STABILITY AND CSCK METRICS ON POLARISED DEL PEZZO SURFACES (Joint with Cheltsov)

1. MOTIVATION

$\boxed{\text{THANKS!!}}$

$X \equiv$ projective variety (smooth/$\mathbb{C}$)
$L \equiv$ ample ($\mathbb{Q}$-)line bundle.
$n = \dim X$.

Calabi ('50s–60s)
When does (X,L) admit a constant scalar curvature Kähler metric ω? (i.e. $d\omega = 0$, $\omega \in c_1(L)$)

CONJECTURE (YTD: YAU–TIAN–DONALDSON)

(X,L) is csck $\Leftrightarrow$ (X,L) is k-polystable
(analytic notion) (algebraic notion)

CAVEAT!
Not always True: Many obstructions; e.g.:
- Non-reductive $Aut_0(X,L)$. (Matsushima)
- Non-trivial holomorphic vector fields. (Futaki)
- GIT-instability & other stability notions
- ⋮

What is known: ☆ Cases $L = \pm K_X, \mathcal{O}_X$ (Aubin; Yau; Chen–Donaldson–Sun)
☆ "$\Rightarrow$" (Berman–Darvas–Lu).
☆ Toric surfaces (Donaldson; Codogni–Stoppa)

Why is this important for an (algebraic) geometer?

- Interesting PDEs.
- Analogue for varieties of the Kobayashi–Hitchin correspondence for vector bundles and Hermitian-Einstein metrics
- Canonical way of choosing a unique ~~[illegible]~~ Kähler metric

• Moduli Theory: K-stable varieties expected to compactify canonically by ~~k-pol~~ strictly K-polystable varieties. (Known for $L = \pm K_X$) with a canonical ample line bundle for the moduli (Known for $L = K_X$ Li-Xu; Odaka.)

~~• WAKE-UP FELLOW NUMBER THEORIST: K-stability can be defined over any field, even ~~algebraically~~ finite characteristic. Implications yet to be explored.~~

~~How to detect it~~

2. What is K-stability? • Several equivalent definitions.

~~We use one coming from~~

• We use one which feels like GIT under Hilbert-Mumford

A test-configuration of (X, L) is $(\mathcal{X}, \mathcal{L}, p)$ where

~~• $p: \mathcal{X} \to \mathbb{P}^1$ is a $\mathbb{G}_m$ equivariant~~

• $\mathcal{X}$ is a normal projective variety with a $\mathbb{G}_m$-action.

• $p: \mathcal{X} \to \mathbb{P}^1$ is a flat $\mathbb{G}_m$-equivariant morphism st $p^{-1}(\tau) \cong X$ $\forall \tau \in \mathbb{P}^1 \setminus \{0\}$

• $\mathcal{L} \to \mathcal{X}$ is a $\mathbb{G}_m$-linearised $\mathbb{G}_m$-equivariant p-ample line bundle st $\mathcal{L}|_{p^{-1}(\tau)} \cong L^{\otimes r}$ $\forall \tau \in \mathbb{P}^1 \setminus \{0\}$

$(\mathcal{X}, \mathcal{L}, p)$ is trivial iff $(\mathcal{X}, \mathcal{L}) \cong (X \times L)$ (product T.C) and $\mathbb{G}_m$ acts trivially on fibres.

The Donaldson-Futaki invariant of $(\mathcal{X}, \mathcal{L}, p)$ is $(r=1)$

$$DF(\mathcal{X}, \mathcal{L}, p) = \frac{1}{r^n}\left(\frac{n}{n+1}\,\frac{1}{r}\,\frac{-K \cdot L^{n-1}}{L^n}\,\mathcal{L}^{n+1} + \mathcal{L}^n \cdot K_{\mathcal{X}/\mathbb{P}^1}\right).$$

③

(X,L) is K-semistable $\iff$ $DF(\mathcal{X},\mathcal{L},p) \geq 0$ $\forall$ TC.

$\Uparrow$

(X,L) is K-polystable $\iff$ $DF(\mathcal{X},\mathcal{L},p) \geq 0$ $\forall$ non-trivial T.C and $DF(\mathcal{X},\mathcal{L},p) = 0$ only if $(\mathcal{X},\mathcal{L},p)$ is product.

$\Uparrow$

(X,L) is K-stable $\iff$ $DF(\mathcal{X},\mathcal{L},p) > 0$ $\forall$ non-trivial T.C.

(X,L) is K-unst $\iff$ not K-ss.

Notice analogue with GIT.

