Work in progress w S Lamy ①

Bir transf. of the plane over a non-closed field

We look at $\mathrm{Bir}_k\mathbb{P}^2$ = grp of birat. transf. of $\mathbb{P}^2$ defined over a field k.

$\cup$

$f: [x:y:z] \dashrightarrow [f_0(x,y,z) : \ldots : f_2(x,y,z)]$ $\quad f_i \in k[x,y,z]$ homog of eq deg with. ~~common~~ no common factors

If f not def at $p \Rightarrow f$ not def at all conjugates of p
(orbit of $\mathrm{Gal}(L/k)$ where L/k smallest ext. s.t. p def over L)

$p \in \mathbb{P}^2 \rightsquigarrow$ set of all conjugates is a pt of dgr d
has card. d

Examples: $PGl_3(k)$

$(xyz) \mapsto (yz : xz : xy)$ stnd quadr. invol.
contracts 3 lines def / k $z = x = y = 0$

$(xyz) \mapsto (xz : yz : x^2+y^2)$ " —" 1 lines/k $z=0$ + 2 conjugate lines $x \pm iy = 0$

transf. preserving a pencil of lines thr. a pt
$\rightsquigarrow$ bpts of any degree are possible. $(x, \frac{1}{\alpha(x)y}) \rightsquigarrow [\alpha(x)xy : z^d : \alpha(x)yz]$ $\begin{matrix}[1:0:0]\\ [0:1:0]\\ \rightsquigarrow [x_i:1:0]\end{matrix}$ $(d-1)$

over $\bar{k}$, can send onto a pencil of conics

" — " — " conics thr. a pt of deg 4
2 pts of deg 2

Bertini involutions

$$\begin{tikzcd} & dP_1 \arrow[dl, "\deg 8"'] \arrow[dr, "\exists!"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\text{invol}", "b"'] & & \mathbb{P}^2 \end{tikzcd}$$

thr. deg 8 pt is a pencil of cubic curves $\rightsquigarrow$ has a 9th base point (of deg 1) For general $x \in \mathbb{P}^2$, $\rightsquigarrow$ unique cubic in pencil (smooth)
gen member is smooth
~~fixed~~ 9th pt $= 0$ $\rightsquigarrow b(x) = -x$ on the cubic

What do we know abt the generators?

If k alg closed: $\mathrm{Bir}_k\mathbb{P}^2 = \langle$ stnd $PGl_3(k)\rangle$ (Noether-Castelnuovo)

If k not closed: $\mathrm{Bir}_k\mathbb{P}^2 \supsetneq \langle$ stnd, $PGl_3(k)\rangle$ [redacted]
only contains transf that contracts curves def /k.

$k = \mathbb{R}$: Blanc-Mangolte (2014):

$\mathrm{Bir}_{\mathbb{R}}\mathbb{P}^2 = \langle PGl_3(\mathbb{R}), (yz : xz : xy), (xz : yz : x^2+y^2), \text{deg 5 well defined on } \mathbb{P}^2(\mathbb{R})\rangle$

1991 ②

In general: Iskovskikh gives a list of families that generate $Bir_k \mathbb{P}^2 = \langle Aut\,\mathbb{P}^2, \mathcal{E}\rangle$

it includes the transf preserving a pencil of lines thr a pt of deg 1, ~~the above gen~~

· ~~the above generators~~ families of transf.

· ~~Bertini~~ invol, Geiser ~~invol~~, ... blowing up pts of $d \leq 8$ $\mathbb{P}^2$

as cor: If k has no ext. of degs $\Rightarrow Bir_k \mathbb{P}^2 = \langle$ transf. pres pencil of lines thr. k-pt, $PGl_3(k)\rangle$

What do we know about the structure of $Bir_k \mathbb{P}^2$?

