Maximal connected algebraic subgroups of Bir($\mathbb{P}^3$)

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Definitions and reminders

In this talk, the base-field is k, *algebraically closed field, of characteristic* 0 (positive characteristic in preparation).

Aim

Study connected algebraic subgroups of the Cremona group $\mathrm{Bir}(\mathbb{P}^3)$.

The works of Enriques (1893) and Umemura (1980-85) yield

connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^2)$	$\leftrightarrow$	$\mathrm{Aut}^\circ(X)$, X minimal rational surface ($X \simeq \mathbb{P}^2$ or $X \simeq \mathbb{F}_n$: $\mathbb{P}^1$-bundles over $\mathbb{P}^1$).
connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$	$\leftrightarrow$	$\mathrm{Aut}^\circ(X)$, X *some* minimal rational 3-fold (some of them are $\mathbb{P}^1$-bundles over rational surface, e.g. over $\mathbb{P}^2$ or $\mathbb{F}_n$).

Today: the first part

Aim

We want to study $\mathrm{Aut}^\circ(X)$, when $X \to S$ is a $\mathbb{P}^1$-bundle over a projective rational surface S.

Other cases coming from the classification (to appear):

1. Quadric fibrations $X \to \mathbb{P}^1$ (locally $w^2 f(t) + x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^3 \times \mathbb{A}^1$, with an action of $\mathrm{O}(3)$)
2. $\mathbb{P}^2$-bundles over $\mathbb{P}^1$;
3. Finite family of Fano varieties ($\mathbb{P}^3$, quadric $Q \subset \mathbb{P}^4$, $X_{22}, \ldots,$)

The result

Theorem

Let $\pi : X \to S$ be a $\mathbb{P}^1$-bundle over a projective rational surface S. Then, there exists a surface S', being either equal to $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_a$, $a \geq 0$, a $\mathbb{P}^1$-bundle $X' \to S'$, and a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & X' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ S & \underset{\eta}{\dashrightarrow} & S' \end{array}$$

such that ψ, η are $\mathrm{Aut}^\circ(X)$-equivariant birational maps (for some biregular actions of $\mathrm{Aut}^\circ(X)$ on X', S, S') and such that the $\mathbb{P}^1$-bundle $X' \to S'$ is of the following type:

(1)	*a decomposable*	$\mathbb{P}^1$-*bundle*	$\mathcal{F}_a^{b,c}$	$\longrightarrow$ $\mathbb{F}_a$	$a, b \geq 0$, $c \in \mathbb{Z}$;
(2)	*a decomposable*	$\mathbb{P}^1$-*bundle*	$\mathcal{P}_b$	$\longrightarrow$ $\mathbb{P}^2$	$b \geq 0$;
(3)	*an Umemura*	$\mathbb{P}^1$-*bundle*	$\mathcal{U}_a^{b,c}$	$\longrightarrow$ $\mathbb{F}_a$	$a, b \geq 1, c \geq 2$;
(4)	*a Schwarzenberger*	$\mathbb{P}^1$-*bundle*	$\mathcal{S}_b$	$\longrightarrow$ $\mathbb{P}^2$	$b \geq 1$.

Presentations of $\mathbb{P}^2$, $\mathbb{F}_a$

1. The *projective plane* is $\mathbb{P}^2 = (\mathbb{A}^3 \setminus \{0\})^2/(\mathbb{G}_m)$

$$\begin{array}{ccc} \mathbb{G}_m \times (\mathbb{A}^3 \setminus \{0\})^2 & \to & (\mathbb{A}^3 \setminus \{0\})^2 \\ (\mu, (y_0, y_1, y_2)) & \mapsto & (\mu y_0, \mu y_1, \mu y_2) \end{array}$$

