THE GEOMETRY OF THE LINEAR FRACTIONAL GROUP $LF(4,2)$

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THE smallest of all finite fields, F let us call it, consists of only the two marks 0 and 1; on F we can base, for the different integers n, finite geometries of $2^{n+1}-1$ points in space of n dimensions. Each point has $n+1$ coordinates which are all either 0 or 1, and every set of $n+1$ marks save the set of $n+1$ zeros serves as coordinates of a point. We propose to study the three-dimensional geometry in some detail, and what we say first about the geometries in one and two dimensions is selected as being relevant to what we are to say later. The three-dimensional geometry has a group of 20,160 projectivities which is known (Jordan (**9**), Moore (**12**), Dickson (**5**) and (**6**)) to be isomorphic to $\mathfrak{A}_8$, the alternating group of degree 8. It is natural to ask what, if any, are the 8 objects which undergo permutation. This question was discussed at length by Moore in (**12**). But, while there is no thought either of controverting Moore's claim to have answered it or of disputing his priority, the question is primarily a geometrical one, and abstention from geometrical terminology or reasoning in discussing it imposes a somewhat laboured and periphrastic style. Moore's paper, notwithstanding his criticizing Jordan, on p. 418, for lack of clarity, smothers the investigation with an agglomeration of hieroglyphics whose significance is not easy to grasp however expertly they may be marshalled. In this paper we obtain various geometrical constructs first and, in virtue of their interrelations, attach symbols to them afterwards. We carry our investigation as far as the discovery of 8 sets of 7 mutually non-apolar linear complexes. The set of 7 symbols in the centre of p. 439 of (**12**) can be thus interpreted. Indeed it is in this § 7 of (**12**) that the essential information is assembled, and all Moore's symbols answer to geometrical constructs that we shall encounter. Whereas Moore speaks of triads as associated or separated, we speak of lines as intersecting or skew, and Moore's 56 systems of 5 mutually separated triads correspond to the 56 quintuples of skew lines (see § 7 below); and so on.

The necessary minimum of preliminary exposition occupies §§ 1–5; the account of the three-dimensional geometry opens in § 6 and linear complexes appear in § 8. §§ 6–16 give many geometrical facts all of which, it is submitted, are interesting in themselves; but line geometry and linear complexes are brought into prominence because they afford the clearest explanation of the effect of $LF(4, 2)$. There is, of course, a Klein representation of the lines but any account of it is, with other topics, omitted so as not to lengthen the paper. In § 17 we introduce and explain the symbolism and in § 18 establish the isomorphism between $LF(4, 2)$ and $\mathfrak{A}_8$. Moore claims (**12**, 419) to give a 'pure group-theoretic' proof of this isomorphism; our proof is 'pure geometric', and although its appearance comes more than eighty years after Jordan's original proof it is hoped that others, as well as geometers, will agree that it is the most natural and appropriate proof. We also establish the isomorphism between the symplectic group $C(4, 2)$ and $\mathfrak{S}_6$; this too was originally done by Jordan.

$\mathfrak{A}_8$ has many subgroups and one, of order 1,344, has now a history of almost a century. Of the many properties of this and other subgroups some are almost intuitive when viewed in the light of the geometry expounded here. In §§ 19–23 some of these subgroups are defined as groups of projectivities that belong to $LF(4, 2)$ and are represented as groups of matrices all of whose elements belong to F, that is are all either 0 or 1.

1. On a line there are 3 points, namely those whose coordinates constitute the column vectors

$$\begin{matrix} . & 1 & 1 \\ 1 & . & 1 \end{matrix} \tag{1.1}$$

They undergo all 3! permutations when subjected to projectivities. A projectivity operates by premultiplying the vectors (1.1) by *non-singular* matrices whose elements all belong to F. The determinant of any matrix whose elements are all marks of F is itself a mark of F; singular matrices have determinant 0, non-singular matrices have, all of them, determinant 1. The first column may be any of (1.1), but when it has been chosen the stipulation of non-singularity debars the same column being chosen again. The matrices which impose projectivities are then

$$\begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix} \begin{bmatrix} . & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & . \end{bmatrix} \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix} \begin{bmatrix} 1 & . \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ . & 1 \end{bmatrix}, \tag{1.2}$$

the last three all having I as their square, the latent root 1 repeated, and one latent column vector. The second and third matrices are of period 3 and inverse to each other; these have no latent root in F. Note always, here and hereafter, that the mark 1 satisfies $1+1 = 0$ and is equal to its negative -1.

2. In a plane S_2 are the 7 points

$$\begin{matrix} 1 & . & . & . & 1 & 1 & 1 \\ . & 1 & . & 1 & . & 1 & 1 \\ . & . & 1 & 1 & 1 & . & 1 \end{matrix} \tag{2.1}$$

They lie 3 on each of 7 lines, 3 lines passing through each point. We represent lines by row vectors, a point x and a line u being incident whenever $ux = 0$ but not incident when $ux = 1$. In building a non-singular matrix M we can take the first column to be any of (2.1) and the second to be any other among (2.1). Both these columns, and also their sum—the only column, other than themselves, that is linearly dependent on them—are ineligible for the last column of M, for which therefore there remain 4 choices. Thus there are $7.6.4 = 168$ non-singular matrices; they constitute, as was remarked by Weber,† a Klein group K. K is doubly transitive on the 7 points. In order, for instance, to transform the first vertex of the triangle of reference into a point P we take the first column of M to be the column vector of coordinates of P; thus any point is transformed into any other point of S_2, or indeed into itself, by $6.4 = 24$ operations of K.

We suppose, as is customary, that whenever the points of S_2 are permuted by premultiplying their column vectors by a non-singular matrix M the lines of S_2 are simultaneously permuted by postmultiplying their row vectors by M^{-1}. Should a point and line be incident so are the point and line that arise by subjecting them to any projectivity, and non-incidence is maintained also. A line, like a point, is invariant for 24 operations of K; if the point $(a, b, c)'$ is unchanged on premultiplication by M, then the line (a, b, c) is unchanged on postmultiplication by M'.

3. The 4 points which do not lie on a given line l form a *quadrangle* L; no three are collinear since the join of any two has its third point on l. The number of quadrangles in S_2 is 7, the same as the number of lines. Those 24 projectivities for which l is invariant form an octahedral group Ω and impose all 4! permutations on the vertices of L, so that there is a set of 7 conjugate octahedral subgroups of K (they are conjugate because K is transitive on the lines). There are also in S_2 7 *quadrilaterals* p, each consisting of the 4 lines which do not pass through some point P. The 24 projectivities for which P is invariant form an octahedral group and subject the sides of p to all 4! permutations, so that there is a second set of 7 conjugate octahedral subgroups of K. Suppose, for example, that P is $x = y = z$; each row of any matrix which leaves the column vector of

† (15), 371. Note the occurrence of the matrices (1.2) on p. 370.

coordinates of P invariant under premultiplication must have its three elements summing to 1, and conversely. The only such rows are

$$1\ .\ .,\quad .\ 1\ .,\quad .\ .\ 1,\quad 1\ 1\ 1,$$

and $4.3.2 = 24$ non-singular matrices can be formed from them. Likewise those 24 matrices each of whose columns sums to 1 leave invariant the line $x+y+z = 0$, although of course they may permute the 3 points on it.

4. While a given line l is invariant for the 24 projectivities of Ω there are, as we saw in § 1, only 6 projectivities on l; each of the 6 permutations is imposed on the points of l by 4 operations of Ω. In particular, the identity permutation on l is imposed by 4 projectivities that form a self-conjugate subgroup Ω_0 of Ω, and those 4 projectivities that impose any one permutation on the points of l belong to the same coset of Ω_0 in Ω. For example: if l is $x+y+z = 0$, Ω is the set of matrices each of whose columns sums to 1, while Ω_0 consists of the matrices

$$\begin{bmatrix} 1+a & a & a \\ b & 1+b & b \\ c & c & 1+c \end{bmatrix} \quad \text{with } a+b+c = 0.$$

These 4 matrices answer to

$$(a,b,c) = (0,0,0); \qquad (0,1,1); \qquad (1,0,1); \qquad (1,1,0);$$

each of the last three has the first, namely I, as its square and they form a non-cyclic abelian 4-group. The same, or rather the dual, situation occurs in the octahedral groups of the other conjugate set.

