Smooth weighted hypersurfaces that are not stably rational (Edge days 2017) ①

A variety X is stably rational $\overset{\text{def}}{\Leftrightarrow}$ $X \times \mathbb{P}^m$ is rational for some $m \geq 0$

Thm (Totaro '16)

$X = X_d \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ very general hypersurface of degree d, $n \geq 3$.

if $d \geq 2\lceil \frac{n+2}{3} \rceil$, then X is not stably rational.

Aim: is to generalize this result to smooth weighted hypersurfaces.

- Smooth weighted hypersurfaces ($/\mathbb{C}$)

A general weighted hypersurface $X = X_d \subset \mathbb{P}_{\mathbb{C}}(a_0, \cdots, a_{n+1})$ is smooth

$\Leftrightarrow \begin{cases} \cdot\ \gcd\{a_i, a_j\} = 1 \text{ for } \forall i \neq j. \\ \cdot\ d \text{ is divisible by } a_0 \cdots a_{n+1} \end{cases}$

For smooth $X_d \subset \mathbb{P}(a_0, \cdots, a_{n+1})$,

$\alpha = \alpha(X) = \sum_{i=0}^{n+1} a_i - d$, so that $\mathcal{O}_X(-K_X) \cong \mathcal{O}_X(\alpha)$

X: Fano $\Leftrightarrow$ $\alpha > 0$.

Example $\dim X = 3$, X: Fano

- $X_2 \subset \mathbb{P}^4$ ($\alpha = 3$); rational
- $X_3 \subset \mathbb{P}^4$ ($\alpha = 2$); not rational (Clemens-Griffiths)
- $X_4 \subset \mathbb{P}^4$ ($\alpha = 1$); not stably rat (Colliot-Thélène–Pirutka)
- $X_6 \subset \mathbb{P}(1,1,1,1,3)$ ($\alpha = 1$); not stably rat (Beauville)
- $X_4 \subset \mathbb{P}(1,1,1,1,2)$ ($\alpha = 2$); — " — (Voisin)
- $X_6 \subset \mathbb{P}(1,1,1,2,3)$ ($\alpha = 1$); — " — (Hassett-Tschinkel)

Example cyclic covers of $\mathbb{P}^n$

$$\underset{\ell m}{\overset{\parallel}{X_d}} = (y^m - f(x_0, \cdots, x_n) = 0) \subset \mathbb{P}(\overset{x_0}{1}, \cdots, \overset{x_n}{1}, \overset{y}{\ell})$$

$$\downarrow m:1$$

$$\underset{\text{branch}}{(f=0)} \subset \mathbb{P}^n$$

Thm (Colliot-Thélène–Pirutka '16 [m: prime], O. '16 [m: arbitrary])

if $d \geq n+1$, then a very general X is not stably rational.

Main Thm

$X = X_d \subset \mathbb{P}(a_0, \cdots, a_{n+1})$; smooth, very general.

Assume that one of the following holds

(a) $\exists$ prime number p s.t. $p \mid \frac{d}{a_0 \cdots a_{n+1}}$ and $d \geq \frac{p}{p+1} \sum_{i=0}^{n+1} a_i$

(b) $d = a_0 \cdots a_{n+1}$ and $\underbrace{\sum_{i=0}^{n+1} a_i - d}_{\alpha} \leq \max\{a_0, \cdots, a_{n+1}\}$

Then X is not stably rational.

Cor

X is not stably rational

- if $\alpha = 1$, or
- if $\alpha = 2$ and except for $X_3 \subset \mathbb{P}^4$, $X_5 \subset \mathbb{P}^6$, or
- if $\alpha = 3$ and except for $X_2 \subset \mathbb{P}^4$, $X_3 \subset \mathbb{P}^5$, $X_4 \subset \mathbb{P}^6$, $X_5 \subset \mathbb{P}^7$, $X_7 \subset \mathbb{P}^9$ and $X_{2p} \subset \mathbb{P}(\underbrace{1, \cdots, 1}_{2p+1}, 2)$ for $p \geq 3$ prime.

Degeneration method

X : projective variety / a field k

- X is universally CH_0-trivial $\overset{\text{def}}{\iff}$ $\forall F \supset k$, $\deg : CH_0(X_F) \to \mathbb{Z}$ is an isom.
- a proj. morphism $\varphi : Y \to X$ is univ CH_0-trivial $\overset{\text{def}}{\iff}$ $\forall F \supset k$, $\varphi_* : CH_0(Y_F) \to CH_0(X)$ is an isom.

e.g. resol. $\varphi : Y \to X$ of isolated singularities whose exc divisor is a SNC with rational components is univ. CH_0-trivial

Thm (Colliot-Thélène – Pirutka '16)

A : DVR with frac. field K, residue field $k = \bar{k}$.

