simeq \mathbb{P}^2$ and f is given by polynomials of degree 2. Hence, in the exceptional case of a cuspidal curve $B \subset \mathbb{P}^2$ with $\deg B = 6$ and $c = 9$, each of the three inequivalent generic coverings f_i with $\deg f_i = 4$ ramified over B is a generic projection of $\mathbb{P}^2$ embedded in $\mathbb{P}^5$ by the Veronese embedding. It is easy to see that the fourth exceptional generic covering $f_4\colon S \to \mathbb{P}^2$ with $\deg f_4 = 3$ is not a generic projection (see the case $m = 3$ in the proof of the theorem). Hence we get the following corollary.

Corollary 2. *Let $f\colon S \to \mathbb{P}^2$ be a generic linear projection branched over a cuspidal curve $B \subset \mathbb{P}^2$. Then the surface S is uniquely determined (up to isomorphism) by the curve B.*

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