L-equivalence of K3 surfaces

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Introduction: Geometric meaning of D-equivalence

Grothendieck ring of varieties and L-equivalence

Quadrics, quadric fibrations and K3 surfaces

D-equivalence

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- Some flops [Bondal-Orlov, Bridgeland...]

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- For instance, it is known in dimension 3 [Bridgeland, Kawamata]

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- $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y, \alpha)$ [Mukai, Caldararu...].

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If X and Y are related by a flop then $[X] = [Y]$.

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Corollary

If X, Y are non-uniruled (e.g. K3s or Calabi-Yau) such that $[X] \equiv [Y] \pmod{\mathbb{L}}$ then X and Y birational.

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- Borisov 2014 (improved by Martin 2016): $\mathbb{L}^6([X] - [Y]) = 0$ for Calabi-Yau threefolds X and Y in the Pfaffian-Grassmannian correspondence

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It is easy to see that L-equivalent varieties have the same Hodge numbers, so this would imply invariance of Hodge numbers under D-equivalence.

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- $Y \to \mathbb{P}^1$ double cover ramified in Z; Y is an elliptic curve.

Properties of X and Y

- Over an algebraically closed field, X and Y are D-equivalent, L-equivalent, and in fact isomorphic.
- Over non-algebraically closed field, X and Y don't have to be isomorphic, but are unlikely to be D-equivalent or L-equivalent either

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Mukai's approach: $Y = M_X(2, H, 2)$, the non-fine moduli space of spinor bundles on X and $\alpha_Y = 0$ if and only if the moduli space is fine.

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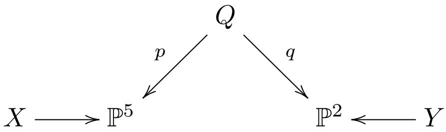

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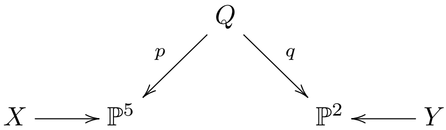

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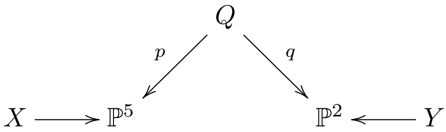

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- Use hyperbolic reduction of quadrics to relate $[H]$ to the double cover Y

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- We get a relation $[Q] = 1 + \mathbb{L}^{\dim(Q)} + \mathbb{L}[\overline{Q}] \in K_0(Var/k)$.

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If $Q \to S$ is a quadric fibration of relative dimension n and $s\colon S \to Q$ is a smooth section, then

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Let $Q \to \mathbb{P}^m$ be a linear system of quadrics in $\mathbb{P}^{n+1}$. Let X be the base locus of this system. Any smooth point $x \in X$ determines a section of Q and the Lemma allows us to relate $[Q]$ and $[\overline{Q}]$.

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This applies in particular to our K3 surface $X = Q_1 \cap Q_2 \cap Q_3$ and gives us a quadric fibration $\overline{Q} \to \mathbb{P}^2$ of relative dimension 2 with the same Brauer class α_Y.

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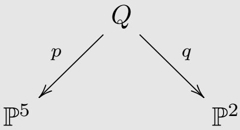

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- Finally: canceling matching terms gives $\mathbb{L}^2[X] = \mathbb{L}^2[Y]$

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Remark: refining the argument one can show that $\alpha_Y = 0 \implies \mathbb{L}([X] - [Y]) = 0$.

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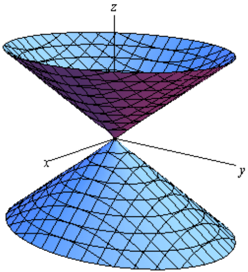

THE END