Projectivities which leave invariant a side or a vertex of the triangle of reference also afford clear, though less symmetrical, representations. Those for which $x = 0$ is invariant are imposed by matrices

$$\begin{bmatrix} 1 & . & . \\ \alpha & \gamma & \epsilon \\ \beta & \delta & \zeta \end{bmatrix}$$

wherein $\begin{bmatrix} \gamma & \epsilon \\ \delta & \zeta \end{bmatrix}$, having to be non-singular, is one of the 6 matrices (1.2) while both α and β can be either 0 or 1. The number of such matrices is thus $6.2^2 = 24$ and they form a group under multiplication. The three points on $x = 0$ undergo permutation save for the 4 matrices

$$\begin{bmatrix} 1 & . & . \\ \alpha & 1 & . \\ \beta & . & 1 \end{bmatrix};$$

these again form a 4-group, self-conjugate in the octahedral group. There are analogous results for the other two sides and, involving the transposed matrices, for the three vertices of the triangle of reference.

5. We add a few words concerning the maximal subgroups of K other than the 14 octahedral groups; they are of order 21, and we shall refer to them again in § 20.

There must be, not only in K but in any subgroup of order 21, operations of period 7. Any such projectivity permutes the 7 points in a single cycle; no point can be invariant, so that the corresponding matrix cannot have a latent root in F. Its characteristic function is a cubic and a factor of

$$(\lambda^7-1)/(\lambda-1) \equiv \lambda^6+\lambda^5+\lambda^4+\lambda^3+\lambda^2+\lambda+1 \equiv (\lambda^3+\lambda^2+1)(\lambda^3+\lambda+1).$$

If we extend F by adjoining to it any one root of either factor it is thereby amplified to a field of 8 marks of which the non-zero ones behave, so far as their *multiplicative* properties are concerned, exactly as the seventh roots of 1 in the field of complex numbers; the three roots of either factor have 1 for their product and are the reciprocals of the roots of the other. We expect a matrix τ of period 7 to have, with τ^2 and τ^4, one of these two factors as its characteristic function while τ^3, τ^5, τ^6 have the other.

Take, with Weber,

$$\tau = \begin{bmatrix} 1 & . & 1 \\ 1 & . & . \\ . & 1 & . \end{bmatrix},$$

whereupon $|\tau-\lambda I| = 1+\lambda^2+\lambda^3$. Then $M^{-1}\tau M = \tau^2$ provided that

$$M = \begin{bmatrix} a & b & b+c \\ b & c & a+b+c \\ c & a+b & a+c \end{bmatrix};$$

these $2^3 = 8$ matrices M are all, save the one with $a = b = c = 0$, non-singular and of period 3 (note that Weber's χ occurs for $a = 1, b = c = 0$). These 7 non-zero matrices, together with their inverses and the powers of τ, constitute a group of order 21. There are 8 such subgroups in K; each is transitive on the 7 points, τ permuting them in a single cycle.

6. We consider now a three-dimensional space S wherein the four coordinates x, y, z, t of any point are marks of F; it consists of 15 points which lie 7 in each of 15 planes, while through each point pass 7 of the planes.† We arrange plane coordinates as row vectors, and use column vectors of point coordinates.

† There is a brief allusion to this geometry in Fano (7), 114.

Let us take four vertices of a tetrahedron, four points, that is, which are not coplanar. The freedom of choice is

15 positions for the first vertex,
14 for the second (any point other than the first),
12 for the third (any point not on the join of the first two),
8 for the fourth (any point not in the plane of the first three),

so that there are $15.14.12.8 \div 4! = 840$ tetrahedra in S.

Each projectivity in S is imposed by some non-singular matrix whose elements all belong to F, and conversely. Since the columns of such a matrix are linearly independent they represent the vertices of a tetrahedron; they may, however, the matrix remaining non-singular, undergo any permutation, so that the number of projectivities is

$$840 \times 4! = \tfrac{1}{2}.8!.$$

They form the linear fractional group $LF(4, 2)$; here we call it Γ; it is known to be isomorphic to $\mathfrak{A}_8$, the alternating group of degree 8, and we shall establish this isomorphism *de novo* in § 18. But we must not postpone the acknowledgement of its first having been established by Camille Jordan in § 516 of (9). These are significant paragraphs of the *Traité*, though in places somewhat cryptic. Jordan does not use geometrical terms, but it strains one's credulity to suppose that geometry was not ancillary to his reasoning. His calculation on p. 382 of the order of Γ is nothing more or less than our calculation of the number of tetrahedra, having regard to the ordering of their vertices. Moreover, there are displayed conspicuously on p. 381 the 15 row vectors of coordinates of planes of S and, in addition to this, Jordan notices,† also on p. 381, what for us are the 35 lines of S. We now proceed to discuss the geometry of these lines.

7. Each pair of points of S is joined by a line and on this line is a third point; hence there are $\tfrac{1}{3}{}^{15}C_2 = 35$ lines in all. They are self-dual and may equally well be determined as intersections of planes; each pair of planes of S has a line of intersection and through it there passes a third plane. In every plane of S are 7 lines whose geometry is that of §§ 2–5. Through each point of S pass 7 lines each containing 2 of the other 14 points. Since any line contains 3 points through each of which pass 6 other lines, the number of lines skew to it is $35-1-18 = 16$, and so the number of pairs of skew lines in S is $\tfrac{1}{2}.35.16 = 280$. Every skew pair has 9 transversals, and the 9 points not on either line of the pair lie one on each transversal. There are 6 different ways of choosing from these 9 transversals a set of 3

† There are two misprints. At the foot of p. 380 only seven of eight triplets are given, 101 having been omitted. And in line 9 of p. 381 *abced* ought to be *abecd*.

all skew to one another; moreover, when a *triple* of skew lines m, m', m'' is given the transversals from the points on m to m' and m'' form another triple *associated* with the former. Hence the number of triples in S is $280 \times 6 \div 3 = 560$, and they consist of 280 associated pairs.

That the number of pairs of associated triples is equal to the number of pairs of skew lines is no accident, and there is a $(1, 1)$ correspondence between them. In the first place there are 6 lines skew to both members n, n' of a given skew pair; they complete with n and n' the triples associated with the 6 triples among their transversals, and two of them intersect or not according as the two triples of transversals share a line or not. Hence each of the 6 lines skew to n and n' meets 3 and is skew to 2: they form a pair of associated triples. Conversely: take a pair of associated triples. We can argue just as one does with Dandelin's skew hexagon in classical projective geometry† and thus derive 6 points of concurrence of sets of 3 lines, the 6 points lying 3 on each of two skew lines n, n'. These 6 points are those other than the 9 intersections of the two triples, and either triple determines the other and n, n' uniquely. We say that n, n' and either triple are associated.

A pair of skew lines and either of its associated triples form a *quintuple*, or set of 5 mutually skew lines; it accounts, with 3 points on each line, for every point of S. Each skew pair belongs to 2 quintuples. Since each quintuple admits 10 different separations into a skew pair and an associated triple the number of quintuples in S is‡

$$280 \times 2 \div 10 = 56.$$

Each triple belongs to one, and only one, quintuple. The number of quintuples that include a given line is

$$56 \times 5 \div 35 = 8.$$

8. We now introduce the linear complexes, or *screws* as we may call them, in S. They can be obtained from the null polarities: correlations $u' = Bx$ wherein B is a skew matrix. Since every element of B is to be a mark of F we can also say that B is symmetric, but has its four diagonal elements all zero:

$$B = \begin{bmatrix} . & c & b & a' \\ c & . & a & b' \\ b & a & . & c' \\ a' & b' & c' & . \end{bmatrix}.$$

† For Dandelin's figure see, for instance, Baker (**2** *a*), 45. This figure can serve as a sketch for geometry in S; it portrays all 15 points and 26 of the lines. The other 9 lines do not appear visually rectilinear; they are the transversals ALX', BMX', CNX', $A'LX$, $B'MX$, $C'NX$, DLO, EMO, FNO of LMN and OXX'.

