A differential equation corresponding to Bridgeland stability

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Edge Days 2021, Explicit K-stability and Moduli Problems, 29 Nov - 3 Dec

2nd of December, 2021

Introduction

Mirror symmetry

Mirror symmetry predicts that to each Calabi–Yau threefold X there should exist a mirror threefold $X^\vee$. This operation should exchange the symplectic and holomorphic structures of X and $X^\vee$.

- The symplectic data is encoded in the category of Lagrangian submanifolds, the Fukaya category $\mathrm{Fuk}(X)$.
- The holomorphic data is encoded in the derived category of coherent sheaves, $\mathcal{D}^b \mathrm{Coh}(X)$.

The (homological) mirror symmetry conjecture asserts

$$\mathrm{Fuk}(X) \cong \mathcal{D}^b \mathrm{Coh}(X^\vee), \qquad \mathcal{D}^b \mathrm{Coh}(X) \cong \mathrm{Fuk}(X^\vee).$$

or maybe $\mathcal{D}^b \mathrm{Fuk}(X)$.

Stable objects

For physical reasons, the category of coherent sheaves $\mathrm{Coh}(X)$ is too small, but the full derived category of coherent sheaves $\mathcal{D}^b\,\mathrm{Coh}(X)$ is too large! One should restrict to a certain class of "stable" objects.

This requires a notion of stability for complexes of sheaves which makes sense on the derived category.

Such a notion was invented by Douglas (2000) and Bridgeland (2002), now known as Bridgeland stability.

Stability conditions

Complexes of coherent sheaves

Let X be a smooth projective variety $/\mathbb{C}$. A bounded complex of coherent sheaves is given by

$$\mathcal{E} : \cdots \to E_{i-1} \xrightarrow{\phi_{i-1}} E_i \xrightarrow{\phi_i} E_{i+1} \to \cdots$$

of coherent sheaves with morphisms satisfying $\phi_i \circ \phi_{i-1} = 0$, such that $E_i = 0$ for $i \ll 0$ and $i \gg 0$.

The ith cohomology sheaf of $\mathcal{E}$ is given by

$$H^i(\mathcal{E}) = \frac{\ker \phi_i : E_i \to E_{i+1}}{\operatorname{im} \phi_{i-1} : E_{i-1} \to E_i}$$

and the cohomology complex $H^\bullet(\mathcal{E})$ is

$$H^\bullet(\mathcal{E}) : \cdots \to H^{i-1}(\mathcal{E}) \xrightarrow{0} H^i(\mathcal{E}) \xrightarrow{0} H^{i+1}(\mathcal{E}) \to \cdots .$$

A morphism $f : \mathcal{E} \to \mathcal{E}'$ of complexes is called a *quasi-isomorphism* if the induced morphism $H(f) : H^\bullet(\mathcal{E}) \to H^\bullet(\mathcal{E}')$ is an isomorphism.

The derived category

Call two complexes $\mathcal{E}, \mathcal{E}'$ *quasi-isomorphic* if there exists a quasi-isomorphism from $\mathcal{E}$ to $\mathcal{E}'$. The bounded derived category $\mathcal{D}^b \operatorname{Coh}(X)$ has objects given by bounded complexes of coherent sheaves.

For every quasi-isomorphism $f : \mathcal{E} \to \mathcal{E}'$ one can formally add an inverse f^{-1}, making quasi-isomorphic complexes isomorphic in the derived category.

The derived category $\mathcal{D}^b \operatorname{Coh}(X)$ is the category generated by this formal inversion of quasi-isomorphisms.

Stability conditions

To define stability on the derived category, one chooses a *central charge*

$$Z : K(X) \to \mathbb{H} \subset \mathbb{C},$$

an additive group homomorphism.Defining

$$Z(\mathcal{E}) = \sum_i (-1)^i Z(E_i)$$

one obtains a group homomorphism

$$Z : K(\mathcal{D}^b \operatorname{Coh}(X)) \to \mathbb{C}$$

and if $\mathcal{E}, \mathcal{E}'$ are quasi-isomorphic then $Z(\mathcal{E}) = Z(\mathcal{E}')$.