The class of (y_0, y_1, y_2) is written $[y_0 : y_1 : y_2]$

2. Let $a \in \mathbb{Z}$. The *a-th Hirzebruch surface* is $\mathbb{F}_a = (\mathbb{A}^2 \setminus \{0\})^2/(\mathbb{G}_m)^2$, where

$$\begin{array}{ccc} (\mathbb{G}_m)^2 \times (\mathbb{A}^2 \setminus \{0\})^2 & \to & (\mathbb{A}^2 \setminus \{0\})^2 \\ ((\mu, \rho), (y_0, y_1, z_0, z_1)) & \mapsto & (\mu\rho^{-a} y_0, \mu y_1, \rho z_0, \rho z_1) \end{array}$$

The class of (y_0, y_1, z_0, z_1) will be written $[y_0 : y_1; z_0 : z_1]$. The projection

$$\tau_a \colon \mathbb{F}_a \to \mathbb{P}^1, \quad [y_0 : y_1; z_0 : z_1] \mapsto [z_0 : z_1]$$

identifies $\mathbb{F}_a$ with $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1})$ as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

$\mathbb{F}_a$ is isomorphic to $\mathbb{F}_{-a}$, via $[y_0 : y_1; z_0 : z_1] \mapsto [y_1 : y_0; z_0 : z_1]$

Decomposable $\mathbb{P}^1$-bundles over $\mathbb{P}^2$

Let $b \in \mathbb{Z}$. We define $\mathcal{P}_b = (\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^3 \setminus \{0\})/(\mathbb{G}_m)^2$ given by

$$\begin{array}{ccc} (\mathbb{G}_m)^2 \times (\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^3 \setminus \{0\}) & \to & (\mathbb{A}^2 \setminus \{0\}) \times (\mathbb{A}^3 \setminus \{0\}) \\ ((\mu,\rho),(y_0,y_1;z_0,z_1,z_2)) & \mapsto & (\mu\rho^{-b}y_0,\mu y_1;\rho z_0,\rho z_1,\rho z_2) \end{array}$$

The class of $(y_0, y_1, z_0, z_1, z_2)$ will be written $[y_0 : y_1; z_0 : z_1 : z_2]$. The projection

$$\mathcal{P}_b \to \mathbb{P}^2, \quad [y_0 : y_1; z_0 : z_1 : z_2] \mapsto [z_0 : z_1 : z_2]$$

identifies $\mathcal{P}_b$ with

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(b) \oplus \mathcal{O}_{\mathbb{P}^2}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-b))$$

$\mathcal{P}_b \xrightarrow{\simeq} \mathcal{P}_{-b}$ *via* $[y_0 : y_1; z_0 : z_1 : z_2] \leftrightarrow [y_1 : y_0; z_0 : z_1 : z_2]$.

Decomposable $\mathbb{P}^1$-bundles over $\mathbb{F}_a$

Let $a, b, c \in \mathbb{Z}$. We define $\mathcal{F}_a^{b,c} = (\mathbb{A}^2 \setminus \{0\})^3/(\mathbb{G}_m)^3$ where

$$\begin{array}{ccc} (\mathbb{G}_m)^3 \times (\mathbb{A}^2 \setminus \{0\})^3 & \to & (\mathbb{A}^2 \setminus \{0\})^3 \\ ((\lambda,\mu,\rho),(x_0,x_1,y_0,y_1,z_0,z_1)) & \mapsto & (\lambda\mu^{-b}x_0, \lambda\rho^{-c}x_1, \mu\rho^{-a}y_0, \mu y_1, \rho z_0, \rho z \end{array}$$