‡ See, for an interpretation in terms of the elementary abelian group of order 16, (**3**), 118, Ex. 2.

If in B we transpose a with a', b with b', c with c', and multiply the two matrices we find that

$$B(a,b,c;a',b',c')B(a',b',c';a,b,c) \equiv (aa'+bb'+cc')I,$$

so that whenever $aa'+bb'+cc' = 1$ the inverse of B is got simply by this transposition. But when $aa'+bb'+cc' = 0$ B is singular, and indeed its rank sinks to 2.

The same screw may also be defined as the aggregate of lines whose Plücker coordinates satisfy a linear condition

$$x'B\xi \equiv \sum (ap_{23}+a'p_{14}) = 0; \tag{8.1}$$

the symbol $\sum$ implies the sum of those three terms which arise, from the one written, by simultaneous cyclic permutations of the letters a, b, c and of the suffixes 1, 2, 3; it is not necessary for both letters and suffixes to occur but, whenever suffixes do occur, 4 is never permuted. The three points of a line have coordinates of the form

$$(x,y,z,t), \quad (\xi,\eta,\zeta,\tau), \quad (x+\xi,y+\eta,z+\zeta,t+\tau),$$

and the Plücker coordinates of this line are

$$p_{12} = x\eta-y\xi = x\eta+y\xi, \qquad p_{34} = z\tau-t\zeta = z\tau+t\zeta,$$

and so on; they always satisfy the identity $\sum p_{23}p_{14} \equiv 0$, and can be expressed also in terms of the planes which contain the line.

Any six marks p_{ij} of F which satisfy $\sum p_{23}p_{14} = 0$ are coordinates of a line, and this affords another means of calculating how many lines there are in S. For either all three terms of the identity are 0 or else one is 0 and the others both 1. The first alternative can happen in $3^3-1 = 26$ ways, since we do not allow all six p_{ij} to be 0; the second alternative can happen in 9 ways, since there are 3 choices for the zero term and, once it has been chosen, either or both of its factors may be 0. The number of lines is thus $26+9 = 35$.

Two lines meet when their moment

$$\varpi(m,n) \equiv \sum (m_{23}n_{14}+m_{14}n_{23})$$

is 0; when $\varpi = 1$ they are skew.

9. We use the term *screw* only when B is non-singular and so $\sum aa' = 1$. When $\sum aa' = 0$ we say that (8.1) defines a *sheaf*; the lines of the sheaf are those 18 which meet the line

$$p_{23} = a', \quad p_{14} = a, \quad p_{31} = b', \quad p_{24} = b, \quad p_{12} = c', \quad p_{34} = c,$$

together with this line itself, the axis of the sheaf. The number of screws in S is easy to find, for when $\sum aa' = 1$ one of four possibilities must be realized:

$$\text{(i)} \quad aa' = bb' = cc' = 1,$$

$$\text{(ii)} \quad aa' = 1,\ bb' = cc' = 0, \qquad \text{(iii)} \quad bb' = 1,\ cc' = aa' = 0,$$

$$\text{(iv)} \quad cc' = 1,\ aa' = bb' = 0,$$

and under these respective headings occur 1, 9, 9, 9 screws. Hence there are 28 screws in all. They all have similar geometrical properties; that one of them is apparently singled out in (i) is a consequence of the introduction of coordinates and choice of tetrahedron of reference. The number of screws to which a given line belongs is

$$28 \times 15 \div 35 = 12$$

(since, as we show immediately, every screw contains 15 lines).

Those lines which belong to a given screw σ and pass through a given point P are the 3 which lie in the null plane of P; σ therefore contains 3 lines through every one of the 15 points of S, and as there are 3 points on every line the number of lines which belong to σ is 15. The remaining 20 are paired as polar lines in regard to σ, and all 9 transversals of any pair of polar lines belong to σ. Let σ be given by (8.1) and let m be any line not satisfying (8.1). Then, if n both belongs to σ and meets m,

$$\sum \{(a+m_{14})n_{23}+(a'+m_{23})n_{14}\} = \sum (an_{23}+a'n_{14})+\varpi(m,n) = 0+0 = 0.$$

But

$$\sum (a+m_{14})(a'+m_{23})$$
$$= \sum aa' + \sum (am_{23}+a'm_{14}) + \sum m_{23}m_{14} = 1+1+0 = 0.$$

so that
$$m'_{23} = a'+m_{23},\quad m'_{14} = a+m_{14},\ \ldots$$

are coordinates of a line m' which is met by n. Every line of σ which meets m also meets m' and, since m' does not satisfy (8.1) when m does not, and $m_{23} = a'+m'_{23}$, etc., every line of σ which meets m' also meets m. Any pair of polar lines of σ satisfies

$$m_{23}+m'_{23} = a',\quad m_{14}+m'_{14} = a,\ \ldots$$

and σ is linearly dependent on the two sheaves whose axes are any pair of polar lines. Every pair of skew lines determines in this way the unique screw for which they are polars, and the 280 pairs of skew lines are distributed as 28 sets of ten pairs, the pairs of any set being the pairs of polar lines for some screw. The 15 lines of the screw consist of the 9 transversals of any pair m, m' of its polar lines and the two triples associated with m, m'. The occurrence of ten pairs of associated triples among the 15 lines, and the passing of these lines 3 by 3 through 15 points, are paralleled by known phenomena in classical projective geometry (see, e.g. (**2** *b*), 114–15 and frontispiece).

Note that, whereas the third member of the pencil determined by two sheaves with skew axes is a screw, the third member of the pencil determined by two sheaves with intersecting axes is a sheaf. The axis of this sheaf is, of course, the line that is both concurrent and coplanar with the axes of the other two.

10. Two screws may, or may not, be so related that the corresponding null polarities commute. Let two non-singular null polarities be

$$u' = B_1 x, \qquad u' = B_2 x,$$

or, expressing the same equations differently,

$$x = B_1^{-1} u', \qquad x = B_2^{-1} u'.$$

The pole y in the second screw of the polar plane of x in the first is

$$y = B_2^{-1} B_1 x,$$

and

$$B_2^{-1}(a_2, b_2, c_2; a_2', b_2', c_2') B_1(a_1, b_1, c_1; a_1', b_1', c_1')$$
$$= B_2(a_2', b_2', c_2'; a_2, b_2, c_2) B_1(a_1, b_1, c_1; a_1', b_1', c_1').$$

The non-diagonal elements in this product of two matrices are all unaltered by transposing the suffixes 1 and 2, whereas the four diagonal elements are all, in general, changed. But these four diagonal elements are seen also to be unaffected by the transposition, provided only that

$$\varpi_{12} \equiv \sum (a_1 a_2' + a_2 a_1') = 0.$$

Screws which satisfy this relation are said, in classical projective geometry, to be *apolar* or *in involution*; here we shall say that they are *syzygetic*, while screws for which $\varpi_{12} = 1$ are *azygetic*. These are the terms used by Frobenius (**8**, 82) to describe relations between period characteristics of theta functions.† The presence or absence of syzygy between screws is, of course, unaffected by projective transformations.

Although ϖ_{12} was obtained as a function of elements of two non-singular matrices the same function is available, and has its geometrical interpretation, when either or both matrices are singular. A screw and a sheaf are syzygetic when the axis of the sheaf is a line of the screw; otherwise they are azygetic. Two sheaves are syzygetic when their axes intersect, azygetic when they are skew.

We may denote by σ not merely a screw but also the linear expression

† When there are 3 variables the period characteristic is $\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$ wherein all 6 marks are either 0 or 1, and is even or odd with $\sum aa'$. Thus the even characteristics other than the zero one answer to the lines, and so to the sheaves, in S, the odd characteristics to the screws in S.

in the line coordinates whose vanishing determines the screw. If, then, σ_1 and σ_2 are syzygetic $\sigma_1+\sigma_2$ is a sheaf; for

$$\sum (a_1+a_2)(a_1'+a_2') = \sum a_1 a_1' + \varpi_{12} + \sum a_2 a_2' = 1+0+1 = 0.$$

Moreover the axis of $\sigma_1+\sigma_2$ belongs both to σ_1 and σ_2 since, for instance,

$$\sum \{a_1(a_1'+a_2')+a_1'(a_1+a_2)\} = \sum (a_1 a_2'+a_1' a_2) = \varpi_{12} = 0.$$

Conversely: let m be any line of a screw σ. Then

$$\sum \{(a+m_{14})p_{23}+(a'+m_{23})p_{14}\} = 0$$

is a screw, because

$$\sum (a+m_{14})(a'+m_{23}) = \sum aa' + \sum m_{14} m_{23} = 1+0 = 1,$$

and is syzygetic to σ because

$$\sum \{(a+m_{14})a'+(a'+m_{23})a\} = \sum (am_{23}+a'm_{14}) = 0.$$

Hence 15 screws are syzygetic and 12 azygetic to σ. When σ_1 and σ_2 are syzygetic any line of either which meets the axis of $\sigma_1+\sigma_2$ belongs to the other: any point on this axis has the same null plane in σ_1 and σ_2.

