THE OSCULATING SPACES OF A CERTAIN CURVE IN [n]

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1

The curve in question is the non-singular intersection Γ of the $n-1$ quadric primals

$$\sum_{j=0}^{n} a_j^k x_j^2 = 0, \quad k = 0, 1, 2, \ldots, n-2, \tag{1.1}$$

where it is presumed that no two of the $n+1$ numbers a_j are equal. Define

$$f(\phi) \equiv (\phi - a_0)(\phi - a_1)\ldots(\phi - a_n);$$

then it will be seen that the osculating prime of Γ at $x = \xi$ is

$$\Sigma\{f'(a_j)\}^{n-2}\xi_j^{2n-3}x_j = 0. \tag{1.2}$$

Indeed, equations will be given for all the osculating spaces [s], such a space being determined by $n-s$ linearly independent linear equations. But (1.2) is mentioned at the outset because equations for the osculating plane in [3] and for the osculating solid in [4] are already known. The equation of the osculating plane of an elliptic quartic curve in [3] is given by Salmon (3, p. 380); the coefficients $f'(a)\xi^3$ appear there on taking $a' = b' = c' = d' = 1$. The equation of the osculating solid of a special canonical curve in [4] is given by Edge (**2**, p. 278), whose more prolix equation is seen, for $n = 4$, to be equivalent to (1.2) here on substituting, from what is there labelled (2.2), in the equation written as $\Sigma(p+a_jq)^2\xi_jx_j = 0$. It was the belated perception of this that suggested (1.2), and there is no difficulty in an *a posteriori* verification. This relies on two circumstances.

(1) If $\sigma_k = \Sigma a_j^k/f'(a_j)$ then

$$\sigma_0 = \sigma_1 = \ldots = \sigma_{n-1} = 0. \tag{1.3}$$

This is proved by using the partial fractions for $\phi^k/f(\phi)$.

(2) The equations (1.1) may be regarded as $n-1$ linear equations for the $n+1$ " unknowns " x_j^2; they are, no two a_j being equal, linearly independent and so have $n+1-(n-1) = 2$ linearly independent solutions. Clearly, in virtue of (1.3), two such solutions are

$$x_j^2 = 1/f'(a_j) \quad \text{and} \quad x_j^2 = a_j/f'(a_j).$$

Hence, whatever number θ may be, other than the $n+1$ critical values $-a_j$, the $n+1$ equations

$$\xi_j^2 f'(a_j) = \theta + a_j \tag{1.4}$$

give, by the alternative signing of $n+1$ square roots, a batch of 2^n points on Γ. One such batch has $\theta = \infty$. If, differentiation being imminent, one scruples to treat this batch as on a par with others there is the alternative use of

$$\xi_j^2 f'(a_j) = 1+a_j\phi \tag{1.4'}$$

when the batch corresponds to $\phi = 0$.

2

Equation (1.2) is established if it can be shown that, for $p = 0, 1, \ldots, n-1$,

$$\Sigma\{f'(a_j)\}^{n-2}\xi_j^{2n-3}d^p\xi_j = 0;$$

this means that, for all these n values of p,

$$\Sigma(\theta+a_j)^{n-2}\xi_j d^p\xi_j = 0. \tag{2.1}$$

For $p = 0$ this is so, by (1.1). Otherwise one repeatedly differentiates

$$\xi_j\sqrt{f'(a_j)} = (\theta+a_j)^{\frac{1}{2}};$$

this determines, for each j, one of the two analytic branches of $(\theta+a_j)^{\frac{1}{2}}$ according to the square root chosen and, having made the choice, one adheres thereto in the subsequent differentiations. Then, on this understanding,

$$d^p\xi_j\sqrt{f'(a_j)} = A_p(\theta+a_j)^{\frac{1}{2}-p}(d\theta)^p$$

with A_p a non-zero constant, so that

$$(\theta+a_j)^{n-2}\xi_j d^p\xi_j . f'(a_j) = A_p(\theta+a_j)^{n-p-1}(d\theta)^p$$

and, so long as $p \leqq n-1$, (2.1) holds because of (1.3). This establishes the validity of (1.2), at least for finite values of θ. But one can also differentiate the square roots of the two sides of (1.4') and so arrive at

$$d^p\xi_j\sqrt{f'(a_j)} = A_p a_j^p(1+a_j\phi)^{\frac{1}{2}-p}(d\phi)^p$$

and

$$(1+a_j\phi)^{n-2}\xi_j d^p\xi . f'(a_j) = A_p a_j^p(1+a_j\phi)^{n-p-1}(d\phi)$$

from which the desired conclusion follows. Henceforward we may refrain from glossing the text by references to (1.4').