TODAY: We focus on constructing destabilising test-config $(DF(\mathcal{X},\mathcal{L},p) < 0)$ for del Pezzo surfaces.

Since we have (X,L) cscK $\overset{BDL}{\Rightarrow}$ (X,L) K-polyst $\Rightarrow$ (X,L) K-ss.

We'll provide obstructions to $\exists$ of cscK metrics.

SETTING: let (S,L) be a smooth polarised surface, $Z \subseteq S$ a curve. Ross-Thomas constructed a test-configuration by deformation to the normal cone:

KEEP.

$$\begin{tikzcd}
 & S\times\mathbb{P}^1 \arrow[dl, "p_S"'] \arrow[ddd, "\pi_{\mathbb{P}^1} = \text{blow-up of } O_1\times\mathbb{P}^1,\dots,O_k\times\mathbb{P}^1"] & \mathcal{X} \arrow[l, "\pi_Z = \text{blow-up } Z\times\{0\}"'] \arrow[d, dashed, "p"] \arrow[dddd, bend left, "p"] \\
S \arrow[d, "\text{blow-up of } O_1,\dots,O_k \equiv \pi"'] & & \tilde{\mathcal{X}} \arrow[d, "\pi_\pi"] \arrow[ddd, bend left, "q"] \\
\bar{S} & & \bar{\mathcal{X}} \arrow[dl, "\pi_{\bar{Z}}"'] \arrow[dd, "\bar{q}"] \\
 & \bar{S}\times\mathbb{P}^1 \arrow[ul, "q_{\bar{S}}"] \arrow[dr, "q_{\mathbb{P}^1}"'] & \\
 & & \mathbb{P}^1
\end{tikzcd}$$

$\mathbb{P}^1$

Seshadri constant: $\sigma(X,L,Z)=\sup\{\lambda \mid L-\lambda Z \text{ is ample}\}$. $\sigma \geq \tau$.
Pseudo-effective Threshold: $\tau(X,L,Z)=\sup\{\mu \mid L-\mu Z \text{ is pseudo-eff}\}$.

E_Z: Π_Z = exceptional divisor, $\mathcal{L}_\lambda := (p_S \circ \Pi_Z)^*(L)-\lambda E_Z$ is p-ample iff $\lambda < \sigma(S,L,Z)$.

$$\mathrm{DF}(\lambda) = \mathrm{DF}(\mathcal{X},\mathcal{L}_\lambda)=\underbrace{\frac{2}{3}\frac{KL}{L^2}\left(\lambda^3 Z^2-3\lambda^2 L\cdot Z\right)+\lambda^2(2-2g(Z))+2\lambda L Z}_{\text{polynomial}}.$$

If $\mathrm{DF}(\sigma(S,L,Z))<0 \Rightarrow (X,L)$ is K-unstable. KEEP

What Z to choose? Ross-Panov: Z is (-1)-curve.

Think of S as:
~~PROBLEM~~: If $\pi: S \to \bar{S}$ is a blow-up, $\bar{Z}=\pi_*(Z)$ and π is blow-up of $O_1,\dots,O_k \in \bar{Z}$, $\bar{L}=\pi_*(L)$,

Problem: $0<\lambda<\sigma(S,L,Z)\leq\sigma(\bar{S},\bar{L},\bar{Z})$ (usually $\ll$.)

For del Pezzo surfaces this only works for $S \simeq \mathbb{F}_1 \downarrow \bar{S}=\mathbb{P}^2$, L arbitrary;
$S\simeq dP_7 \xrightarrow{Bl_{p_1,p_2}} \mathbb{P}^2\simeq\bar{S}$, L very special

The reason is that λ cannot be very large the more you blow-up:

CHELTSOV-RUBINSTEIN.

Let $C_1,\dots,C_k$ be the π-exceptional curves and ~~identify~~ ~~then~~ let $\mathcal{C}_1,\dots,\mathcal{C}_k$ be their proper transforms in the central fibre of $\mathcal{X}$. It can be shown that $\mathcal{C}_i\simeq\mathbb{P}^1$ ~~$N_{\mathcal{C}_i/\mathcal{X}}\simeq O(-1)$~~

$N_{\mathcal{C}_i/\mathcal{X}}\simeq O(-1)\oplus O(-1) \rightsquigarrow$ We can flop them!!

$$\begin{tikzcd} \mathcal{X} \arrow[r, dashed, "\rho"] \arrow[dr, "p"'] & \hat{\mathcal{X}} \arrow[d, "q"] \\ & \mathbb{P}^1 \end{tikzcd}$$

$\hat{\mathcal{L}}_\lambda=\rho_*(\mathcal{L}_\lambda)$

Problem: flops do not (necessarily) preserve projectivity

Why is $\hat{\mathcal{X}}$ projective?

