Complete intersections with a pt of high multiplicity (1)

V a primitive Fano variety: $\mathbb{Q}$-fact terminal sing., $\operatorname{rk}\operatorname{Pic} V = 1$, $-K_V$ ample

__bir. superrigid__ for every mobile linear system $\Sigma \subset |-nK_V|$, $n \in \mathbb{Q}_+$

every bir. morphism $\varphi: \widetilde{V} \to V$ with $\widetilde{V}$ non-sing. proj.

every φ-exceptional prime divisor $E \subset \widetilde{V}$

$$\operatorname{ord}_E \varphi^* \Sigma \leqslant n \cdot a(E, V)$$

[$=$ Σ has no maximal singularities]

__Known & easy__ V bir. superrigid $\Longrightarrow$ for every bir. map

$$\begin{tikzcd} V \arrow[r, dashed, "\chi"] & V' \arrow[d, "\text{Mfs}"] \\ & S' \end{tikzcd}$$

$S' = *$ a pt

χ is an isomorphism

($\Rightarrow V$ non-rational, $\operatorname{Bir} V = \operatorname{Aut} V$, etc.)

TECHNIQUES: the $4n^2$-inequality (a local fact)

X non-singular $\supset B$ irr. subvariety of codim ≥ 3
Σ mobile LS with a MS E (a divisor over X): $\mathrm{ord}_E \Sigma > n\, a(E)$
($n > 0$ some)

$\Longrightarrow$ the self-intersection $Z = (D_1 \circ D_2)$ for general $D_1, D_2 \in \Sigma$

satisfies $\mathrm{mult}_B Z > 4n^2$

Remarks
- reduce to the case $B = o$ a pt
- application $V = V_4 \subset \mathbb{P}^4$ three-dim. quartic
  here $\Sigma \subset |nH|$ cut out on V by hypers. of degree n
  Z is a 1-cycle on V of degree $4n^2$ ∎
  ↳ exclusion of a MS : need higher local estimates

the $4n^2$-inequality for complete int. sing. $o \in X \subset \mathbb{C}^{M+k}$ given by

$$\begin{cases} 0 = q_{1,\mu_1} + \dots \\ \dots \\ 0 = q_{k,\mu_k} + \dots \end{cases} \qquad \mu_i \geqslant 2, \quad M \geqslant k + |\underline{\mu}| + 3, \quad \underline{\mu} = (\mu_1, \dots, \mu_k)$$

$$|\underline{\mu}| = \mu_1 + \dots + \mu_k$$

general position for a general linear $\mathbb{P} \subset \mathbb{C}^{M+k}$, $\mathbb{P} \ni o$, $\dim \mathbb{P} = 2k + |\underline{\mu}| + 3$:

- the singularity $o \in X_{\mathbb{P}} = X \cap \mathbb{P}$ isolated,
- for its blow up $\varphi_{\mathbb{P}}: X_{\mathbb{P}}^{+} \to X_{\mathbb{P}}$, $Q_{\mathbb{P}} = \varphi_{\mathbb{P}}^{-1}(o)$ is a non-singular complete intersection $\{q_{1,\mu_1} = \dots = q_{k,\mu_k} = 0\}$ of type $\underline{\mu}$ in $\mathbb{P}^{2k+|\underline{\mu}|+2}$ so that $X_{\mathbb{P}}^{+}$ is non-singular around $Q_{\mathbb{P}}$

<u>Assume</u> Σ mobile LS on X s.t.

$(X, \frac{1}{n}\Sigma)$ not canonical at o, but canonical outside o for some $n > 0$

$\equiv$ for some E over X, $\operatorname{centre}(E, X) = o$ $\quad \operatorname{ord}_E \Sigma > n\,a(E, X)$

for every E over X, $\operatorname{centre}(E, X) \neq o$ $\quad \operatorname{ord}_E \Sigma \leq n\,a(E, X)$