Suppose now that σ_1 and σ_2 are azygetic. Then σ_3, where $\sigma_1+\sigma_2+\sigma_3 \equiv 0$, is a screw because

$$\sum (a_1+a_2)(a_1'+a_2') = \sum a_1 a_1' + \varpi_{12} + \sum a_2 a_2' = 1+1+1 = 1,$$

and is azygetic both to σ_1 and σ_2 since, for instance,

$$\sum \{a_1(a_1'+a_2')+a_1'(a_1+a_2)\} = \sum (a_1 a_2'+a_1' a_2) = \varpi_{12} = 1.$$

Such a set of three linearly dependent screws every pair of which is azygetic we call a *trio*.† The 12 screws azygetic to σ consist of 6 pairs, each pair forming a trio with σ, and as each screw belongs to 6 trios the number of trios is

$$28 \times 6 \div 3 = 56.$$

11. Although there are 28 screws there are, once a point P is given, only the 7 planes through P to serve as its null planes; we expect that each plane through P is its null plane in 4 screws, and this is indeed so. For let σ be given by (8.1) and let q, r, s be those lines which pass through P and lie in the null plane π of P in σ. Then π is also the null plane of P in

$$\sum \{(a+q_{14})p_{23}+(a'+q_{23})p_{14}\} = 0;$$

we saw in § 10 that this is indeed a screw and that it is syzygetic to σ. The statement follows because both r and s belong to this screw as well as to σ; indeed

$$\sum \{(a+q_{14})r_{23}+(a'+q_{23})r_{14}\} = \sum (ar_{23}+a'r_{14}) + \varpi(q,r) = 0$$

† Jordan (**9**, 232, 233) uses this term with a much wider connotation.

because r belongs to σ and intersects q; similarly with s. The other two screws in which π is the null plane of P are

$$\sum \{(a+r_{14})p_{23}+(a'+r_{23})p_{14}\} = 0,$$
$$\sum \{(a+s_{14})p_{23}+(a'+s_{23})p_{14}\} = 0;$$

and every two of the 4 screws are syzygetic because, for instance,

$$\sum \{(a+r_{14})(a'+s_{23})+(a'+r_{23})(a+s_{14})\}$$
$$= \sum (ar_{23}+a'r_{14}) + \sum (as_{23}+a's_{14})+\varpi(r,s) = 0+0+0 = 0.$$

Moreover, the 4 screws are linearly dependent, the sum of their left-hand sides being identically zero. The sheaf linearly dependent on any two of them is the same as that which is linearly dependent on the other two.

That any two screws in which some point has the same null plane must be syzygetic also follows because if $u' = B_1 x = B_2 x$ then $(B_1+B_2)x = 0$ so that B_1+B_2, having linearly dependent columns, is singular and $\sigma_1+\sigma_2$ is a sheaf whose axis passes through x. Then, indeed, B_1+B_2 has rank 2 and there are linearly independent columns x and y making

$$(B_1+B_2)x = (B_1+B_2)y = 0;$$

x, y, $x+y$ are the points on the axis of $\sigma_1+\sigma_2$.

Suppose, on the other hand, that σ_1, σ_2, σ_3 are mutually syzygetic screws; then $\sigma_4 \equiv \sigma_1+\sigma_2+\sigma_3$ is also (we suppress the routine details of the proof) a screw and every pair of the 4 screws is syzygetic. Suppose that

$$q \text{ is the axis of } \sigma_2+\sigma_3 \equiv \sigma_1+\sigma_4,$$
$$r \text{ is the axis of } \sigma_3+\sigma_1 \equiv \sigma_2+\sigma_4,$$
$$s \text{ is the axis of } \sigma_1+\sigma_2 \equiv \sigma_3+\sigma_4.$$

Then q, r, s are linearly dependent. But they intersect because, for instance,

$$\varpi(r,s) = \sum \{(a_3+a_1)(a_1'+a_2')+(a_3'+a_1')(a_1+a_2)\}$$
$$= \varpi_{23}+\varpi_{31}+\varpi_{12} = 0;$$

hence q, r, s are concurrent at a point P and lie in a plane π. This plane is the null plane of P in all 4 screws, and P are π are uniquely determined by them. There must then be $15\times 7 = 105$ sets of 4 mutually syzygetic screws, and if we add to any set of 4 the 3 sheaves determined by the pairs of them we obtain a set of 7 of which every two members are syzygetic.

12. Let us give an example. The point $x = y = z = t$ is the pole of $x+y+z+t = 0$ in any screw for which

$$b+c+a' = c+a+b' = a+b+c' = a'+b'+c' = 1.$$

Then

$$1 = aa'+bb'+cc' = a(1+b+c)+b(1+c+a)+c(1+a+b) = a+b+c,$$

so that $$a = a', \quad b = b', \quad c = c',$$

and the screws answer to the matrices

$$\begin{bmatrix} . & 1 & 1 & 1 \\ 1 & . & 1 & 1 \\ 1 & 1 & . & 1 \\ 1 & 1 & 1 & . \end{bmatrix}, \quad \begin{bmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{bmatrix}, \quad \begin{bmatrix} . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \\ . & 1 & . & . \end{bmatrix}, \quad \begin{bmatrix} . & 1 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \end{bmatrix}$$

The sum of any two of these is singular, and equal to the sum of the other two, the sum of all four being the zero matrix. Each of the three singular matrices has for its elements the coordinates of one of the three lines that pass through $x = y = z = t$ and lie in $x+y+z+t = 0$. These lines are common to all four screws and their coordinates are as follows:

p_{23}	p_{14}	p_{31}	p_{24}	p_{12}	p_{34}
.	.	1	1	1	1
1	1	.	.	1	1
1	1	1	1	.	.

13. Take now a trio $\sigma_1, \sigma_2, \sigma_3$; any line common to two members belongs also to the third. Any plane e has poles, one in each screw; no two of these three poles can coincide because no two of the trio are syzygetic. Moreover, the three poles are collinear because B_1, B_2, B_3 are linearly dependent. The line m on which they lie is the only line in e common to all three screws; hence each of the 15 planes contains one and only one base line of the trio. Likewise, by the dual arguments, through each of the 15 points passes one and only one base line of the trio. Wherefore these base lines constitute a quintuple. We have already seen that the number of trios is 56, the same as the number of quintuples. Not only does each trio have a quintuple for its base but each quintuple is the base of a trio.

The 35 lines of S are separated by a trio into 4 disjoint batches:

(i) the base quintuple Q, (ii) 10 lines of σ_1,

(iii) 10 lines of σ_2, (iv) 10 lines of σ_3.

Each of these last three batches has two of its lines through any point P of S, their plane being the null plane of P in the appropriate screw. Let m_1, m_2, m_3 be any three lines of Q. Were two members of their associated triple to belong to the same screw of the trio so would the third because the three planes through m_1 are null planes, in any screw to which m_1 belongs, of the three points on m_1. The 15 lines of the screw would then consist of the two associated triples and of one line through each of their 9 intersections, and this cannot occur for any of $\sigma_1, \sigma_2, \sigma_3$ because the two other lines of Q have to belong to it. Hence the members of the triple associated with m_1, m_2, m_3 belong one to each of $\sigma_1, \sigma_2, \sigma_3$; since reciprocation in any of these leaves every line of Q unchanged, the two members of the associated triple that do not belong to one of the screws are polars in regard to it. All this holds for each of the ten triples included in Q.