Let $E_{\bar{Z}}$ be $\Pi_{\bar{Z}}$-exc. divisor $\subset\bar{\mathcal{X}}$.

Let $F_i = \Pi_{\bar{Z}}$-transform of $O_i\times\mathbb{P}^1 \subseteq \bar{\mathcal{X}}$.

Answer: ~~proj~~

FACT (C-R)
$\Pi_\Gamma \equiv$ blow-up of $\Gamma_1, \ldots, \Gamma_k$ $\Pi_\Gamma : \hat{\mathcal{X}} \longrightarrow \overline{\mathcal{X}}$.

$$\begin{tikzcd} \hat{X} \arrow[r, "\Pi_\Gamma"] \arrow[dr, "q"'] & \overline{\mathcal{X}} \arrow[d, "\bar{q}"] \\ & \mathbb{P}^1 \end{tikzcd}$$

and general fibre of q is S.

So $(\hat{\mathcal{X}}, \hat{\mathcal{L}}_\lambda, q)$ is a test-configuration ~~flop-slope T.C. centred at Z with flopping curves $C_1, \ldots, C_r$.~~ whenever $\hat{\mathcal{L}}_\lambda$ is q-ample:

FACT (C-R) $\hat{\mathcal{L}}_\lambda$ is q-ample $\iff$ $L C_i < \lambda < \tau(S, L, Z)$.

ADVANTAGE For most del Pezzos $\tau(S, L, Z) = \sigma(\bar{S}, \bar{L}, \bar{Z}) \geq \sigma(S, L, Z)$.

Moreover any 3-divisors in proj. 3-fold, ρ flop: $\rho(H_1)\rho(H_2)\rho(H_3) = H_1 H_2 H_3$.

We deduce from here and the fact $K_{\mathcal{X}/\mathbb{P}^1} \cdot$ floppable curve $= 0$ that

$$DF(\hat{\mathcal{X}}, \hat{\mathcal{L}}, q) = DF(\mathcal{X}, \mathcal{L}, p) + \underbrace{\frac{2}{3}\frac{-KL}{L^2}\left(\sum_{i=1}^{k} (\lambda - LC_i)^3\right)}_{\text{ADD}}$$

In all examples of (S, L) S del Pezzo, this ~~includes~~ improves bounds a lot.

~~3. DEL PEZZO~~

3. POLARISED DEL PEZZO SURFACES (smooth) $S \equiv$ smooth, $-K_S > 0$, $9 \geq d = K^2 \geq 1$, $S = dP_d$

What was known:

- ~~$L = -K_S$~~ (S, L) $\begin{matrix}(\mathbb{P}^2, L)\\(\mathbb{P}^1\times\mathbb{P}^1, L)\end{matrix}$ always K-polystable.
- $(S, -K_S)$ K-polyst $\iff$ $S \not\cong \mathbb{F}_1$, $K^2 \neq 7$. (Tian)
- $S \simeq \mathbb{F}_1$, $S \simeq dP_7$, $L > 0$ $\Rightarrow$ (S, L) unst. (Matsushima: $A.T._0(S)$ non-reductive)
- $S \simeq dP_6 \xrightarrow{\gamma} \mathbb{P}^2$. $L \sim \gamma^*(\varepsilon \gamma) - \sum \varepsilon_i E_i > 0$. (S, L) K-polyst. $\iff$ $\varepsilon = \sum \varepsilon_i$ or $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$.

- (dP_1, L)
  (dP_2, L), K-polystable if $-K - \frac{2}{3}\frac{(-K)L}{L^2}L$ nef. extra condition for (dP_3, L).
  (Cheltsov-MG, dP_1 by Hong-Won).

- If $A_iT(S) < \infty$. Then (S,L) K-polystable in sense Euclidean neighbourhood of $-K_S$. (LeBrun-Simanca).

~~$\Rightarrow$~~ Note most of this results are qualitative.

$\rightsquigarrow$ We introduce new language for —"— & quantitative.

Let $\mu_L = \inf \{ \lambda \in \mathbb{Q}_{>0} \mid K + \lambda L \in \overline{NE}(S) \} \in \mathbb{Q}_{>0}$.

