REMARKS ON ALGEBRAIC GEOMETRY

IVAN CHELTSOV

Abstract. The purpose of these notes is to explain what is Algebraic Geometry in a simple geometric way. They are not very precise and cannot be considered as an introduction to Algebraic Geometry. All results I mentioned here are either left without proofs or their proofs are briefly sketched. The reader is encouraged to fill the gaps. Let me mention few books that can be quite helpful for doing so. Frances Kirwan's book *Complex algebraic curves* is an excellent introduction to complex algebraic curves (see [5]). Whenever possible I have included a page reference to the book, in the form [5]. Another beautiful book on this subject is Rick Midanda's book *Algebraic curves and Riemann surfaces* (see [6]). To get a feeling what is higher-dimensional complex algebraic geometry, see the book *Undergraduate algebraic geometry* by Miles Reid (see [8]).

1. Algebraic varieties

Algebraic Geometry deals with geometrical objects that are given by finitely many polynomial equations. These objects are called algebraic varieties. Algebraic varieties give us many explicit examples that we use almost every day. Here are some them.

Example 1.1. What is a line? Perhaps this question is a bit philosophical. Lets me ask a more explicit question: what is a line in $\mathbb{R}^2$? Let A, B, and C be real numbers such that $(A, B) \neq (0, 0)$. Then the equation

$$Ax + By + C = 0$$

defines a line L in $\mathbb{R}^2$. You can consider this as a definition of a line in $\mathbb{R}^2$. Let L' be another line in $\mathbb{R}^2$. Then L' is given by the equation $A'x + B'y + C' = 0$ for some real numbers A', B', and C' such that $(A', B') \neq (0, 0)$. When $L = L'$? This is easy:

$$L = L' \iff (A, B, C) = \lambda\big(A', B', C'\big)$$

for some non-zero real number λ. It is visually clear that the intersection $L \cap L'$ consists of at most one point. Similarly, the line L does intersect L' if and only if they are not parallel (this part I do not like, since it would be much better if two lines on a plane always intersect each other). However, we do not need to *see* anything here: *we can split the harmony by Algebra.* Indeed, Linear Algebra tells us that if $L \neq L'$, then

$$L \cap L' \neq \varnothing \iff \det\begin{pmatrix} A & B \\ A' & B' \end{pmatrix} \neq 0.$$

Example 1.2. The curves in $\mathbb{R}^2$ given by $x = y^2$, $xy = 1$, and $x^2 + 2y^2 = 1$ are called *parabola*, *hyperbola*, and *ellipse* respectively. These curves are special cases of curves known as *conic sections*. Conic sections are *nice* curves given by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, C, D, E, and F are some real numbers such that $(A, B, C) \neq (0, 0, 0)$. Sometimes this equation defines something *not nice* like $x^2 + y^2 = -1$ or $xy = 0$. Up to translations, rotations, and scaling, every (nice) conic section is either given by $x = y^2$, or by $xy = 1$, or by $x^2 + y^2 = 1$ (see [8]).

Note that we meet conics sections every day, e.g. when you are reading this text you are orbiting around the sun by an ellipse (not by a circle!).

Example 1.3. What is a line in $\mathbb{R}^3$? Let A, B, C, and D be real numbers such that $(A, B, C) \neq (0, 0, 0)$. Then the equation $Ax + By + Cz + D = 0$ defines a plane in $\mathbb{R}^3$. You can consider this as a definition of a plane in $\mathbb{R}^3$. Then every line is a non-empty intersection of two distinct planes in $\mathbb{R}^3$. Once again, you can consider this as a definition of a line in $\mathbb{R}^3$. Let us describe another, though equivalent, way of defining a line in $\mathbb{R}^3$. Pick two distinct points P and Q in $\mathbb{R}^3$. Let $P = (x_P, y_P, z_P)$ and $Q = (x_Q, y_Q, z_Q)$. The subset in $\mathbb{R}^3$ spanned by points

$$\Big(x_P + \lambda(x_Q - x_P), y_P + \lambda(y_Q - y_P), z_P + \lambda(z_Q - z_P)\Big)$$

when λ varies in $\mathbb{R}$ is also a line. This way of defining a line is called *parametric*.

Exercise 1.4. *Let S be a subset in $\mathbb{R}^3$ that is given by $x = zy$. Then S looks like this roof*

and is called hyperbolic paraboloid. *Try to prove that for every point $P \in S$, there exist exactly two lines contained in S that pass through P. Can you see these lines on this picture?*

If you failed to see lines in the hyperbolic paraboloid roof in Exercise 1.4, try to see them here

Exercise 1.5. *Let S be a subset in $\mathbb{R}^3$ that is given by $z^2+1=x^2+y^2$. Then S is called* hyperboloid of one sheet*. It looks like the Shukhov radio tower in Moscow designed by Vladimir Shukhov and built in 1919–1922 during the Russian Civil War:*

Try to prove that for every point $P \in S$, there exist exactly two lines contained in S that passes through P. Can you see these lines in the Shukhov radio tower?

2. Complex algebraic varieties

In all examples above, algebraic varieties are defined using real numbers. Such algebraic varieties are called *real algebraic varieties*. Despite clear relation to real life problems, it is *better* to consider algebraic varieties that are defined using complex numbers. Why this is better? The reasons is simple: complex numbers are better than reals.

Example 2.1. Every non-zero non-constant polynomial $f(x)$ with complex coefficients is always a product of polynomials of degree one. Every non-zero non-constant polynomial $g(x)$ with real coefficients is not necessary a product of polynomials of degree one (but it is always a product of polynomials of degree at most two).

Example 2.2. Let C be the subset in $\mathbb{C}^2$ that are given by

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0,$$

where A, B, C, D, E, and F are some complex numbers such that $(A,B,C)\neq(0,0,0)$. One can show (see [8]) that up to translations and linear transformation, C can be given by one of the following equations: $x=y^2$, $xy=1$, $xy=0$, $x^2=1$, or $x^2=0$ (cf. Example 1.2).

Example 2.3. Let S be the subset in $\mathbb{C}^3$ that are given by

$$Ax^2+Bxy+Cy^2+Dxz+Eyz+Fz^2+Gx+Hy+Iz+K=0,$$

where A, B, C, D, E, F, G, H, I, K are some complex numbers such that $(A,B,C,D,E,F)\neq(0,0,0,0,0,0)$. One can show that up to translations and linear transformation, S can be given by one of the following equations: $x^2+1=yz$, $x=yz$, $x^2=yz$, $x=y^2$, $xy=1$, $xy=0$, $x^2=1$, or $x^2=0$.

Exercise 2.4. *Define lines in $\mathbb{C}^3$ similar to the way I defined them in Example 1.3. Let S be a subset in $\mathbb{C}^3$ that is given by one of the following equations: $x^2+1=yz$, $x=yz$, $x^2=yz$, $x=y^2$, or $xy=1$. Let P be a point in S. Prove that there exist exactly two lines contained in S that pass through P (cf. Exercises 1.4 and 1.5) if S is given either by $x^2+1=yz$ or by $x=yz$. Prove that there exists exactly one line contained in S that passes through P if S is given either by $x=y^2$ or by $xy=1$. Prove that there exists exactly one line contained in S*

that passes through P if S is given by $x^2 = yz$ and $P \neq (0,0,0)$. If S is given by $x^2 = yz$ and $P = (0,0,0)$, prove that there exist infinitely many lines in S that pass through P.

Algebraic varieties that are defined using $\mathbb{C}$ are called *complex algebraic varieties*. Complex algebraic varieties play a very important role in Geometry.

3. Closed subvarieties

Real algebraic varieties can be considered as complex as well. For instance, the equation $x^2+y^2=1$ defines a unit circle C in $\mathbb{R}^2$. The same equation defines a complex algebraic variety in $\mathbb{C}^2$. Identifying $\mathbb{C}^2$ with $\mathbb{R}^4 = (\mathrm{Re}(x), \mathrm{Im}(x), \mathrm{Re}(y), \mathrm{Im}(y))$, we see than

$$x^2+y^2=1 \iff \begin{cases} \mathrm{Re}\big(x^2+y^2\big)=1, \\ \mathrm{Im}\big(x^2+y^2\big)=0, \end{cases}$$

which shows that $x^2+y^2=1$ is something two-dimensional. In fact, we can say more about this something. If you know what is *manifold*, then it is a two-dimensional manifold in $\mathbb{R}^4$.

Lemma 3.1. *Let $f(x,y)$ be a polynomial with real coefficients such that $f(a,b)=0$ for every $(a,b) \in C$. Then*

$$(x^2+y^2-1)g(x,y)$$

for some polynomial with real coefficients $g(x,y)$.

Proof. This is not obvious. But this is true. It follows, for example, from the Bezout's theorem (see Theorem 12.3). It also follows from the proof of Lemma 3.3. □

Vice versa, every polynomial in $\mathbb{R}[x,y]$ that is divisible by x^2+y^2-1 must vanish at every point of the circle C (it must be zero at every point of the curve C). The same holds over complex numbers.

Does C contain any real algebraic *subvarieties* that are cut out on C by finitely many polynomial equations? Yes, of course. Say, point $(0,1)$ is given by two polynomial equations

$$\begin{cases} x=0 \\ y-1=0 \end{cases}$$

Does C have any complex algebraic *subvarieties* that are cut out on C by finitely many polynomial equations? I.e. is there any complex variety Y such that $Y \subset C$ and Y is cut out on C by finitely many polynomial equations with complex coefficients? Yes, of course. Say, the set

$$(i, \sqrt{2}) \cup (-i, \sqrt{2})$$

is cut out on C by $y-\sqrt{2}=0$. In fact, any real line $L \subset \mathbb{R}^2$ intersects C either in two real points (if L is tangent to C at some point, then this point must be counted with multiplicity two) or in two complex conjugate points!

Definition 3.2. A subset Y of a complex algebraic variety $V \subset \mathbb{C}^n$ is called *closed complex subvariety* if it is cut out on V by finitely many polynomial equations with complex coefficient.

We see that complex points in C and C itself are closed subvarieties of the complex variety C. Does C contains any other subvarieties that looks differently? No.