Thm (Cornulier, 2013) If k is alg closed, then $Bir_k \mathbb{P}^2$ is not an non-trivial amalgam

Best that can be obtained Blanc, 2011 $Bi = \ast_{\cap} Aut$ / relation ;

Supp. k has an extension of degree 8; If b is a Bertini invol then aba', $a \in PGl_3$ is one too

B = set of representatives of Bertini invol with bps = orbit of 8 pts (up to left-and right mult by autom)

$\mathcal{E}$ = set of repres of generaters of Iskovskikh up to left & right mult by autom

Set $G_b = \langle PGl_3(k), b\rangle$, $b \in B$

$G_e = \langle PGl_3(k), \mathcal{E} \setminus$ Bertini w. bp = orbit of 8 pts $\rangle$

Thm (Lamy-) If k has an extension of deg 8, then $|B| \geq |k|$

$$Bir_k \mathbb{P}^2 = \mathop{\ast}_{PGl_3(k)} G_i \quad i \in I = B \cup \{e\}, \quad G_i \cap G_j = PGl_3(k) \; \forall i \neq j \text{ in } I$$

In particular

$$Bir_k \mathbb{P}^2 = \langle PGl_3(k), B\rangle \mathop{\ast}_{PGl_3(k)} G_e .$$

" $\ast G_b$ "

Furthermore, $G_b = PGl_3(k) \ast \underbrace{\langle b\rangle}_{= \mathbb{Z}/2\mathbb{Z}} \quad \forall b \in B$

This yields a surj homom

$$Bir_k \mathbb{P}^2 \twoheadrightarrow \mathop{\ast}_B \mathbb{Z}/2\mathbb{Z}$$

where each gen. is sent onto the resp. gen of $\mathbb{Z}/2\mathbb{Z}$ and the kernel is $\langle\langle G_e\rangle\rangle$

~~Moreover~~ .

The abelian contains a subgrp of $\bigoplus_B \mathbb{Z}/2\mathbb{Z}$.

If $[\bar{k}:k]<\infty \Rightarrow [\bar{k}:k]=2$ and char $k=0$ Then.

Thm (–) If $\bar{k}/k$ is of deg 2, then

$$\mathrm{Bir}_k\mathbb{P}^2 = \left\langle \begin{array}{c} PGL_3(k), \\ \text{transf preserving} \\ \text{the pencil of lines thr.} \\ (001) \end{array} \right\rangle \ast_{\cap} \left\langle \begin{array}{c} PGL_3(k) \\ \text{transf pres the pencil} \\ \text{of conics thr 2 pairs} \\ \text{of conj pts} \end{array} \right\rangle$$

(We expect a similar result for extension of deg 7 (write Geiser invol) but dont know yet   list of rel by Isk.-Tregub-Kabdykairov 1993)

While the thm for $k=\mathbb{R}$ issues an explicit description of $\mathrm{Bir}_{\mathbb{R}}\mathbb{P}^2$ by generators and relation, the thm for fields having an extension of deg 8 consists of constr. a tree on which $\mathrm{Bir}_k\mathbb{P}^2$ acts, and which turns out to be the Bass Serre tree of $\ast_{PGL_3(k)} G_i$ $I = B \cup \{e\}$

($\varphi_i: G_i \to H \Rightarrow \exists$ homo $\psi: G \to H$ that extends all φ_i (universal property!))

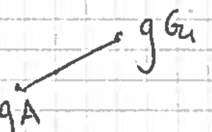
Bass-Serre tree:

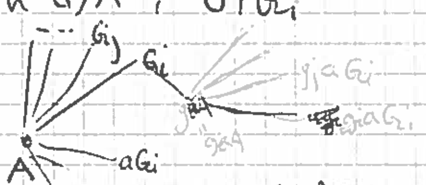
let G be a grp, $A \subset G$ subgrp, $\{G_i\}_{i\in I}$ a collection of subgrps of G generating G.

$G_i \cap G_j = A \quad \forall i \neq j$

to the amalgam $\ast_A G_i$ we constr. a graph:

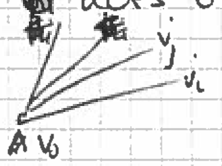
vertices $\hat{=}$ left cosets gA, gG_i in G/A, G/G_i

edge 



$G \circlearrowleft$ graph by $f\,gA = (fg)A$, $f(gG_i) = (fg)G_i$

If $G = \ast_A G_i \Rightarrow$ graph is a tree and $\mathrm{stab}_G(A) = A$, $\mathrm{Stab}_G(G_i) = G_i$ by constr

