Non-solidity of high index Fano 3-folds

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Overview

1. MMP: What is it all about?
2. Birational Non-solidity
3. Conjecture

MMP: The goal

Guiding Problem

Classify Algebraic Varieties up to Birational equivalence.

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MMP: Building Blocks

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Conjecturally, the variety W' falls into one of the following types:

- **Fano** if $-K_X$ is ample;
- **Calabi-Yau** if K_X is numerically trivial;
- **Canonically polarised** if K_X is ample.

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However W' is not necessarily smooth.

MMP: Singularities

Definition

A prime divisor D on X is $\mathbb{Q}$-**Cartier** if there is a non-zero multiple m such that mD is Cartier. A normal variety X is $\mathbb{Q}$-**factorial** if every divisor on X is $\mathbb{Q}$-Cartier.

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Example

The cone $(xy - uv = 0) \subset \mathbb{C}^4$ is not $\mathbb{Q}$-factorial. On the other hand, $(xy + zw + z^3 + w^3 = 0) \subset \mathbb{C}^4$ is $\mathbb{Q}$-factorial.

MMP: Singularities

Definition

A normal $\mathbb{Q}$-factorial variety X has **terminal singularities** if for any resolution $\varphi\colon Y \to X$ of X we have,

$$K_Y = \varphi^* K_X + \sum a_i E_i, \quad a_i > 0,$$

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Let X be the cone over the Veronese Surface in $\mathbb{P}^6$. Then $X \simeq \mathbb{P}(1,1,1,2)$. The singularity at the vertex is a non-smooth, terminal singularity of the 3-fold X. It has discrepancy $\frac{1}{2}$.

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Theorem (BCHM)

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A **Mori Fibre Space** is a contraction $f\colon X \to S$ of fibre type with connected fibres between normal varieties. That is,

1. X has at most $\mathbb{Q}$-factorial and terminal singularities.
2. $-K_X$ is $f-$ample.
3. $\rho(X/S) = 1$ and $\dim S < \dim X$.

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If $\dim X = 3$, there are three cases:

1. If $\dim S = 0$, X is a $\mathbb{Q}$-Fano 3-fold.
2. If $\dim S = 1$, X is a del Pezzo fibration.
3. If $\dim S = 2$, X is a conic bundle.

MMP: Objects

Suppose X is a $\mathbb{Q}$-Fano 3-fold. Since $\rho_X = 1$, there is $q \in \mathbb{Z}_+$ such that $-K_X \sim qA$, where $A \in Cl(X)$. The maximum of such numbers is called the **Fano index** of X and is denoted by ι_X.

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Example (See grdb.co.uk)

- A smooth cubic or quartic in $\mathbb{P}^4$
- The smooth complete intersection of two quadrics or of a quadric and a cubic in $\mathbb{P}^5$
- A quintic inside $\mathbb{P}(1, 1, 1, 1, 2)$ with a $\frac{1}{2}(1, 1, 1)$ cyclic quotient singularity and smooth otherwise.
- ...

Uniqueness of Output

Definition (Corti)

A $\mathbb{Q}$-Fano variety X is **birationally rigid** if the existence of a birational map $\sigma\colon X \dashrightarrow Y/S$ to a Mori fibre space Y/S implies that $X \simeq Y$ (and S is therefore a point). It is **birationally super-rigid** if in addition $\mathrm{Aut}(X) = \mathrm{Bir}(X)$.

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Let $X_{n+1} \subset \mathbb{P}^{n+1}$ be an n-dimensional smooth hypersurface of degree $n+1$ where $n \geq 3$. Then X_{n+1} is birationally super-rigid.

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Proof.

See Kollár's beautiful survey article! □

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Theorem (Corti-Mella, '01)

Let $Z_4 \subset \mathbb{P}^4$ *be a quartic threefold with a singularity*

$$0 \in (xy + z^3 + t^3 = 0) \subset \mathbb{C}^4$$

and general otherwise. Then Z_4 *has exactly two Mori fibre space structures. These are* Z_4 *itself and a complete intersection of a cubic and a quartic in* $\mathbb{P}(1,1,1,1,2,2)$*.*

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Notice that birational rigidity of the Fano X implies that X is strongly irrational.