Lemma 3.3. *Let W be a closed subvariety in C that contains infinitely many points. Then $W=C$.*

Proof. Let $h(x,y)$ be a polynomial of degree d that is zero at every point of W. Then the system

$$\begin{cases} x^2+y^2=1 \\ h(x,y)=0 \end{cases}$$

has infinitely many solutions. Is it OK? No, this is not OK. This means that $h(x, y)$ must be divisible by $x^2 + y^2 - 1$ (one can use Theorem 12.3 to prove this) and, hence, the system

$$\begin{cases} x^2 + y^2 = 1 \\ h(x, y) = 0 \end{cases}$$

defines the same circle C as well (and we never ever get W by taking common zeroes of finitely many polynomial equations!). Indeed, expressing x as $\pm\sqrt{1 - y^2}$ and plugging it into $h(x, y) = 0$ and simplifying the obtained equation a bit, we see that the above system has finitely many solutions (we must use Fundamental Theorem of Algebra here) unless the obtained equation degenerates to $0 = 0$, which is exactly the case when $h(x, y)$ is divisible by $x^2 + y^2 - 1$. □

What about consider C as real subvariety?

Exercise 3.4. *Define what is a real closed subvariety of a real algebraic variety. Google to check whether your definition is OK or not. Find all real subvarieties in C (in your definition) when we consider C as a real algebraic variety.*

Note that the complex case looks slightly better (aesthetically).

4. Irreducible complex varieties

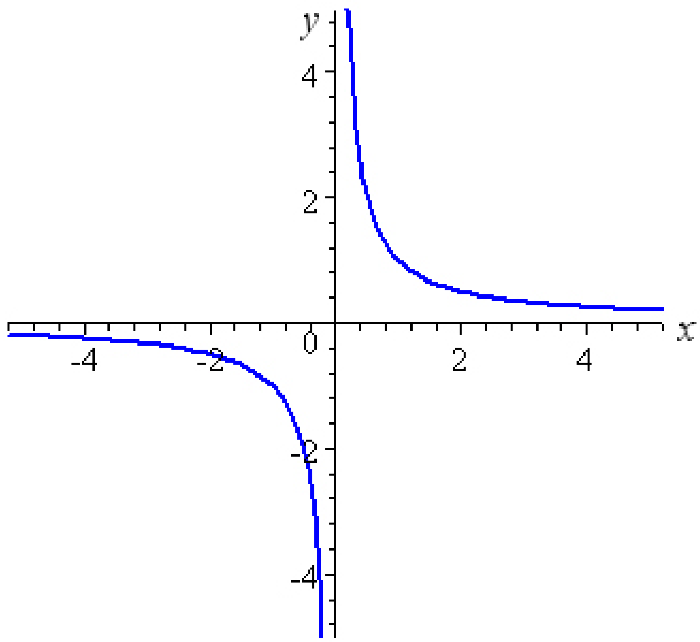
Take a real number λ. Then the equation

$$xy = \lambda$$

defines a hyperbola H_λ in $\mathbb{R}^2$. The same equation defines *complex hyperbola* in $\mathbb{C}^2$. For the sake of simplicity, let us denote the complex hyperbola also by the same symbol H_λ (but this may be a little bit confusing to use the same notation for real and complex objects).

Is H_λ always a hyperbola? Almost always. If $\lambda \neq 0$, then H_λ is a hyperbola. What is H_λ if $\lambda = 0$? It is just a union of two lines! Not just any lines. These lines, given by $x = 0$ and $y = 0$, are asymptotes of the hyperbola H_λ for every non-zero $\lambda \in \mathbb{R}$.

Which closed subvariety is better H_1 or H_0? Of course, the answer depends on your taste. Note that H_0 consists of two lines. So may be this makes it look worse than H_1? Yes, it does. The variety H_0 is a union of two different algebraic varieties, i.e. the line $x = 0$ and the line $y = 0$, which are both closed subvarieties of H_0 (see Definition 3.2 and Exercise 3.4). While H_1 can not be represented like this no matter how we consider H_1 (as a real variety or as a complex variety). Is this really true? Drawing a picture, we vividly see that the real algebraic variety H_1 consists of two parts: the *left* part and the *right* part on this picture:



Algebraically, the left part can be given by

$$\begin{cases} xy = 1 \\ x < 0 \end{cases}$$

and the right part can be given by

$$\begin{cases} xy = 1 \\ x > 0 \end{cases}$$

But these parts are NOT closed subvarieties in $\mathbb{R}^2$. Indeed, we can argue as above (bit informal but this is OK) and check that every polynomial $h(x, y)$ that is vanish at every point of the left part must by divisible by $xy-1$ and, hence, vanish at every point of the hyperbola H_1. So what is wrong here? Nothing really. The problem is that we used > 0, which is not allowed by the rules of our game (we can use only polynomial EQUALITIES when define varieties). Note that over complex numbers H_1 is connected and > 0 does not make any sense, which makes our life a bit simpler.

This example can be put into a definition.

Definition 4.1. A complex algebraic variety is said to be *irreducible* if it is not a union of two different complex closed algebraic subvarieties.

Naturally, it is much easier to work with irreducible varieties. In fact, in many books algebraic varieties are assumed to be irreducible! I will not assume this. But I will use mostly irreducible subvariety.

What are *proper* irreducible complex closed subvarieties in $\mathbb{C}^2$? I.e. what are irreducible complex closed subvarieties in $\mathbb{C}^2$ that are different from $\mathbb{C}^2$?

Theorem 4.2. *Irreducible proper closed subvarieties in* $\mathbb{C}^2$ *are either points or subvarieties that are given by*

$$f(x, y) = 0,$$

where $f(x, y)$ *is an irreducible polynomial with complex coefficients.*

Proof. The required assertion is pure commutative algebra. Its proof is not very complicated (see [9, Chapter 1.5] for details). Try to prove it by using Bezout's theorem (see Theorem 12.3) and the fact that $\mathbb{C}[x, y]$ is Unique Factorization Domain. $\square$

What is $x^2 = 0$? It is a line $x = 0$, of course. But it is given by zeroes of REDUCIBLE polynomial. Is it OK? Yes, it is OK. If we take care only about zeroes of finitely many polynomial equations, then we must live with this subtle problem, i.e. irreducible varieties in $\mathbb{C}^2$ can be given by reducible polynomials. Another way of sorting out this problem is to consider the variety given by $x^2 = 0$ as a double line (a line *multiplied* by two) not just as a line. I will not do this for the sake of simplicity (see [8] if you curios).

5. Real pathologies

How to define irreducible real algebraic varieties? We can try the same way as we defined irreducible complex algebraic varieties. Namely, we can say that a real algebraic variety is *irreducible* if it is not a union of two different real closed algebraic subvarieties (see Exercise 3.4).

What are irreducible real closed subvarieties in $\mathbb{R}^2$? Points and varieties that are given by $f(x, y) = 0$, where $f(x, y)$ is an irreducible polynomial with real coefficients now. There are two minor problems here. Let me describe them.

First problem is related to something like

$$x^2 + y^2 + 1 = 0$$

that defines an empty set. The polynomial $x^2 + y^2 + 1$ is just fine (irreducible). But it is not zero at any real point in $\mathbb{R}^2$. What does $x^2 + y^2 + 1 = 0$ really define? You can either say that it defines an empty set (I already did this) and forget about this. Or you can say that it defines a circle without real points (circle of imaginary radius). If you chose the second option, the set of zeroes of every irreducible polynomial in $\mathbb{R}[x, y]$ is an irreducible closed algebraic subvariety in $\mathbb{R}^2$.

The second problem is more subtle. It is about *being irreducible*. Note that the equation

$$x^2 + y^2 = 0$$

defines just the point $(0, 0)$ when we consider it as a real equation. When we consider it over complex numbers, it splits as

$$(x + \sqrt{-1}y)(x - \sqrt{-1}y) = 0,$$

which means that $x^2+y^2 = 0$ is a union of two complex (conjugated) lines. So it is not irreducible over complex numbers. Are we happy with this? What should we do about this? Again, we have two options. Either we can consider $x^2 + y^2 = 0$ as a point $(0, 0)$, which is clearly an irreducible algebraic variety. Or, alternatively, we can consider $x^2 + y^2 = 0$ is a union of two complex (conjugated) lines that intersect at the real point $(0, 0)$. Which option is closer to your taste?

We see over and over again that complex numbers are better than reals. Over $\mathbb{R}$, algebra and geometry do not match well. Over $\mathbb{C}$, algebra and geometry do match well.

6. Dimension

Let V be an irreducible complex algebraic variety V in $\mathbb{C}^n$. Then V has *dimension*, which is usually denoted as $\dim(V)$. What it is? How to define it? You may find these questions silly. But they are not silly.

The dimension $\dim(V)$ can be defined purely algebraically (see [8]).

Definition 6.1. Let R be the commutative ring of all functions $V \to \mathbb{C}$ obtained as restrictions of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ (the ring of all polynomial functions $\mathbb{C}^n \to \mathbb{C}$) to the variety V. Then R is a domain, i.e. there are no two non-zero functions f and g in R such that fg is a zero function. Thus, there exists a field of fractions of R, which we can denote by K. Note that K contains $\mathbb{C}$ as a subfield. Then $\dim(V)$ is the transcendence degree of the field K, i.e. the smallest number of functions $f_1, f_2, \dots, f_k$ in K such that the field K is a finite extension of the field $\mathbb{C}(f_1, f_2, \dots, f_k)$, i.e. for every $h \in K$, we have

$$h^m + g_1(f_1, f_2, \dots, f_k)h^{m-1} + \cdots + g_{m-2}(f_1, f_2, \dots, f_k)h^2 + g_{m-1}(f_1, f_2, \dots, f_k)h + g_m(f_1, f_2, \dots, f_k) = 0,$$

where $g_i(f_1, f_2, \dots, f_k)$ is a rational function in $f_1, f_2, \dots, f_k$.

The dimension $\dim(V)$ can also be defined topologically if we identify $\mathbb{C}$ with $\mathbb{R}^2$ and consider complex algebraic variety V as a real subset in $\mathbb{R}^{2n}$. Do these definitions match? Yes, they do. But we should keep in mind that $\mathbb{C}$ has dimension two over real numbers. So over real numbers, the dimension of the variety V is $2\dim(V)$, because the number $\dim(V)$ is the dimension of the variety V over complex numbers!

Is there any cheap way to define dimension? Yes, it is.

Definition 6.2. The dimension $\dim(V)$ is the maximal d such that we have a sequence

$$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d,$$

where $V_0, V_1, \dots, V_d$ are complex irreducible algebraic subvarieties in V.

It is not quite obvious that Definitions 6.1 and 6.2 define the same thing. But this is true (see [1, Chapter 11]).