3

The same reasoning, however, applies to the equation

$$\Sigma\{f'(a_j)\}^{r-2}\xi_j^{2r-3}d^p\xi_j = 0, \tag{3.1}$$

or

$$\Sigma(\theta+a_j)^{r-2}\xi_j d^p\xi_j = 0,$$

for any r such that $0<r \leqq n$. Since this last relation is an identity in θ for $p = 0, 1, \ldots, r-1$ the points

$$\xi,\ d\xi,\ d^2\xi,\ \ldots,\ d^s\xi$$

all satisfy those s equations (3.1) for which $r = n, n-1, \ldots, s+1$. These therefore, with x_j replacing $d^p\xi_j$, are the $n-s$ equations determining the osculating [s] of Γ at ξ. That they do determine the [s] is consequent on their linear independence; that they are linearly independent follows once (4.3) below, and the proceedings relating to it, have been noted.

4

One can now calculate R_s, the sth rank of Γ, i.e. the number of spaces [s] that osculate Γ and meet a given [$n-s-1$]. If this [$n-s-1$] is determined by the $s+1$ linear equations

$$\alpha_{i,0}x_0+\alpha_{i,1}x_1+\ldots+\alpha_{i,n}x_n=0 \quad (i=n-s+1, \ldots, n+1) \tag{4.1}$$

it is met by those [s] which osculate Γ at points whose coordinates ξ cause a certain $(n+1)$-rowed determinant Δ to be zero: column $j+1$ of Δ consists of

$$\{f'(a_j)\}^{n-2}\xi_j^{2n-3}, \{f'(a_j)\}^{n-3}\xi_j^{2n-5}, \ldots, \{f'(a_j)\}^{s-1}\xi_j^{2s-1} \tag{4.2}$$

followed by $\alpha_{n-s+1,j}\ldots\alpha_{n+1,j}$. The $n-s$ numbers (4.2) all have the factor $\{f'(a_j)\}^{s-1}\xi_j^{2s-1}$; the residual factors are, in virtue of (1.4),

$$(\theta+a_j)^{n-s-1}, (\theta+a_j)^{n-s-2}, \ldots, \theta+a_j, 1. \tag{4.3}$$

If these are now multiplied in order by

$$1, \binom{n-s-1}{1}(-\theta), \binom{n-s-1}{2}(-\theta)^2, \ldots, \binom{n-s-1}{n-s-2}(-\theta)^{n-s-2}, (-\theta)^{n-s-1}$$

and the products added, the sum is

$$(\theta+a_j-\theta)^{n-s-1}=a_j^{n-s-1}.$$

One next performs a similar operation that does not involve the leading member in (4.2); omit the leader in (4.3) and multiply the others in order by

$$1, \binom{n-s-2}{1}(-\theta), \binom{n-s-2}{2}(-\theta)^2, \ldots, \binom{n-s-2}{n-s-3}(-\theta)^{n-s-3}, (-\theta)^{n-s-2}$$

and add the products; the sum is a_j^{n-s-2}. And so on. The whole procedure transforms Δ, without changing its value, into a determinant having, so long as $i \leqq n-s$, in row i and column $j+1$ the element

$$a_j^{n-s-i}\{f'(a_j)\}^{s-1}\xi_j^{2s-1}.$$

The remaining $s+1$ rows are still filled, as originally, by the coefficients of (4.1). It now appears, by Laplace expansion on these s + 1 rows, that the degree of Δ in the coordinates ξ_j is $(n-s)(2s-1)$.

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When ξ is replaced by x, $\Delta=0$ becomes the equation of a primal whose $2^{n-1}(n-s)(2s-1)$ intersections with Γ are those points at which the osculating [s] intersects the [$n-s-1$] given by (4.1). And so

$$R_s=2^{n-1}(n-s)(2s-1).$$

In particular: the class of Γ, or the number of its osculating primes passing through an arbitrary point is

$$R_{n-1} = 2^{n-1}(2n-3),$$

a classical result for $n = 3$ (there are 12 osculating planes of an elliptic quartic through an arbitrary point in [3]) and obtained for $n = 4$ in (**2**). Also: the order of the primal generated by the osculating $[n-2]$'s of Γ is

$$R_{n-2} = 2^{n}(2n-5),$$

of course classical for $n = 3$ (the tangents of an elliptic quartic generate a scroll of order 8). For $n = 4$ it follows that the osculating planes of the canonical model of Humbert's plane sextic generate a threefold of order 48. There will be, for each n, a single equation for R_{n-2}, presumably obtainable by some process of elimination.