Thm (Bass-Serre) If G (a grp) acts on a tree T with fund. domain st 

$A = \mathrm{stab}_G(v_0)$

$G_i = \mathrm{stab}_G(v_i)$

$\Rightarrow G = \ast_A G_i$ and T is isom to Bass-Serre tree of $\ast_A$

For $|I|=2$ this is $\underset{G_1}{\bullet}\!-\!\underset{A}{\bullet}\!-\!\underset{G_2}{\bullet}$ $\rightsquigarrow$ remove vertex A

How do we find such a tree?

simplicial, all faces are squares

① $\mathrm{Bir}_k\mathbb{P}^2$ acts on a square complex:

let
- S rat surface, B a pt or $\mathbb{P}^1$ ~~smooth curve~~
- $\pi: S \xrightarrow{\text{morph}} B$ ~~rel picard~~ ≤ 3

S/B rank r-fibr. st
- rel. Picard rank r ~~number~~ ≤ 3
- $-K_S$ is π-ample ($K_S\cdot C<0$ $\forall$ contr. curves)

denote by S/B

~~Complex: vertices $(S/B, \ldots)$~~ ~~$\ldots \overset{\text{birat}}{\dashrightarrow} \mathbb{P}^2$~~

Complex: vertices classes $(S/B, \varphi)$ $\quad \varphi: S \overset{\text{birat}}{\dashrightarrow} \mathbb{P}^2$

$(S/B,\varphi) = (S'/B',\varphi')$ $\iff$

$$\begin{tikzcd} S \arrow[r, "\sim", "\varphi'^{-1}\circ\varphi"'] \arrow[d] \arrow[dr, phantom, "G"] & S' \arrow[d] \\ B \arrow[r, leftrightarrow] & B' \end{tikzcd}$$

$(\mathbb{P}^2, a) \sim (\mathbb{P}^2, \mathrm{id})$ $\forall a \in PGl_3(k)$

edges $(S'/B', \varphi\pi) \xrightarrow{\pi} (S'/B', \varphi)$

If $\begin{tikzcd} S' \arrow[r, "\text{blowup}"] \arrow[d] & S \arrow[d] \\ B' & B \arrow[l] \end{tikzcd}$, or isom. over different bases $\begin{tikzcd} S' \arrow[r, "\sim"] \arrow[d] & S \arrow[d] \\ \mathbb{P}^1 & pt \arrow[l] \end{tikzcd}$ $\begin{tikzcd} \mathbb{F}_1 \arrow[r] \arrow[d] & \mathbb{F}_1 \arrow[d] \\ \mathbb{P}^1 \arrow[r, leftrightarrow] & * \end{tikzcd}$

($B=pt \Rightarrow$ cond = empty, If $B'=\mathbb{P}^1 \Rightarrow B=\mathbb{P}^1$)

$\begin{tikzcd} \mathbb{F}_1 \arrow[r, "1pt"] \arrow[d] & \mathbb{P}^2 \arrow[d] \\ \mathbb{P}^1 & * \arrow[l] \end{tikzcd}$ $\begin{tikzcd} S \arrow[r] \arrow[d] & \mathbb{F}_n \arrow[d] \\ \mathbb{P}^1 & \mathbb{P}^1 \arrow[l] \end{tikzcd}$

$$\begin{tikzcd} & (S_3/B_3, \varphi_3) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ (S_2'/B_2', \varphi_2') \arrow[dr, "p"'] & & (S_2/B_2, \varphi_2) \arrow[dl, "q"] \\ & (S_1/B_1, \varphi_1) & \end{tikzcd}$$

~~remove this edge~~ $\rightsquigarrow$ square complex

$$\begin{tikzcd} & (S_3/\mathbb{P}^1, \varphi) \arrow[dr, "p"] & \\ (S_1/pt, \varphi) \arrow[ur] \arrow[dr, "p"'] & & (S_3/\mathbb{P}^1, \varphi) \\ & (S_2/pt, \varphi) \arrow[ur] & \end{tikzcd}$$

Thm (Kaloghiros, 2013; case for surfaces over any field)

This square complex is connected and simply connected.

