From Kähler-Einstein metrics with prescribed singularities to K-stability

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EXPLICIT K-STABILITY AND MODULI PROBLEMS

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For a fixed h_0 positive metric on $-K_X$ with curvature form ω, this correspondence is given by $h = h_0 e^{-\varphi} \to \omega + dd^c\varphi$ ($dd^c = \frac{i}{2\pi}\partial\bar{\partial}$).

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The set of positive singular metrics are described by $PSH(X,\omega) := \{u \in L^1(X) \,:\, u \text{ is ups and } \omega + dd^c u \geq 0 \text{ in a weak sense}\}$.

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$\Rightarrow \mathcal{M}^+_{klt}$ represents the set of all admissible prescribed singularities, and we are looking for $[\psi]-$KE metrics.

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- define the set $\mathcal{M}^+_{D,klt} \subsetneq \mathcal{M}^+_{klt}$ as the closure under decreasing limits of the set of prescribed singularities with algebraic singularities types. Any element $\psi \in \mathcal{M}^+_{D,klt}$ is recovered by a decreasing sequence $\{\mathcal{I}_k^{c_k}\}$, or equivalently by a tower of weak log Fano pairs $(Y_0, \Delta_0) := (X, 0) \xleftarrow{p_0} (Y_1, \Delta_1) \xleftarrow{p_1} \dots \xleftarrow{p_{k-1}} (Y_k, \Delta_k) \xleftarrow{p_k} (Y_{k+1}, \Delta_{k+1}) \xleftarrow{p_{k+1}} \dots$ such that $p_k^*(\Delta_k + K_{Y_k}) \leq \Delta_{k+1} + K_{Y_{k+1}}$.

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$\mathcal{M}^+_{D,klt}$ is the biggest "algebraic" set of admissible prescribed singularities.

Analytic Picture for a fixed singularity type

It is possible to define $\psi-$*relative Ding and Mabuchi functionals* D_ψ, M_ψ on $\mathcal{E}^1(X, \omega, \psi)$ for a fixed $\psi \in \mathcal{M}^+_{D,klt}$.

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Theorem ([Tru20b])

Let $\psi \in \mathcal{M}^+_{D,klt}$. Then the following statements are equivalent:

- there exists a $[\psi]$-KE metric $\omega + dd^c u$
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Moreover the uniqueness of $[\psi]$-KE metrics holds modulo $\mathrm{Aut}(X,[\psi])^\circ := \mathrm{Aut}(X)^\circ \cap \mathrm{Aut}(X,[\psi])$ where $\mathrm{Aut}(X,[\psi]) := \{F \in \mathrm{Aut}(X) \,:\, F^*\psi - \psi \text{ is globally bounded}\}$

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$\mathcal{E}^1(X,\omega,\psi)$ is naturally endowed of a metric *strong topology* given by a distance d ([Tru19], [Tru20c]), which generalises to the prescribed singularity setting the Darvas' d_1-distance ([Dar15]).

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Remark: By [CDS15] if $0 \in \mathcal{M}_{KE}$ then $t\psi_D \in \mathcal{M}_{KE}$ for any $D \in |-rK_X|$ smooth and for any $t \in [0,1)$. Moreover, by [Zho21] (Theorem 3.3), considering any divisor $D \in |-rK_X|$ and keeping assuming $0 \in \mathcal{M}_{KE}$, the set of all $t \in [0,1)$ such that $t\psi_D \in \mathcal{M}_{KE}$ is connected.

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- The solution to the Yau-Tian-Donaldson conjecture in [CDS15] can be thought as a variant of the classical continuity method in which *KE metrics with cone singularities* along a suitable smooth divisor are deformed into a genuine KE metric;
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Let $\{\psi_t\}_{t\in[0,1]} \subset \mathcal{M}^+_{D,klt}$ be an increasing continuous segment such that

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New tools: The set $\bigsqcup_{t\in[0,1]} \mathcal{E}^1(X, \omega, \psi)$ has a natural metric structure given by a distance d, and $P_{t,s} : \big(\mathcal{E}^1(X, \omega, \psi_t), d\big) \to \big(\mathcal{E}^1(X, \omega, \psi_s), d\big)$ for $t \geq s$ are Lipschitz and have other nice properties ([Tru19], [Tru20c]).

α-invariant function

Let $\alpha_\omega : \mathcal{M}^+_{klt} \to (0, +\infty)$ be the α-invariant function given as

$$\alpha_\omega(\psi) := \sup\Big\{\alpha > 0 \, : \sup_{u \in PSH(X,\omega), u \leq \psi, \sup u = 0} \int_X e^{-\alpha u} \omega^n < +\infty\Big\}.$$

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- $\alpha_\omega(\cdot)$ increases when ψ decreases (i.e. when the singularities increases), and there are less functions/divisors to test.

Sufficient numerical conditions for K-polystability

Recall that $\mathcal{M}_{KE} := \{\psi \in \mathcal{M}^+_{klt} \,:\, \text{there exists a } [\psi]\text{-KE metric}\}$, that $\psi \in \mathcal{M}_{KE} \cap \mathcal{M}^+_{klt,D}$ is equivalent to the log K-stability of a weak Fano pair, and that $0 \in \mathcal{M}_{KE}$ is equivalent to K-polystability.

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More precisely,

$$\alpha_\omega(\psi) > C(\psi) = \min\left\{ \max\left\{1, \frac{n^2 A_\psi + 1}{n+1}\right\}, \max\left\{\frac{A_\psi \mathrm{lct}(X,\psi) + 1}{\mathrm{lct}(X,\psi) + A_\psi}, \frac{n(nA_\psi + 1)}{n+1}\right\}\right\}.$$

About the constant $C(\text{lct}(X,\psi), A_\psi, n)$

Assume $\psi \in \mathcal{M}^+_{klt,D}$ with algebraic singularities such that $lct(X,\psi) \gg 1$ big enough (i.e. mild singularities). Then

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Recall that $A_\psi = \left(1 - \frac{V_\psi}{V_0}\right)^{1/2}$ for

- $V_0 = \mathrm{Vol}_X(-K_X) = (-K_X)^n$;
- $V_\psi = \mathrm{Vol}_Y\big(\pi^*(-K_X) - cD\big) = \big(\pi^*(-K_X) - cD\big)^n$ where
  - the singularities of ψ are encoded in $\mathcal{I}^c$;
  - $\pi : Y \to X$ is a log-resolution of the ideal sheaves $\mathcal{I}$;
  - $\pi^{-1}\mathcal{I} = \mathcal{O}_Y(-D)$.

Estimating $\alpha_\omega(\psi)$ for isolated singularities

- Consider N different points $p_1, \dots, p_N$ in X, and let $\psi_{N,\delta} \in \mathcal{M}^+_{klt,D}$ with isolated logarithmic singularities at $p_1, \dots, p_N$,

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Thank you!

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