$\mathfrak{X}$: flat proper scheme / A with geometrically integral fibers

Assume $\left\{ \begin{array}{l} \cdot\ X_{\bar{K}} \text{ is smooth} \\ \cdot\ \exists \varphi : \tilde{Y} \to Y \text{ univ } CH_0\text{-trivial resolution.} \end{array} \right.$

$$\begin{tikzcd}
 & & & & \tilde{Y} \arrow[d, "\varphi"] \\
X_{\bar{K}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \arrow[r] & \mathfrak{X} \arrow[d] & & Y \arrow[ll] \arrow[d] \\
\operatorname{Spec}\bar{K} \arrow[r] & \operatorname{Spec}K \arrow[r] & \operatorname{Spec}A & & \operatorname{Spec}k \arrow[ll]
\end{tikzcd}$$

If $\tilde{X}$ is univ. CH_0-trivial, then so is $\tilde{Y}$.

Fact X : smooth, projective variety / a field k

- $H^0(X, \Omega_X^i) \neq 0$ for some $i > 0$ $\Rightarrow$ X is not univ. CH_0-trivial $\Rightarrow$ X is not stably rational.

Kollár's covering technique

Z : smooth variety / $k = \bar{k}$, char $= p > 0$

$\mathcal{L}$: line bundle on Z.

$m \in \mathbb{Z}_{>0}$ s.t. $p \mid m$.

$s \in H^0(Z, \mathcal{L}^{\otimes m})$

$\rightsquigarrow$ $Y = (y^m - \pi^* s = 0) \subset \operatorname{Spec}(\bigoplus_{i \geq 0} \mathcal{L}^{-i})$ $(= U)$

$$\begin{tikzcd}
Y \arrow[d, "\pi"] & U \arrow[dl, "\pi"] \\
Z &
\end{tikzcd}$$

Fact

Suppose Y has only "nondegenerate double points"

$\left(\text{e.g. when } p \neq 2, \quad y^p = \underbrace{g(x_1, \cdots, x_n)}_{\text{nondeg. quadric}} + (\text{higher}) \right)$

Then,

Construction of M

$$\begin{tikzcd}
0 \arrow[r] & \pi^*\Omega_Z^1 \arrow[r] & \Omega_U^1 \arrow[r] \arrow[d, "|_Y"] & \pi^*\mathcal{L}^{-1} \arrow[r] & 0 \\
 & \square & & 0 \arrow[d] & \\
0 \arrow[r] & \pi^*\Omega_Z^1 \arrow[r] \arrow[u] & \Omega_U^1|_Y \arrow[r] & \pi^*\mathcal{L}^{-1} \arrow[r] & 0 \\
 & I_Y/I_Y^2 \arrow[r] \arrow[u] & \Omega_U^1|_Y \arrow[r] \arrow[u, equal] & \Omega_Y^1 \arrow[r] \arrow[u] & 0 \\
 & \pi^*\mathcal{L}^{-m} \arrow[u, phantom, "\cong"] & & \square \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

$M = (\wedge^{n-1} \square)^{\vee\vee} \hookrightarrow (\Omega_Y^{n-1})^{\vee\vee}$.

$\exists$ a line bundle $\mathcal{M} \subset (\Omega_Y^{n-1})^{\vee\vee}$, and $\exists$ a univ. CH_0-trivial resol. $\varphi : \tilde{Y} \to Y$

s.t. $\begin{cases} \cdot \ \mathcal{M} \cong \pi^*(\omega_Z \otimes \mathcal{L}^{\otimes n}) \\ \cdot \ \varphi^*\mathcal{M} \hookrightarrow \Omega_{\tilde{Y}}^{n-1} \end{cases}$

Proof of Main Thm

$X = X_d \subset \mathbb{P}_{\mathbb{C}}(a_0, \cdots, a_{n+1})$; smooth, Fano ($\Rightarrow \# \{ i \mid a_i = 1 \} > \frac{n}{2}$)

Case (a) $\exists p$ s.t. $p \mid \frac{d}{a_0 \cdots a_{n+1}}$, $d \geq \frac{p}{p+1} \sum_{i=0}^{n+1} a_i$.

Consider $\left(y^p - g(x_0, \cdots, x_{n+1}) = ty - h(x_0, \cdots, x_{n+1}) = 0 \right) \subset \mathbb{P}_{\mathbb{C}}(\overset{x_0}{a_0}, \cdots, \overset{x_{n+1}}{a_{n+1}}, \overset{y}{b}) \times \mathbb{A}^1_t$, $b = \frac{d}{p}$

$\downarrow$

$\mathbb{A}^1_t$.

- General fiber X
- Central fiber $X' = (y^p - g = h = 0) \subset \mathbb{P}_{\mathbb{C}}(a_0, \cdots, a_{n+1}, b)$

reduction mod p

$Y = (y^p - g = h = 0) \subset \mathbb{P}_k(a_0, \cdots, a_{n+1}, b)$ $k = \bar{k}$, char $k = p$

$\pi \downarrow \deg = p$ corresp. to $g \in H^0(Z, \mathcal{L}^{\otimes p})$, $\mathcal{L} = \mathcal{O}_Z(b)$

$Z = (h = 0) \subset \mathbb{P}(a_0, \cdots, a_{n+1})$

Lem $\cdot$ Z is smooth ($\Leftarrow \deg Z = b = \frac{d}{p}$ is divisible by $a_0 \cdots a_{n+1}$ and $\# \{ i \mid a_i = 1 \} > \frac{n}{2}$)