$\Rightarrow$ for the self-intersection $Z = (D_1 \circ D_2)$, $\quad D_1, D_2 \in \Sigma$ general

$$\operatorname{mult}_o Z > 4n^2\mu$$

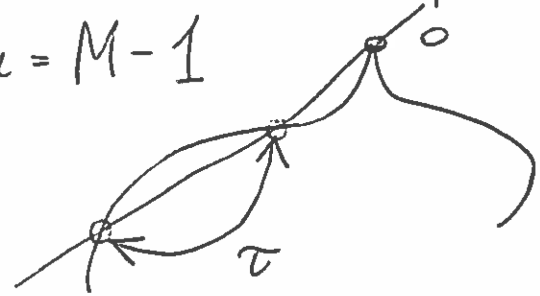
$$\mu = \mu_1 \cdot \ldots \cdot \mu_k = \operatorname{mult}_o X$$

<u>Ex.</u> $k = 1$ hypersurface $X \subset \mathbb{C}^{\mu+5}$ given by $f = q_\mu + q_{\mu+1} + \ldots$

$\mathbb{C}^{\mu+5} = \mathbb{P}$ isolated singularity, $\{q_\mu = 0\}$ non-singular

general case: $X \subset \mathbb{C}^{M+1}$, $M \geq \mu + 4$.

Ex. (P., 2003) $V \subset \mathbb{P}^{M+1}$, $\deg V = M+1$ general
with the only singular pt $o \in V$ of multiplicity $\mu \in \{2, 3, \dots, M-1\}$.
when $\mu \leq M-2$ bir. superrigid

$\mu = M-1$  bir. rigid: $\mathrm{Bir}\, V = \langle \tau \rangle \cong C_2$ cyclic.

cases $\mu = 3, 4$ hard now $\mathrm{mult}_o Z > 12n^2, 16n^2$ resp
easy to exclude ∎

Ex. (D. Evans, P, work in progress) $V \subset \mathbb{P}^{M+k}$ complete intersections of type $\underline{d} = (d_1, \dots, d_k)$, $2 \leq d_1 \leq \dots \leq d_k$, $|\underline{d}| = M+k$ with at most multi-quadratic singularities: birationally superrigid the complement in the param. space $\mathcal{P}(\underline{d})$ of codim $\sim \frac{1}{2} M^2$ ∎

Complete int. with a pt of high multiplicity $k \geq 2$, $M \geq 2k+1$

- $\mathbb{P} = \mathbb{P}^{M+k}$, $\underline{d} = (d_1, \dots, d_k) \in \mathbb{Z}^k_+$, $2 \leq d_1 \leq \dots \leq d_k$, $|\underline{d}| = M+k$
- $\underline{\xi} = (\xi_1, \dots, \xi_k) \in \mathbb{Z}^k_+$, $1 \leq \xi_i \leq d_i$, satisfying:

$$(1)\quad \sum_{i=1}^{k} \left[(d_i+1)(d_i+2) - \xi_i(\xi_i+1) \right] \geq 4M + 2d_k + 2c_* - 2k$$

where $c_* = \# \{ i \mid \xi_i = d_i ,\ 1 \leq i \leq k \}$

$$(2)\quad M \geq 3 + \sum_{\xi_i \geq 2} (\xi_i + 1)$$

General tuple $\underline{f} = (f_1, \dots, f_k)$ of homogen. poly.'s of degree $d_1, \dots, d_k$ such that $f_i(o) = 0$ for some $o \in \mathbb{P}$, $i = 1, \dots, k$

$$\operatorname{mult}_o f_i = \operatorname{mult}_o \{ f_i = 0 \} = \xi_i ,\quad i = 1, \dots, k$$

so that $V = V(\underline{f}) = \{ f_1 = \dots = f_k = 0 \}$ is a complete int. of codim k in $\mathbb{P}$

- non-singular outside o
- $\mathrm{mult}_o V = \mu = \xi_1 \cdots \xi_k$: a complete int. singularity of type $(\xi_{i_1}, \dots, \xi_{i_\ell})$ where $\{i_1, \dots, i_\ell\} = \{i \mid \xi_i \geqslant 2\}$

Note. $\sup(\mu/d) = 1$ for all tuples $\underline{\xi}, \underline{d}$ satisfying (1), (2).

THEOREM For a Zariski general tuple $\underline{f}$: V is bir. superrigid.

