Blown-up toric surfaces with non-polyhedral effective cone

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Moduli space of stable rational curves

- $\mathcal{M}_{0,n} = \left\{ p_1, \ldots, p_n \in \mathbb{P}^1 \mid p_i \neq p_j \right\} / \text{PGL}_2$
- $\mathcal{M}_{0,3} = \text{pt (send } p_1, p_2, p_3 \to 0, 1, \infty\text{)}$
- $\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ via cross-ratio
- $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$
- $\overline{\mathcal{M}}_{0,n}$ functorial compactification
- $\overline{\mathcal{M}}_{0,5} = \text{dP}_5$ (del Pezzo of degree 5)
- $\overline{\mathcal{M}}_{0,6} = \text{blow-up of the Segre cubic at the 10 nodes } (-K \text{ is big and nef})$
- $\overline{\mathcal{M}}_{0,n}, n \geq 8: -K \text{ not pseudo-effective}$
The effective cone of $\overline{M}_{0,n}$

- (Kapranov models) $\overline{M}_{0,n} = \ldots \text{Bl}
  \binom{n-1}{3} \text{Bl}
  \binom{n-1}{2} \text{Bl}_{n-1} \mathbb{P}^{n-3}$
  (blow-up $n-1$ points, all lines, planes,... spanned by them)

- Every boundary divisor is contracted by a Kapranov map
  $\overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ and generates an extremal ray of $\text{Eff}(\overline{M}_{0,n})$

- $\text{Eff}(\overline{M}_{0,5})$ is generated by the 10 boundary divisors ($-1$ curves)

- $\text{Eff}(\overline{M}_{0,6})$ is generated by boundary and Keel–Vermeire divisors
  (Hassett–Tschinkel 2002)
The effective cone of $\overline{M}_{0,n}$

- $\overline{\text{Eff}}(\overline{M}_{0,n})$ has many extremal rays, generated by hypertree divisors, contractible by birational contractions (C.–Tevelev 2013)

**Theorem (C.–Laface–Tevelev–Ugaglia 2020)**

The cone $\overline{\text{Eff}}(\overline{M}_{0,n})$ is not polyhedral for $n \geq 10$, both in characteristic 0 and in characteristic $p$, for an infinite set of primes $p$ of positive density (including all primes up to 2000).
RATIONAL CONTRACTIONS

DEFINITION
A rational contraction $X \to Y$ between $\mathbb{Q}$-factorial, normal projective varieties, is a rational map that can be decomposed into a sequence of

- small $\mathbb{Q}$-factorial modifications,
- surjective morphisms between $\mathbb{Q}$-factorial varieties.

THEOREM
Let $X \to Y$ be a rational contraction. If $X$ has any of these properties then $Y$ does as well:

- Mori Dream Space (Keel–Hu 2000, Okawa 2016)
- (rational) polyhedral effective cone (BDPP 2013)
\( \overline{M}_{0,n} \) AND BLOW-UP-UPS OF TORIC VARIETIES

**Philosophy (Fulton)**

\( \overline{M}_{0,n} \) is similar to a toric variety.

Not quite true. Instead, \( \overline{M}_{0,n} \) is similar to a blown up toric variety:

**Theorem (C.–Tevelev 2015)**

There are rational contractions

\[
\text{Bl}_e \overline{L M}_{0,n+1} \rightarrow \overline{M}_{0,n} \rightarrow \text{Bl}_e \overline{L M}_{0,n},
\]

where \( \overline{L M}_{0,n} \) is the Losev-Manin moduli space of dimension \( n - 3 \),

\( e \) = identity point of the open torus \( \mathbb{G}_m \subset \overline{L M}_{0,n} \).

Kapranov description: \( \overline{L M}_{0,n} \) = \( \ldots \text{Bl}_{n-2} \text{Bl}_{n-2} \text{Bl}_{n-2} \mathbb{P}^{n-3} \) (blow-up \( n - 2 \) points, all lines, planes,... spanned by them)
The Losev-Manin moduli space $\overline{LM}_{0,n}$

The Losev-Manin moduli space $\overline{LM}_{0,n}$ is the Hassett moduli space of stable rational curves with $n$ markings and weights $1, 1, \epsilon, \ldots, \epsilon$.

trees of $\mathbb{P}^1$'s \quad \rightarrow \quad \text{chains of } \mathbb{P}^1$'s
THEOREM

$X$ projective $\mathbb{Q}$-factorial toric variety. For $n \gg 0$

- there exists a toric rational contraction $\overline{LM}_{0,n} \rightarrow X$
- there exists a rational contraction $\text{Bl}_e \overline{LM}_{0,n} \rightarrow \text{Bl}_e X$

COROLLARY (C.–Tevelev, 2015)

$\overline{M}_{0,n}$ is not a MDS in characteristic 0 for $n \gg 0$. There exists a rational contraction

$$\overline{M}_{0,n} \rightarrow \text{Bl}_e \mathbb{P}(a, b, c)$$

for some $a, b, c$ such that $\text{Bl}_e \mathbb{P}(a, b, c)$ has a nef but not semi-ample divisor (Goto–Nishida–Watanabe 1994).