$\cdot$ Y has only nondegenerate double pts

$\rightsquigarrow \cdot \ (\Omega_Y^{n-1})^{\vee\vee} \supset \exists \mathcal{M} \simeq \pi^*(\underbrace{\omega_Z}_{\mathcal{O}_Z(b - \Sigma a_i)} \otimes \underbrace{\mathcal{L}^{\otimes p}}_{\mathcal{O}_Z(pb = d)}) = \mathcal{O}_X\Big(\underbrace{\tfrac{p+1}{p} d - \sum_{i=0}^{n+1} a_i}_{\geq 0}\Big)$

- $\exists$ univ. CH_0-trivial resol s.t. $\varphi^*\mathcal{M} \hookrightarrow \Omega_{\tilde{Y}}^{n-1}$

$\rightsquigarrow H^0(\tilde{Y}, \Omega_{\tilde{Y}}^{n-1}) \neq 0$

Case (b) $d = a_0 \cdots a_{n+1}$ and $\sum_{i=0}^{n+1} a_i - d \leq \max\{a_0, \cdots, a_{n+1}\}$

Assume $a_0 \leq \cdots \leq a_n \leq a_{n+1}$

may assume $2 \leq a_n$ (otherwise X is a cyclic covering)

p : prime number s.t. $p \mid a_n$, $e := a_0 \cdots a_{n-1} = \frac{d}{a_n a_{n+1}}$

$X \subset \mathbb{P}_{\mathbb{C}}(\overset{x_0}{a_0}, \cdots, \overset{x_n}{a_n}, \overset{y}{a_{n+1}}) \rightsquigarrow X' \subset \mathbb{P}_{\mathbb{C}}(a_0, \cdots, a_n, a_{n+1})$

very general

$X' = \left(y^{e a_n} + y^{(e-1)a_n} f_{a_n a_{n+1}} + \cdots + y^{a_n} f_{(e-1)a_n a_{n+1}} + f_{e a_n a_{n+1}} = 0 \right)$, $e a_n a_{n+1} = d$

$$\xrightarrow{\text{mod } p} \; Y \subset \mathbb{P}_k(a_0, \dots, a_n, a_{n+1})$$

$$\| $$

$$\left(y^{e a_n} + y^{(e-1)a_n} f_{a_n a_{n+1}} + \dots + f_d = 0 \right)$$

$\pi \downarrow \deg = a_n$, corresp to $z \in H^0(Z, \mathcal{L}^{\otimes a_n})$, $\mathcal{L} = \mathcal{O}_Z(a_{n+1})$

$x_0 \quad x_n \quad z$

$$Z \subset \mathbb{P}_k(a_0, \dots, a_n, a_n a_{n+1})$$

$$\|$$

$$\left(z^{e} + z^{e-1} f_{a_n a_{n+1}} + \dots + f_d = 0 \right)$$

<u>Lem</u> Y has only nondegenerate double pts

$\rightsquigarrow$ $\cdot$ $(\Omega_Y^{n-1})^{\vee\vee} \supset \exists \mathcal{M} \cong \pi^*(\omega_Z \otimes \mathcal{L}^{\otimes a_n}) = \mathcal{O}_Y(\underbrace{d - \sum_{i=0}^{n} a_i}_{\geq 0})$

$\omega_Z = \mathcal{O}_Z(d - \sum_{i=0}^{n} a_i - a_n a_{n+1})$, $\mathcal{L}^{\otimes a_n} = \mathcal{O}_Z(a_n a_{n+1})$

$\cdot$ $\exists$ univ. CH_0-trivial resol $\varphi : \tilde{Y} \to Y$ s.t. $\varphi^* \mathcal{M} \hookrightarrow \Omega_{\tilde{Y}}^{n-1}$

$\rightsquigarrow$ $H^0(\tilde{Y}, \Omega_{\tilde{Y}}^{n-1}) \neq 0$ $\square$

By a similar argument, I can prove $\quad$ $\downarrow$ I did NOT talk!

<u>Thm</u>

n, m, r positive integers s.t. $n \geq 3$, $n \geq m + r$.

Then a very general hypersurface X_{n+1} of degree $n+1$ in $\mathbb{P}_{\mathbb{C}}^{n+1}$ containing an m-ple r-plane is not stably rational

$\left(\text{i.e. } X_{n+1} \in \left| I_L^m \cdot \mathcal{O}_{\mathbb{P}^{n+1}}(n+1) \right|, \; L \cong \mathbb{P}^r \right)$

~~"proof" $Bl_L X$ (hypersurf) $\hookrightarrow \mathbb{P}_{\mathbb{P}^{n-r}}(\mathcal{O}(-1) \oplus \dots \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}) = \begin{pmatrix} 1 \cdots 1 & 1 \cdots 1 & 0 \\ 0 \cdots 0 & 1 \cdots 1 & 1 \end{pmatrix}$ ($w\; x_i \; x$)~~

~~$\downarrow \qquad \downarrow$~~

~~$X \dashrightarrow \mathbb{P}^{n-r}$~~