The polars in σ_1 of the lines (iii) are the lines (iv) and so, since σ_1 is linearly dependent both on the sheaves whose axes are any pair of polar lines and on the other two screws of any trio of which it is a member, the screw dependent on σ_2 and a line (iii) is also dependent on σ_3 and a line (iv). Thus 10 screws are syzygetic both to σ_2 and σ_3. The other 5 screws syzygetic to σ_2 are azygetic to σ_3, the other 5 screws syzygetic to σ_3 are azygetic to σ_2. Hence there are

$$26-10-5-5 = 6$$

screws azygetic to both σ_2 and σ_3. One of these is σ_1, and there are 5 others.

14. Any screw σ belongs to 6 trios each of which has a base quintuple in σ. It is not possible for two of these quintuples to have two lines m, n in common: m and n belong only to two quintuples, namely those that arise on adding to m and n the two associated triples, and it has just been seen that m, n and their associated triples cannot belong all to the same screw. It follows, since there are 15 lines in σ and no more, that every pair of the 6 quintuples share one line, and indeed that the lines of σ are determined as common one to each of the ${}^6C_2$ pairs of these quintuples.

Take, as an example, for σ

$$p_{23}+p_{14}+p_{31}+p_{24}+p_{12}+p_{34} = 0.$$

Since an even number of the p_{ij} must be 1 while $\sum p_{23}p_{14} = 0$ the coordinates of the 15 lines of this screw are as under:

p_{23}	p_{14}	p_{31}	p_{24}	p_{12}	p_{34}		
.	.	1	.	1	.	II	III
.	.	1	.	.	1	I	III
.	.	.	1	1	.	II	IV
.	.	.	1	.	1	I	IV
1	.	.	.	1	.	I	V
.	1	.	.	1	.	I	VI
1	.	.	.	.	1	II	V
.	1	.	.	.	1	II	VI
1	.	1	.	.	.	IV	VI
1	.	.	1	.	.	III	VI
.	1	1	.	.	.	IV	V
.	1	.	1	.	.	III	V
.	.	1	1	1	1	V	VI
1	1	.	.	1	1	III	IV
1	1	1	1	.	.	I	II

The duads of roman numerals on the right show how these lines are distributed among 6 quintuples. Since intersecting lines never belong to the same quintuple those three of the above 15 lines that pass through any point of S must, through the three pairs of quintuples to which they belong, together account for all 6 quintuples; thus each point of S corresponds to a syntheme of the 6 quintuples, and all 15 synthemes are thus accounted for. Here one may again refer to (2*b*), 114.

15. Let
$$m_1,\ m_2,\ m_3,\ m_4,\ m_5$$
be a quintuple Q of lines of a screw σ. The screw that is linearly dependent on σ and the sheaf whose axis is m_i is, as explained in § 10, syzygetic to σ; we now show that the 5 screws so determined are all azygetic to each other. For suppose that σ is given by (8.1); the screws syzygetic to σ and determined by m_i and m_j are
$$\sum \{(a+m_{14}^{(i)})p_{23}+(a'+m_{23}^{(i)})p_{14}\} = 0,$$
$$\sum \{(a+m_{14}^{(j)})p_{23}+(a'+m_{23}^{(j)})p_{14}\} = 0,$$
and, since both m_i and m_j satisfy (8.1),
$$\sum \{(a+m_{14}^{(i)})(a'+m_{23}^{(j)})+(a'+m_{23}^{(i)})(a+m_{14}^{(j)})\} = \varpi(m_i, m_j) = 1.$$
Further: these 5 mutually azygetic screws are all azygetic to both screws, other than σ, of the trio based on Q. For, if either of these be
$$\sum (Ap_{23}+A'p_{14}) = 0,$$
then
$$\sum \{(a+m_{14}^{(i)})A'+(a'+m_{23}^{(i)})A\}$$
$$= \sum (aA'+a'A)+ \sum (Am_{23}^{(i)}+A'm_{14}^{(i)}) = 1+0 = 1.$$
We have thus obtained a heptad $\mathfrak{H}$ of 7 mutually azygetic screws. Any 2 screws of $\mathfrak{H}$, being azygetic, share a quintuple; each screw of $\mathfrak{H}$ shares with the others 6 quintuples any 2 of which have a common line. Three screws of $\mathfrak{H}$ share a single line and no line can belong to more than 3 screws of $\mathfrak{H}$; indeed the 35 lines of S are obtainable one from each of the ${}^7C_3 = 35$ sets of 3 screws of $\mathfrak{H}$. Note that 3 screws of $\mathfrak{H}$, although mutually azygetic, do not constitute a trio because they are not linearly dependent. The 21 screws extraneous to $\mathfrak{H}$ are those which complete trios with the 21 pairs of screws that belong to $\mathfrak{H}$. It is not possible for 2 pairs among these 21 to be enlarged to trios by the same screw; this would imply that 4 screws of $\mathfrak{H}$ were linearly dependent, and their being mutually azygetic prevents this. Whenever $\sigma_1+\sigma_2+\sigma_3+\sigma_4 \equiv 0$ then $\varpi_{11}+\varpi_{12}+\varpi_{13}+\varpi_{14} = 0$ and, ϖ_{11} being identically zero, this cannot hold if $\varpi_{12} = \varpi_{13} = \varpi_{14} = 1$. Similar reasoning serves to show that no 5 or 6 screws of $\mathfrak{H}$ can be linearly dependent: the identity $\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5 = 0$ cannot hold because
$$\varpi_{16}+\varpi_{26}+\varpi_{36}+\varpi_{46}+\varpi_{56} = 1.$$

16. It is clear from the construction of $\mathfrak{H}$ that there is a heptad containing any given pair of azygetic screws σ_1 and σ_2; their common quintuple can serve as base for the construction, the part of σ being played by $\sigma_1+\sigma_2$.

Moreover we saw, at the end of § 13, that there are, apart from the screw which completes the trio, 5 screws simultaneously azygetic to σ_1 and σ_2. These can only be the other members of $\mathfrak{H}$, which is thus uniquely determined by σ_1 and σ_2. Now there are 168 pairs of azygetic screws, and 21 of these pairs occur in $\mathfrak{H}$; hence there are 8 heptads. Since two azygetic screws determine a unique heptad no two heptads can share more than one screw; they must, however, always share one because there are only 28 screws to furnish the 8 heptads. Indeed the 8 heptads identify the screws, one screw being common to each of the ${}^8C_2$ pairs of heptads. The heptads are cardinal features of the figure and essential to an understanding of its geometry.

An instance of 7 screws that form a heptad is the following:

$$\begin{array}{lll}
p_{23}+p_{14}+p_{31}+p_{24}+p_{12}+p_{34} & = 0 & \\
p_{23}+p_{14} \qquad\qquad\quad +p_{12} & = 0 & \mathrm{I} \\
p_{23}+p_{14} \qquad\qquad\qquad\qquad +p_{34} & = 0 & \mathrm{II} \\
p_{23} \qquad +p_{31}+p_{24} & = 0 & \mathrm{VI} \\
\qquad p_{14}+p_{31}+p_{24} & = 0 & \mathrm{V} \\
\qquad\qquad p_{31} \qquad +p_{12}+p_{34} & = 0 & \mathrm{III} \\
\qquad\qquad\qquad p_{24}+p_{12}+p_{34} & = 0 & \mathrm{IV}
\end{array}$$

The first of these screws has been considered in § 14; the roman numerals which appear here to the right of the equations of the other six signify, with the same numeration as in § 14, those quintuples of the first screw that belong respectively to these other six. A second heptad that includes the first screw is found at once on replacing each of these other six screws by the screw which completes a trio with it and the first. The equation of the new screw is got by adding those of the other two screws in the trio, and so we find the heptad

$$\begin{array}{lll}
p_{23}+p_{14}+p_{31}+p_{24}+p_{12}+p_{34} & = 0 & \\
\qquad\qquad p_{31}+p_{24} \qquad +p_{34} & = 0 & \mathrm{I} \\
\qquad\qquad p_{31}+p_{24}+p_{12} & = 0 & \mathrm{II} \\
\qquad p_{14} \qquad\qquad +p_{12}+p_{34} & = 0 & \mathrm{VI} \\
p_{23} \qquad\qquad\qquad\quad +p_{12}+p_{34} & = 0 & \mathrm{V} \\
p_{23}+p_{14} \qquad +p_{24} & = 0 & \mathrm{III} \\
p_{23}+p_{14}+p_{31} & = 0 & \mathrm{IV}
\end{array}$$

Since each of the 7 screws of a heptad can be used as a pivot in this way we obtain 7 more heptads from a given one. All heptads are hereby accounted for.