$\Rightarrow \mu_L L \sim_{\mathbb{Q}} -K + bF + \sum_{i=1}^{m} a_i E_i$

~~...~~ $\{E_i\}$ disjoint exc. curves

$F \sim$ class of a 0-curve. $F \simeq \mathbb{P}^1$.

$0 \leq a_1 \leq a_2 \ldots \leq a_m$. , $b \geq 0$

either $m = 9 - K^2$ and $b = 0$ or

$m = 8 - K^2$ ~~and~~ and $b > 0$

The contraction of $\{E_1, \ldots, E_m, \cancel{E}\}$ induces (MMP).

$S \longrightarrow \hat{S}$ and either.

$\hat{S} \simeq \mathbb{P}^2$. ($L$ of $\mathbb{P}^2$-Type). (~~Contraction of F gives~~

$\hat{S} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (L of $\mathbb{P}^1 \times \mathbb{P}^1$-Type)] Contract also F

$\hat{S} \simeq \mathbb{F}_1$ (L of $\mathbb{F}_1$-Type).] for $S \longrightarrow \mathbb{P}^1$.

How to choose $C_1, \dots, C_r$? ~~Recall $L \sim_{\mathbb{Q}} -K + \sum a_i E_i$~~

$\mathbb{P}^2$-Type (other cases more complex, but also done.

Recall $L \sim_{\mathbb{Q}} -K + \sum_{i=1}^{r} a_i E_i$ $\quad 0 \le a_1 \le \dots \le a_r < 1$ $\quad r = 9 - K^2$.

~~Usi~~ ~~IT~~ By intersecting with (-1)-curves is easy to see that $\sigma(\bar{S}, \bar{L}, \bar{Z})$ is larger for $Z = E_1$ over all (-1)-curves. ~~Indeed~~

Let $\mu = \tau(S, L, Z)$, $\quad u : S \to \mathbb{P}^2$ contraction of $E_1, \dots, E_r$.

L_{1i} proper transform of line through $\pi(E_1), \pi(E_i)$

~~Using Zariski decomposition~~ $\pi : S \to \hat{S} \simeq \mathbb{F}_1$ contraction or $\mathbb{P}^1 \times \mathbb{P}^1$

of $L_{12}, \dots, L_{1r}$. ~~$L - \mu Z \sim$~~ Using Zariski decomp., if $u_3 \ge \frac{2}{3}$

$$L - \mu Z \sim_{\mathbb{Q}} \pi^*(\bar{L} - \mu \bar{Z}) + \sum r_i L_{1i}$$

So hypothesis are satisfied. ~~if~~ and $\mu = \sigma(\bar{S}, \bar{L}, \bar{Z})$ $\quad \bar{Z} = 2 + a_1$

We plug a_i on $DF(\lambda = \mu) = \dfrac{p(a_1, \dots, a_r)}{q}$

~~no dege~~ $q \equiv$ positive polyn. $\quad p \equiv$ deg 4 polyn.

Find conditions on $a_1, \dots, a_r$ to make $p < 0$.

We get many results ~~(Cheltsov-MG)~~ including <u>all</u> previously known obstructions (Toric, Ross-Thomas, ~~Matsushima~~) and new ones:

~~THM (CHELTSOV-MG) (S, L) K-unstable~~ if:

(i) L is of $\mathbb{F}_1$-Type / $\mathbb{P}^1 \times \mathbb{P}^1$-Type and

$$a_1^2 + 6 - K^2 < \sum_{i \ge 2} a_i^2$$

(ii)

(8)

(ii) L is $\mathbb{P}^2$ and $a_2 - a_1 \geq \left\{\begin{array}{ll} 0.8717, & K^2 = 1 \\ 0.8469, & K^2 = 2 \\ \vdots & \vdots \\ 0.6248, & K^2 = 5 \end{array}\right.$

(We actually can find them in terms of the only root $\in [0,1)$ of certain deg 4 polyn's.

(Similar for $a_3 - a_1 \geq$, for $\mathbb{F}_1$-Type, $\mathbb{P}^1 \times \mathbb{P}^1$-Type)

(iii) (Qualitative.

L is of $\mathbb{P}^2$-Type and $1 > a_4 \gg a_3 > a_1$

$\mathbb{P}^1 \times \mathbb{P}^1$ $\quad 1 > a_3 \gg a_2 > a_1$

$\mathbb{F}_1$- $\quad a_3 \gg a_2$.

Why not $a_4 - a_1$, $a_4 - a_3$?

If $a_1 = a_2 = a_3$ by Arezzo Pacard we can lift from $S = dP_6$ to a cscK metric in $\mathbb{P}^2$.