REMARK

This argument cannot work in characteristic $p$, where, by Artin’s contractibility criterion, a nef divisor on $\text{Bl}_e \mathbb{P}(a, b, c)$ is semi-ample.
Blown up toric surfaces

Theorem (C.-Laface-Tevelev-Ugaglia 2020)

There exist projective toric surfaces $\mathbb{P}_\Delta$, given by good polygons $\Delta$, such that $\overline{\operatorname{Eff}}(\text{Bl}_e \mathbb{P}_\Delta)$ is not polyhedral in characteristic 0.

For some of these toric surfaces, $\overline{\operatorname{Eff}}(\text{Bl}_e \mathbb{P}_\Delta)$ is not polyhedral in characteristic $p$ for an infinite set of primes $p$ of positive density.

Corollary

For $n \geq 10$, the space $\overline{M}_{0,n}$ is not a MDS both in characteristic 0 and in characteristic $p$ for an infinite set of primes of positive density, including all primes up to 2000.
Example of a good polygon
**Example of a good polygon**

There is a rational contraction $\overline{M}_{0,10} \to \text{Bl}_e \overline{LM}_{0,10} \to \text{Bl}_e \mathbb{P}_\Delta$:

Red $\to$ normal fan of $\Delta$  
Black $\to$ projection of fan of $\overline{LM}_{0,10}$
Elliptic Pairs

A good polygon will correspond to an elliptic pair \((\text{Bl}_e \mathbb{P}_\Delta, C)\).

**Definition**

An elliptic pair \((C, X)\) consists of
- a projective rational surface \(X\) with log terminal singularities,
- an arithmetic genus 1 curve \(C \subseteq X\) such that \(C^2 = 0\),
- \(C\) disjoint from singularities of \(X\).

Restriction map \(\text{res} : C^\perp \rightarrow \text{Pic}^0(C),\ D \mapsto \mathcal{O}(D)|_C\)

\(C^\perp \subseteq \text{Cl}(X)\) orthogonal complement of \(C\), \(C^\perp\) contains \(C\)

**Definition**

The order \(e(C, X)\) of the pair \((C, X)\) is the order of \(\text{res}(C)\) in \(\text{Pic}^0(C)\).

In characteristic \(p\), we have \(e(C, X) < \infty\).
**Order of an elliptic pair**

The order $e(C, X)$ is the smallest integer $e > 0$ such $h^0(eC) > 1$.

**Lemma**

- If $e = e(C, X) < \infty$, then $h^0(eC) = 2$ and $|eC| : X \to \mathbb{P}^1$ is an elliptic fibration with $C$ a multiple fiber.
- If $e(C, X) = \infty$, then $C$ is rigid:
  
  $$h^0(nC) = 1 \quad \text{for all} \quad n \geq 1.$$  

  In this case, $\overline{\text{Eff}}(X)$ is not polyhedral if $\rho(X) \geq 3$.

**Proof.**

Observation (Nikulin): If $\rho(X) \geq 3$ and $\overline{\text{Eff}}(X)$ is polyhedral, then

- $\overline{\text{Eff}}(X)$ is generated by negative curves,
- every irreducible curve with $C^2 = 0$ is contained in the interior of a facet; in particular, a multiple moves.
**Minimal elliptic pairs**

Polyhedrality when $e(C, X) < \infty$? In general, for any $e(C, X)$:

**Definition**

An elliptic pair $(C, X)$ is called **minimal** if there are no smooth rational curves $E \subseteq X$ such that $K \cdot E < 0$ and $C \cdot E = 0$.

**Theorem**

For an elliptic pair $(C, X)$, there exists a minimal elliptic pair $(C, Y)$ and a morphism $\pi : X \to Y$, which is an isomorphism in a neighborhood of $C$.

In particular, $e(C, X) = e(C, Y)$.

**Proof.**

$\mathcal{O}(K + C)|_C \simeq \mathcal{O}_C \Rightarrow K \cdot C = 0$

$(C, X)$ is minimal $\iff K + C$ is nef $\iff K + C \sim \alpha C$, $\alpha \in \mathbb{Q}$

Run $(K + C)$-MMP: contract all curves $E \subseteq X$ with $K \cdot E < 0$, $C \cdot E = 0$. 

