30/06/17
Edinburgh

$P \in X$ sm.

$\Delta \geqslant 0$ $\mathbb{R}$-div. (X, Δ) pair

E a divisor over X $\overset{\text{def}}{\Leftrightarrow}$ $\underset{\text{prime}}{E} \subset \underset{\text{normal}}{{}^{\exists}Y} \xrightarrow[\text{birat.}]{\pi} X$

log discrep. $a_E(X, \Delta) := 1 + \mathrm{Coef}_E(K_{Y/X} - \pi^*\Delta)$

centre $C_X(E) := \overline{\pi(E)}$

min. log disc. $\mathrm{mld}_P(X, \Delta) := \inf\limits_{\substack{E \text{ div}/X \\ C_X(E) = P}} \{a_E(X, \Delta)\} \in \underbrace{\mathbb{R}_{\geqslant 0}}_{\Leftrightarrow \text{ lc}} \cup \{-\infty\}$

Def E div./X, computes $\mathrm{mld}_P(X, \Delta)$ $\overset{\text{def}}{\Leftrightarrow}$
$$\left| \begin{array}{l} C_X(E) = P. \\ \text{AND} \\ a_E(X, \Delta) \begin{cases} = \mathrm{mld}_P(X, \Delta) & \text{lc} \\ \text{OR} & \\ < 0 & \text{not lc} \end{cases} \end{array} \right.$$

MMP: term. of flips $\overset{\text{Shokurov}}{\Leftarrow}$ $\underset{\text{dim } 3}{\text{lower s.c.}}$ + $\underset{\text{dim } 2}{\text{ACC}}$ for mld

for X sm, by the theory of generic limit:

- ACC for a-lc thresholds
- (Mustaţă) m-adic s.c.
- (Nakamura) boundedness conj: $P \in X$ sm. $I \subset \mathbb{R}_{>0}$ finite

$\Rightarrow$ ${}^{\exists}\ell$ s.t. ${}^{\forall}\Delta \in I$ (i.e. ${}^{\forall}$coef in Δ $\in I$).

${}^{\exists}E$ div / X s.t. $\left\{ \begin{array}{l} E \text{ computes } \mathrm{mld}_P(X, \Delta) \\ a_E(X) \leqslant \ell \end{array} \right.$

dim 3: the above holds for mld $\geqslant 1$.

remaining case: (X, Δ) has a unique lc centre, of dim 1

W to understand properties of E which comp's mld.

Thm $P \in X$ sm. dim 2. (X, Δ) pair

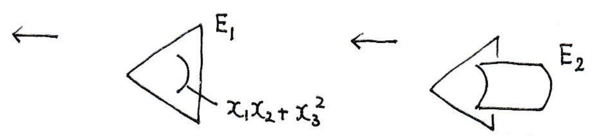
(1) (X, Δ) lc $\Rightarrow$ ${}^{\forall}E$ comp'g $\mathrm{mld}_P(X, \Delta)$ is obtained by a wtd bl-up

(2) not lc $\Rightarrow$ ${}^{\exists}E$ — // —

i.e. ${}^{\exists}x_1, x_2 \in \mathcal{O}_{X,P}$ RSOP. ${}^{\exists}w_1, w_2 \in \mathbb{N}$

$E \subset \mathrm{WBl}(X) \longrightarrow X$ w.r.t. $\mathrm{wt}(x_1, x_2) = (w_1, w_2)$

false in dim 3: Ex $P \in \mathbb{A}^3$ x_1, x_2, x_3 Δ def'd by $(x_1x_2 + x_3^2) + m^3$

$\mathbb{A}^3 \ni P \longleftarrow$ 

$a_{E_1} = 1/3$ log resol. $a_{E_2} = 0 = \mathrm{mld}_P(X, \Delta)$

pf) (2) easily reduced to (1)

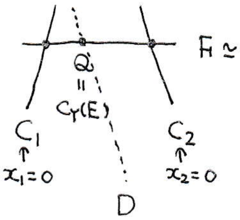
(1) E comp'g $\mathrm{mld}_P(X, \Delta)$

set $\quad w_1 := \max_{x_1 \in m \setminus m^2} \{\mathrm{ord}_E\, x_1\} \qquad = g w_1'$