A heptad corresponds to a set of 7 period characteristics

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$$

all of which are odd and every two azygetic, and in this context the 8 heptads were found long ago; they are displayed on pp. 308–9 of (**1**), the notation there serving to identify the appropriate odd period characteristics on pp. 305–6. The two heptads that we have displayed above are the sixth and third of the seven that appear on p. 309 of Baker's book.

17. Since each screw is determined by the pair of heptads to which it belongs we label the screws by binary symbols; (ij) signifies the screw common to $\mathfrak{H}_i$ and $\mathfrak{H}_j$ where i, j are two among the digits 0, 1, 2, 3, 4, 5, 6, 7. Screws whose symbols share a digit are in the same heptad and therefore azygetic; all 12 screws azygetic to (ij) are accounted for by the 6 others in $\mathfrak{H}_i$ and the 6 others in $\mathfrak{H}_j$. Screws whose symbols do not share a digit are syzygetic.

Four mutually syzygetic screws must, by their four binary symbols, account for all 8 digits; an example is (01), (23), (45), (67). There are 105 sets of this type and they answer, in accordance with § 11, one to each pencil of lines.

The only screws azygetic to (ij) and not in $\mathfrak{H}_j$ are in $\mathfrak{H}_i$; hence there is only one screw, namely (ik), azygetic to both (ij) and (jk) and yet not in $\mathfrak{H}_j$. Thus

$$(ij), \quad (jk), \quad (ki)$$

constitute a trio and we thereby account for all ${}^8C_3 = 56$ trios, each with its base quintuple. We therefore label any trio, or its base quintuple, by a ternary symbol (ijk). Quintuples whose symbols share two digits both belong to the same screw symbolized by these two digits and so share a single line. The line shared by (123) and (124) belongs to

$$(23) \quad (31) \quad (12) \quad (14) \quad (24) \quad (34);$$

to the first five of these obviously, and to (34) because (34) completes a trio with (31) and (14), or with (23) and (24), and so includes any line common to them. The same line belongs also to (134) and (234); it is the only line common to the six screws because two skew lines belong to only two quintuples. Each of the six screws is syzygetic to one other, and the axis of the sheaf determined by such a pair of syzygetic screws is the line common to all six. Take, for instance, (23) and (14); the resulting axis belongs to both of them. But since (23) and (14) are both azygetic to (31) the sheaf is syzygetic to (31) and the axis of the sheaf belongs to (31), and similarly to the other screws. We have, however, seen in § 11 that this axis belongs also to

$$(56) \quad (67) \quad (75) \quad (50) \quad (60) \quad (70),$$

so that the line common to these six screws is the same as that common to the former six. It may be labelled by either of the symbols $(1234) = (5670)$;

each of the $35 = \frac{1}{2}{}^8C_4$ lines of S is identified by a separation of the 8 heptads into complementary sets of 4. The 8 quintuples which include a line m are got by omitting any one of the digits from either of the two equivalent quaternary symbols. These 8 quintuples fall into opposite sets of 4; while two quintuples in the same set share only m two in the opposite sets also share a line skew to m. The 16 lines skew to m are all accounted for in this way. The 12 screws which include m are got by omitting any pair of the four digits from either of the two equivalent quaternary symbols. The 12 screws fall into opposite sets of six, each screw being syzygetic to all six in the opposite set and to one in its own set.

Intersecting lines m, n cannot both belong to any quintuple; hence the four digits in either quaternary symbol for m must occur two in each symbol for n. When n does not belong to (ij) i and j cannot both be present in either symbol for n and so must occur one in each symbol. The polar line n' is then got by transposing i and j. Take, to fix ideas, (17) and let n be (1234); then n' is (2347). For if a line of (17) meets (1234) one of its symbols is $(17\times\times)$ where one cross is one of 2, 3, 4 and the other cross not, and this line also meets (2347). A pair of polar lines for (ij) thus answers to the separation of the six digits other than i and j into complementary triads, and the ten such separations yield the ten pairs of polar lines.

18. It is, then, small wonder that Γ, the group of $\frac{1}{2}.8!$ projectivities in S, is isomorphic to $\mathfrak{A}_8$. Any projectivity turns screws into screws, azygetic screws into azygetic screws, trios into trios, quintuples into quintuples, heptads into heptads. Γ is a permutation group on the heptads. Should every heptad be invariant for a projectivity so must every screw, as common to two heptads, be invariant, as must every quintuple and every line of each screw, and therefore every point and plane of S; the projectivity can only be the identity operation. Hence Γ imposes a group of $\frac{1}{2}.8!$ distinct permutations on the 8 heptads and must be isomorphic to $\mathfrak{A}_8$. This geometrical discussion has not only demonstrated the fact of the isomorphism but has disclosed what is surely its *raison d'être*.

If we amplify Γ by adjoining to it a reciprocation in one of the screws the resulting group of order 8! is isomorphic to $\mathfrak{S}_8$ and imposes all permutations on the heptads. For let us reciprocate in (ij). Since each pair of polar lines is transposed so, by § 13, are the members (ik) and (jk) of any one of the 6 trios to which (ij) belongs. Thus $\mathfrak{H}_i$ and $\mathfrak{H}_j$ are transposed while every other heptad $\mathfrak{H}_k$ is unaltered. Reciprocations in the 28 screws impose the 28 transpositions on the heptads and generate $\mathfrak{S}_8$.

Since Γ, as the group $\mathfrak{A}_8$, is certainly doubly transitive on the heptads it is transitive on the screws: hence there are 6! projectivities of Γ for which a given screw σ is invariant. This subgroup of Γ is a permutation group on those 6 quintuples in σ that are bases of the trios to which σ belongs; whenever all these quintuples are invariant so is every line of σ, as common to two of them, and so therefore is every point of S. Thus any projectivity that leaves the 6 quintuples all invariant must be the identity; the 6! projectivities of Γ for which σ is invariant impose the whole symmetric group of permutations on the quintuples.

A group of projectivities for which a linear complex is invariant is a *symplectic group*, so that this symplectic group (in three dimensions and over F) is isomorphic to $\mathfrak{S}_6$. This has been proved by Jordan (**9**, 237, 240) and by Dickson (**5**, 99), but the above proof seems simpler and more natural. The ten pairs of lines that are polars of one another in σ afford another representation of $\mathfrak{S}_6$ as a permutation group, this time of degree ten.

19. Γ permutes the planes, as it does the points, of S transitively, and so possesses two conjugate sets each of 15 subgroups of order

$$\tfrac{1}{2}.8!\div 15 = 1344;$$

a subgroup of one set consists of the projectivities that leave a given plane invariant, one of the other set consists of projectivities that leave a given point invariant.

Let us, for example, stipulate that the point $x = y = z = t$ is invariant; then each row of the matrix imposing a projectivity must have its four elements summing to 1. Such a row must include either one or three zeros, and there are eight such rows. Once we have chosen a row there are seven choices for the second and six for the third; the sum of the first two rows is a row summing to 0 and so not among those from which the third is chosen. When the first three rows have been chosen all of them, as well as their sum, are debarred from the last row by the prescription of non-singularity, so that only four choices are possible and the total number of matrices is $8.7.6.4 = 1344$. This is a subgroup of one conjugate set; one of the other set is afforded by those projectivities, for which $x+y+z+t = 0$ is invariant, which are imposed by non-singular matrices each of whose columns sums to 1.