The class of $(x_0, x_1, y_0, y_1, z_0, z_1)$ is written $[x_0 : x_1; y_0 : y_1; z_0 : z_1]$. The projection

$$\mathcal{F}_a^{b,c} \to \mathbb{F}_a, \quad [x_0 : x_1; y_0 : y_1; z_0 : z_1] \mapsto [y_0 : y_1; z_0 : z_1]$$

identifies $\mathcal{F}_a^{b,c}$ with

$$\mathbb{P}(\mathcal{O}_{\mathbb{F}_a}(bs_a) \oplus \mathcal{O}_{\mathbb{F}_a}(cf)) = \mathbb{P}(\mathcal{O}_{\mathbb{F}_a} \oplus \mathcal{O}_{\mathbb{F}_a}(-bs_a + cf))$$

as a $\mathbb{P}^1$-bundle over $\mathbb{F}_a$, where $s_a, f \subset \mathbb{F}_a$ are given by $y_1 = 0$ and $z_1 = 0$.
$\mathcal{F}_a^{b,c} \xrightarrow{\simeq} \mathcal{F}_a^{-b,-c}$ via $x_0 \leftrightarrow x_1$.

Transition functions on $\mathbb{F}_a$

$$\mathbb{F}_a = \{[y_0 : y_1; z_0 : z_1] \mid [y_0 : y_1; z_0 : z_1] = [\mu\rho^{-a} y_0 : \mu y_1 : \rho z_0 : \rho z_1]\}$$

two open subsets $U_0, U_1 \subset \mathbb{F}_a$ (given by $z_0 \neq 0, z_1 \neq 0$) isomorphic to $\mathbb{P}^1 \times \mathbb{A}^1$:

$$\begin{array}{rcl} \mathbb{P}^1 \times \mathbb{A}^1 & \stackrel{\simeq}{\longrightarrow} & U_0 \subset \mathbb{F}_a \\ ([y_0 : y_1], z) & \mapsto & [y_0 : y_1; 1 : z] \\ \mathbb{P}^1 \times \mathbb{A}^1 & \stackrel{\simeq}{\longrightarrow} & U_1 \subset \mathbb{F}_a \\ ([y_0 : y_1], z) & \mapsto & [y_0 : y_1; z : 1] \end{array}$$

glueing function:

$$([y_0 : y_1], z) \mapsto ([z^a y_0 : y_1], z)$$

Every $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ is isomorphic to $\mathbb{F}_a$ for some $a \geq 0$:
One way to see / prove this: for each element $A \in \mathrm{GL}_2(\mathrm{k}[z, \frac{1}{z}])$, there are $B \in \mathrm{GL}_2(\mathrm{k}[\frac{1}{z}])$, $C \in \mathrm{GL}_2(\mathrm{k}[z])$, such that

$$BAC = \begin{pmatrix} z^m & 0 \\ 0 & z^n \end{pmatrix} \rightsquigarrow a = |m - n|$$

Transition functions on $\mathcal{F}_a^{b,c}$

$$\mathcal{F}_a^{b,c} = \{[x_0 : x_1; y_0 : y_1; z_0 : z_1] = [\lambda\mu^{-b}x_0 : \lambda\rho^{-c}x_1 : \mu\rho^{-a}y_0 : \mu y_1 : \rho z_0 : \rho z_1]\}$$
$$\pi\colon \mathcal{F}_a^{b,c} \to \mathbb{F}_a, \quad [x_0 : x_1; y_0 : y_1; z_0 : z_1] \mapsto [y_0 : y_1; z_0 : z_1]$$

$$\mathbb{F}_a = \{[y_0 : y_1; z_0 : z_1] \mid [y_0 : y_1; z_0 : z_1] = [\mu\rho^{-a}y_0 : \mu y_1 : \rho z_0 : \rho z_1]\}$$

two open subsets $V_0 = \pi^{-1}(U_0), V_1 = \pi^{-1}(U_1) \subset \mathcal{F}_a^{b,c}$ (given by $z_0 \neq 0, z_1 \neq 0$) isomorphic to $\mathbb{F}^b \times \mathbb{A}^1$:

$$\begin{array}{ccc} \mathbb{F}_b \times \mathbb{A}^1 & \xrightarrow{\simeq} & V_0 \subset \mathcal{F}_a^{b,c} \\ ([x_0 : x_1; y_0 : y_1], z) & \mapsto & ([x_0 : x_1; y_0 : y_1; 1 : z]) \\ \mathbb{F}_b \times \mathbb{A}^1 & \xrightarrow{\simeq} & V_1 \subset \mathcal{F}_a^{b,c} \\ ([x_0 : x_1; y_0 : y_1], z) & \mapsto & ([x_0 : x_1; y_0 : y_1; z : 1]) \end{array}$$