□
**Minimal + Du Val singularities**

**Definition**

Since $K \cdot C = 0$, define on $\text{Cl}_0(X) = C^\perp / \langle K \rangle$ the reduced restriction map

$$\overline{\text{res}} : \text{Cl}_0(X) \to \text{Pic}^0(C) / \langle \text{res}(K) \rangle$$

**Theorem**

Let $(C, Y)$ be an elliptic pair such that $Y$ has Du Val singularities. Let $Z$ be the minimal resolution of $Y$. Then

$$(C, Y) \text{ minimal } \iff (C, Z) \text{ minimal } \iff \rho(Z) = 10.$$  

In this case $\text{Cl}_0(Z) \simeq \mathbb{E}_8$.

Assume $(C, Y)$ minimal elliptic pair with $\rho(Y) \geq 3$ and $e(C, Y) < \infty$:

$$\overline{\text{Eff}}(Y) \text{ polyhedral } \iff \overline{\text{Eff}}(Z) \text{ polyhedral } \iff \text{Ker}(\overline{\text{res}}) \text{ contains 8 linearly independent roots of } \mathbb{E}_8.$$
Upshot

$(C, Y) = \text{minimal model of elliptic pair } (C, X)$

- $e(C, X) = \infty \Rightarrow \overline{\text{Eff}}(X), \overline{\text{Eff}}(Y) \text{ not polyhedral (if } \rho \geq 3)$
  - In this case, $Y$ is Du Val: $\mathcal{O}(C)|_C$ not torsion implies $-K_Y \sim C$
- $e(C, X) < \infty \text{ and } Y$ is Du Val $\Rightarrow$ polyhedrality criterion for $\overline{\text{Eff}}(Y)$

Problem

- Suppose $C, X, \text{Cl}(X)$ are defined over $\mathbb{Q}$, $e(C, X) = \infty$
- $X \rightarrow Y$ extends to the morphism of integral models $\mathcal{X} \rightarrow \mathcal{Y}$ over $\text{Spec} \mathbb{Z}$ (outside of finitely many primes of bad reduction)
- $(C_p, Y_p)$ is still the minimal elliptic pair associated to $(C_p, X_p)$
- $e(C_p, X_p) < \infty$. Study distribution of “polyhedral” primes
Blown up toric surfaces

Lattice polygon $\Delta \subseteq \mathbb{R}^2 \implies (\mathbb{P}_\Delta, \mathcal{L}_\Delta)$ associated polarized toric surface
Set $X = \text{Bl}_e \mathbb{P}_\Delta$ and let $m > 0$ integer. Then $X, \text{Cl}(X)$ are defined over $\mathbb{Q}$.

Definition
A lattice polygon $\Delta$ with at least 4 vertices is good if there exists

$$C \in |\mathcal{L}_\Delta - mE|$$

irreducible such that $(C, X)$ is an elliptic pair with $e(C, X) = \infty$:

I. The Newton polygon of $C$ coincides with $\Delta$ ($\iff C \subseteq X^{\text{smooth}}$),

II. $\text{Vol}(\Delta) = m^2$ and $|\partial \Delta \cap \mathbb{Z}^2| = m$ ($\iff C^2 = 0$, $p_a(C) = 1$),

III. The restriction $\text{res}(C) = \mathcal{O}_X(C)|_C$ is not torsion in $\text{Pic}^0(C)$ over $\mathbb{Q}$. 
**Theorem**

If $\Delta$ is a good polygon, then $\overline{\text{Eff}}(X)$ is not polyhedral in characteristic 0.

**Example**

\[
\text{Vol}(\Delta) = 36, \quad |\partial\Delta \cap \mathbb{Z}^2| = 6
\]
Example of a good polygon

\[ \text{Vol}(\Delta) = 36, \quad |\partial \Delta \cap \mathbb{Z}^2| = 6 \]

The linear system \(|L_\Delta - 6E| \) contains a unique curve \( C \) with equation

\[
x^4y^6 + 6x^5y^4 - 2x^4y^5 - 14x^5y^3 - 17x^4y^4 - 4x^3y^5 +
+ x^6y + 11x^5y^2 + 38x^4y^3 + 26x^3y^4 - 9x^5y - 27x^4y^2 -
- 34x^3y^3 + 22x^4y + 16x^3y^2 - 10x^2y^3 - 24x^3y +
+ 10x^2y^2 + 15x^2y + 5xy^2 - 11xy + 1 = 0.
\]
The curve $C$ is a smooth elliptic curve labelled 997.a1 in the LMFDB database. It has the minimal equation

$$y^2 + y = x^3 - x^2 - 24x + 54$$

The Mordell-Weil group $C(\mathbb{Q})$ is $\mathbb{Z} \times \mathbb{Z}$, with generators

$$Q = (1, 5), \quad P = (6, -10)$$

Computation: $\text{res}(C) = -Q$ (not torsion, so $\Delta$ is good)
Example - Minimal resolution

