THE CONTACT NET OF QUADRICS

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It is proposed to obtain, in the following pages, some properties of a net of quadric surfaces which is so specialised that its eight base points coincide in pairs; in other words, the quadrics of the net circumscribe a tetrahedron T and have a definite tangent line at each vertex of T. Such a net may perhaps fittingly be called a *contact net*, and it is so described in this paper.

Suppose then that A, B, C, D are the base points of a contact net $\mathfrak{C}$, with the fixed tangent lines α, β, γ, δ respectively. Four such lines are always tangents of a pencil, but not generally of a net, of quadrics. The figure cannot be set up arbitrarily, and the restriction to which it is subjected is expressed by the following

THEOREM. *A necessary and sufficient condition for the lines α, β, γ, δ to be tangent lines, at the vertices A, B, C, D of a tetrahedron T, of a net of quadrics, is that their respective intersections A', B', C', D' with the opposite faces of T form, with the vertices of T, a pair of Möbius tetrads.*

Thus it appears that the contact net is closely related to the Möbius net, and that until some account has been given of its properties the geometry of the Möbius configuration is not complete.

Now there is also an intimate relationship, between the contact net and the Möbius net, apart from that expressed by the theorem; for each net is derivable from the other by means of a simple Cremona transformation. Indeed, if one of the tetrads A, B, C, D of a Möbius configuration is taken to be the set of vertices of the tetrahedron of reference T for a set of homogeneous coordinates x, y, z, t, it is only necessary to replace these coordinates by their reciprocals; whereupon the tetrad A', B', C', D', whose points lie one in each of the faces of T, is made to coincide with the other, the transformation of A' being associated with a definite direction α through A, and so on. And the quadrics of the

Möbius net, passing through the eight points A, B, C, D, A', B', C', D', are transformed into quadrics which circumscribe T and have α, β, γ, δ as tangent lines at its respective vertices.

The properties of the Möbius net have been the subject of a preceding communication*, which will be referred to here as M.T.; so that material from which properties of $\mathfrak{C}$ are immediately deducible by means of this transformation, which will be denoted by $\mathfrak{S}$, lies ready to hand. Thus it is found, in § 2, that the cones of $\mathfrak{C}$ fall into two families (a fact which has been alluded to† before), and that the envelope of the cones of either of the two families is a quartic scroll. These two quartic scrolls are a cardinal feature of the configuration, which may be built up from either of them; this is the standpoint of §§ 4–6, where it is shown how the contact net plays an essential part in the geometry of the quartic scroll, and where, in § 5, it is emphasised that, through its agency, quartic scrolls occur in pairs.

Yet, although properties of the contact net are so readily obtainable from those of the Möbius net once these latter are on record, it should not be forgotten that the contact net exists in its own right. So, before introducing the Möbius net, we give a proof of the theorem from first principles which displays its algebraical content to advantage. This is not pursued further; but, the theorem having been established independently, the canonical form I can be obtained as a corollary and properties of the net derived without the intervention of the Cremona transformation.

1. Take T as tetrahedron of reference for a system of homogeneous coordinates x, y, z, t; then the equations of α, β, γ, δ have the respective forms

$$\frac{y}{m_1}=\frac{z}{n_1}=\frac{t}{p_1};\quad \frac{z}{n_2}=\frac{t}{p_2}=\frac{x}{l_2};\quad \frac{t}{p_3}=\frac{x}{l_3}=\frac{y}{m_3};\quad \frac{x}{l_4}=\frac{y}{m_4}=\frac{z}{n_4}.$$

The matrix

$$M=\begin{pmatrix} \cdot & m_1 & n_1 & p_1 \\ l_2 & \cdot & n_2 & p_2 \\ l_3 & m_3 & \cdot & p_3 \\ l_4 & m_4 & n_4 & \cdot \end{pmatrix}$$

which is formed from the direction ratios of these four lines has, for the

* "The net of quadric surfaces associated with a pair of Möbius tetrads", *Proc. London Math. Soc.* (2), 41 (1936), 337–360.

† *Acta Mathematica*, 64 (1934); see pp. 237 and 238.

constituents of any one row, the coordinates of one of the four points A', B', C', D'. The vanishing of the four principal elements in M expresses the fact that these four points lie one in each of the faces of T. If they form a pair of Möbius tetrads with the vertices of T, the plane of any three of them passes through a vertex of T; the conditions for this to happen are that the four principal minors of M should vanish. Thus, in order to prove the theorem, the vanishing of these principal minors must be established.