The Z-slope μ_Z and Z-phase φ_Z of $\mathcal{E}$ are

$$\mu_Z(\mathcal{E}) := \frac{\operatorname{Im} Z(\mathcal{E})}{\operatorname{Re} Z(\mathcal{E})}, \quad \varphi_Z(E) := \arg Z(\mathcal{E}).$$

Stable objects

To define stable objects, one must pick a *heart* $\mathcal{A}$ of $\mathcal{D}^b \operatorname{Coh}(X)$, a full Abelian subcategory satisfying various assumptions. For example the most obvious choice is $\mathcal{A} = \operatorname{Coh}(X)$.

An object $\mathcal{E} \in \mathcal{A}$ is called $(Z, \mathcal{A})$-(semi)stable if

$$\mu_Z(\mathcal{F}) \underset{(=)}{\leq} \mu_Z(\mathcal{E})$$

for every subobject $\mathcal{F} \to \mathcal{E}$ in $\mathcal{A}$ (or equivalently using φ_Z).

A stability condition $\sigma = (Z, \mathcal{A})$ must satisfy various properties which make $\operatorname{Coh}(X)$ a bad choice of heart.

Examples of stability conditions

Examples of central charges can be easily constructed. For example if

$$Z(E) = -\deg E + i \operatorname{rk} E$$

then μ_Z-stability is slope stability.

More generally we can take

$$Z(E) = \int_X \sum_{d=0}^{n} \rho_d [\omega]^d . \operatorname{Ch}(E).U$$

where $\rho_d \in \mathbb{C}^*$ satisfy $\operatorname{Im}(\rho_{d+1}/\rho_d) > 0$ and $U = 1 + N$ is a cohomology class with $N \in H^{>0}(X, \mathbb{R})$. These are called *polynomial central charges*.

In general finding a heart $\mathcal{A}$ compatible with a central charge Z which satisfies the required properties of a stability condition (existence of Harder–Narasimhan filtrations and the support property) is very difficult. The obvious $\operatorname{Coh}(X)$ is not a valid heart with respect to either central charge above.

Examples of stability conditions

Two more important examples of stability conditions are:

- The *deformed Hermitian Yang–Mills* (dHYM) stability condition with B-field
$$Z_{\mathrm{dHYM}}(E) = -\int_X e^{-i\omega - B}\,\mathrm{Ch}(E).$$

- The dHYM/Todd stability condition
$$Z_{\mathrm{Todd}}(E) = -\int_X e^{-i\omega - B}\,\mathrm{Ch}(E)\sqrt{\mathrm{Td}(X)}$$

  important in string theory.

Z-critical metrics

A differential equation?

Recall the famous Donaldson–Uhlenbeck–Yau theorem:

Theorem

A holomorphic vector bundle $E \to (X, \omega)$ over a compact Kähler n-manifold is slope polystable if and only if it admits a Hermitian metric h solving the Hermite–Einstein equation

$$F(h) \wedge \omega^{n-1} = \lambda(E)\mathbf{1}_E \otimes \omega^n$$

for some constant $\lambda(E) \in \mathbb{C}$.

By direct analogy, one may ask:

Question

Does there exist a differential equation depending on a choice of stability condition $(Z, \mathcal{A})$ such that an object $\mathcal{E} \in \mathcal{A}$ is μ_Z-stable if and only if $\mathcal{E}$ admits a solution to the differential equation?

Z-critical metrics

Let us consider the simplified setting $E \to (X, \omega)$ is a single holomorphic vector bundle over a compact Kähler manifold. We will also consider a limiting situation $\omega \mapsto k\omega$ with $k \to \infty$, called the *large volume limit*.

Let Z_k be a polynomial central charge with respect to $k\omega$:

$$Z_k(E) = \int_X \sum_{d=0}^{n} \rho_d k^d [\omega]^d . \mathrm{Ch}(E).U.$$

Using Chern–Weil theory, if h is a Hermitian metric on E, there is a differential-geometric representative

$$\tilde{Z}_k(h) := \left[\sum_{d=0}^{n} \rho_d k^d \omega^d \wedge \exp\left(\frac{i}{2\pi} F(h) \right) \wedge \tilde{U} \right]^{(n,n)}$$

where $F(h)$ is the Chern curvature of h and $\tilde{U}$ is any form representing U. This satisfies

$$Z_k(E) = \int_X \mathrm{tr}\, \tilde{Z}_k(h)$$

for any h.