Although a plane π is invariant for a group G of 1,344 projectivities we saw in § 2 that only 168 different projectivities can be induced in π. Each of these is induced by 8 different projectivities of G. Those 8 which induce identity in π form a self-conjugate subgroup A of G, and each of the 168 cosets of A in G consists of 8 operations that induce the same projectivity in π. Suppose, for example, that π is $x+y+z+t = 0$: which non-singular matrices, each of whose columns sums to 1, leave invariant not only the whole plane but every point in it? It is enough if some three non-collinear points in π are each invariant, and by remarking the effect of premultiplying such column vectors as $(1,1,0,0)'$, $(1,0,1,0)'$, and so on, we see that the three non-diagonal elements in any row must be equal to each other but not to the diagonal element. The matrices therefore are (cf. § 4)

$$M(a,b,c,d) \equiv \begin{bmatrix} a+1 & a & a & a \\ b & b+1 & b & b \\ c & c & c+1 & c \\ d & d & d & d+1 \end{bmatrix}$$

wherein $a+b+c+d = 0$. Since three of a, b, c, d may be chosen to be either 0 or 1 the number of such matrices is $2^3 = 8$. Direct multiplication shows at once that

$$M(a_1,b_1,c_1,d_1)M(a_2,b_2,c_2,d_2) = M(a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2);$$

the matrices form an abelian group and are all, save $M(0,0,0,0) = I$, of period 2.

Projectivities which leave the plane $t = 0$ invariant are imposed by non-singular matrices which, when they postmultiply $(0,0,0,1)$, leave it unchanged. Such matrices are of the form

$$\begin{bmatrix} & \mathbf{N} & & \alpha \\ & & & \beta \\ & & & \gamma \\ . & . & . & 1 \end{bmatrix} \tag{19.1}$$

where $\mathbf{N}$ is one of the 168 matrices of K; since there are 168 choices for $\mathbf{N}$ and two for each of α, β, γ we have the 1,344 projectivities for which $t = 0$ is invariant. The 1,344 matrices have the group property; those for which not only the whole plane $t = 0$ but every point of it is invariant have $\mathbf{N} = I$, and we have another representation of the self-conjugate abelian subgroup. There are analogous statements concerning projectivities for which any one face of the tetrahedron of reference is invariant, as there are, but with the matrices now transposed, concerning projectivities for which any one vertex of the tetrahedron of reference is invariant.

20. The group G of order 1,344 was found by Mathieu (**11**, 290) as a triply transitive permutation group of degree 8. If we take it to be the subgroup of operations of Γ which leave a plane π invariant we represent it (not only as a permutation group of the 8 heptads but also) as a permutation group of the 8 points of S outside π. The first set of 7 functions on p. 291 of (**11**) then answers to the 7 pairs of planes that pass one pair through each line of π, while the second set of 7 functions answers to the 7 sets of 4 concurrent lines, one set concurring at each point of π. That G is then triply transitive follows because those of its operations which leave invariant not only π but also one point V outside π form a Klein group doubly transitive, by § 2, on the points of π; any two points other than V outside π are now permuted like the intersections with π of the lines joining them to V. If, for instance, π is $t = 0$, so that G consists of the matrices (19.1), the point $x = y = z = 0$ outside π is not invariant unless $\alpha = \beta = \gamma = 0$, and we obtain the matrices $\mathsf{N} \dot{+} 1$ of a Klein group. Incidentally, by taking π as $x+y+z+t = 0$ and V to be a vertex of the tetrahedron of reference, we see that those non-singular matrices all of whose columns sum to 1 constitute a Klein group whenever one particular column is constrained to have its three non-diagonal elements all zero.

Mathieu observes (**11**, 292) that there are two distinct types of subgroups of order 168 in G, both of them doubly transitive on those 8 objects on which G is triply transitive; and describes this circumstance as *remarquable*. But in the light of the geometry in S it loses some of the surprise it might otherwise create. One of the two types is the Klein group and Mathieu gives, at the foot of p. 292, its representation as a congruence group, modulo 7, of bilinear transformations. The other type is formed by 21 cosets of A in G whenever their operations induce in π the projectivities of one of the maximal subgroups, say k, of order 21 in K. The matrices (19.1) form such a subgroup when N is restricted to the 21 matrices of k, say to those of the group generated by M and τ in § 5. This subgroup is doubly transitive on the 8 points outside $t = 0$ because those of its operations which leave one of these points V invariant are transitive on the 7 points in $t = 0$; if, for instance, V is $x = y = z = 0$ then $\alpha = \beta = \gamma = 0$ and we obtain the group of order 21 which was remarked in § 5 to be transitive.

21. The next appearance of G seems to be in (**9**) and there, on p. 305, its self-conjugate abelian subgroup A appears too. Jordan's notation is equivalent to using the matrices (19.1), and he gives these 1,344 matrices again on p. 380. His procedure in deriving A on p. 305 is tantamount to replacing N by I.

G and A then appear in (**13**), and Mathieu's two sets of 7 functions occur with them,† on pp. 93–95. The geometry in S throws some light on this paper of Noether's. Noether's symbol [01, 23, 45, 67] can be interpreted as a set of four mutually syzygetic screws (01), (23), (45), (67) wherein a plane π has the same pole P. Noether's problem of listing seven symbols which together account for all 28 pairs ij has thus two salient solutions: we may take either those seven sets of four syzygetic screws wherein π has the seven points in it as poles or, dually, those seven sets of four syzygetic screws wherein P has the seven planes through it as polars.

† They occur again, 20 years later, in (**12**), 432, where G and A also reappear.

Let P be the pole of π in each of

$$(01),\quad (23),\quad (45),\quad (67). \tag{A}$$

Any other point P' of π is its pole in four syzygetic screws none of which occurs in (A); we may suppose (02) to be one of these. Since (01), (02), (12) are a trio the pole of π in (12) is P'', the third point on PP'; then, since (12), (23), (31) are a trio, P' is the pole of π in (31). Indeed we may choose the notation so that P' is the pole of π in

$$(02),\quad (31),\quad (64),\quad (57), \tag{B}$$

whereupon P'' is the pole in

$$(12),\quad (03),\quad (56),\quad (47). \tag{C}$$

The sets A, B, C together constitute what Noether calls a *Tripel*; the 7 *Tripeln* yielded one by each line of π constitute a *Tripelsystem*. It is determined by π, and Noether's 30 such systems consist of 15 determined by the planes and 15 determined dually by the points of S.

22. The isomorphism between Γ and $\mathfrak{A}_8$ affords a geometrical representation of $\mathfrak{A}_8$ that renders many of its properties almost visually obvious. Take this example, which we now transcribe and solve, from (**3**, 230).

Ex. 2. Show that the alternating group of degree 8 contains 30 regular Abelian subgroups of order 8 and type (1, 1, 1), forming two conjugate sets of 15 subgroups each.

If H_1, H_2 are any two subgroups belonging to the same conjugate set of 15, prove that $\{H_1, H_2\}$ is a subgroup of order $2^6.3^2$, permuting the symbols in 2 imprimitive systems of 4 each; and that $\{H_1, H_2\}$ contains just one other subgroup H_3 belonging to the same set. Hence show that from the 15 conjugate subgroups a complete set of 35 triplets may be formed, which is invariant when the subgroups are transformed by any operation of the alternating group. Prove also that when the subgroups of the second set of 15 are transformed by the operations of H, 7 are transformed into themselves and the other 8 are permuted regularly.

The abelian subgroups consist either of projectivities for which every point of some plane in S is invariant or else of projectivities for which every plane through some point of S is invariant, and since Γ is transitive on either the points or the planes of S the 15 subgroups of either kind are conjugate. But subgroups of different kinds are not.

Let H_1 be the subgroup for which every point of a plane π_1 is invariant, H_2 that for which every point of a plane π_2 is invariant. Then $\{H_1, H_2\}$ is the subgroup for which every point of the line n common to π_1 and π_2 is invariant; since all three points of n are invariant whenever two of them are, and since Γ is doubly transitive on the points of S, the order of $\{H_1, H_2\}$ is $\frac{1}{2}.8!\div(15\times 14) = 2^5.\dot{3}$. Burnside's $2^6.3^2$ seems to be a slip; a group of this larger order occurs when the points of n undergo permutation and its order is $\frac{1}{2}.8!\div 35$ because Γ is transitive on the lines of S. However, the fact that the symbols are permuted 'in 2 imprimitive systems of 4 each' is true of this larger group as well as of its self-conjugate subgroup $\{H_1, H_2\}$; Burnside's 'symbols' label the 8 objects that are permuted, and we have seen that n may be denoted, say, by $(1234) \equiv (5670)$. H_3 is, of course, the subgroup of projectivities for which every point of π_3, the other plane through n, is invariant, and there are 35 such triplets of subgroups, one for each line of S.