Question (Abban, Okada):

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A $\mathbb{Q}$-Fano variety X is **birationally solid** if there is no birational map from X to a strict Mori fibre space, that is, to a Mori fibre space $f: Y \to S$ where $\dim S > 0$.

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Question: Do birationally solid Fano varieties exist in higher codimensions?

Partial Answer (Campo, DG, 2021):

Theorem (Campo - DG, 2021)

Let X be a $\mathbb{Q}$-Fano 3-fold with Fano index $\iota_X \geq 2$ embedded in a weighted projective space $X \subset \mathbb{P}(a_0, a_1, a_2, \ldots, a_N)$ such that $a_i \leq a_{i+1}$. Suppose that $l := \mathrm{lcm}(a_0, a_1) < \iota_X$. Then, X is birational to a Mori fibre space $Y \to S$ where $\dim S > 0$.

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Proof.

Consider the projection $\mathbb{P}(a_0, \ldots, a_N) \dashrightarrow \mathbb{P}(a_0, a_1)$ and call π its restriction to X. The generic fibre is a surface in X of degree $l := \operatorname{lcm}(a_0, a_1)$.

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$$-K_S = (-K_X - S)|_S + \mathrm{Diff} \sim (\iota_X - l)H|_S + \mathrm{Diff}\,.$$

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Hence $-K_S$ is big. Take the minimal resolution of S, $\varphi\colon \tilde{S} \to S$. Then,

$$-K_{\tilde{S}} = \varphi^*(-K_S) - \sum a_i E_i, \qquad a_i \leq 0.$$

Hence $-K_{\tilde{S}}$ is also big and therefore $H^0(\tilde{S}, mK_{\tilde{S}}) = 0$, i.e., $\kappa(\tilde{S}) = -\infty$ and $\tilde{S}$ is uniruled. □

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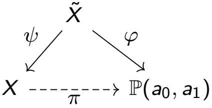
We resolve the indeterminacy of π to get a commutative diagram



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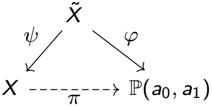

Let S' be the proper transform of S in $\tilde{X}$. Then S' is uniruled as well. Apply relative MMP to the fibration $\varphi\colon \tilde{X} \to \mathbb{P}(a_0, a_1)$. The uniruledness of the fibres is preserved under the MMP.

Explicit examples

Question: Suppose there is a birational map $\sigma\colon X \dashrightarrow Y/S$ from a Fano variety to a Mori fibre space. Can we retrieve this map explicitly?

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Theorem (Sarkisov, Corti - 1995, Hacon-McKernan - 2013)

Two Mori fibre spaces are birational if and only if they are related by a finite sequence of elementary operations called Sarkisov links. A birational map from a $\mathbb{Q}$-Fano threefold always starts with a blowup of a centre.

Explicit examples

Theorem (Okada, 2014; Abban-Cheltsov-Park, 2020; DG, 2021)

Suppose X is a quasismooth $\mathbb{Q}$-Fano 3-fold weighted complete intersection and $\mathbf{p} \in X$ a singular point. Then either

1. *There is a blowup of $\mathbf{p}$ which initiates a Sarkisov link and a complete breakdown of the steps is written or*
2. *There is no blowup of $\mathbf{p}$ which initiates a Sarkisov link.*

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Example

Let $X_{6,8} \subset \mathbb{P}(1,2,2,3,3,5)$ with homogeneous variables x, y, z, t, v, w be the Fano 3-fold given by the complete intersection of the hypersurfaces

$$wx + vt + f_6 = 0$$
$$w(v+t) + v^2 z + vg_5 + g_8 = 0.$$

Then blowing up the point $\mathbf{p} = (0:0:0:0:0:1)$ initiates a Sarkisov link to a singular quartic threefold $Z_4 \subset \mathbb{P}^4$.

Codimension 4

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Let $S \subset \operatorname{Proj} \mathbb{C}[x, y, z, w] \simeq \mathbb{P}^3$ be a cubic surface containing the line $L\colon (x = y = 0)$. Then, S is defined by $xB - yA = 0$, where A, B are general quadratic forms. Define

$$S'\colon (xs = A, ys = B) \subset \mathbb{P}^4 = \operatorname{Proj} \mathbb{C}[x, y, z, w, s].$$

We have a projection $\pi\colon S' \dashrightarrow S$ from $\mathbf{p_s} := (0 : 0 : 0 : 0 : 1) \in S'$ whose inverse is the *unprojection*

$$\pi^{-1}\colon S \dashrightarrow S'.$$

This map contracts L to $\mathbf{p_s} \in S'$.