Let the equations of three linearly independent quadrics which circumscribe T be

$$q_i \equiv f_i yz + g_i zx + h_i xy + u_i xt + v_i yt + w_i zt = 0 \quad (i = 1, 2, 3).$$

If every quadric $\xi q_1 + \eta q_2 + \zeta q_3 = 0$ touches α at A, then

$$m_1 h_i + n_1 g_i + p_1 u_i = 0 \quad \text{for} \quad i = 1, 2, 3;$$

hence the determinant (hgu) vanishes, its columns being linearly dependent according to the multipliers m_1, n_1, p_1. Corresponding considerations for the other vertices of T show that, when the quadrics of the net all have α, β, γ, δ for tangent lines at A, B, C, D, the relations

$$(hgu) = (fhv) = (gfw) = (uvw) = 0 \tag{1.1}$$

must be satisfied. The three columns of any one of these determinants are linearly dependent according to multipliers which occur in one of the rows of M, and all these relations of linear dependence are given by the matrix equation

$$\begin{pmatrix} f_1 & g_1 & h_1 & u_1 & v_1 & w_1 \\ f_2 & g_2 & h_2 & u_2 & v_2 & w_2 \\ f_3 & g_3 & h_3 & u_3 & v_3 & w_3 \end{pmatrix} \begin{pmatrix} \cdot & n_2 & m_3 & \cdot \\ n_1 & \cdot & l_3 & \cdot \\ m_1 & l_2 & \cdot & \cdot \\ p_1 & \cdot & \cdot & l_4 \\ \cdot & p_2 & \cdot & m_4 \\ \cdot & \cdot & p_3 & n_4 \end{pmatrix} = 0.$$

Let this be written

$$FL = 0.$$

We now appeal to the known fact* that the sum of the ranks of F

* If X denotes a column vector of six elements, the equation $FX = 0$ has three linearly independent solutions. Thus a matrix such as L, formed by juxtaposition of solutions X, has rank 3 at most. Its rank is actually equal to 3 when its columns include a *complete* solution of $FX = 0$, and this the columns of L actually do. The distribution of zero elements in L makes it manifest that no three of its columns can be linearly dependent.

and L is 6 (the number of columns in F and of rows in L). Since the three quadratic forms q_i are presumed to be linearly independent, the rank of F is 3; wherefore the rank of L must also be 3. Hence every determinant of four rows and columns which occurs in L must vanish.

The vanishing of these determinants yields the desired conditions. For such a determinant is obtained when any two rows of L are omitted; in particular, two of the three rows containing the non-vanishing constituents of any one column may be omitted, and so it follows that four three-rowed determinants which, upon examination, are seen to be the four principal minors of M, must vanish.

It only remains to say, in order to complete the proof of the theorem, that not only do the principal minors of M vanish in consequence of L being of rank 3, but that, conversely, the rank of L reduces to 3 provided that the four principal minors of M vanish. For these conditions, by the manner in which they were found, ensure that twelve of the fifteen four-rowed determinants of L vanish; that the remaining three also vanish in consequence is quickly seen on closer scrutiny.

A contact net $\mathfrak{C}$ gives rise, in consequence of this theorem, to a unique Möbius net; but the same Möbius net arises from eight different contact nets, corresponding to the eight different ways (*cf.* M.T., p. 349) in which the points of a Möbius configuration can be grouped as a pair of Möbius tetrads.

The derivation of a contact net from a Möbius net.

2. From the equations (1.1) of M.T., and the equations for the three quadrics on p. 341 of that paper, it is seen, on replacing the homogeneous coordinates by their reciprocals, that the quadrics of a contact net are linearly dependent on three quadrics whose equations may be taken to be

$$\left.\begin{aligned} Q_1 &\equiv m(zx+yt)+n(xy-zt) = 0,\\ Q_2 &\equiv n(xy+zt)+l(yz-xt) = 0,\\ Q_3 &\equiv l(yz+xt)+m(zx-yt) = 0.\end{aligned}\right\} \qquad \text{(I)}$$

This is a canonical form for the quadrics of $\mathfrak{C}$. The three quadrics $Q_1 = 0$, $Q_2 = 0$, $Q_3 = 0$ contain each a pair of opposite edges of T. The four quadrics

$$Q_2+Q_3-Q_1 = 0, \quad Q_3+Q_1-Q_2 = 0, \quad Q_1+Q_2-Q_3 = 0, \quad Q_1+Q_2+Q_3 = 0,$$

are cones, their vertices being A, B, C, D respectively; each cone contains the three edges of T which pass through its vertex.

