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On singularities of K-moduli spaces of Fano varieties

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Theorem (Murphy's law by Vakil '04)

Moduli of surfaces of general type can be **arbitrarily badly** *singular.*

Toric varieties can easily give examples of obstructed algebraic varieties $\Rightarrow$

Theorem (Kaloghiros–P. '20)

Moduli of K-polystable Fano varieties of dimension ≥ 3 *can be singular, in particular non-reduced and/or non-unibranch.*

Theorem (P. '21)

Moduli of K-polystable Fano varieties of dimension ≥ 3 *can be* **quite** *singular, in particular can have arbitrarily many local branches.*

X normal algebraic variety over $\mathbb{C}$

Def_X is the **deformation space** of X:

- analytic germ, it is the base of the semiuniversal (a.k.a. Kuranishi) deformation of X

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "\text{flat}"] \\ \operatorname{Spec}\mathbb{C} \arrow[r, phantom, "\in"] & T \end{tikzcd} = \mathrm{Def}_X$$

when it is the semiuniversal def.

- the corresponding complete noetherian local $\mathbb{C}$-algebra

$$\widehat{\mathcal{O}}_{\mathrm{Def}_X, 0} = R$$

$$\mathbb{T}^i_X := \mathsf{Ext}^i(\Omega_X, \mathcal{O}_X) \text{ for } i = 0, 1, 2.$$

Then

- $\mathbb{T}^0_X = \mathrm{T}_{\mathrm{id}}\mathsf{Aut}(X)$
- there exists a holomorphic map of analytic germs

  $$\mathrm{ob}_X \colon (\mathbb{T}^1_X, 0) \longrightarrow (\mathbb{T}^2_X, 0)$$

  such that ob_X has zero linear part and

  $$\mathsf{Def}_X = \mathrm{ob}_X^{-1}(0).$$

- $\mathbb{T}^1_X = \mathrm{T}_0\,\mathsf{Def}_X$

X is rigid if $\mathrm{Def}_X = \text{point}$ $\iff T^1_X = 0$

X is unobstructed if Def_X smooth $\iff ob_X = 0 \Longleftarrow T^2_X = 0$

$f \in \mathbb{C}[x_1, \ldots, x_n]$, $X = V(f) \subset \mathbb{A}^n$, I ideal of $X \hookrightarrow \mathbb{A}^n$

$$0 \to I/I^2 \longrightarrow \Omega_{\mathbb{A}^n}|_X \longrightarrow \Omega_X \longrightarrow 0$$

$$\begin{array}{ccc} I/I^2 & & \Omega_{\mathbb{A}^n}|_X \\ \| & & \| \\ \mathcal{O}_X & \xrightarrow[\left(\frac{\partial f}{\partial x_i}\right)_i]{} & \mathcal{O}_X^{\oplus n} \end{array}$$

$\mathrm{Hom}(\cdot, \mathcal{O}_X)$

$$\begin{array}{ccccccc} \mathrm{Hom}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_X) & \to & \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) & \longrightarrow & \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) & \longrightarrow & \mathrm{Ext}^1(\mathcal{O}_X^{\oplus n}, \mathcal{O}_X) \\ \| & & \| & & \| & & \| \\ \mathcal{O}_X^{\oplus n} & \xrightarrow{\nabla f} & \mathcal{O}_X & & T_X^1 & & 0 \end{array}$$

$$\Rightarrow T_X^1 = \frac{\mathcal{O}_X}{\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)} = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)}$$

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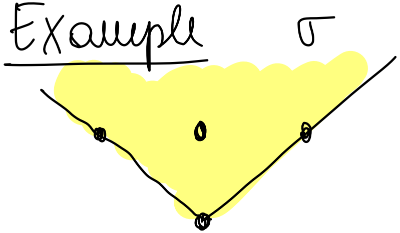
$$T_X^2 = 0$$

Example



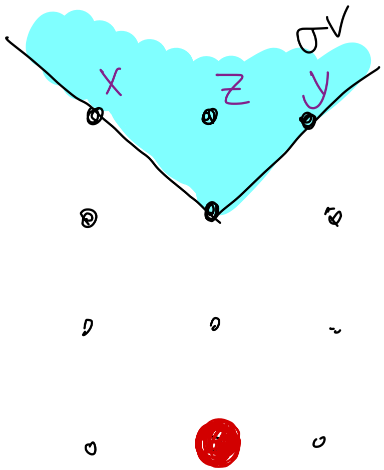
$N = \mathbb{Z}^2$

X = toric affine variety associated to σ

$= \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$

A_1 surface sing.