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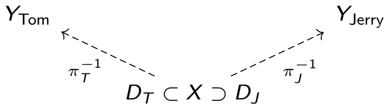
Any quasismooth Fano 3-fold $X \subset w\mathbb{P}^6$ (except the smooth complete intersection of three quadrics) is given by Pfaffian equations on a 5×5 skew symmetric matrix. The idea to get families in codimension 4 is to force a divisor $D \subset X$ and *unproject*:



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Using the graded ring approach of Reid, there are 34 candidates for (deformation families of) Fano 3-folds with $-K_X = 2H$ embedded as codimension 4 in weighted projective space. Livia has constructed these families explicitly

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Example

GRDB ID: #39961: $X \subset \mathbb{P}^7(1,2,2,3,4,5,5,7)$

$$\begin{cases} x_2\xi y_1 - y_1^2 - y_4x_1 + x_2y_2 = 0 \\ -x_2^7t^3 - x_2^3\xi^2 t - x_2\xi^3 + x_2^2\xi y_1 t + x_2\xi x_1 - x_2^2 y_2 t - y_1x_1 + y_4 s = 0 \\ \xi^3 y_4 - y_4^4 t^3 + x_2\xi y_2 - y_1y_2 + x_2y_3 = 0 \\ x_2^2\xi^3 t + x_2y_4^2y_1t^3 + x_2^2\xi x_1 t + x_2^2 y_4x_1t^2 - x_1^2 - y_1 s = 0 \\ x_2^6y_1t^3 - x_2y_4^4t^4 + \xi^3 y_1 - y_4^3y_1t^3 - x_2y_4^2x_1t^2 + x_2^2y_3t + x_1y_2 = 0 \\ x_2^5y_4y_1t^3 - y_4^5t^4 - y_4^3x_1t^2 + x_2y_4y_3t + y_2^2 - y_1y_3 = 0 \\ x_2^7y_1t^4 - x_2^6x_1t^3 + x_2^2\xi^4 t - x_2^2y_4^4t^5 - x_2y_4^3y_1t^4 + x_2^2\xi y_4x_1t^2 \\ -x_2^2y_4^2x_1t^3 - \xi^3x_1 + y_4^3x_1t^3 - x_2y_4y_1x_1t^2 - x_2y_4^2y_2t^3 + x_2^3y_3t^2 + y_2 = 0 \\ x_2^5y_4x_1t^3 - x_2^6y_2t^3 - x_2\xi y_4^4t^4 + y_4^4y_1t^4 - x_2\xi y_4^2x_1t^2 \\ +y_4^2y_1x_1t^2 - \xi^3y_2 + y_4^3y_2t^3 + x_2^2\xi y_3 t - x_2y_1y_3t - x_1y_3 = 0 \\ -x_2^6\xi^3t^3 + x_2^6y_4^3t^6 - x_2^6\xi x_1t^3 + x_2^2\xi^2y_4^3t^4 + x_2^5y_1x_1t^3 - \xi^6 + \xi^3y_4^3t^3 - x_2\xi y_4^3y_1t^4 \\ +x_2^2\xi^2y_4x_1t^2 - y_4^4x_1t^4 - x_2\xi y_4y_1x_1t^2 - x_2\xi y_4^2y_2t^3 + x_2y_4^3y_2t^4 \\ -y_4^2x_1^2t^2 + y_4^2y_1y_2t^3 + x_2y_4x_1y_2t^2 + x_2x_1y_3t - y_3 s = 0 \end{cases}$$

Codimension 4

Example (Continuation)

Actually... This is birational to a del Pezzo fibration $Y/\mathbb{P}(1,2)$ of degree 2. The generic fibre is

$$\left(t\xi^3 + t^2y_2 + t\xi y_2 - y_2^2 + t^2\xi y_1 - \xi^3 y_1 - ty_2y_1 - \xi y_1y_2 - t^2y_1^2 + y_2y_1^2 = 0\right) \subset \mathbb{P}(1_t, 1_\xi, 1_{y_1}, 2_{y_2})$$