It follows from Theorem 4.2 that the only irreducible varieties in $\mathbb{C}^2$ are points (a, b) (given by $x - a = y - b = 0$), $\mathbb{C}^2$ (it is given by very funny equation $0 = 0$ in $\mathbb{C}^2$), and closed subvarieties that are given by

$$f(x, y) = 0,$$

where $f(x, y)$ is an irreducible polynomial with complex coefficients. This gives $\dim(\mathbb{C}^2) = 2$ (which we already knew intuitively).

Remark 6.3. By the Fundamental Theorem of Algebra (see Example 2.1), the only irreducible closed subvarieties in $\mathbb{C}$ are points and $\mathbb{C}$. So $\dim(\mathbb{C}) = 1$.

How to prove that $\dim(\mathbb{C}^n) = n$? This is less easy if $n \geqslant 3$. Is there another way of defining dimension of a complex algebraic variety? Yes. There are plenty and all of them give the same number (google it).

Exercise 6.4. *There is a geometric way of defining the dimension of a complex algebraic subvariety V in $\mathbb{C}^n$. Let $\dim(V)$ be the number of hyperplanes (hyperplane in $\mathbb{C}^n$ is an irreducible algebraic variety that are given by zeroes of a polynomial of degree 1) in generic position which are needed to have an intersection with V which is reduced to a finite number of points. Check that this definition gives the same answer for $\mathbb{C}$ and $\mathbb{C}^2$. Prove that this definition implies that $\dim(\mathbb{C}^n) = n$.*

Note that assuming that V is irreducible is quite handy for defining dimension. Indeed, irreducible components of a reducible algebraic varieties can have different dimensions (e.g. union of a line and a plane).

Algebraic geometers reserve special funny words for complex irreducible algebraic varieties of low dimensions. To us, *curve* means complex irreducible algebraic variety of dimension one. *Surface* means complex irreducible algebraic variety of dimension two. *Threefold* means complex irreducible algebraic variety of dimension three. *Fourfold* means complex irreducible algebraic variety of dimension four. This may be confusing sometimes, because $\mathbb{C}^2$ is a surface in Algebraic Geometry, but it has real (topological) dimension four when we identify $\mathbb{C}^2$ with $\mathbb{R}^4$.

Example 6.5. The equation $x^2 + y^2 = 1$ defines a complex irreducible algebraic curve in $\mathbb{C}^2$, which has (topological) dimension two since

$$x^2 + y^2 = 1 \iff \begin{cases} \mathrm{Re}\big(x^2 + y^2\big) = 1, \\ \mathrm{Im}\big(x^2 + y^2\big) = 0, \end{cases}$$

where we identify $\mathbb{C}^2$ with $\mathbb{R}^4 = (\mathrm{Re}(x), \mathrm{Im}(x), \mathrm{Re}(y), \mathrm{Im}(y))$.

To avoid confusion, it would be better to say complex irreducible algebraic curve instead of just using the word *curve*. But I always forget to do this. And most of algebraic geometers forget to do this too

What about irreducible real algebraic varieties? How to define dimension of an irreducible real variety in $\mathbb{R}^n$. One way is to make it equal to the dimension of its complexification (you just consider the same defining equation in $\mathbb{C}^n$). This is a good way (it matches all algebraic definitions). But in this case, the dimension of

$$x^2 + y^2 = -1$$

would be 1. I am happy with this. Are you?

7. Compactness

Recall then $x^2 + y^2 = 1$ defines a circle in $\mathbb{R}^2$, which is compact. The very same equation $x^2 + y^2 = 1$ defines an irreducible complex algebraic subvariety in $\mathbb{C}^2 \cong \mathbb{R}^4$.

Exercise 7.1. *Let C be a subset in $\mathbb{C}^2 \cong \mathbb{R}^4$ that is given by $x^2 + y^2 = 1$. Equip C with induced Euclidean topology from $\mathbb{C}^2 \cong \mathbb{R}^4$. Prove that C is not compact.*

Algebraic geometers always prefer to work with compact sets. In fact, we all prefer to work with compact sets. We live on a sphere (real ellipsoid actually), which is compact. Unfortunately, irreducible complex algebraic varieties (that we defined already) are not compact except the very trivial example of a point. How to get rid of this non-compactness? We have to *compactify* complex irreducible algebraic varieties. And we better do this in a natural way.

Let V be an irreducible complex closed algebraic subvariety in $\mathbb{C}^n$. Then V is given by

$$f_1(z_1, \ldots, z_n) = f_2(z_1, \ldots, z_n) = f_2(z_1, \ldots, z_n) = \cdots = f_m(z_1, \ldots, z_n) = 0,$$

where $f_i(z_1, \ldots, z_n)$ is a polynomial of degree d_i for every i, and $(z_1, \ldots, z_n)$ are coordinates on $\mathbb{C}^n$. Replace every $f_i(z_1, \ldots, z_n)$ by its homogenization

$$F_i(x_0, \ldots, x_n) = x_0^{d_i} f_i\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

Note that $F_i(x_0, \ldots, x_n)$ is a homogeneous polynomial of degree d_i (for every i).

Definition 7.2. Let $\sim$ be the equivalent relation on $\mathbb{C}^n \setminus (0, \ldots, 0)$ such that

$$\big(x_0, \ldots, x_n\big) \sim \big(x_0', \ldots, x_n'\big) \iff \big(x_0, \ldots, x_n\big) = \big(\lambda x_0', \ldots, \lambda x_n'\big)$$

for some non-zero complex number λ. The projective space $\mathbb{CP}^n$ is $(\mathbb{C}^n \setminus (0, \ldots, 0))/\sim$.

We refer to the elements of the set $\mathbb{CP}^n$ as points, and we denote by $[x_0 : \ldots : x_n]$ the equivalence class of $(x_0, \ldots, x_n)$. Consider points in $\mathbb{CP}^n$ as $(n+1)$-tuples $[x_0 : \ldots : x_n]$ such that

$$\big[x_0 : \ldots : x_n\big] = \big[x_0' : \ldots : x_n'\big] \iff \big(x_0, \ldots, x_n\big) = \big(\lambda x_0', \ldots, \lambda x_n'\big)$$

for some non-zero complex number λ. We must exclude the $(n+1)$-tuple $[0 : 0 : \ldots : 0]$ (by definition)

Remark 7.3. We can identify $\mathbb{C}^n$ with points $[x_0 : \ldots : x_n]$ in $\mathbb{CP}^n$ such that $x_0 \neq 0$. Indeed, the map

$$(z_1, \ldots, z_n) \mapsto [1 : z_1 : \ldots : z_n] \in \mathbb{CP}^n$$

is one-to-one (injective). Note that $\mathbb{CP}^n$ is equipped with a natural structure of a topological spaces that induces usual topology on $\mathbb{C}^n \subset \mathbb{CP}^n$. Moreover, the projective space $\mathbb{CP}^n$ is equipped with a natural structure of a *complex manifold* (see [6, Definition 1.24]).

Let us consider the closed subset $\bar{V}$ in $\mathbb{CP}^n$ that is given by

$$F_1(x_0, \ldots, x_n) = F_2(x_0, \ldots, x_n) = \cdots = F_m(x_0, \ldots, x_n) = 0.$$

Exercise 7.4. *Prove that $\bar{V}$ is compact.*

Since we can identify $\mathbb{C}^n$ with points $[x_0 : \ldots : x_n]$ in $\mathbb{CP}^n$ such that $x_0 \neq 0$, we can identify V with points $[x_0 : \ldots : x_n]$ in $\bar{V}$ such that $x_0 \neq 0$. I.e. $\bar{V}$ is the desired compactification of V. Is it natural? I am not sure. Sometimes it is quite natural.

Exercise 7.5. *Let C be a subset in $\mathbb{C}^2$ that is given by*

$$y^2 - x(x-1)(x-2)(x-3) = 0.$$

Prove that C is irreducible complex algebraic variety of dimension one (it is a curve!). Compactify C in two different ways: using $\mathbb{CP}^2$ and using $\mathbb{CP}^1 \times \mathbb{CP}^1$. Which compactification is better? Try to find another compactification.

So there are many different ways of compactifying V. Can we chose some special compactification of V that is the best? Sometimes the answer is yes (see [8]). But the answer is NO in general.

Remark 7.6. So we compactified V by $\bar{V} \setminus V$, i.e. so we have

$$\bar{V} = V \sqcup \Big(\bar{V} \setminus V\Big),$$

where $\sqcup$ means disjoint union. What is $\bar{V} \setminus V$? This is the subset in $\mathbb{CP}^n$ that is given by

$$F_1(0, \ldots, x_n) = F_2(0, \ldots, x_n) = \cdots = F_m(0, \ldots, x_n) = 0,$$

i.e. $\bar{V} \setminus V$ is the intersection of $\bar{V}$ and the subset given by $x_0 = 0$. What is the subset in $\mathbb{CP}^n$ given by $x_0 = 0$? It is the subset in $\mathbb{CP}^n$ that is given by $x_0 = 0$. Just joking. But indeed, this is just the subset in $\mathbb{CP}^n$ that is given by $x_0 = 0$. Usually, people call this subset *infinite* hyperplane (or infinite point if $n = 1$, or infinite line if $n = 2$, or infinite plane if $n = 3$).

So we compactified V by adding $\bar{V} \cap H_0$, where H_0 is the subset in $\mathbb{CP}^n$ that is given by $x_0 = 0$.

Exercise 7.7. *Use Definition 4.1 to define complex irreducible projective varieties in $\mathbb{CP}^n$ (cf. Definition 8.4). Construct an example of and irreducible algebraic variety V such that $\bar{V}$ is not irreducible.*

When I say projective variety, I usually (almost always) mean complex projective variety. Of course, one can define and use $\mathbb{RP}^n$ and real projective varieties as well. But I will not do this here.

8. Projective varieties

What are complex projective varieties? Subsets in $\mathbb{CP}^n$ that given by finitely many homogeneous polynomial equations, i.e. a subset in $\mathbb{CP}^n$ that is given by

$$F_1(x_0, \ldots, x_n) = F_2(x_0, \ldots, x_n) = \cdots = F_m(x_0, \ldots, x_n) = 0, \tag{8.1}$$

where $F_i(x_0, \ldots, x_n)$ is a homogeneous polynomial (sometimes called *form*) of degree d_i for every i.

Remark 8.2. If the only solution of (8.1) is $x_0 = x_1 = x_2 = \cdots = x_n = 0$, then (8.1) defines an empty subset in $\mathbb{CP}^n$ (see Definition 7.2).