$\quad w_2 := \min_{x_2 \in m \setminus m^2} \{\mathrm{ord}_E\, x_2\} \ (= \mathrm{ord}_E\, m) \quad = g w_2'$

$$\begin{array}{ccc} Y = \mathrm{WBl}(X) & \longrightarrow & X \\ \cup & & \Psi \\ F & \longrightarrow & P \end{array} \qquad \mathrm{wt}(x_1, x_2) = (w_1', w_2')$$

enough to show : $E = F$



$F \simeq \mathbb{P}(w_1', w_2')$

str. tf of $x_1^{w_2'} + \lambda_0 x_2^{w_1'} = 0$ $\quad$ ($\lambda_0 \in k^\times$)

$w_i = \mathrm{ord}_E F \cdot \underbrace{\mathrm{ord}_F x_i}_{= w_i'} + \mathrm{ord}_E C_i$

$\dfrac{w_2'}{w_1'} = \dfrac{w_2}{w_1} = \dfrac{w_2' \,\mathrm{ord}_E F + \mathrm{ord}_E C_2}{w_1' \,\mathrm{ord}_E F + \mathrm{ord}_E C_1} \quad \to \quad C_Y(E) \not\subset F \cap C_i$

$a_E \leq a_F \Rightarrow (Y, F + \Delta_Y)$ not plt at Q

$\overset{\text{inv. adj.}}{\Leftrightarrow} (F, \Delta_Y|_F)$ not klt at $Q \quad \Leftrightarrow \mathrm{ord}_Q \Delta_Y|_F \geq 1$

$\Delta_Y|_F = x_1^{s_1} x_2^{s_2} \prod_{\lambda \in k^\times} (x_1^{w_2'} + \lambda x_2^{w_1'})^{t_\lambda}$

$\mathrm{ord}_Q \Delta_Y|_F = t_{\lambda_0} \leq 1/w_1' + 1/w_2' \qquad w_2' = 1$

$t_{\lambda_0} \leq \dfrac{\mathrm{ord}_F \Delta_Y}{w_1' w_2'} \leq \dfrac{w_1' + w_2'}{w_1' w_2'}$

$\mathrm{ord}_E (x_1 + \lambda_0 x_2^{w_1'}) = \underbrace{\mathrm{ord}_E F \cdot w_1'}_{w_1} + \underbrace{\mathrm{ord}_E D}_{> 0} > w_1 \quad \lightning$ q.e.d.

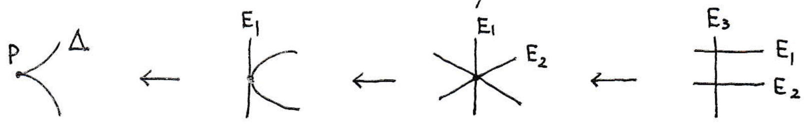
relevant inv : lc threshold

easier, focusing a boundary B s.t. $\mathrm{mld}_P(X, B) = 0$

$P \in (X, \Delta)$ lc, want to construct Γ s.t. $\mathrm{mld}_P(X, \Delta + \Gamma) = 0$

Q $P \in (X, \Delta)$ lc $\Rightarrow \exists ?\ \Gamma$ s.t. some div. E computes $\begin{cases} \text{mld} : \mathrm{mld}_P(X, \Delta) \\ \text{AND} \\ \text{lct} : \mathrm{mld}_P(X, \Delta + \Gamma) = 0 \end{cases}$ $\xrightarrow{\text{MMP}} \oplus \mathcal{O}(-iE)$ f.g.

false in dim 2 : Ex $P \in \mathbb{A}^2$ x_1, x_2 Δ by $(x_1^2 + x_2^3)^{\frac{2}{3}}$



$a_{E_1} = 2/3 \qquad a_{E_2} = 1 \qquad a_{E_3} = 1$

$\mathrm{mld}_P(\mathbb{A}^2, \Delta + \Gamma) = 0 = a_{E_1}(\mathbb{A}^2, \Delta + \Gamma) \quad \to \mathrm{ord}_{E_1} \Gamma = 2/3$

$\to \mathrm{ord}_{E_3} \Gamma \geq 4/3 \qquad a_{E_3}(\mathbb{A}^2, \Delta + \Gamma) \leq 1 - 4/3 < 0 \quad \lightning$

in any dim : Prop $P \in X$ sm. (X, Δ) pair $\quad E \subset \mathrm{Bl}_P X \longrightarrow X$

(1) $\mathrm{mult}_P \Delta \leq 1 \quad \Rightarrow \quad E$ comp's $\mathrm{mld}_P(X, \Delta)$

(2) $\qquad\quad < 1 \quad \Rightarrow \quad E$ unique div. which comp's $\mathrm{mld}_P(X, \Delta)$

$\to a_{E_{i+1}} \geq a$