The application of the transformation $\mathcal{S}$ to the Möbius net affords an instantaneous proof of the theorem. For, the base points of the Möbius net being given by the rows of

$$\begin{pmatrix} . & -n & m & -l \\ n & . & -l & -m \\ -m & l & . & -n \\ l & m & n & . \end{pmatrix},$$

the lines α, β, γ, δ which arise on applying the transformation meet the opposite faces of T in the points given by the rows of

$$\begin{pmatrix} . & -n^{-1} & m^{-1} & -l^{-1} \\ n^{-1} & . & -l^{-1} & -m^{-1} \\ -m^{-1} & l^{-1} & . & -n^{-1} \\ l^{-1} & m^{-1} & n^{-1} & . \end{pmatrix},$$

and this matrix is also of the standard form for the Möbius configuration.

A cone of the Möbius net becomes, when subjected to $\mathcal{S}$, a cone of $\mathfrak{C}$, the vertices of the two cones being corresponding points in the transformation. Thus those cones whose vertices lie on the line e (one of the two transversals, e and f, of AA', BB', CC', DD'; *cf.* M.T., § 6) become cones whose vertices lie on the curve which is the transformation of e, namely a twisted cubic k_1 circumscribing T. And the cones whose vertices lie on f become cones whose vertices lie on k_2, a second twisted cubic circumscribing T. The Jacobian curve of $\mathfrak{C}$ is therefore constituted by k_1 and k_2.

Again: the surface which is the envelope of the cones of the Möbius net whose vertices are on e is transformed into the surface which is the envelope of the cones of $\mathfrak{C}$ whose vertices are on k_1. Now (*cf.* M.T., § 7) the cones of the Möbius net whose vertices lie on e envelop a Plücker surface U having e as a nodal line and nodes at the eight points of the configuration. The transformation by $\mathcal{S}$ of any quartic surface which has nodes at the vertices of T is another quartic surface likewise having nodes at the vertices of T, so that, in particular, U is transformed into a quartic surface. This quartic surface must, since e is a nodal line on U, have the twisted cubic k_1, which is the transformation of e, for a nodal curve; it is therefore a scroll R_1^4, and the cones of $\mathfrak{C}$ whose vertices lie on k_1 are the tangent cones to this scroll from the points of its nodal curve. Those cones of $\mathfrak{C}$ whose vertices lie on k_2 likewise envelope a quartic scroll R_2^4, and are the tangent cones to this scroll from the points of its nodal curve. The envelope of the cones of $\mathfrak{C}$ consists of R_1^4 and R_2^4; these two scrolls, being both of class four, make up an envelope of total class eight, which is known to be the correct class for the envelope of the cones which belong to a net of quadrics.

3. Consider now the scroll generated by the trisecants of the Jacobian curve ϑ. Since ϑ here consists of k_1 and k_2, any trisecant must be a chord of one curve and a secant of the other; the question therefore arises, whether the line conjugate to a point of k_1 is a chord of k_1 and a secant of k_2, or whether it is a secant of k_1 and a chord of k_2. This is answered at once by appealing to a known property of ϑ; namely that the trisecants which are conjugate to the vertices of any tetrahedron which form a canonical set on ϑ lie in the opposite faces of this tetrahedron*. For, since (M.T., p. 342) a canonical set on the Jacobian curve of the Möbius net consists of two points on e and two points on f, a canonical set on the Jacobian curve of the contact net must consist of two points on k_1 and two points on k_2. The face of the tetrahedron, whose vertices are such a set of four points, which is opposite to one of the points P on k_1 has two of its three vertices on k_2; thus the remaining three intersections of this face with the Jacobian curve, which lie on the trisecant conjugate to P, can only include one point of k_2. So the trisecants conjugate to the points of k_1 are chords of k_1 and secants of k_2, while those conjugate to the points of k_2 are chords of k_2 and secants of k_1.

The scroll of order eight generated by the trisecants of the Jacobian curve thus consists of the quartic scroll K_1 generated by those chords of k_1 which meet k_2, and of the quartic scroll K_2 generated by those chords of k_2 which meet k_1. Those trisecants which are conjugate to the common points A, B, C, D of k_1 and k_2 are common generators of K_1 and K_2; these are in fact the lines $\alpha, \beta, \gamma, \delta$, since the polar plane of A with respect to any quadric of $\mathfrak{C}$ contains α, and similarly for the other vertices of T. The common curve of K_1 and K_2 consists of the four lines $\alpha, \beta, \gamma, \delta$ and of the two curves k_1 and k_2, both counted twice, since each curve is nodal on one scroll and simple on the other.

The derivation of a contact net from a quartic scroll.