Note that in the way we defined *projective variety*, it goes together with *embedding* into $\mathbb{CP}^n$. Can we *forget* about the ambient projective space $\mathbb{CP}^n$? Yes, we can. In order to do this, we have to be able to say when two projective varieties lying in different projective spaces are the same. And more generally we have to define maps between projective varieties. This is not hard. But these tasks go beyond the topic of this notes (see [8]). By the way, we should have done the same for complex algebraic varieties to get rid of the ambient $\mathbb{C}^n$.

Definition 8.3. A subset Y of a complex projective variety $V \subset \mathbb{CP}^n$ is called *closed project subvariety* if it is cut out on V by finitely many homogeneous polynomial equations with complex coefficients.

Definition 8.4. A complex projective variety is said to be *irreducible* if it is not a union of two different complex projective closed subvarieties.

Let V be an irreducible complex projective variety in $\mathbb{CP}^n$. How to define the dimension of the variety V?

Definition 8.5. The dimension of the projective variety V, denoted as $\dim(V)$, is the maximal d such that we have a sequence

$$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d,$$

where $V_0, V_1, \ldots, V_d$ are irreducible complex projective subvarieties in V.

Example 8.6. Since the polynomial $x^n + y^n - z^n$ is irreducible, the equation $x^n + y^n = z^n$ defines an irreducible projective variety in $\mathbb{CP}^2$ of dimension one.

There are plenty of alternative ways of defining $\dim(V)$.

Exercise 8.7. *Use Definition 6.2 and Remark 7.6 to give an alternative definition of* $\dim(V)$*. Check in some cases that this definition matches Definition 8.5.*

Exercise 8.8. *Let us call* hyperplane *the closed subvariety in* $\mathbb{CP}^n$ *that is given by* $\sum_{i=0}^n a_i x_i = 0$ *for some* $[a_0 : a_1 : \cdots : a_n]$ *in* $\mathbb{CP}^n$*. Define* $\dim(V)$ *to be the biggest number* m *such that*

$$V \cap H_1 \cap H_2 \cap \cdots \cap H_m \neq \varnothing$$

for any sufficiently general hyperplanes $H_1, H_2, \ldots, H_m$ *in* $\mathbb{CP}^n$*. In the latter case, show that the intersection* $V \cap H_1 \cap H_2 \cap \cdots \cap H_m$ *consists of finitely many points (cf. Exercise 6.4). Denote by* $\deg(V)$ *(the degree of* $V \subset \mathbb{CP}^n$*) the number of points in this intersection. Check that* $\dim(V) = n - 1$ *and* $\deg(V) = d$ *(in this definition) provided that* V *is given by*

$$F(x_0, x_1, \ldots, x_n) = 0$$

where $F(x_0, x_1, \ldots, x_n)$ *is an irreducible homogeneous polynomial of degree* d *with complex coefficients.*

Projective varieties are equipped with topology induced from $\mathbb{CP}^n$. Since $\mathbb{CP}^n$ is compact, all projective complex varieties are compact.

Remark 8.9. When I say projective variety, I usually (almost always) mean complex projective variety. But we can define $\mathbb{RP}^n$ and define real projective varieties. They are also compact.

Complex projective varieties in $\mathbb{CP}^n$ are usually considered up to *projective transformations*, i.e. we do not make any distinction between two closed subvarieties in $\mathbb{CP}^n$ that can be obtained from each other by means of *projective transformations*. What are *projective transformations*?

Definition 8.10. A mapping $\phi\colon \mathbb{CP}^n \to \mathbb{CP}^n$ is said to be a *projective transformation* if there exists a $(n+1) \times (n+1)$ matrix M with complex entries and $\det(M) \neq 0$ that is given by

$$[x_0 : x_1 : \cdots : x_n] \mapsto [x_0 : x_1 : \cdots : x_n]M.$$

Note that projective transformations are bijective (they are homeomorphisms of course). They map complex projective varieties to complex projective varieties. Moreover, they preserve all basic properties of projective varieties, e.g. irreducible varieties are mapped to irreducible ones etc. Simply speaking, projective transformations do not change anything. Applying appropriate projective transformation, we can sometimes find slightly better defining equations for a given projective subvariety in $\mathbb{CP}^n$.

Lemma 8.11 (cf. Examples 1.2)**.** *Suppose that* C *is an irreducible curve in* $\mathbb{CP}^2$ *of degree* 2*. Then, up to projective transformations,* C *can be given by* $zx = y^2$*.*

Proof. The defining equation of the curve C looks like this

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0,$$

where A, B, C, D, E, and F are complex numbers such that some of them is not zero. In fact, since C is irreducible, at least three of them is not zero. So, we can rewrite the equation of the curve C in the matrix form as

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0,$$

and we can find a projective transformation $\phi\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ such that the corresponding matrix for $\phi(C)$ is diagonal. Since C is irreducible, $\phi(C)$ can be given by $x^2 + y^2 + z^2 = 0$. The rest of the proof is left to the reader (it is very easy). $\square$

Irreducible complex projective curves in $\mathbb{CP}^2$ of degree 2 are usually called *conics*. Their two-dimensional analogues are called *quadric surfaces*.

Example 8.12 (cf. Example 2.3)**.** Let S be the projective variety in $\mathbb{CP}^3$ that are given by

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 + Gxw + Hyw + Izw + Kw^2 = 0,$$

where A, B, C, D, E, F, G, H, I, K are some complex numbers such that $(A, B, C, D, E, F, G, H, I, K) \neq (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$, and $[x : y : z : w]$ are projective coordinates on $\mathbb{CP}^3$. Suppose that the polynomial in the equation above is irreducible. Then, up to projective translations, either S is given by one $xw = yz$ (generic case), or S is given by $x^2 = yz$ (quadric cone).

Exercise 8.13 (cf. Exercise 2.4)**.** *Define lines in $\mathbb{CP}^3$ as compactification of the lines in $\mathbb{C}^3$ (see Remark 7.6). Let S be the projective variety in $\mathbb{CP}^3$ that are given by either by $xw = yz$, or by $x^2 = yz$, where $[x : y : z : w]$ are projective coordinates on $\mathbb{CP}^3$. Let P be a point in S. Show that there exist exactly two lines contained in S that pass through P (cf. Exercises 1.4 and 1.5) if S is given by $xw = yz$. Show that this is no longer true in the case when S is given by $x^2 = yz$. Try to explain this.*

Projective transformations of $\mathbb{P}^n$ is an analogue of linear transformation of a vector space.

9. Topology of projective spaces

When we consider $\mathbb{CP}^n$ over real numbers, it is a compact topological space (even a compact real manifold) of (real) dimension $2n$. Let us consider its topology.

Example 9.1. What is topology of $\mathbb{CP}^1$? Put $P_x = [0 : 1] \in \mathbb{CP}^1$, and put $P_y = [1 : 0] \in \mathbb{CP}^1$. Then $P_x \neq P_y$. Put $U_x = \mathbb{CP}^1 \setminus P_x$, and put $U_y = \mathbb{CP}^1 \setminus P_y$. Then

$$\mathbb{CP}^1 = U_x \sqcup P_x = U_y \sqcup P_y = U_x \sqcup U_y,$$

and there are good bijections $U_x \to \mathbb{C}$ and $U_y \to \mathbb{CP}^1$. Indeed, for every point $[x : y] \in \mathbb{CP}^1$, we have

$$[x : y] = \begin{cases} \left[1 : \dfrac{y}{x}\right] & \text{if } x \neq 0 \\ \left[\dfrac{x}{y} : 1\right] & \text{if } y \neq 0 \end{cases}$$

which implies that the map $U_x \to \mathbb{C}$ given by $[x : y] \mapsto y/x$ is a bijection, and the map $U_y \to \mathbb{C}$ given by $[x : y] \mapsto x/y$ is a bijection. This implies that $\mathbb{CP}^1$ is homeomorphic to a sphere (see [5, Lemma 4.1]).

Now let me consider the complex projective plane $\mathbb{CP}^2$ in more details. Let me start with reminding you what is $\mathbb{CP}^2$.

Remark 9.2. Repetition is a mother of learning.

Let $\sim$ be a relation on $\mathbb{C}^3 \setminus (0,0,0)$ such that

$$(x,y,z) \sim (x',y',z') \iff \exists\, \lambda \in \mathbb{C} \setminus 0 \bigm| (x,y,z) = (\lambda x', \lambda y', \lambda z')$$

for any (x,y,z) and (x',y',z') in $\mathbb{C}^3 \setminus (0,0,0)$. Then $\sim$ is an equivalence relation.

Definition 9.3. The projective plane $\mathbb{CP}^2$ is $(\mathbb{C}^3 \setminus (0,0,0))/\sim$.

We refer to the elements of the set $\mathbb{CP}^2$ as points. Let $[x:y:z]$ be the equivalence class of $(x,y,z) \neq (0,0,0)$. Then we can consider points in $\mathbb{CP}^2$ as 3-tuples $[x:y:z]$ such that

$$[x:y:z] = [x':y':z'] \iff \exists\, \lambda \in \mathbb{C} \setminus 0 \bigm| (x,y,z) = (\lambda x', \lambda y', \lambda z'),$$

excluding the 3-tuple $[0:0:0]$ (bad point!) Note that

$$[1:2:3] = [2:4:6] = [1973:3946:5919] = [2012:4024:6036] = \cdots.$$

Definition 9.4 (cf. Exercise 8.8)**.** A line in $\mathbb{CP}^2$ is the subset given by

$$Ax + By + Cz = 0$$

for some (fixed) point $[A:B:C] \in \mathbb{CP}^2$.

Up to projective transformations (see Definition 8.10), all lines in $\mathbb{CP}^2$ are the same. In particular, we see that every line in $\mathbb{CP}^2$ is homeomorphic to $\mathbb{CP}^1$ (cf. [5, Lemma 4.1]).

Exercise 9.5. *Let P and Q be two points in $\mathbb{CP}^2$ such that $P \neq Q$. Prove that there is a unique line $L \subset \mathbb{CP}^2$ such that $P \in L$ and $Q \in L$.*

Exercise 9.6. *Let L and L' be two lines in $\mathbb{CP}^2$ such that $L \neq L'$. Prove that $L \cap L'$ consists of exactly one point.*

Let L_x, L_y, and L_z be the lines in $\mathbb{CP}^2$ given by $x = 0$, $y = 0$, and $z = 0$, respectively. Put $U_x = \mathbb{CP}^2 \setminus L_x$, put $U_y = \mathbb{CP}^2 \setminus L_y$, and put $U_z = \mathbb{CP}^2 \setminus L_z$. Then $\mathbb{CP}^2 = U_x \cup U_y \cup U_z$, since $L_x \cap L_y \cap L_z = \varnothing$.