yields a $\mathbb{F}_b$-bundle $\mathcal{F}_a^{b,c} \to \mathbb{P}^1$, with a transition function

$$([x_0 : x_1; y_0 : y_1], z) \mapsto \left([x_0 : x_1 z^c; y_0 z^a : y_1], \tfrac{1}{z}\right).$$

Not all $\mathbb{P}^1$-bundles over $\mathbb{F}_a$ are of this type!

Begining of the proof: descent lemma

Lemma (Descent lemma)

Let $\eta\colon \hat{S} \to S$ be a birational morphism between two smooth projective surfaces and let $\hat{\pi}\colon \hat{X} \to \hat{S}$ be a $\mathbb{P}^1$-bundle. Then, there exists a $\mathbb{P}^1$-bundle $\pi\colon X \to S$, unique up to isomorphism, and a birational map $\psi\colon \hat{X} \to X$ making the following diagram commutative. Moreover, the group $\mathrm{Aut}^\circ(\hat{X})$ acts on the four varieties and the four rational maps are equivariant.

$$\begin{array}{ccc} \hat{X} & \overset{\psi}{\dashrightarrow} & X \\ {\scriptstyle\hat{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \hat{S} & \underset{\eta}{\longrightarrow} & S \end{array}$$

One can then reduce the study to $\mathbb{P}^1$-bundles over $\mathbb{P}^2$ or Hirzebruch surfaces $\mathbb{F}_n$, $n \geq 0$ (and could avoid $n = 1$).

$\mathbb{P}^1$-bundles over $\mathbb{F}_a$: removing jumping fibres

Lemma (Removal of jumping fibres)

Let $a \geq 0$, let $\pi\colon X \to \mathbb{F}_a$ be a $\mathbb{P}^1$-bundle. There is an integer $b \geq 0$ and a dense open subset of $U \subset \mathbb{P}^1$ such that $(\tau_a\pi)^{-1}(p)$ is a Hirzebruch surface $\mathbb{F}_b$ for each $p \in U$. Moreover, we have:

1. *If $U = \mathbb{P}^1$, then $\tau_a\pi\colon X \to \mathbb{P}^1$ is a $\mathbb{F}_b$-bundle which is trivial on every affine open subset of $\mathbb{P}^1$. In this case we say that η has no jumping fibre.*
2. *If one fibre $(\tau_a\pi)^{-1}(\{p\})$ is isomorphic to $\mathbb{F}_c$ for some $c \neq b$, then $c - b$ is a positive even integer (we say that $\tau_a^{-1}(\{p\})$ is a jumping fibre), and the blow-up of the (unique) exceptional section of $\mathbb{F}_c$ followed by the contraction of the strict transform of $\mathbb{F}_c$ gives an $\mathrm{Aut}^\circ(X)$-equivariant birational map $X \dashrightarrow X'$ to another $\mathbb{P}^1$-bundle over $\mathbb{F}_a$. After finitely many such steps, one gets case 1.*

$\rightsquigarrow$ *Use transition matrices given by generic fibre being $\mathbb{F}_n$ over $\mathrm{k}(x)$, induction on determinant of matrices B, C*

Proposition

Let $a \geq 0$ and let $\pi\colon X \to \mathbb{F}_a$ be a $\mathbb{P}^1$-bundle.