4. If a surface F has a nodal curve g there are, on this nodal curve, a certain number of *pinch-points*; at any point of g there are two tangent planes to F, both passing through the tangent of g; the pinch-points are

* *Proc. London Math. Soc.* (2), 44 (1938), 467.

those points of g for which these two planes coincide. The tangent plane of F at a pinch-point meets the surface in a curve having a triple point there; of the three branches of the curve, two touch the tangent of g at the pinch-point while the third has another tangent, this latter being called by Cayley* the *cotangent*. The tangent cone to F from any point P (apart from the cone which projects g from P) touches F along the *polar curve*† of P, and this polar curve, whatever the position of P, passes through each pinch-point and touches the cotangent there.

Suppose now that F is a scroll. The two tangent planes of F at a point of its nodal curve are the planes which join the tangent of the nodal curve to the two generators which pass through the point. But, whereas through a general point of the nodal curve there pass two generators, through a pinch-point there passes only one. This is a torsal generator, and is the cotangent at the pinch-point. Hence the polar curve of any point P passes through the pinch-points and has the torsal generators for its tangents there.

When F is a quartic scroll R^4 its nodal curve is a twisted cubic k, and there are four pinch-points A, B, C, D; through these pass the respective torsal generators α, β, γ, δ. The polar curve of any point P is a sextic, touching α, β, γ, δ at A, B, C, D. If, however, P lies on the nodal curve, its polar curve includes the two generators through P; the remaining part is a quartic, having a node at P and touching α, β, γ, δ at A, B, C, D. The tangents to R^4 from P are the chords of this curve which pass through P, so that they generate a quadric cone which passes through A, B, C, D and has α, β, γ, δ for tangent lines there. That this cone of tangents is of the second order is also readily seen on taking a plane section through P, for there are two tangents to a trinodal plane quartic from one of its nodes.

There has thus arisen a singly-infinite set of quadric cones, namely the tangent cones to R^4 from the points of k, all of which pass through A, B, C, D and have α, β, γ, δ as tangent lines there. These cones must belong either to a contact net or to a pencil; in the latter event they would constitute the whole of the pencil. But there does not exist a pencil of quadric cones the locus of whose vertices is a twisted cubic; therefore the cones belong to a contact net $\mathfrak{C}$.

Since a quartic scroll gives rise in this way to a contact net, the theorem may be stated as a property (as yet apparently unnoticed) of a quartic

* *Collected Mathematical Papers*, 6, 335.

† This is the curve called the *pure polar* by Miss Hudson in *Cremona transformations* (Cambridge, 1927), 372, and the *proper curve of contact* by Baker in *Principles of geometry*, 6 (Cambridge, 1933), 150.

scroll:

The four points in which the torsal generators of a quartic scroll meet the faces of the tetrahedron whose vertices are the pinch-points form, when taken with the pinch-points, a pair of Möbius tetrads.

5. It is now evident that *quartic scrolls occur in pairs.* For let a quartic scroll R_1^4 be given. It has a nodal twisted cubic k_1, and the tangent cones to R_1^4 from the points of k_1 belong to a contact net $\mathfrak{C}$. The cones of $\mathfrak{C}$, however, fall into two families, of which this set of tangent cones of R_1^4 is but one; the Jacobian curve of $\mathfrak{C}$ consists of k_1 and of a second twisted cubic k_2 meeting k_1 at the base points of $\mathfrak{C}$ (*i.e.*, the pinch-points of R_1^4). The envelope of this second family of cones is a second quartic scroll R_2^4, which is thus obtainable when R_1^4 is given.

These two scrolls, together constituting the envelope of the cones of $\mathfrak{C}$, are symmetrically related to one another; R_1^4 may be derived from R_2^4 exactly as R_2^4 has been derived from R_1^4. The base points of $\mathfrak{C}$ are pinch-points, and the fixed tangent lines of $\mathfrak{C}$ are torsal generators, *on both scrolls.*

6. Let the nodal curve k of a quartic scroll R^4 be given by

$$x:y:z:t=\theta^3:\theta^2:\theta:1.$$

Then if P is the point on k whose parameter is θ, the equation of any quadric cone whose vertex is P is given by equating to zero some quadratic form in the three variables

$$x-\theta y, \quad y-\theta z, \quad z-\theta t.$$

If this cone is the tangent cone from P to R^4, the coefficients in the quadratic form depend only on R^4 itself, and do not vary with θ; thus the equation of this tangent cone has the form

$$\theta^2 S_0+2\theta S_1+S_2=0, \tag{6.1}$$

where S_0, S_1, S_2 are certain quadratic forms, fixed when R^4 is given in x, y, z, t. These cones clearly belong to a net of quadrics.