$\frac{1}{2}(1,1)$



$M = \operatorname{Hom}(N, \mathbb{Z})$

$\mathbb{N}^3 \twoheadrightarrow \sigma^\vee \cap M$

$\mathbb{C}[x,y,z] = \mathbb{C}[\mathbb{N}^3] \twoheadrightarrow \mathbb{C}[\sigma^\vee \cap M]$

$\mathbb{A}^3 \hookleftarrow X$ closed emb.

$X = V(xy - z^2)$

$$T^1_X = \frac{\mathbb{C}[x,y,z]}{(xy-z^2, y, x, -2z)} = \frac{\mathbb{C}[x,y,z]}{(x,y,z)} = \mathbb{C}, \quad T^2_X = 0$$

The semiuniv. def of X is $(xy - z^2 - t = 0) \subset \mathbb{A}^3_{x,y,z} \times \mathbb{A}^1_t$

$$\downarrow$$

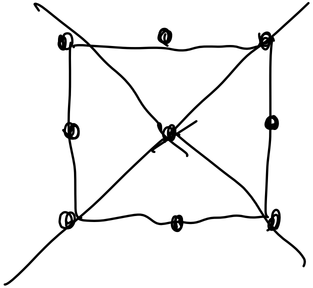
$$\mathbb{A}^1_t$$

$T = N \otimes_{\mathbb{Z}} \mathbb{C}^* \curvearrowright X$. M = character lattice of T

$\deg x = (-1,1) \in M$

$\deg z = (0,1)$, $\deg y = (1,1) \Rightarrow \deg t = (0,2) \Rightarrow T^1_X = \mathbb{C}$ with weight $(0,-2) \in M$

$P =$



in $N = \mathbb{Z}^2$ cochar. latt. of T

X = toric variety associated to the face fan of P

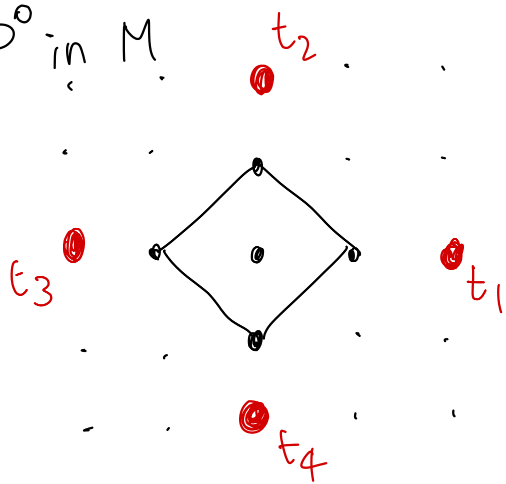
X del Pezzo surface with $4 \times A_1$

$K_X^2 = \mathrm{Vol}(P^\circ) = 4$

$h^0(-K_X) = \#(P^\circ \cap M) = 5$

$\mathrm{Bary}(P^\circ) = 0 \Rightarrow X$ K-ps

P° in M



$$\left.\begin{array}{l} H^1(\mathcal{T}_X^0) = 0 \\ H^2(\mathcal{T}_X^0) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} T_X^2 = 0 \\ T_X^1 = H^0(\mathcal{T}_X^1) = \mathbb{C}^4 \end{array}$$

$$\mathrm{Def}_X = \mathbb{A}^4_{t_1, t_2, t_3, t_4}$$

$$\mathrm{Aut}(X) = \underset{(\mathbb{C}^*)^2}{\underset{\parallel}{T}} \rtimes \underset{D_4}{\underset{\parallel}{\mathrm{Aut}(P)}}$$

generated by

ρ = rotation | σ refl.