Fan of the minimal resolution $\tilde{\mathbb{P}}_\Delta$ of $\mathbb{P}_\Delta$:

The proper transforms $C_1, C_2$ of 1-parameter subgroups $\{v = 1\}, \{u = 1\}$

- have self-intersection $-1$ on $\text{Bl}_e \tilde{\mathbb{P}}_\Delta$, and also on $X = \text{Bl}_e \mathbb{P}_\Delta$
- have $C \cdot C_1 = C \cdot C_2 = 0$
**Example - Minimal elliptic pair**

\((C, X)\) elliptic pair, \(X = \text{Bl}_e \mathbb{P}_\Delta\)

Zariski decomposition \(K_X + C = N + P, \ N = 3C_1 + 2C_2, \ P = 0\)

To get minimal elliptic pair \((C, Y)\), contract \(C_1, C_2\).

\[
\begin{array}{c}
\text{Bl}_e \mathbb{P}_\Delta \\
\downarrow \\
X
\end{array} \quad \longrightarrow \quad \begin{array}{c}
Z \\
\downarrow \\
Y
\end{array}
\]

\(Z \to Y\) minimal resolution, \(\rho(X) = 5, \ \rho(Y) = 3, \ \rho(Z) = 10\)

\(T = \) sublattice spanned by classes of \((-2)\) curves above singularities of \(Y\)

Computation : \(T = \mathbb{A}^7\)
Example - Minimal resolution

$Z \to Y$ minimal resolution of $Y$, $\text{Cl}(Z) = \text{Cl}(Y) \oplus T$

$T =$ sublattice spanned by classes of $(-2)$ curves above singularities of $Y$

$\text{Cl}_0(Y) = \text{Cl}_0(Z)/T = \mathbb{E}_8/\mathbb{A}^7 \cong \mathbb{Z}$

Reduced restriction map $\overline{\text{res}} : \text{Cl}_0(Y) \to \text{Pic}^0(C)/\langle Q \rangle$, $Q = (1, 5)$

$\overline{\text{Eff}}(Y)$ is not polyhedral in characteristic $p \iff$

$\iff \overline{\text{res}}(\beta) \neq 0$ for all $\beta =$ image in $\text{Cl}_0(Y)$ of a root in $\mathbb{E}_8 \setminus T$

If $\alpha \in \text{Cl}_0(Y)$ generator $\implies$ Images of roots of $\mathbb{E}_8$ are $\pm k\alpha$, for $0 \leq k \leq 3$

Computation : $\text{res}(\alpha) = P - Q$, where $P = (6, -10)$

$\overline{\text{Eff}}(Y)$ not polyhedral in characteristic $p \iff k\overline{P} \notin \langle \overline{Q} \rangle$ for $k = 1, 2, 3$
**Example - Non-polyhedral primes**

Prove that the set of primes $p$ such that

$$\overline{P}, 2\overline{P}, 3\overline{P} \not\in \langle \overline{Q} \rangle \subseteq C(\mathbb{F}_p)$$

has positive density.

Fix $q$ prime. It suffices to prove that the set of primes $p$ such that

- $q$ divides the index of $\langle \overline{Q} \rangle \subseteq C(\mathbb{F}_p)$
- $q$ does not divide the index of $\langle 6\overline{P} \rangle \subseteq C(\mathbb{F}_p)$

has positive density.

Apply Chebotarev’s Density theorem + a theorem of Lang-Trotter
Lang-Trotter Criterion

C elliptic curve defined over \( \mathbb{Q} \), without complex multiplication over \( \overline{\mathbb{Q}} \). Fix \( q \) prime and let \( C[q] \subset C(\overline{\mathbb{Q}}) \) be the \( q \)-torsion points of \( C \).

For \( x \in C(\mathbb{Q}) \), choose \( x/q \in C(\overline{\mathbb{Q}}) \) and consider the Galois extension of \( \mathbb{Q} \)

\[
K_x = \mathbb{Q}(C[q], x/q)
\]
Lang-Trotter Criterion

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Fix a prime and let \( C[q] \subset C(\overline{\mathbb{Q}}) \) be the \( q \)-torsion points of \( C \).