So much is clear without using the explicit equation of R^4; but now suppose that this equation is

$$(a, b, c, f, g, h\between zx-y^2, xt-yz, yt-z^2)^2=0. \tag{6.2}$$

Then the chord which joins the points θ_1 and θ_2 on k is a generator of R^4 provided that

$$(a, b, c, f, g, h \between \theta_1\theta_2, \theta_1+\theta_2, 1)^2 = 0. \tag{6.3}$$

The enveloping cone from P to R^4 is the envelope of those planes which join P to the generators of R^4. If P lies on k and has the parameter θ, the equation of the plane which joins it to the chord joining the points θ_1 and θ_2 of k is

$$x-y(\theta+\theta_1+\theta_2)+z(\theta\theta_1+\theta\theta_2+\theta_1\theta_2)-t\theta\theta_1\theta_2 = 0,$$

which may be written

$$\theta_1\theta_2(z-\theta t)+(\theta_1+\theta_2)(\theta z-y)+x-\theta y = 0.$$

The envelope of this plane, subject to (6.3), is, by clear analogy with the theory of conic sections, the quadric cone

$$(A, B, C, F, G, H \between z-\theta t, \theta z-y, x-\theta y)^2 = 0,$$

where $A = bc-f^2, \ldots, H = fg-ch$.

This, then, is the equation of the cone of tangent lines from P to R^4; it is in complete agreement with (6.1), as is seen by writing

$$S_0 \equiv (A, B, C, F, G, H \between -t, z, -y)^2,$$

$$S_1 \equiv (A, B, C, F, G, H \between -t, z, -y \between z, -y, x),$$

$$S_2 \equiv (A, B, C, F, G, H \between z, -y, x)^2.$$

It was remarked by Cayley* that the three quadrics $S_i = 0$ all meet k at the pinch-points; and, since the four points of k whose parameters satisfy

$$(A, B, C, F, G, H \between 1, -\theta, \theta^2)^2 = 0$$

lie on all these quadrics, they must be the four pinch-points.

Incidentally, since R^4 is the envelope of the tangent cones from the points of k, its equation, by (6.1), must be $S_0 S_2 = S_1^2$. With the above forms for S_0, S_1, S_2 this is the same equation as (6.2).

Since every cone (6.1) must touch every generator of R^4, and every cone (6.1) passes through the pinch-points, the torsal generators of R^4 must touch every one of the cones at the pinch-points. This shows that the net of quadrics to which the cones belong is a contact net.

The two points θ and ϕ of k are found to be conjugate with respect to every cone (6.1), and therefore with respect to every quadric of the

* *Collected Mathematical Papers*, 6, 124.

contact net to which these cones belong, provided only that

$$(A,\ B,\ C,\ F,\ G,\ H \between 1,\ -\theta,\ \theta^2 \between 1,\ -\phi,\ \phi^2) = 0. \tag{6.4}$$

Hence the polar planes, of any point ϕ of k, with respect to the tangent cones to R^4 from all the different points of k, have in common a line which is a chord of k. This is in accordance with the result previously found for $\mathfrak{C}$, that the trisecant conjugate to a point of either twisted cubic which is part of the Jacobian curve is a chord of this cubic (and a secant of the other).

If θ_1 and θ_2 are the two roots of (6.4) for any given value of ϕ, then

$$\theta_1\theta_2 : \theta_1+\theta_2 : 1 = A-H\phi+G\phi^2 : H-B\phi+F\phi^2 : G-F\phi+C\phi^2,$$

so that, whatever the value of ϕ,

$$\{a\theta_1\theta_2+h(\theta_1+\theta_2)+g\}\{g\theta_1\theta_2+f(\theta_1+\theta_2)+c\} = \{h\theta_1\theta_2+b(\theta_1+\theta_2)+f\}^2.$$

This is of the same form as (6.3), with the constants

$$a,\quad b,\quad c,\quad f,\quad g,\quad h$$

of (6.3) replaced respectively by

$$ag-h^2,\ hf-b^2,\ cg-f^2,\ \tfrac{1}{2}(ch+fg)-bf,\ \tfrac{1}{2}(ac+g^2)-hf,\ \tfrac{1}{2}(af+gh)-bh.$$

If this replacement is carried out in (6.2), there results the equation of the quartic scroll K generated by those chords of k which are conjugate to the points of k. The chord which is conjugate to a pinch-point of R^4 is the corresponding torsal generator, so that the four torsal generators of R^4 are also (non-torsal) generators of K.

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