6

There is a more sophisticated procedure for determining the ranks R_s, and perhaps it should be described. In order to apply it one must know the genus π of Γ and a certain formula for the number of points, in the sets of a linear series g on Γ, of multiplicity exceeding the freedom r of g; and indeed a precise rule for calculating the number of times a multiple point of specified singularity has to be counted.

As for the genus of Γ it is known (**5**, p. 83) that the canonical series of grade $2\pi-2$ is cut, on the complete non-singular intersection of genus π of $n-1$ primals in $[n]$, by primals of order $n_1+n_2+\ldots+n_{n-1}-(n+1)$, where the n_i are the orders of the primals through the curve. Since, for Γ, each n_i is 2 the canonical series is cut by primals of order $n-3$ and so

$$2\pi-2 = 2^{n-1}(n-3),$$

$$\pi = 1+2^{n-2}(n-3)$$

as stated by Baker (**1**, p. 185).

Take now any $[n-s-1]$; the primes through it cut on Γ a linear series g of grade 2^{n-1} and freedom s. If a prime contains an osculating $[s]$ of Γ, then the contact with Γ counts $s+1$ times in the corresponding set of g. The standard formula (**4**, p. 85) for the number of points of multiplicity $s+1$ in a linear series of grade 2^{n-1} and freedom s on a curve of genus π is

$$(s+1)\{2^{n-1}+(\pi-1)s\};$$

for Γ this is

$$2^{n-2}(s+1)\{2+(n-3)s\}. \tag{6.1}$$

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This, however, is not R_s because, whatever $[n-s-1]$ is chosen, there are certain points W on Γ where osculating spaces of dimension *less than s* have $(s+1)$-point intersection; these spaces can be joined to points in $[n-s-1]$

by spaces of dimension less than n. One has to know two things: how many points W there are, and how much each contributes to the number (6.1).

Identification of these W is easy; they are the intersections of Γ with the $n+1$ bounding primes $x = 0$ of the simplex of reference. Once this has been proved it follows that there are $2^{n-1}(n+1)$ of them, so that if each contributes m to (6.1)

$$R_s = 2^{n-2}[(s+1)\{2+(n-3)s\}-2m(n+1)]. \tag{7.1}$$

First, then, to note the special attributes of the points W.

Γ is its own harmonic inverse in each vertex X and opposite bounding prime $x = 0$ of the simplex of reference: if P is on Γ, then XP is a chord of Γ since it contains the image P' of P in the inversion. So the tangent of Γ at any point W contains a vertex X. But the osculating plane ω at W is the limiting position of the plane joining this tangent to a neighbouring point P of Γ; since this plane contains both X and P it contains P'; ω has 4-point intersection with Γ at W. Similar reasoning shows the osculating solid to have 6-point intersection, and so on, the osculating $[s]$ having $2s$-point intersection.

Now let B be an $[n-s-1]$. The join $[n-s]$ of B to any W lies in ∞^{s-1} primes; of these, some, to be accounted for in a moment, are special, but the " general " prime among these ∞^{s-1} has only a single intersection with Γ at W. However, the ∞^{s-2} primes containing B and the tangent of Γ at W all have 2-point intersection, the ∞^{s-3} primes containing B and the osculating plane of Γ at W all have 4-point intersection, and so on, until one has the single prime, spanned by B and the osculating $[s-1]$ of Γ at W, having $(2s-2)$-point intersection. Then the rule, due to Corrado Segre (**4**, p. 86; for a textbook reference see **6**, p. 131) prescribes that, in such circumstances, W contributes

$$m = 1+2+4+\ldots+2(s-1)-\tfrac{1}{2}s(s+1) = \tfrac{1}{2}(s-1)(s-2)$$

to the number (6.1). When $\frac{1}{2}(s-1)(s-2)$ is substituted for m in (7.1) one finds, as obtained by more elementary methods earlier,

$$R_s = 2^{n-1}(n-s)(2s-1).$$

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