Now: $\mathrm{Bir}_k\mathbb{P}^2$ acts on it; $f\cdot(S/B,\varphi) = (S/B, f\circ\varphi)$

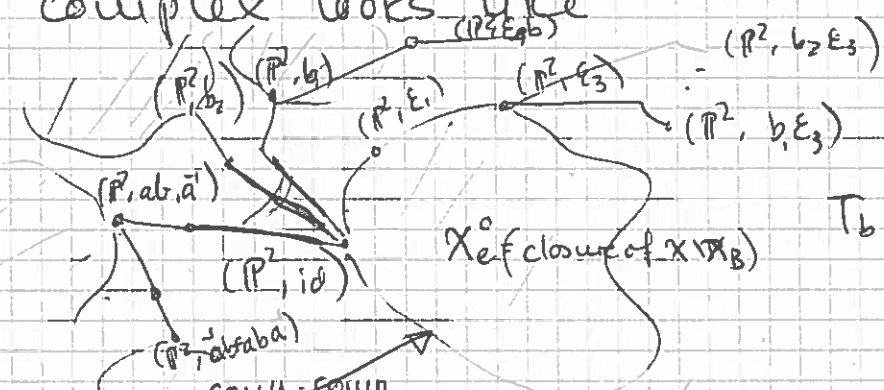
Note:

$$\begin{tikzcd} & (S/pt, \varphi_{\eta_b}) \arrow[dl, "8"'] \arrow[dr, "8"] & \\ (\mathbb{P}^2, \varphi) \arrow[rr, dashed, "b"'] & & (\mathbb{P}^2, b) \end{tikzcd}$$

~~does~~ is not contained in a square

So the complex looks like

- $\mathcal{X}_B^\circ$ = tree
- $\mathcal{X}_e^\circ$ simply conn

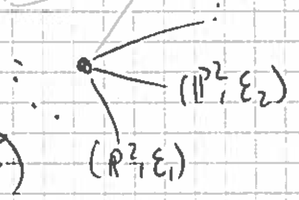


$\mathcal{X}_B$ = complex whose edges are 4 of 8 pts,
$\mathcal{X}_e = \mathcal{X}\setminus\mathcal{X}_B$

T_b = convex hull of $G_b\cdot(\mathbb{P}^2, id)$
$\ni(\mathbb{P}^2, g)\quad g\in G_b$
$\ni[S, g\eta_{a b a^{-1}}]\quad a\in PGL_3(k)$
= (union of) tree(s)

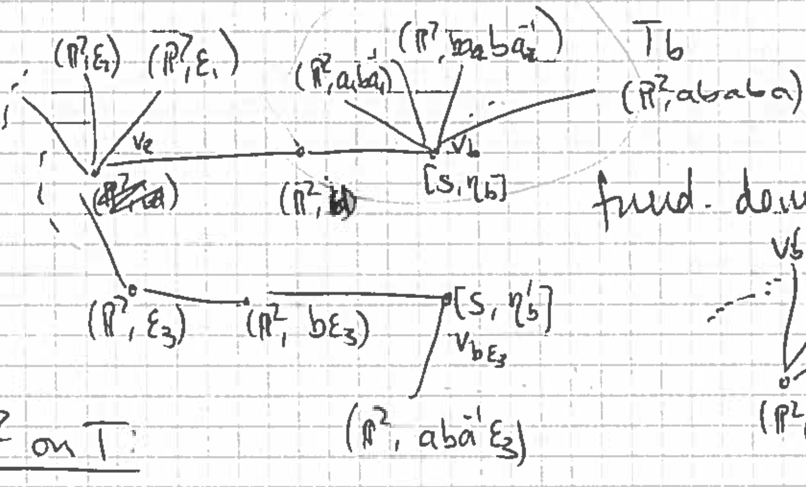
② Tree:

- We contract conn. comp. onto
- We identify all vert. of $\mathcal{X}_e(\mathbb{P}^2, id)$ ~~(S, η_{ab})~~ ~~(S, η_{aba})~~
  $g(T_b)$ & conn. comp. $g(\mathcal{X}_e^\circ)$ $g\in Bir_k\mathbb{P}^2$

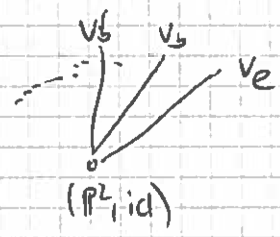


$g(T_b)\cap g(T_b') = \{(\mathbb{P}^2, id)\}$
$g(\mathcal{X}_e^\circ)\cap g(T_b) = \{(\mathbb{P}^2, id)\}$

this yields a tree T and $\mathcal{X}\longrightarrow T$



fund. domain:



Action of $Bir_k\mathbb{P}^2$ on T:

Explain stablisers ~~G_e ... $g(\mathcal{X}_e^\circ) = \mathcal{X}_e^\circ$~~

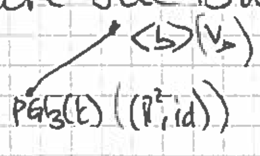
$G_e = Stab_{Bir_k}(\text{conn. comp. } \mathcal{X}_e^\circ)$
$G_b = Stab_{Bir_k}(\{(S, \eta_{aba^{-1}})\mid a\in PGL_3(k)\})$
$PGL_3(k) = Stab_{Bir_k}(\mathbb{P}^2, id)$

$\Longrightarrow Stab_{Bir_k}(v_b) = G_b\ \forall b,\quad Stab_{Bir}(v_e) = G_e,\quad Stab(\mathbb{P}^2, id) = PGL_3(k)$

Bass-Serre $\Longrightarrow$ $Bir_k\mathbb{P}^2 = \underset{PGL_3(k)}{\ast} G_i \qquad I = B\cup\{e\}$

Since $\underset{Aut}{\ast} G_b = \langle Aut, B\rangle \Rightarrow Bir = \langle Aut, B\rangle \underset{Aut}{\ast} G_e$

The T_b are the Bass-Serre trees of G_b with fund domain



$\Rightarrow G_b = PGL_3(k) \ast \underbrace{\langle b\rangle}_{\cong\mathbb{Z}/2\mathbb{Z}}$

we use the universal property to get the homom.

$$Bir_k\mathbb{P}^2 \longrightarrow \underset{|B|}{\ast}\mathbb{Z}/2\mathbb{Z}$$

6

hat er das nun auch für $C_n(\mathbb{C})$ bewiesen?

$n=2$

mein Resultat für $n=2$

~~veröffentlicht?~~ ~~leistet~~ braucht Arbeit

$(S''/B'', \varphi)$

$(S'/B', \varphi')$

$(S/B, \varphi)$

$G = \mathrm{Aut}(\mathbb{P}^2(k))$ B

Stab

$(a^2)^2 = (1+a)^2 = a$

$a^3 = 1$

$\mathbb{P}^2$

$-K_S = -3L + 3E = 3(E-L)$

$p_E = \mathbb{Q}_S$

$[x, \frac{1}{\alpha(x)y}]$

$[\alpha(x)xy : z^d]$

$\alpha(x) \in \mathbb{Q}(x)$

$\alpha(x)yz^2]$

$[\bar{\alpha}(x)xy : z^d : \bar{\alpha}(x)yz^2]$

red. mod p

$[:co]$

vielleicht nach Laij

$x^2+x+1 = (x-a)(x-(1+a))$

$a\ b\ Ge = b\ Ge$

$b\ a\ b\ c\ Ge \Rightarrow a = id$

$\alpha(x) \neq 0$ in $\mathbb{F}_p$

$x^{p^s} - x = \prod_{a \in \mathbb{F}_{p^s}} (x-a)$

$= x(x^{p^s-1}-1)$

$\mathbb{F}_{2^8}$

$\#(\mathbb{F}_2)$

$\mathbb{F}_{p^4}$

$xyz = x^3 - z^3$

$\left(a, \frac{a^3-1}{a}\right)$

$a \in k$ ~~von grad~~ 8

ordnet grad 8

$\mathbb{F}_{2^8}$

$\mathbb{F}_{2^8} \setminus \mathbb{F}_{2^4}$

deg 2

$\mathrm{Gal}(\mathbb{F}_{2^8}/\mathbb{F}_2) = \mathbb{Z}/8\mathbb{Z} = \langle x \mapsto x^{(2)} \rangle$

$x \mapsto x^p \mapsto (x^p)^p = x^{p^2}$

$x^{p^8} = x$

$a_1, \ldots, a_8$ dicht

$\mathbb{F}_2(a_1) = \mathbb{F}_{2^8}$