1. *For all integers $b \geq 0$, $c \in \mathbb{Z}$, the following are equivalent.*
   1. *X is the gluing $Z_a^{b,c,P}$ of two copies of $\mathbb{F}_b \times \mathbb{A}^1$ along $\mathbb{F}_b \times \mathbb{A}^1 \setminus \{0\}$ by the automorphism $\nu_{c,P} \in \mathrm{Aut}(\mathbb{F}_b \times \mathbb{A}^1 \setminus \{0\})$ given by* $\nu_{c,P}\colon ([x_0 : x_1; y_0 : y_1], z) \mapsto ([x_0 : x_1 z^c + x_0 P(y_0, y_1, z); y_0 z^a : y_1], \frac{1}{z})$ *for some $P \in \mathrm{k}[y_0, y_1, z, \frac{1}{z}]$, homogeneous of degree b in y_0, y_1, such that $\pi\colon X \to \mathbb{F}_a$ sends $([x_0 : x_1; y_0 : y_1], z) \in \mathbb{F}_b \times \mathbb{A}^1$ onto resp. $[y_0 : y_1; 1 : z] \in \mathbb{F}_a$ and $[y_0 : y_1; z : 1] \in \mathbb{F}_a$ on the 2 charts.*
   2. *$\pi\colon X \to \mathbb{F}_a$ is the projectivisation of a rank two vector bundle $\mathcal{E}$ which fits in a short exact sequence*
   $$0 \to \mathcal{O}_{\mathbb{F}_a} \to \mathcal{E} \to \mathcal{O}_{\mathbb{F}_a}(-bs_a + cf) \to 0$$
   where $f, s_a \subset \mathbb{F}_a$ are given by $y_1 = 0$ and $z_1 = 0$.
2. *If there exists $b \geq 0$ such that the preimage of each fibre of the $\mathbb{P}^1$-bundle $\tau_a\colon \mathbb{F}_a \to \mathbb{P}^1$ is isomorphic to $\mathbb{F}_b$ (no jumping fibre), then there is an integer $c \in \mathbb{Z}$ such that the above properties are satisfied. If $b > 0$, c is unique. If $b = 0$, then $\pi\colon X \to \mathbb{F}_a$ is decomposable.*

The integers a, b, c are called *numerical invariants*.

Proposition

Let $a, b, c \in \mathbb{Z}$ with $a, b \geq 0$, such that $c \geq 0$ if $b = 0$. (always possible)

1. *There is a unique isomorphism class of decomposable $\mathbb{P}^1$-bundle $X \to \mathbb{F}_a$ with numerical invariants (a, b, c): $\mathcal{F}_a^{b,c} \to \mathbb{F}_a$.*
2. *If $b = 0$ or $c \leq 1$ every $\mathbb{P}^1$-bundle $X \to \mathbb{F}_a$ with numerical invariants (a, b, c) is decomposable, and thus isomorphic to $\mathcal{F}_a^{b,c} \to \mathbb{F}_a$.*
3. *If $b \geq 1$ and $c \geq 2$, every $\mathbb{P}^1$-bundle with numerical invariants (a, b, c) is isomorphic to $Z_a^{b,c,P} \to \mathbb{F}_a$ (glueing), where*

$$P(y_0, y_1, z) = \sum_{i=0}^{b} {y_0}^i {y_1}^{b-i} P_i(z) z^{ai+1}$$

and $P_i(z) \in \mathrm{k}[z]_{\leq c-2-ai}$ (hence $P_i = 0$ if $c < ai + 2$), for $i = 0, \ldots, b$.
The isomorphism class of the $\mathbb{P}^1$-bundle is determined by the class of P, up to scalar multiplication by an element of k^*. The $\mathbb{P}^1$-bundle is decomposable if and only if $P = 0$.