For \( x \in C(\mathbb{Q}) \), choose \( x/q \in C(\overline{\mathbb{Q}}) \) and consider the Galois extension of \( \mathbb{Q} \)

\[
K_x = \mathbb{Q}(C[q], x/q)
\]

For almost all primes \( q \), we have \( \text{Gal}(K_x/\mathbb{Q}) \cong \text{GL}_2(\mathbb{Z}/q\mathbb{Z}) \ltimes (\mathbb{Z}/q\mathbb{Z})^2 \)

For any \( L/\mathbb{Q} \) Galois, for almost all primes \( p \), there is a Frobenius element \( \sigma_p \in \text{Gal}(L/\mathbb{Q}) \) of \( p \) in \( L/\mathbb{Q} \) (well-defined up to conjugacy).

Lang-Trotter (1976): \( q \) divides the index of \( \langle x \rangle \subseteq C(F_p) \) \iff 

\[ \iff \text{the Frobenius element } \sigma_p = (\gamma_p, \tau_p) \in \text{GL}_2(\mathbb{Z}/q\mathbb{Z}) \ltimes (\mathbb{Z}/q\mathbb{Z})^2 \]

with \( \gamma_p \) with 1 as an eigenvalue, and either \( \gamma_p = 1 \), or \( \tau_p \in \text{Im}(\gamma_p - 1) \).
Non-polyhedral primes

$C$ elliptic curve defined over $\mathbb{Q}$, without complex multiplication over $\overline{\mathbb{Q}}$. For $x, y \in C(\mathbb{Q})$, let $K_{x,y} = \mathbb{Q}(C[q], x/q, y/q)$ (Galois extension of $\mathbb{Q}$). The Frobenius element $\sigma_p$ of $p$ in $K_{x,y}/\mathbb{Q}$ is

$$\sigma_p = (\gamma_p, \tau_p, \tau'_p) \in \text{Gal}(K_{x,y}/\mathbb{Q}) \simeq \text{GL}_2(\mathbb{Z}/q\mathbb{Z}) \ltimes ((\mathbb{Z}/q\mathbb{Z})^2)^2$$

where $(\gamma_p, \tau_p) \in \text{Gal}(K_x/\mathbb{Q})$, $(\gamma_p, \tau'_p) \in \text{Gal}(K_y/\mathbb{Q})$ (Frobenius elements).

By Lang-Trotter, the set of primes $p$ such that
- $q$ divides the index of $\langle \overline{x} \rangle \subseteq C(\mathbb{F}_p)$
- $q$ does not divide the index of $\langle \overline{y} \rangle \subseteq C(\mathbb{F}_p)$

is the set of primes $p$ such that:
- $\gamma_p$ has 1 as an eigenvalue, $\tau_p \in \text{Im}(\gamma_p - 1)$, $\tau'_p \notin \text{Im}(\gamma_p - 1)$

This condition is closed under conjugacy (and such elements exist).
Non-polyhedral primes

The set of non-polyhedral primes \( p < 2000 \) for our running example of a good polygon:


This gives 18% of the primes under 2000.
Further Examples

There are:

- 135 toric surfaces corresponding to good polygons with volume $\leq 49$;
- Infinite sequences of good pentagons with all primes polyhedral;
- Infinite sequences of good heptagons. For all but finitely many, the set of non-polyhedral primes has positive density.
An infinite sequence of pentagons

Polygon \( \Delta \) is a pentagon with vertices

\[
(0, 0), \quad (2k, 0), \quad (2k + 4, 1), \quad (2k + 2, 2k + 4), \quad (2k + 1, 2k + 3)
\]

\[
\text{Vol}(\Delta) = (2k + 4)^2, \quad |\partial \Delta \cap \mathbb{Z}^2| = 2k + 4
\]

Then \( \Delta \) is good for every \( k \geq 1 \).
An infinite sequence of pentagons

Equation of $C$ is:

\[
(uv + 2x_0^{k+2})(u - 2x_0^{k+1})^{2k+3} - 2u^{k+1}(v + x_0)^{k+2}(u - 2x_0^{k+1})^{k+2} - \\
- u^{2k+1}(v + x_0)^{2k+3}(uv + u(x_0 - x_1) + 2x_1x_0^{k+1}) = 0,
\]

where

\[
x_0 = 2(k + 1)(3k + 2), \quad x_1 = 2(k + 1)(3k + 4).
\]
An infinite sequence of pentagons

The curve $C$ has Weierstrass equation

$$y^2 = x(x^2 + ax + b), \quad \text{where}$$

$$a = -(12k^2 + 24k + 11), \quad b = 4(k + 1)^2(3k + 2)(3k + 4).$$