Z-critical metrics

Definition

A hermitian metric h on $E \to (X, \omega)$ is called Z_k-*critical* if

$$\operatorname{Im}\left(e^{-i\varphi_k(E)}\tilde{Z}_k(h)\right) = 0$$

where Im denotes the skew-Hermitian part with respect to h ($F(h)$ is skew-Hermitian).

- If E is a line bundle, then $F(h)$ is an imaginary two-form and Im takes the imaginary part of the (n, n)-form inside.
- If E is a line bundle, the Z_k-critical equation asks that the phase of the function $\tilde{Z}_k(h)/\omega^n \in C^\infty(X, \mathbb{C})$ is constant equal to its required topological average $\varphi_k(E)$.
- If E is a line bundle and $Z = Z_{\mathrm{dHYM}}$, this equation can be derived using mirror symmetry from the special Lagrangian equation.

Large volume limit

In the limit as $k \to \infty$, the leading order term of the Z_k-critical equation is the Hermite–Einstein equation

$$F(h) \wedge \omega^{n-1} = \lambda(E)\mathbf{1}_E \otimes \omega^n.$$

Analogously, for a polynomial central charge Z_k, μ_{Z_k}-stability is given by slope stability to leading order in k as $k \to \infty$.One can attempt to perturb from $k = \infty$ to solve the Z_k-critical equation for $k \gg 0$ by assuming *asymptotic Z-stability*:$E \to (X, \omega)$ is asymptotically Z-stable if for every subsheaf $F \subset E$ we have

$$\mu_{Z_k}(F) < \mu_{Z_k}(E)$$

for all $k \gg 0$.

A correspondence

Theorem (Dervan–M.–Sektnan)

A holomorphic vector bundle $E \to (X, \omega)$ on a compact Kähler manifold admits a Z_k-critical Hermitian metric for all $k \gg 0$ if and only if it is asymptotically Z-stable.

- We make the assumption that E is slope semistable and its polystable degeneration $\mathrm{Gr}(E)$ is locally-free.
- In the large volume limit the Z_k-critical equation is elliptic and one can try to apply perturbation theory to obtain a solution.
- Good enough approximate solutions for the inverse function theorem to apply exist precisely when the algebraic stability condition is satisfied.
- The $\Longleftarrow$ direction uses the principle that *curvature decreases in subbundles*.

An example

Let $E \to \mathbb{P}^2$ be a simple rank three bundle on $\mathbb{P}^2$ appearing as an extension

$$0 \to F \to E \to \mathcal{O}_{\mathbb{P}^2} \to 0$$

where F is slope stable of rank 2 and $c_1(F) = 0$. Then $c_1(E) = 0$ and $c_2(E) = c_2(F)$. The stability of F implies $c_1(E) \geq 0$ by the Bogomolov inequality. Fix a B-field $B \in H^2(\mathbb{P}^2, \mathbb{R})$ and consider $B.[\omega] \in \mathbb{R}$.

- If $c_2(E) > 0$ then E is asymptotically $Z_{\mathrm{dHYM},B}$-stable whenever $B.[\omega] < 0$ and admits a dHYM metric in this case. E is unstable when $B.[\omega] \geq 0$.
- E^* is asymptotically Z-stable when $B.[\omega] > 0$ and admits a dHYM metric in this case.
- If $c_2(E) > 0$ then E is asymptotically $Z_{\mathrm{Todd},B}$-stable whenever $B.[\omega] < \frac{3}{4}$ and admits a Z-critical metric.
- For E^* stability holds when $B.[\omega] > \frac{3}{4}$.
- When $c_2(E) = 0$ both E and E^* are Z-semistable.

Complexes

Z-critical metrics on complexes

Can the notion of a Z-critical metric be defined on elements of $\mathcal{D}^b \operatorname{Coh}(X)$? Such an equation should satisfy the following:

- If $\mathcal{E}$ admits a solution to the Z-critical equation and $\mathcal{E}'$ is quasi-isomorphic to $\mathcal{E}$, then $\mathcal{E}'$ should also admit a solution.
- If $\mathcal{E}$ admits a solution to the Z-critical equation and $\mathcal{A}$ is a heart with $\mathcal{E} \in \mathcal{A}$ and $(Z, \mathcal{A})$ forms a Bridgeland stability condition, then $\mathcal{E}$ should be μ_Z-stable.