PROOF Recall the non-singular case. Except for 3 infinite series:

$$(2, \dots, 2) \qquad (2, \dots, 2, 3) \qquad (2, \dots, 2, 4)$$

+ finitely many families of $\dim \leq 12$, generic complete inters of index 1 are bir. superrigid.

for $\dim V = M \geqslant 2k+1$ means

<u>Regularity conditions</u> $p \in V$ any: $(z_1, \dots, z_{M+k})$ on $\mathbb{A}^{M+k} \subset \mathbb{P}^{M+k}$

$p = (0, \dots, 0)$

$$f_1 = q_{1,1} + \dots + q_{1,d_1}$$

$$\dots$$

$$f_k = \underbrace{q_{i,1}}_{} + \dots + q_{k,d_k}$$

lin. independent

$\longrightarrow$ $q_{i,j}$ form a regular sequence

$\{ q_{i,j} = 0 \mid (i,j) \neq (k, d_k) \}$ gives finitely many lines through p

($M+k-1$ polynomials)

Needs to be true for every point $p \in V$

Therefore, to prove THM, we need:

1. For a general tuple $\underline{f} = (f_1, \dots, f_k)$ the reg. condition holds at every <u>non-singular</u> pt $p \neq 0$ Argue as in the non-sing case

2. The centre of a MS E can not be the <u>singular</u> pt o

For 2: assume the converse $\Sigma \subset |nH|$ mobile LS

H ← hyperplane section

$$\operatorname{ord}_E \Sigma > n\, a(E)$$

for some E over V s.t. centre$(E) = o$

Then by the <u>new</u> $4n^2$-inequality $\operatorname{mult}_o Z > 4n^2 \mu$

$$f_1 = q_{1,\xi_1} + \dots + q_{1,d_1}$$

$$\dots \qquad \longleftarrow f_i = q_{i,\xi_i} + q_{i,\xi_i+1} + \dots + q_{i,d_i}$$

$$f_k = q_{k,\xi_k} + \dots + q_{k,d_k}$$

$$d_i - \xi_i \geq 1.$$

So we get <u>hypertangent divisors</u> $\left(q_{i,\xi_i}+\ldots+q_{i,\alpha}\right)\Big|_V = \left(-q_{i,\alpha+1}+\ldots\right)\Big|_V$

$$\cap$$

$$|\alpha H - (\alpha+1)Q|$$

the except. div. of blow up of o

$$\frac{\mathrm{mult}_o}{\deg} Z > \frac{4\mu}{d}, \quad d = d_1 \cdot \ldots \cdot d_k = \deg V$$

so for some irreducible Y of codim 2

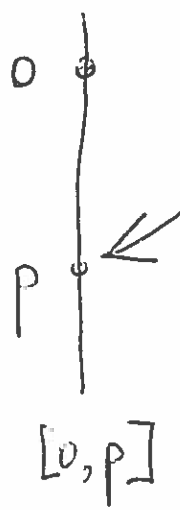
$$\frac{\mathrm{mult}_o}{\deg} Y > \frac{4\mu}{d}$$

and intersecting with all hypertangent divisors except 2:

$$\frac{\mathrm{mult}_o}{\deg}\left(\begin{array}{c}\text{positive-}\\ \text{dim}\\ \text{subvariety}\end{array}\right) > \frac{4\,\xi_1\ldots\xi_k}{d_1\ldots d_k} \underbrace{\text{“}\prod_{i=1}^{k}\prod_{j=\xi_i}^{d_i-1}\frac{j+1}{j}\text{”}}_{\text{with 2 factors omitted} \leq 2\cdot 2=4} = 1 \quad \blacksquare$$

For 1 : non-trivial as $q_{i,j}$ arbitrary, **BUT** we need

re-calculate $f_i = \Phi_{i,1} + \Phi_{i,2} + \ldots + \Phi_{i,d_i}$ at a point $p \neq 0$, $p \in V$



$\Phi_{i,j}$ all except for Φ_{k,d_k} must form a reg. sequence

not arbitrary independent poly.'s

- express in terms of $q_{i,j}$
- at a fixed point $p \neq 0$

violation imposes $\geq M+k+1$ indep. cond. on $q_{i,j}$ ∎

- easy case: $[0,p] \not\subset T_p V$

harder case: $[0,p] \subset T_p V$