Codimension 4

Example

GRDB ID #40671: $X \subset \mathbb{P}(1,1,1,2,2,2,3,3)$. Consider the two consecutive projections $X \dashrightarrow X' \dashrightarrow X''$ where $X \dashrightarrow X'$ is the projection away from $\mathbf{p}_s \in X$ and $X' \dashrightarrow X''$ is the projection away from $\mathbf{p}_{y_3} \in X'$. The equations of X'' are given explicitly by

$$\begin{pmatrix} -y_1 & y_2 & y_2^2x_2 + y_1x_2^2 - x_2y_4 \\ y_4 - y_1^2 - y_1y_2 - y_1x_2 - y_2x_2 & -y_4 & -y_1^4 + y_1^3y_2 + y_2^2y_4 + y_1x_2y_4 - y_4^2 \end{pmatrix} \begin{pmatrix} x_1 \\ \xi \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let $\Gamma \subset \mathbb{P}(1,1,1,2)$ be defined by the three 2×2 minors of the matrix above. The curve Γ has two irreducible and reduced components: one is rational, and the other has genus 4. It turns out that X can be constructed from the blowup of $\Gamma \subset \mathbb{P}(1,1,1,2)$.

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Theorem (Campo-DG, 2021)

With the possible exception of 3 deformation families, each of the 34 admits a realisation as a deformation family Fano 3-fold which is non-solid.

Sarkisov links via toric varieties

By Hu and Keel, birational contractions in a Mori Dream Space arise from toric geometry.

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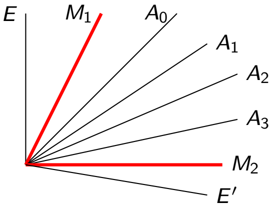

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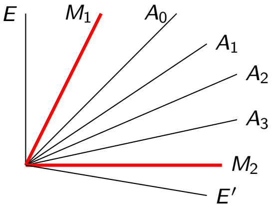

1. $\mathrm{Nef}(T) = \mathbb{R}_+[M_1] + \mathbb{R}_+[A_0]$.

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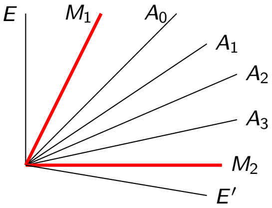

1. $\mathrm{Nef}(T) = \mathbb{R}_+[M_1] + \mathbb{R}_+[A_0]$.
2. If T_i and T_{i+1} are ample models in adjacent chambers, they are related by a small $\mathbb{Q}$-factorial modification.

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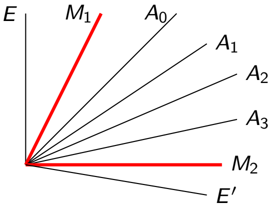

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3. We have $M_2 \sim_{\mathbb{Q}} E'$.

Sarkisov links via toric varieties: Restriction

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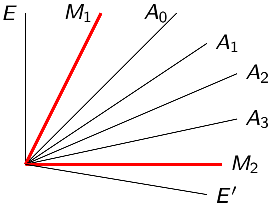

Sarkisov links via toric varieties: Restriction

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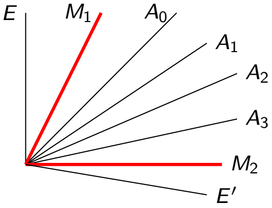

1. $\mathrm{Nef}(Y) = \mathbb{R}_+[M_1] + \mathbb{R}_+[A_1]$.
2. If Y_i and Y_{i+1} are ample models in adjacent chambers, they are related by a flip.
3. $-K_Y \in \mathrm{Int}\,\mathrm{Mov}(Y)$
4. We have $M_2 \sim_{\mathbb{Q}} E'$.

Conjecture

- **Fano** if $-K_X$ is ample;
- **Calabi-Yau** if K_X is numerically trivial;
- **Canonically polarised** if K_X is ample.

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Conjecture

Let W be a smooth uniruled variety. Then W is birational to a Mori fibre space whose general fibre is K-stable.

Thanks

Thank you!