Hermitian metric on a complex

Let $\mathcal{E}$ be a complex

$$\mathcal{E} : \cdots \to E_{i-1} \xrightarrow{\phi_{i-1}} E_i \xrightarrow{\phi_i} E_{i+1} \to \cdots .$$

Say $\mathcal{E}$ is *formally locally free* if E_i is locally free for every i, and $\ker \phi_i, \operatorname{im} \phi_i$ are locally free for every i. Then the cohomology sheaves $H^i(\mathcal{E})$ are also holomorphic vector bundles.

Definition

An *admissible Hermitian metric h* on $\mathcal{E}$ is given by Hermitian metrics h_i on E_i such that

$$h_i|_{\ker \phi_i^{\perp(h_i)}} = \phi_i^* \left(h_{i+1}|_{\operatorname{im} \phi_i} \right).$$

Denote the space of admissible Hermitian metrics $\mathcal{H} = \mathcal{H}(\mathcal{E})$.

Gauge transformations

A gauge transformation g of $\mathcal{E}$ is given by automorphisms $g_i : E_i \to E_i$ such that

$$g_{i+1} \circ \phi_i = \phi_i \circ g_i$$

for every i. Denote the space of gauge transformations $\mathcal{G}^{\mathbb{C}}$. If h is an admissible metric, let $\mathcal{G}(h)$ denote the gauge transformations in $\mathcal{G}^{\mathbb{C}}$ which are h_i-unitary for each i.

Proposition

If $h \in \mathcal{H}$ is an admissible Hermitian metric on $\mathcal{E}$ and $g \in \mathcal{G}^{\mathbb{C}}$ is a gauge transformation, then $g^* h$ is an admissible Hermitian metric. If $h \in \mathcal{H}$ is fixed, then $\mathcal{G}^{\mathbb{C}}$ acts transitively on $\mathcal{H}$ and $\mathcal{H} = \mathcal{G}^{\mathbb{C}}/\mathcal{G}(h)$.

An example

Let $X = \Sigma$ be a compact Riemann surface and consider a complex

$$\mathcal{E} : E_0 \xrightarrow{\phi} E_1$$

where ϕ is surjective and $\ker \phi$ is locally free. Then the inclusion $\ker \phi \hookrightarrow E_0$ gives a quasi-isomorphism

$$H^\bullet(\mathcal{E}) = (\ker \phi \to 0) \xrightarrow{\sim} \mathcal{E}.$$

Notice that $E_0 / \ker \phi \cong \operatorname{im} \phi = E_1$. If h is an admissible Hermitian metric on $\mathcal{E}$, then using the isomorphism $E_0 / \ker \phi \cong \ker \phi^\perp$, one may identify $\ker \phi^\perp$ with E_1 using ϕ. With respect to the (smooth!) splitting $E_0 \cong \ker \phi \oplus \ker \phi^\perp \cong \ker \phi \oplus E_1$ we have

$$h_0 = \begin{pmatrix} h_{\ker \phi} & 0 \\ 0 & h_1 \end{pmatrix}.$$

Hermite–Einstein equation for a complex

Given the holomorphic subbundle $\ker \phi \subset E_0$, we obtain a *second fundamental form* β measuring the extent to which the short exact sequence

$$0 \to \ker \phi \to E_0 \to E_0/\ker \phi \to 0$$

splits. One has $\beta \in \Omega^{0,1}(\Sigma, \mathrm{Hom}(E_0/\ker \phi, \ker \phi))$.

Say h is Hermite–Einstein on $\mathcal{E}$ if

$$(F(h_0) - (d\beta - d\beta^* + \beta \wedge \beta^* + \beta^* \wedge \beta) - F(h_1)) = \lambda(\mathcal{E})(\mathbf{1}_{E_0} - \mathbf{1}_{E_1}) \otimes \omega.$$

Here we use the isomorphism $\ker \phi^\perp \cong E_1$ to make sense of the subtraction $F(h_0) - F(h_1)$ and $\mathbf{1}_{E_0} - \mathbf{1}_{E_1}$.

Proposition

The complex $\mathcal{E}$ admits a Hermite–Einstein metric if and only if the vector bundle $\ker \phi$ admits a Hermite–Einstein metric in the sense of vector bundles.

Outlook

- Can this definition be extended to more general formally locally free complexes?
- Does this definition make sense for complexes which are not quasi-isomorphic to their cohomology?
- Can one define a Hermitian metric on an arbitrary coherent sheaf?
- What is the Z-critical equation for an arbitrary complex of coherent sheaves?

Thank you for listening!