Suppose that π_1 is $y = 0$ and π_2 $z = 0$; then matrices of the forms

$$\begin{bmatrix} 1 & \alpha & . & . \\ . & 1 & . & . \\ . & \beta & 1 & . \\ . & \gamma & . & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & . & \delta & . \\ . & 1 & \epsilon & . \\ . & . & 1 & . \\ . & . & \zeta & 1 \end{bmatrix}$$

impose the projectivities of H_1 and H_2 respectively. Each set consists of 8 commuting matrices, all 7 other than I having I for their square; in other words they form an abelian group of order 8 and type $(1, 1, 1)$. The group generated by these two groups of matrices consists of matrices

$$\begin{bmatrix} 1 & \times & \times & . \\ . & \times & \times & . \\ . & \times & \times & . \\ . & \times & \times & 1 \end{bmatrix}$$

where each cross denotes either 0 or 1. The choice of these 8 elements, however, is not completely free because the central block of four has to be non-singular and so one of the six matrices (1.2); we thus have a group of order $6.2^4 = 2^5.3$. This imposes the projectivities for which every point on $y = z = 0$ is invariant. If, on the other hand, we allow these three points to undergo permutations among themselves, then the four corner elements can form any one of the six non-singular matrices and need not be the unit matrix: this gives the group of order $2^6.3^2$ for which $y = z = 0$ is invariant. The subgroup H_3, for which every point of $y+z = 0$ is invariant, consists of the projectivities imposed by the eight matrices

$$\begin{bmatrix} 1 & \eta & \eta & . \\ . & \multicolumn{2}{c}{\multirow{2}{*}{J}} & . \\ . & & & . \\ . & \theta & \theta & 1 \end{bmatrix}$$

where J is either $\begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix}$ or $\begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix}$.

Let H be the abelian subgroup for which every point of π is invariant. The subgroups of the other conjugate set are associated one with each point of S; 7 of these points lie in π and are invariant under H and so these 7 subgroups are all transformed into themselves by H. The other 8 points are outside π, and Burnside's statement is established by remarking that any operation of H except identity transposes these 8 points in pairs. It must do this because its period is 2, and an operation which leaves invariant some point outside π as well as every point in π leaves every point of S invariant. If an operation of H transposes P and P' then PP' meets π in a point O and the other 6 points outside π lie 2 on each of 3 lines through O and are transposed accordingly.

The geometry also handles expeditiously the properties of $\mathfrak{A}_8$ on pp. 456–7 of (3), properties which appear also in Carmichael (4); see the examples therein on pp. 320–1. Moreover, Carmichael proceeds, in the chapters which follow, to discuss finite geometries and their groups of projectivities; the references relevant to $\mathfrak{A}_8$ and Γ are 336–7 (exx. 4–10), 351 (ex. 1), 353 (ex. 4), 394 (exx. 5–7).

23. We have considered at length an example on p. 230 of (**3**). In the next example on this page Burnside gives a group of order 192; this is indeed a subgroup of G and has been since obtained by Littlewood (**10**, 159) and Todd (**14**). We close by showing how this group too is conspicuous in the geometry, as is also a different type of subgroup of G of this order that was found by Todd, and we give, in addition to geometrical definitions of these groups, matrix representations for them.

G leaves a plane π invariant; the group of Burnside's example is the subgroup of G consisting of those operations which leave invariant not only π but also some point of π, while the group of the same order found by Todd consists of those operations of G which leave invariant not only π but also some line of π. G contains a conjugate set of 7 subgroups of either type.

The first type of subgroup contains, as Burnside says in his example, a self-conjugate operation of period 2. Suppose that π is $x+y+z+t = 0$; G, as remarked in § 19, consists of all those non-singular matrices wherein every column sums to 1. Those of its operations for which $x = y = z = t$, a point in π, is invariant are imposed by matrices wherein not only every column but also every row sums to 1. Every such matrix commutes with

$$\begin{bmatrix} . & 1 & 1 & 1 \\ 1 & . & 1 & 1 \\ 1 & 1 & . & 1 \\ 1 & 1 & 1 & . \end{bmatrix},$$

and this is of period 2. If, on the other hand, π is $t = 0$ then G consists of the matrices (19.1); but the point $y = z = t = 0$ of π is not invariant except for matrices of the form

$$\begin{bmatrix} 1 & \delta & \epsilon & \alpha \\ \cdot & & & \beta \\ & & \mu & \\ \cdot & & & \gamma \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \tag{23.1}$$

wherein μ, being non-singular, is one of the 6 matrices (1.2); the number of matrices of this kind is $6.2^5 = 192$, and they all commute with

$$\begin{bmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

whose period is 2. On the other hand, those matrices (19.1) for which $x = t = 0$ is invariant have the form

$$\begin{bmatrix} 1 & \cdot & \cdot & \alpha \\ \delta & & & \beta \\ & & \mu & \\ \epsilon & & & \gamma \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \tag{23.2}$$

and so constitute a group T of order 192; it is this type of group that was found by Todd, and it has no self-conjugate operation save identity. Some features of T are patent. When given by (23.2) it permutes the points outside $t = 0$ in two imprimitive sets of 4, one set lying in $x = 0$ and the other in $x+t = 0$. These two sets are transposed if $\alpha = 1$; the matrices with $\alpha = 0$ do not transpose the sets, and constitute a self-conjugate subgroup of T of order 96. Another self-conjugate subgroup of order 96 arises on restricting μ to the first 3 matrices of (1.2), say $\mu = \mu^+$, so that the points on $x = t = 0$ undergo only even permutations. The intersection of these two subgroups is self-conjugate and of order 48; for this both $\mu = \mu^+$ and $\alpha = 0$. Other self-conjugate subgroups of T are one of order 32, wherein $\mu = I$, and one of order 16, wherein both $\mu = I$ and $\alpha = 0$. The occurrence of these self-conjugate subgroups accords with the character table of T (**14**, 150, Table A; this table is original with Todd). The conjugate sets of T which make up these subgroups are those of the columns wherein the following sets of characters (in Todd's notation) have their components all equal to their degrees (**3**, 278, Theorem IV):

$$\psi^0, \psi^5; \quad \psi^0, \psi^8; \quad \psi^0, \psi^5, \psi^8; \quad \psi^0, \psi^3, \psi^8;$$
$$\psi^0, \psi^3, \psi^5, \psi^7, \psi^8, \psi^9.$$

The still larger set of characters

$$\psi^0, \psi^1, \psi^2, \psi^3, \psi^5, \psi^7, \psi^8, \psi^9, \psi^{12}, \psi^{13}$$

answers to a self-conjugate group of order 4. It consists of matrices

$$\begin{bmatrix} 1 & . & . & . \\ . & 1 & . & \beta \\ . & . & 1 & \gamma \\ . & . & . & 1 \end{bmatrix}$$

which induce the identity projectivity in $t = 0$ and do not transpose the planes $x = 0$ and $x+t = 0$.

Analogous discussions apply to the matrices (23.1) and the character table of this group (**14**, 150, Table B and **10**, 277; this table was found by Littlewood) and serve to underline the contrasts between it and T.

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Note added in Proof

Since this paper was finished I have found one by G. M. Conwell: *Annals of Mathematics* (2) 11 (1910), 60–76.

Conwell uses the Klein representation of the lines of S by the 35 points of a quadric Q in space of 5 dimensions, and shows that the 56 lines in this space that are (p. 67) skew to Q can be arranged as edges of 8 heptagons (p. 68). The 7 vertices of such a heptagon answer to mutually azygetic screws in S. Furthermore, Conwell denotes these 56 lines by ternary symbols (p. 69) and each line of S by a pair of quaternary symbols (p. 72). This is indeed the ideal apparatus for establishing the isomorphism between $LF(4, 2)$ and $\mathscr{A}_8$ and it is a pity that Conwell suddenly forsakes the geometry and follows in the wake of E. H. Moore.