Moreover, there are homeomorphisms $U_x \to \mathbb{C}^2$, $U_y \to \mathbb{C}^2$, and $U_z \to \mathbb{C}^2$, since

$$[x:y:z] = \begin{cases} \left[1 : \dfrac{y}{x} : \dfrac{z}{x}\right] & \text{if } x \neq 0, \\ \left[\dfrac{x}{y} : 1 : \dfrac{z}{y}\right] & \text{if } y \neq 0, \\ \left[\dfrac{x}{z} : \dfrac{y}{z} : 1\right] & \text{if } z \neq 0. \end{cases}$$

Example 9.7. For any point $[x:y:z] \in U_x$, the map given by

$$[x:y:z] \mapsto \left(\frac{y}{x}, \frac{z}{x}\right) \in \mathbb{C}^2$$

is a bijection. So we can identify U_x with $\mathbb{C}^2$ and consider L_x as an infinite line (cf. Remark 7.6).

Since we know that L_x, L_y, and L_z are homeomorphic to a sphere, we see that

$$H_0(\mathbb{CP}^2, \mathbb{Z}) \cong H_2(\mathbb{CP}^2, \mathbb{Z}) \cong H_4(\mathbb{CP}^2, \mathbb{Z}) \cong \mathbb{Z},$$

and $H_1(\mathbb{CP}^2, \mathbb{Z}) = H_3(\mathbb{CP}^2, \mathbb{Z}) = 0$.

Exercise 9.8. *Compute integral homology groups of $\mathbb{CP}^n$.*

Note that we can choose any line among L_x, L_y, and L_z to be an infinite line (see Example 9.7 and Remark 7.6). Since every point $P \in \mathbb{CP}^2$ lie in one of U_x, U_y, U_z, we always reduce every question about $\mathbb{CP}^2$ near P to a problem about a point in $\mathbb{C}^2$ (which may be easier to handle sometimes).

10. Plane complex curves

Let C be an irreducible complex projective curve in $\mathbb{CP}^2$, i.e. irreducible complex projective variety of dimension one (see Definition 8.5 and Exercise 8.8). Then C is usually called plane (complex projective) curve.

Theorem 10.1. *Irreducible proper closed subvarieties in $\mathbb{CP}^2$ are either points or subvarieties that are given by $f(x,y,z)=0$, where $f(x,y,z)$ is an irreducible homogenous polynomial with complex coefficients.*

Proof. The proof is the same as the proof of Theorem 4.2 and is left to the reader. □

So, since C is not a point (points are zero-dimensional subvarieties), we see that C is given by

$$f_d(x,y,z)=0,$$

where $f_d(x,y,z)$ is an irreducible homogenous polynomial of degree d with complex coefficients.

Definition 10.2 (cf. Exercise 8.8)**.** The degree of the curve C is the number d.

Among irreducible complex projective curves in $\mathbb{CP}^2$, some are slightly better that the others.

Definition 10.3. A point $[a:b:c]\in\mathbb{CP}^2$ is a singular point of the curve C if

$$\frac{\partial f_d(a,b,c)}{\partial x}=\frac{\partial f_d(a,b,c)}{\partial y}=\frac{\partial f_d(a,b,c)}{\partial z}=0.$$

It may look a bit weird that in Definition 10.3 we do not check that the singular point is contained in the curve C. But this is OK thanks to the following

Lemma 10.4 ([5, Lemma 2.32])**.** *We have*

$$x\frac{\partial f_d(x,y,z)}{\partial x}+y\frac{\partial f_d(x,y,z)}{\partial y}+z\frac{\partial f_d(x,y,z)}{\partial z}=df_d(x,y,z),$$

Proof. This formula is called Euler's formula. It is very easy to prove. Indeed, we have

$$f_d\big(\lambda x,\lambda y,\lambda z\big)=\lambda^d f_d\big(x,y,z\big)$$

for every $\lambda\in\mathbb{C}$. Taking derivative by λ, we have

$$d\lambda^{d-1}f\big(x,y,z\big)=xf_x\big(\lambda x,\lambda y,\lambda z\big)+yf_y\big(\lambda x,\lambda y,\lambda z\big)+zf_z\big(\lambda x,\lambda y,\lambda z\big),$$

which implies what we need after plugging in $\lambda=1$. Another way to prove Euler's formula is to notice that its left side and the right side are linear operators on the vector space of homogeneous polynomials of degree d. Thus, it is enough to check the formula for any basis, e.g. for monomials, which is easy. □

The set of singular points of the curve C is denoted by $\mathrm{Sing}(C)$.

Exercise 10.5. *Suppose that C is given by $zx^{d-1}=y^d$. where $d\geqslant 2$. Show that C is smooth for $d=2$. Prove that* $\mathrm{Sing}(C)=[0:0:1]$ *if* $d\geqslant 3$.

Non-singular points of the curve C are called smooth. We say that the complex projective curve C is smooth if $\mathrm{Sing}(C)=\varnothing$. Of course, all lines in $\mathbb{CP}^2$ are smooth curves of degree 1. Moreover, it follows from Lemma 10.2 that every irreducible curve in $\mathbb{CP}^2$ of degree 2 is smooth, i.e. irreducible conics are smooth.

Exercise 10.6. *Show that C is smooth if it is given by $x^d+y^d-z^d=0$.*

Let P be a point in C. Up to a projective transformation, we may assume that $P=[0:0:1]$. Then C is given by

$$z^{d-1}h_1(x,y)+z^{d-2}h_2(x,y)+\cdots+zh_{d-1}(x,y)+h_d(x,y)=0,$$

where $h_i(x,y)$ is a homogenous polynomial of degree i.

Remark 10.7. The polynomial $h_1(x,y)$ is a zero polynomial $\iff$ the curve C is singular at P.

If $h_1(x,y)$ is not a zero polynomial, then the equation $h_1(x,y)=0$ defines a line in $\mathbb{CP}^2$.

Definition 10.8. $h_1(x,y)$ is not a zero polynomial, then the line in $\mathbb{CP}^2$ given by $h_1(x,y)=0$ is said to be the line tangent to C at the point P.

Exercise 10.9. *For every smooth point $[\alpha:\beta:\gamma]\in C$, show that the line*

$$\frac{\partial f_d(\alpha,\beta,\gamma)}{\partial x}x+\frac{\partial f_d(\alpha,\beta,\gamma)}{\partial y}y+\frac{\partial f_d(\alpha,\beta,\gamma)}{\partial z}z=0$$

is the line tangent to the curve C at the point $[\alpha:\beta:\gamma]$.

What if P is a singular point of the curve C? Then either we may assume that the tangent line to C at the point P does not exist, or that every line passing through P is a tangent line to the curve C at the point P. The former option looks very natural, but the latter option behaves better in some situations (see Definition 10.15).

Definition 10.10. If $h_m(x,y)$ is non-zero polynomial for $m\geqslant 2$, but all

$$h_1(x,y),h_2(x,y),\ldots,h_{m-1}(x,y)$$

are zero ones, then we say that P is called a singular point of multiplicity m.

Example 10.11. Suppose that C is given by $zx^{d-1}=y^d$ and $d\geqslant 3$. Then $[0:0:1]$ is a singular point of the curve C of multiplicity $d-1$.

Multiplicity of a singular point is a way to measure how bad the singular point is. Some of them are bad, some of them are very bad, and some of them are just OK (mild singularities).

Definition 10.12. Two irreducible curves in $\mathbb{CP}^2$ intersect transversally at some point in $\mathbb{CP}^2$ if they both pass through this point, their both smooth at this point, and their tangent lines at this point are different.

Definition 10.13. Two irreducible curves in $\mathbb{CP}^2$ intersect transversally if they intersect transversally in every point of their intersection.

Let L be a line in $\mathbb{CP}^2$. Then L intersect transversally the curve C if and only if L is not a tangent line to C at any its point and L does not pass through any singular point of the curve C.

Exercise 10.14. *Prove that $|L\cap C|=d$ if and only if L intersect C transversally.*

If L does not intersect the curve C transversally at the point P, then either L is a tangent line to C at the point P (in this case L is given by $h_1(x,y)=0$), or P is a singular point of the curve C. In the latter case, we know how to measure singularity of the curve C at the point (see Definition 10.10). In the former case, we also can measure how L is tangent C at the point P. But it would be better to combine these measures together to measure how L intersect C at the point P.

Definition 10.15. If $P\not\in L$, we put $\mathrm{mult}_P(L\cdot C)=0$. If $P\in L$, we may assume that L is given by $x=0$ (up to projective transformation. If $L\neq C$, let $\mathrm{mult}_P(L\cdot C)$ be the smallest m such that $h_m(0,y)$ is a non-zero polynomial. If $C=L$, we either must consider $\mathrm{mult}_P(L\cdot C)$ to be undefined, or make it to be equal $+\infty$.

The number $\mathrm{mult}_P(L\cdot C)$ is usually called the (local) multiplicity of the intersection $C\cap L$ at the point P.

Exercise 10.16 (cf. Exercise 10.16 and Theorem 12.3)**.** *Prove that*

$$d=\sum_{O\in C\cap L}\mathrm{mult}_O(L\cdot C).$$

Note that $\mathrm{mult}_P(L \cdot C) = 1$ if and only if L intersects the curve C transversally at the point P. If P is a singular point of the curve C of multiplicity m ($m = 1$ means that the point P is actually smooth point of the curve C), then it follows from Definitions 10.10 and 10.15 that $\mathrm{mult}_P(L \cdot C) \geqslant m$.

Definition 10.17. If P is a smooth point of the curve C, and $\mathrm{mult}_P(L \cdot C) \geqslant 3$, then we say that the point P is an inflection point of the curve C.

Every point of a line in $\mathbb{CP}^2$ is its inflection point. It follows from Lemma 8.11 that smooth irreducible projective curves in $\mathbb{CP}^2$ do not have inflection points.

Lemma 10.18 ([5, Proposition 3.33], [6, Corollary 4.16])**.** *Suppose that C is a smooth irreducible projective complex curve in $\mathbb{CP}^2$ of degree $d > 2$. Then the number of inflection points of the curve C is finite and non-empty (i.e. the curve C has at least one inflection point).*

Lemma 10.19 (Weierstrass)**.** *Suppose that C is a smooth projective irreducible curve in $\mathbb{CP}^2$ of degree 3. Then, up to projective transformations, the curve C is given by*

$$zy^2 = x(x - z)(x - \lambda z)$$

for some $\lambda \in \mathbb{C}$ such that $\lambda \neq 0$ and $\lambda \neq 1$.