case $b \geq 1$, $c \geq 2$: two $\mathbb{P}^1$-bundles $Z_a^{b,c,P} \to \mathbb{F}_a$ and $Z_a^{b,c',P'} \to \mathbb{F}_a$,

$$\begin{array}{rcl} \mathbb{F}_b \times \mathbb{A}^1 & \dashrightarrow & \mathbb{F}_b \times \mathbb{A}^1 \\ \mathbb{F}_b \times \mathbb{A}^1 \setminus \{0\} & \xrightarrow{\simeq} & \mathbb{F}_b \times \mathbb{A}^1 \setminus \{0\} \\ \nu_{c,P}\colon ([x_0:x_1;y_0:y_1],z) & \mapsto & \left([x_0 : x_1 z^c + x_0 P(y_0,y_1,z); y_0 z^a : y_1], \frac{1}{z}\right), \\ \nu_{c',P'}\colon ([x_0:x_1;y_0:y_1],z) & \mapsto & \left([x_0 : x_1 z^{c'} + x_0 P'(y_0,y_1,z); y_0 z^a : y_1], \frac{1}{z}\right) \end{array}$$

$\mathbb{P}^1$-bundles isomorphic $\Leftrightarrow \nu_{c',P'} = \alpha \nu_{c,P} \beta$ for some automorphisms α, β of the $\mathbb{P}^1$-bundle $\mathbb{F}_b \times \mathbb{A}^1 \to \mathbb{P}^1 \times \mathbb{A}^1$. Since $b \geq 1$, α, β are of the form

$$\begin{array}{rrcl} \theta_{\lambda,Q}\colon & \mathbb{F}_b \times \mathbb{A}^1 & \xrightarrow{\simeq} & \mathbb{F}_b \times \mathbb{A}^1 \\ & ([x_0:x_1;y_0:y_1],z) & \mapsto & ([x_0 : \lambda x_1 + Q(y_0,y_1,z); y_0 : y_1], z) \end{array}$$

where $\lambda \in \mathrm{k}^*$ and $Q \in \mathrm{k}[y_0,y_1,z]_b$. The composition $\theta_{\lambda_2,Q_2}\nu_{c,P}\theta_{\lambda_1,Q_1}$ is

$$\begin{array}{rcl} \mathbb{F}_b \times \mathbb{A}^1 \setminus \{0\} & \xrightarrow{\simeq} & \mathbb{F}_b \times \mathbb{A}^1 \setminus \{0\} \\ ([x_0:x_1;y_0:y_1],z) & \mapsto & ([x_0 : z^c \lambda_1\lambda_2 x_1 + x_0 \tilde{P}(y_0,y_1,z); z^a y_0 : y_1], \frac{1}{z}), \end{array}$$

$$\tilde{P}(y_0,y_1,z) = \lambda_2 P(y_0,y_1,z) + \lambda_2 Q_1(y_0,y_1,z) z^c + Q_2(y_0 z^a, y_1, \tfrac{1}{z}).$$

To get a transition function of the form $\nu_{c',P'}$, we get $c' = c$, $\lambda_1\lambda_2 = 1$.

Corollary

Let $a, b, c \in \mathbb{Z}$, with $a \geq 0$, $b \geq 1$ and $c \geq 2$.
The isomorphism classes of non-decomposable $\mathbb{P}^1$-bundles $X \to \mathbb{F}_a$ with numerical invariants (a, b, c) are parametrised by the projective space

$$\mathcal{M}_a^{b,c} = \mathbb{P}\left(\bigoplus_{i=0}^{b} y_0^i y_1^{b-i} \cdot \mathrm{k}[z]_{\leq c-2-ai}\right)$$

1. Explicit description of the action of $\mathrm{Aut}(\mathbb{F}_a)$ on $\mathcal{M}_a^{b,c}$. *Action of* GL_2*, preserves the direct sum!*
2. If $c - 2 - ai = 0$ for some i, then GL_2 preserves one typical element *Umemura bundles*
3. *If one fibre of $\mathbb{F}_a \to \mathbb{P}^1$ is invariant, we can reduce to decomposable case.*
4. If $a = 0$, one can reduce to the case where the diagonal $\mathrm{PGL}_2 \subset \mathrm{PGL}_2 \times \mathrm{PGL}_2 = \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ lifts. *Explicit family $\hat{S}_b$, with $c = b + 2$.*

Classification of $\mathbb{P}^1$-bundles over Hirzebruch surfaces $\mathbb{F}_a$

Proposition

Let $a \geq 0$ and let $\pi\colon X \to \mathbb{F}_a$ be a $\mathbb{P}^1$-bundle. Then, there exist $b, c \in \mathbb{Z}$ such that one of the following holds.