$t_1 \to t_2 \to t_3 \to t_4$ | $t_1 \leftrightarrow t_2$, $t_3 \leftrightarrow t_4$

K-moduli of Fano varieties:

Theorem (Alper, Blum, Halpern-Leistner, Jiang, Li, Liu, Wang, Xu, Zhuang)

$$
\begin{tikzcd}
\mathcal{M}_n^{Kss} \arrow[d] \arrow[r, equal] & \{\textit{K-semistable Fano n-folds}\} \\
M_n^{Kps} \arrow[r, equal] & \{\textit{K-polystable Fano n-folds}\}
\end{tikzcd}
$$

Artin stack

good moduli space

projective scheme

X K-polystable Fano, $\dim X = n$
$\rightsquigarrow$ closed point $\operatorname{Spec} \mathbb{C} \to \mathcal{M}_n^{\mathsf{Kss}} \to M_n^{\mathsf{Kps}}$

(Alper–Hall–Rydh) The local structure at $[X]$ is

$$\begin{tikzcd} {[\mathrm{Def}_X \ / \ \mathrm{Aut}(X)]} \arrow[r, "\text{ét}"] \arrow[d] & \mathcal{M}_n^{\mathsf{Kss}} \arrow[d] \\ \mathrm{Def}_X \ /\!\!/ \ \mathrm{Aut}(X) \arrow[r, "\text{ét}"] & M_n^{\mathsf{Kps}} \end{tikzcd}$$

$$\mathrm{Def}_X \text{ smooth } \Rightarrow \begin{cases} \mathcal{M}_n^{\mathsf{Kss}} \text{ smooth at } [X] \\ M_n^{\mathsf{Kps}} \text{ normal and Cohen–Macaulay at } [X] \end{cases}$$

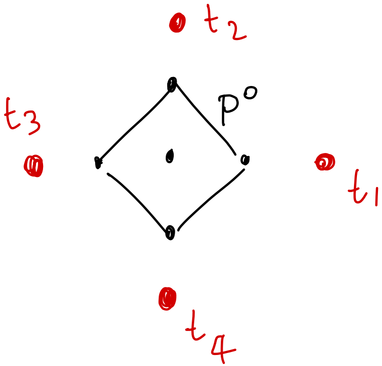
X Fano

- X smooth $\Rightarrow \mathrm{Def}_X$ smooth

- (Hacking–Prokhorov, Odaka–Spotti–Sun, Akhtar–Coates–Corti–Heuberger–Kasprzyk–Oneto–P.–Prince–Tveiten) $\dim X = 2 \Rightarrow \mathrm{Def}_X$ smooth

  $\Rightarrow \mathcal{M}_2^{\mathrm{Kss}}$ smooth and M_2^{Kps} normal

- (Namikawa, Sano) $\dim X = 3$, X terminal $\Rightarrow \mathrm{Def}_X$ smooth



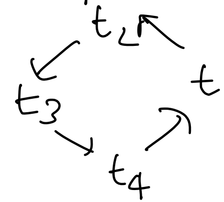
$\mathbb{T}^1_X = \mathbb{C}^4$, $\quad \mathrm{Def}_X = \mathrm{Spec}\, A$, $\quad A = \mathbb{C}[t_1, t_2, t_3, t_4]$

$$\mathrm{Aut}(X) = T \rtimes \underbrace{\mathrm{Aut}(P)}_{= D_4}$$

ρ rotation, $\quad \sigma$ reflection

$t_1 \longleftrightarrow t_2$

$t_3 \longleftrightarrow t_4$



T acts via

t_1	t_2	t_3	t_4
2	0	-2	0
0	2	0	-2

$$A^T = \mathbb{C}[\underbrace{t_1 t_3}_{= a}, \underbrace{t_2 t_4}_{= b}]$$

$\rho : a \Longleftrightarrow b$

$\sigma : a \Longleftrightarrow b$

$$A^{\mathrm{Aut}(X)} = \mathbb{C}[a+b, ab] \simeq \mathbb{C}[x, y] \implies M_2^{Kps} \text{ is smooth of dim } 2 \text{ at } [X]$$

If $t_1 = t_2 = 0$, $t_3 \neq 0$, $t_4 \neq 0$ $\implies$ get strictly K-ss