Proof. It follows from Lemma 15.1 that there exists at least on inflection point of the curve C. Let P be one such point. Let us chose projective coordinates on $\mathbb{CP}^2$ such that $P = [0 : 1 : 0]$, and the tangent line to the curve C at the point P is given by $z = 0$. Then C is given by

$$y^2 l_1(x, z) + y l_2(x, z) + l_3(x, z) = 0,$$

where $l_i(x, y, z)$ is a homogeneous polynomial of degree i. Since $l_1(x, z) = 0$ defines the line tangent to the curve C at the point P, we see that $l_1(x, z) = \lambda z$ for some non-zero complex number λ.

Multiplying the equation above by $1/\lambda$, we see that the curve C is given by $y^2 z + y l_2(x, z) + l_3(x, z) = 0$. Since P is an inflection point, we see that $l_2(x, 0)$ is a zero polynomial. So, we have

$$l_2(x, y) = z g_1(x, z)$$

for some linear form $g_1(x, y)$ is a linear form. So going back to our equation, we see that C is given by

$$y^2 z + yz g_1(x, z) + l_3(x, z) = 0.$$

Replace y be $y + \mu g_1(x, z)$ for some constant μ (to be chosen later). This is projective transformation! Then

$$\Big(y + \mu g_1(x, z)\Big)^2 z + \Big(y + \mu g_1(x, z)\Big) z g_1(x, z) + l_3(x, z) = 0$$

is the equation of the curve C in new projective coordinates. So, we have

$$\big(y^2 + 2\mu y g_1(x, z) + \mu^2 g_1^2(x, z)\big) z + \big(y + \mu g_1(x, z)\big) z g_1(x, z) + l_3(x, z) = 0$$

after simplification. Now we can put $\mu = -1/2$. We get

$$y^2 z = -l_3(x, z) - z g_1^2(x, z)/4 + g_1^2(x, z) z/2,$$

which implies that C is given by the equation $y^2 z = h_3(x, z)$, where $h_3(x, z)$ is a homogeneous polynomial of degree 3.

Since C is irreducible, the polynomial $h_3(x, z)$ does is not divisible by z. Then

$$h_3(x, z) = \epsilon(x + \delta_1 z)(x + \delta_2 z)(x + \delta_3 z)$$

for some non-zero complex number ϵ, and some complex numbers δ_1, δ_2, and δ_2. Now replacing x by $(x + \delta_1 z)$, and scaling y and z, we obtain the required equation. ☐

11. Euler characteristics

Let C be a smooth irreducible complex projective curve in $\mathbb{P}^2$.

Theorem 11.1 ([5, Proposition 1.23]). *The curve C is homeomorphic to a sphere with g handles.*

Let Δ the triangle in $\mathbb{R}^2$ given by $x_1 \geqslant 0$, $x_2 \geqslant 0$, and $x_1 + x_2 \leqslant 1$. Let us denote by Δ^0 its interior.

Recall (see [5, Definition 4.9]) that a triangulation of the complex curve C is the following data:

- a finite nonempty set V of points in C called vertices,
- a finite nonempty set E of continues maps $e\colon [0,1] \to C$ called edges such that
  - the end points of the edges in E are vertices in V,
  - if $e \in E$, then it its to $(0,1)$ is a homeomorphism on its image, and this image contains no points in E or in the image of any other edge in E,
- a finite nonempty set E of continues maps $f\colon \Delta \to C$ called faces such that
  - if $f \in F$, then the restriction of f to Δ^0 is a homeomorphism onto a connected components K_f of $C \setminus \Gamma$, where
  $$\Gamma = \bigcup_{e \in E} e([0,1]),$$
  and if $r\colon [0,1] \to [0,1]$ and $\sigma_i\colon [0,1] \to \Delta$ for $i \in \{1,2,3\}$ are defined as $r(t) = 1-t$, $\sigma_1(t) = (t,0)$, $\sigma_2(t) = (1-t,t)$, and $\sigma_3(t) = (0,1-t)$, then either $f \circ \sigma_i$ or $f \circ \sigma_i \circ r$ is an edge $e_f^i \in E$ for $i \in \{1,2,3\}$,
  - the mapping $f \mapsto K_f$ from F to the set of connected components of $C \setminus \Gamma$ is a bijection,
  - for every $e \in E$, there is exactly one face $f_e^+ \in F$ such that $e = f_e^+ \circ \sigma_i$ for some $i \in \{1,2,3\}$, and exactly one face $f_e^- \in F$ such that $e = f_e^- \circ \sigma_i \circ r$ for some $i \in \{1,2,3\}$,

Theorem 11.2 ([5, Theorem 4.13]). *Given any finite subset Σ in C, there exists a triangulation $\{V, E, F\}$ of the curve C such that $\Sigma \subset V$.*

Put $\chi(C) = |V| - |E| + |F|$.

Exercise 11.3 ([5, Example 4.16], [6, Proposition 4.15]). *Show that $\chi(C) = 2 - 2g$*

The number $\chi(C)$ is usually called *topological Euler characteristic* or simply *Euler characteristic.*

12. Bezout's theorem

Let C and C' be distinct irreducible curves in $\mathbb{CP}^2$ of degree d' such that $C \neq C'$. Then the intersection $C \cap C'$ consists of finitely many points (cf. Lemma 3.3).

How many points $C \cap C'$ has? If C' is a line (i.e. $d' = 1$, then it follows from Exercise 10.16 that the number of points in $C \cap C'$ depends on mutual position of the curves C and C', but it depends only only on d if we count points in $C \cap C'$ with multiplicities.

Exercise 12.1. *Show that C does not have singular points of multiplicity greater then d. Show that if $d > 1$, then C does not have singular points of multiplicity greater then $d-1$ (Hint: use the fact that C is irreducible and apply Exercise 10.16).*

Surprisingly, the assertion similar to Exercise 10.16 holds in general. But we need to define the intersection multiplicities for the curves C and C' similar to Definition 10.15. This is done, for example, in [5, Theorem 3.18]. Namely, for every point $O \in C \cap C'$, there is a positive integer $\mathrm{mult}_O(C \cdot C')$ that measures the non-transversality of the intersection of the curves C and C' at the point O.

I do not want to define $\mathrm{mult}_O(C\cdot C')$ for every point $O\in C\cap C'$, because it is lengthy and bit technical. Instead, let me list (without giving definition) basic properties of these intersection multiplicities.

Lemma 12.2. *Let O be a point in $C\cap C'$. Then*

- *if C is a line then $\mathrm{mult}_O(C\cdot C')$ is defined in Definition 10.15,*
- *$\mathrm{mult}_O(C\cdot C')=1$ if and only if C and C' intersect transversally at the point O,*
- *if O is a singular point of multiplicity m, and O is a singular point of multiplicity m', then*
$$\mathrm{mult}_O\big(C\cdot C'\big)\geqslant mm',$$
where the cases when either $m=1$ or $m'=1$ are also OK.

Now we are ready to count how many points the intersection $C\cap C'$ has.

Theorem 12.3 ([7, Lemma 1], [2], [4, Chapter 5.3], [5, Theorem 3.18])**.** *One has*
$$dd'=\sum_{O\in C\cap C'}\mathrm{mult}_O(C\cdot C').$$

Corollary 12.4 ([5, Theorem 3.9], [8, Theorem 1.19])**.** *One has $|C\cap C'|\leqslant dd'$.*

Corollary 12.5. *If C and C' intersect transversally, then $|C\cap C'|=dd'$.*

Despite the fact that Theorem 12.3 goes back to Isaac Newton (see [7]), it is usually called Bezout's theorem (see [2]).

Exercise 12.6. *Generalize Theorem 12.3 for two possibly reducible curves in $\mathbb{CP}^2$, and prove it in the case when one curve is a union of d distinct lines.*

13. Genus-degree formula

Let C be a complex smooth irreducible projective curve in $\mathbb{CP}^2$ of degree d.

Theorem 13.1. *The complex curve C is homeomorphic is a sphere with g handles attached.*

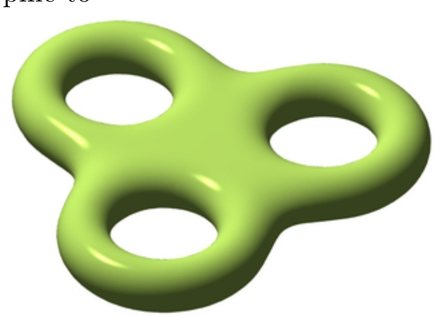
Proof. The proof follows from the fact that C is an oriented compact two-dimensional manifold and classification of oriented compact two-dimensional manifolds. For details see, for example, [5, Proposition 1.23]. □

How to find g?

Theorem 13.2 ([5, Corollary 4.19], [6, Proposition 2.15])**.** *On has $g=(d-1)(d-2)/2$.*

Therefore, for small d we have the following cases:

- $d=1$, and C is homeomorphic to a sphere,
- $d=2$, and C is homeomorphic to a sphere,
- $d=3$, and C is homeomorphic to a torus,
- $d=4$, and C is homeomorphic to



Let us prove Theorem 13.2. We may assume that $d \geqslant 2$, since we know that C is homeomorphic to a sphere if $d = 1$ (see Example 9.1).

Put $O = [0:1:0]$. Let L be a line $y = 0$. Suppose that $O \not\in C$. Define a mapping $\psi\colon C \to L$ as follows:

- take a point $P \in C$ (then $P \neq O$ since we assume that $O \not{i}nC$),
- let $L' \subset \mathbb{CP}^2$ be the unique line that passes through P and O (see Exercise 9.6),
- let $\psi(P)$ be the intersection point $L \cap L'$.

This defines a function $\psi\colon C \to L$. We say that ψ is a (stereographic) projection of the curve C from O to L. For simplicity, we can identify L with $\mathbb{CP}^1$ by $[a:0:c] \mapsto [a:c]$ for every point $[a:0:c] \in L$.

Exercise 13.3. *Show that* $\psi([a:b:c]) = [a:c]$ *for every point* $[a:b:c] \in C$.

Let $\tilde{R}$ be the subset in C consisting of all points such that $Q \in \tilde{R}$ if and only if the tangent line to C at Q passes though O. Put $R = \psi(\tilde{R})$.

Lemma 13.4. *The inequality* $|\tilde{R}| \leqslant d(d-1)$ *holds.*

Proof. Recall that $O = [0:1:0]$. Suppose for a second that $O = [a:b:c]$. Keeping in mind how ψ and $\tilde{R}$ were defined, we see that $\tilde{R}$ is cut out on C by

$$\frac{\partial f_d(x,y,z)}{\partial x}a + \frac{\partial f_d(x,y,z)}{\partial y}b + \frac{\partial f_d(x,y,z)}{\partial z}c = 0.$$

Thus, the set $\tilde{R}$ is given by

$$\frac{\partial f_d(x,y,z)}{\partial y} = f_d(x,y,z) = 0,$$

which implies that $|R| \leqslant |\tilde{R}| \leqslant d(d-1)$ by Theorem 12.3. □

Let $L_1, L_2, \ldots, L_r$ be all lines in $\mathbb{CP}^2$ such that each L_i is tangent to C at an inflection point. Then r is indeed a finite number by Lemma 15.1).

Lemma 13.5. *Suppose that* $O \not\in \cup_{i=1}^r L_i$. *Then* $|\tilde{R}| = d(d-1)$.

Proof. Suppose that $O \not\in \cup_{i=1}^r L_i$. Then

$$\frac{\partial f_d(x,y,z)}{\partial y} = 0$$

intersects the curve C *transversally* at every point of $\tilde{R}$, which implies that all multiplicities of their intersections are 1 (see [5, Remark 4.4] and the proof of [5, Lemma 4.8]), which implies that $|\tilde{R}| = d(d-1)$ by Exercise 10.16 (or, more generally, by Theorem 12.3). □

From now on, I suppose now that $\psi\colon C \to \mathbb{CP}^1$ satisfies the following generality condition: $O \not\in \cup_{i=1}^r L_i$. Then $|\tilde{R}| = d(d-1)$ by Lemma 13.5.

For every point $[a:b:c] \in C$, let $\nu_\psi([a:b:c])$ be the order of the zero of the polynomial $f_d(a,y,b)$ in y at $y = b$.

Exercise 13.6. *For every point* $P = [a:c] \in \mathbb{CP}^2$, *let* L_P *be a line* $ax = cz$. *Prove that*

$$\nu_\psi(O) = \mathrm{mult}_O(L_P \cdot C)$$

for every point $O = [a:b:c] \in C$.

It follows from Exercise 10.16 (or, more generally, from Theorem 12.3) that

$$d = \sum_{Q \in \psi^{-1}(P)} \nu_\psi(Q)$$

for every $P \in \mathbb{P}^1$. It follows from the construction of the map ψ that

$$\nu_\psi\big([a:b:c]\big) \geqslant 2 \iff [a:b:c] \in \tilde{R}.$$

Moreover, since we assume that $O \notin \cup_{i=1}^r L_i$, we have $\nu_\psi([a:b:c]) \leqslant 2$ for every point $[a:b:c] \in C$ (see Definitions 10.15 and 10.17).

It follows from Theorem 11.2 that there is a triangulation $\{\bar{V}, \bar{E}, \bar{F}\}$ of $\mathbb{P}^1$ such that $\bar{V}$ contains all points in R (see Section 12).

Theorem 13.7 ([5, Proposition 4.22])**.** *There exists a triangulation $\{V, E, F\}$ of the complex curve C such that $V = \psi^{-1}(\bar{V})$, $\psi \circ e \in \bar{E}$ for every $e \in E$, and $\psi \circ f \in \bar{F}$ for every $f \in F$.*

Thus, we have $|F| = d|\bar{F}|$, $|E| = d|\bar{E}|$, and

$$|V| = d|\bar{V}| - \sum_{P \in \tilde{R}} (\nu_\phi(P) - 1) = d|\bar{V}| - d(d-1).$$

Now it follows from Exercise 11.3 that

$$2-2g = \chi(C) = |V|-|E|+|F| = d|\bar{V}|-d(d-1)-d|\bar{E}|+d|\bar{F}| = d\chi(\mathbb{CP}^1)-d(d-1)) = 2d-d(d-1) = d(3-d) \tag{13.8}$$

which implies that $g = (d-1)(d-2)/2$. This completes the proof of Theorem 13.2.

The crucial role in the proof of Theorem 13.2 is played by (13.8), which is a very special case of the so-called Riemann–Hurwitz formula (see [6, Theorem 4.16]). Despite its clear geometric nature, the most impressive (from my point of view) elementary application of the Riemann–Hurwitz formula is the following

Theorem 13.9 (Lüroth)**.** *Let F be a subfield in $\mathbb{C}(x)$ that contains $\mathbb{C}$. Then there exists $f(x) \in F$ such that $F = \mathbb{C}(f(x))$.*

Proof. See [10, Chapter 10]. □

14. Hyperelliptic curves

So far, we studied basic properties of complex irreducible projective curves in $\mathbb{CP}^2$. Recall that complex irreducible projective curve in $\mathbb{CP}^2$ of degree d is just a subset in $\mathbb{CP}^2$ that are given by zeroes of some irreducible homogeneous polynomial of degree d.

Now I am going to consider subsets in $\mathbb{CP}^1 \times \mathbb{CP}^1$ that are given by zeroes of irreducible bi-homogeneous polynomials, which we also can call *complex irreducible projective curves*. Everything we did for complex irreducible projective curves in $\mathbb{CP}^2$ (plane curves), can be easily done for irreducible projective curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$. However, I do not want to do this in full generality, since I am mostly interesting in

Theorem 14.1. *For every non-negative integer g, there exists complex irreducible projective curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ that is homeomorphic to a sphere with g handles attached.*

Consider $\mathbb{CP}^1 \times \mathbb{CP}^1$. Let $[x_1 : y_1]$ be the homogenous coordinates on the left $\mathbb{CP}^1$, and let $[x_2 : y_2]$ be the homogenous coordinates on the right $\mathbb{CP}^1$. Let C be the subset in $\mathbb{CP}^1 \times \mathbb{CP}^1$ given by

$${x_1}^2 f(x_2, y_2) + x_1 y_1 g(x_2, y_2) + {y_1}^2 g(x_2, y_2) = 0,$$

where f, g, and h are homogeneous polynomials of degree $m \geqslant 1$. Put

$$\Delta(x_2, y_2) = g^2(x_2, y_2) - 4f(x_2, y_2)g(x_2, y_2).$$

It follows from the Fundamental Theorem of Algebra that

$$\Delta(x_2, y_2) = (\lambda_1 x_2 - \mu_1 y_2)(\lambda_2 x_2 - \mu_2 y_2) \cdots (\lambda_{2m} x_2 - \mu_{2m} y_2)$$

for some points $[\lambda_1 : \mu_1], [\lambda_2 : \mu_2], \ldots, [\lambda_{2m} : \mu_{2m}]$ in $\mathbb{CP}^1$. Put $P_i = [\lambda_i : \mu_i]$ for every $i \in \{1, 2, \ldots, m\}$. Suppose that all points $P_1, P_2, \ldots, P_{2m}$ are distinct.

Exercise 14.2. *Consider the map* $\pi\colon \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^3$ *that are given by*

$$\Big([x_1 : y_1], [x_2 : y_2]\Big) \mapsto \big[x_1x_2 : x_1y_2 : y_1x_2 : y_1y_2\big] \in \mathbb{CP}^3.$$

Show that $\pi(\mathbb{CP}^1 \times \mathbb{CP}^1)$ *is a complex surface that is given by* $xw = yt$*, where* $[x : y : z : t]$ *are projective coordinates on* $\mathbb{CP}^3$*. Show that* $\pi(C)$ *is an irreducible curve in* $\mathbb{CP}^3$ *of degree* $m + 2$*.*

Similar to Theorem 13.1, one can show that C is homeomorphic to a sphere with g handles attached.

Exercise 14.3. *Show that* $g = 0$ *if* $m = 1$*.*

Theorem 14.4 ([6, Lemma 1.7])**.** *One has* $g = m - 1$*.*

Proof. Let $\psi\colon C \to \mathbb{CP}^1$ be a map given by

$$\Big([x_1 : y_1], [x_2 : y_2]\Big) \mapsto [x_2 : y_2] \in \mathbb{CP}^1.$$

When $[x_2 : y_2]$ is fixed, we can solve

$${x_1}^2 f(x_2, y_2) + x_1y_1g(x_2, y_2) + {y_1}^2 g(x_2, y_2) = 0$$

by using formula for the roots of quadratic equation. This formula involves taking $\sqrt{\Delta(x_2, y_2)}$. Then

- $\psi^{-1}(P)$ consists of 1 point if $P \in \{P_1, \ldots, P_{2m}\}$,
- $\psi^{-1}(P)$ consists of 2 points if $P \notin \{P_1, \ldots, P_{2m}\}$.

It follows from Theorem 11.2 that there is a triangulation $\{\bar{V}, \bar{E}, \bar{F}\}$ of $\mathbb{P}^1$ such that $\bar{V}$ contains all points $P_1, \ldots, P_{2m}$ (see Section 12). Then

$$\chi(\mathbb{CP}^1) = |\bar{V}| - |\bar{E}| + |\bar{F}| = 2$$

by Exercise 11.3.

Arguing as in the proof of [5, Proposition 4.22], we see that there exists a triangulation $\{V, E, F\}$ of the complex curve C such that $V = \psi^{-1}(\bar{V})$, $\psi \circ e \in \bar{E}$ for every $e \in E$, and $\psi \circ f \in \bar{F}$ for every $f \in F$. Then

$$|F| = 2|\bar{F}|,\ |E| = 2|\bar{E}|,\ |V| = 2|\bar{V}| - 2m.$$

Note that we defined triangulations only for smooth irreducible projective curves in $\mathbb{CP}^2$ (see Section 12). Of course, the same definition works for C (it works for any compact oriented two-dimensional surface).

Arguing as in the solution to Exercise 11.3, we see that

$$2 - 2g = \chi(C) = |V| - |E| + |F|,$$

which implies that $2 - 2g = 4 - 2m$. Hence, we have $g = m - 1$. □

Note that Theorem 14.4 implies Theorem 14.1.

15. Hesse pencil

Let us illustrate the proof of Theorem 13.2 with one very interesting example. Let C be a curve that is given by

$$f_d(x, y, z) = x^3 + y^3 + z^3 + \lambda xyz,$$

where λ is a generally chosen complex number. One can show that the polynomial $x^3 + y^3 + z^3 + \lambda xyz$ is irreducible and C is smooth.

Put $P = [1 : -1 : 0]$. Since $1^3 + (-1)^3 + 0^3 - \lambda 1 \times (-1) \times 0 = 0$, we see that $P \in C$.

Lemma 15.1. *The point* P *is an inflection point of the curve* C*.*

Proof. It follows from Exercise 10.9 that

$$3(x+y)-\lambda z=0$$

defined the line in $\mathbb{CP}^2$ that is tangent to C at P. It follows from Exercise 10.16 that for every line L in $\mathbb{CP}^2$, we have

$$3=\sum_{O\in C\cap L}\mathrm{mult}_O(L\cdot C),$$

and $\mathrm{mult}_O(L\cdot C)=3$ if and only if the point $O\in C\cap L$ is an inflection point of the curve C and L is the line that is tangent to C at this point O. Now let us find the intersection of the curve C with the line the line in $\mathbb{CP}^2$ that is tangent to C at P. These intersection consists of all points $[x:y:z]$ that satisfy

$$\begin{cases}3(x+y)-\lambda z=0,\\ x^3+y^3+z^3+\lambda xyz=0,\end{cases}$$

which implies (easily) that this intersection consists of the single point P. Indeed, we have $z=3(x+y)/\lambda$ and

$$x^3+y^3+27\frac{(x+y)^3}{\lambda^3}+3xy(x+y)=0,$$

which can be rewritten as

$$(x+y)\left(x^2-xy+y^2+27\frac{x^2+2xy+y^2}{\lambda^3}-3xy\right)=0,$$

which implies that $(x+y)^3(1+27/\lambda^3)=0$. But $\lambda^3\neq 27$, since λ is generally chosen complex number. Then $x+y=0$ and $z=3(x+y)/\lambda=0$, which implies that $[x:y:z]=[1:-1:0]=P$. Thus, the intersection of the curve C with the line the line in $\mathbb{CP}^2$ that is tangent to C at P consists solely of the point P. This implies that P is an inflection point of the curve C. □

Let us find all inflection points of the curve C. This is classically known (see [3]). They are solutions of the system of equations

$$H(x,y,z)=f_d(x,y,z)=0,$$

where $H(x,y,z)$ is the Hessian

$$\det\begin{pmatrix}\frac{\partial^2 f_d(x,y,z)}{\partial x^2} & \frac{\partial^2 f_d(x,y,z)}{\partial x\partial y} & \frac{\partial^2 f_d(x,y,z)}{\partial x\partial z}\\ \frac{\partial^2 f_d(x,y,z)}{\partial x\partial y} & \frac{\partial^2 f_d(x,y,z)}{\partial y^2} & \frac{\partial^2 f_d(x,y,z)}{\partial z\partial y}\\ \frac{\partial^2 f_d(x,y,z)}{\partial x\partial z} & \frac{\partial^2 f_d(x,y,z)}{\partial z\partial y} & \frac{\partial^2 f_d(x,y,z)}{\partial z^2}\end{pmatrix}.$$

Let us compute $H(x,y,z)$. We have

$$H(x,y,z)=\det\begin{pmatrix}6x & \lambda z & \lambda y\\ \lambda z & 6y & \lambda x\\ \lambda y & \lambda x & 6z\end{pmatrix}=(6^3+2\lambda^3)xyz-6\lambda^2(x^3+y^3+z^3).$$

Thus, the equalities $H(x,y,z)=f_d(x,y,z)=0$ imply that

$$(6^3+2\lambda^3)xyz-6\lambda^2(x^3+y^3+z^3)=(6^3+2\lambda^3)xyz+6\lambda^3xyz=8(27+\lambda^3)xyz=0,$$

which implies that $xyz=0$. Since $x^3+y^3+z^3+\lambda xyz=0$, this gives us 9 inflection points:

$$[1:-1:0],[1:-\epsilon_3:0],[1:-\epsilon_3^2:0],$$
$$[1:0:-1],[1:0:-\epsilon_3],[1:0:-\epsilon_3^2],$$
$$[0:1:-1],[0:1:-\epsilon_3],[0:1:-\epsilon_3^2].$$

Lemma 15.2. *There are exactly 3 lines in $\mathbb{CP}^2$ that pass through the point P and are tangent to the curve C at some other point.*

Proof. It follows from the proof of Lemma 10.19 that there is a projective transformation $\pi\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ such that $\pi(C)$ is given by

$$zy^2 = x(x-z)(x-\epsilon z)$$

for some $\epsilon \in \mathbb{C}$ such that $\gamma \neq 0$ and $\epsilon \neq 1$, and $\pi(P) = [0:1:0]$. To prove that there are exactly 3 lines in $\mathbb{CP}^2$ that pass through the point P and are tangent to the curve C at some other point, it is enough to prove the same assertion for $\pi(C)$ and $\pi(P)$. But the latter is very easy to do, since the equation of the curve $\pi(C)$ is very simple. Indeed, every line that passes through the point $\pi(P)$ is either the line tangent to $\pi(C)$ at the point $\pi(P)$ (this line is given by $z = 0$) or given by $x = \mu z$ for some $\mu \in \mathbb{C}$. Since the line the line tangent to $\pi(C)$ at the point $\pi(P)$ intersects the curve $\pi(C)$ only at the point $\pi(P)$ (we already proved this), we only need to consider lines given by $x = \mu z$. Denote the line given by $x = \mu z$ by L_μ. Then the points $L_\mu \cap \pi(C)$ are given by

$$\begin{cases} x = \mu z, \\ zy^2 = \mu z(\mu z - z)(\mu z - \epsilon z), \end{cases}$$

and L_μ is tangent to $\pi(C)$ at some point $Q \in \pi(C)$ that is different from $\pi(P)$ if and only if $L_\mu \cap C$ consists of P and Q (see Exercise 10.16). The latter implies that the equation

$$y^2 = \mu(\mu-1)(\mu-\epsilon)z^2$$

gives us just a single point $[y:z] \in \mathbb{CP}^1$. This only possible if the left hand side is zero, i.e. when $\mu \in \{0, 1, \epsilon\}$. Thus, there are there are exactly 3 lines in $\mathbb{CP}^2$ that pass through the point $\pi(P)$ and are tangent to the curve $\pi(C)$ at some other point. □

Let L_y be a complex projective line in $\mathbb{CP}^2$ that is given by $y = 0$. Let $\psi\colon C \setminus P \to L_y$ be a projection from O to L_y, i.e. for a point $O \in C$ such that $O \neq P$ we have

$$\psi(O) = L \cap L_y,$$

where L is the unique complex projective line in $\mathbb{CP}^2$ that passes through P and O. Put $Q = [\lambda : 0 : 3]$.

Lemma 15.3. *The image of ψ is $L_y \setminus Q$.*

Proof. The point Q is the intersection of the line in $\mathbb{CP}^2$ that is tangent to C at the point P and the line L_y. Every other line in $\mathbb{CP}^2$ that passes through P intersect L_y in some point that is different from Q, because there exists exactly on line in $\mathbb{CP}^2$ that passes through two given different points. On the other hand, every line in $\mathbb{CP}^2$ passing through P that is not tangent to C at the point P intersects C at some point different from P. This means that for every point $O \in L_y \setminus Q$, there exists at least one point $\tilde{O} \in C \setminus P$ such that $\psi(\tilde{O}) = O$, i.e. the image of ψ is $L_y \setminus Q$. □

For every line L in $\mathbb{CP}^2$ that passes through P and is not tangent to C at the point P, we have

$$3 = \sum_{O \in C \cap L} \mathrm{mult}_O(L \cdot C) = 1 + \sum_{O \in C \cap L, O \neq P} \mathrm{mult}_O(L \cdot C),$$

and $\mathrm{mult}_O(L \cdot C) > 1$ if and only if L is tangent to C at the point O (see Exercise 10.16). This implies that for every point $O \in L_y \setminus Q$, there are at most two points in $\psi^{-1}(O)$. Moreover, since we already proved that there are exactly 3 lines in $\mathbb{CP}^2$ that pass through P and are tangent to C at some other point, we see that there are exactly three points (the intersection points of L_y and these 3 lines), say O_1, O_2, and O_3 in $L_y \setminus Q$ such that $\psi^{-1}(O_i)$ consists of a single point.

Let $\phi\colon C \to L_y$ be a map such that is define as

$$\phi(O) = \begin{cases} \psi(O) \text{ if } O \neq P, \\ Q \text{ if } \ O = P. \end{cases}$$

Exercise 15.4. *Show that ϕ is continuous map (when C is equipped with induced topology from $\mathbb{CP}^2$).*

It follows from Theorem 11.2 that there is a triangulation $\{\bar{V}, \bar{E}, \bar{F}\}$ of L_y such that $\bar{V}$ contains points in Q, O_1, O_2, O_3. Since L_y is homeomorphic to a sphere, we know that the equality

$$|\bar{V}| - |\bar{E}| + |\bar{F}| = 2$$

holds (see Exercise 11.3). On the other hand, it follows from Theorem 13.7 that there exists a triangulation $\{V, E, F\}$ of the complex curve C such that $V = \phi^{-1}(\bar{V})$, $\phi \circ e \in \bar{E}$ for every $e \in E$, and $\phi \circ f \in \bar{F}$ for every $f \in F$. Thus, we have $|F| = 2|\bar{F}|$, $|E| = 2|\bar{E}|$, and $|V| = 2|\bar{V}| - 4$. It follows from Exercise 11.3 that

$$2 - 2g = |V| - |E| + |F| = 2|\bar{V}| - 4 - 2|\bar{E}| + 2|\bar{F}| = 2 \times 2 - 4 = 0,$$

where g is the genus of C. Thus, we see that $g = 1$, i.e. C is homomorphic to a torus.

References

[1] M. Atiyah, I. Macdonald, *Commutative algebra*
Addison–Wessley (1969)

[2] E. Bezout, *Theorie generale des equations algebriques*
Paris, Impr. de P.-D. Pierres (1779)

[3] M. Artebani, I. Dolgachev, *The Hesse pencil of plane cubic curves*
L'Enseignement Mathematique **55** (2009), 235-273

[4] W. Fulton, *Algebraic Curves*
Mathematics Lecture Note Series (1974) W.A. Benjamin, 112 pages

[5] F. Kirwan, *Complex algebraic curves*
Cambridge University Press (1992)

[6] R. Miranda, *Algebraic curves and Riemann surfaces*
American Mathematical Society (1995)

[7] I. Newton, *Mathematical Principles of Natural Philosophy, Volume I*
Second edition published by Sir Isaac Newton (1713)

[8] M. Reid, *Undergraduate algebraic geometry*
Cambridge University Press (1988)

[9] M. Reid, *Undergraduate commutative algebra*
Cambridge University Press (1995)

[10] B. L. van der Waerden, *Algebra*
Seventh edition, Springer (1955)