1. *X is isomorphic to a decomposable $\mathbb{P}^1$-bundle $\mathcal{F}_a^{b,c} \to \mathbb{F}_a$;*
2. *X is isomorphic to an Umemura $\mathbb{P}^1$-bundle $\mathcal{U}_a^{b,c} \to \mathbb{F}_a$;*
3. *We have $a = 0$ and there is an isomorphism $\kappa\colon \mathbb{F}_0 \xrightarrow{\simeq} \mathbb{F}_0$ such that $\kappa\pi$ is isomorphic to $\hat{\mathcal{S}}_b \to \mathbb{F}_0$; or*
4. *There exists a $\mathbb{P}^1$-bundle $\tau\colon \mathbb{F}_a \to \mathbb{P}^1$ and a closed point $p \in \mathbb{P}^1$ such that $(\tau\pi)^{-1}(p)$ is invariant by $\mathrm{Aut}^\circ(X)$.*

In cases 3-4, there is a decomposable bundle $\mathcal{F}_a^{b,c} \to \mathbb{F}_a$ and a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & \mathcal{F}_a^{b,c} \\ & \searrow & \downarrow \\ & & \mathbb{F}_a \end{array}$$

where ψ is a birational map satisfying $\psi\,\mathrm{Aut}^\circ(X)\psi^{-1} \subsetneq \mathrm{Aut}^\circ(\mathcal{F}_a^{b,c})$.

The $\mathbb{P}^1$-bundles $\hat{\mathcal{S}}_b \to \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{S}_b \to \mathbb{P}^2$

Definition (Schwarzenberger $\mathbb{P}^1$-bundles)

Let $b \geq -1$ and let $\kappa\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ be the $(2:1)$-covering defined by

$$\begin{array}{cccc} \kappa\colon & \mathbb{P}^1 \times \mathbb{P}^1 & \to & \mathbb{P}^2 \\ & ([y_0 : y_1],[z_0 : z_1]) & \mapsto & [y_0z_0 : y_0z_1 + y_1z_0 : y_1z_1], \end{array}$$

branch locus: diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$, ramification locus: $\Gamma = \{[X : Y : Z] \mid Y^2 = 4XZ\} \subset \mathbb{P}^2$. The *$b$-th Schwarzenberger $\mathbb{P}^1$-bundle* $\mathcal{S}_b \to \mathbb{P}^2$ is the $\mathbb{P}^1$-bundle given by

$$\mathcal{S}_b = \mathbb{P}(\kappa_* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-b-1, 0)) \to \mathbb{P}^2.$$

$\mathcal{S}_{-1} \simeq \mathcal{P}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2})$, $\mathcal{S}_0 \simeq \mathcal{P}_0 = \mathbb{P}^1 \times \mathbb{P}^2$, $\mathcal{S}_1 \simeq \mathbb{P}(T_{\mathbb{P}^2})$.

Proposition

For each $b \geq 1$, there is an involution $\iota \in \mathrm{Aut}(\hat{\mathcal{S}}_b)$ such that $\mathcal{S}_b = \hat{\mathcal{S}}_b / \langle \iota \rangle$.

$$\begin{array}{ccc} \hat{\mathcal{S}}_b & \longrightarrow & \mathcal{S}_b \\ \downarrow & & \downarrow \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow[\kappa]{} & \mathbb{P}^2 \end{array}$$

The $\mathbb{P}^1$-bundles over $\mathbb{P}^2$

Extensive literature. Here we can use the work made for Hirzebruch surfaces, and recover results explicitely.

Proposition

Let $X \to \mathbb{P}^2$ be a $\mathbb{P}^1$-bundle, and let $H \subset \mathrm{PGL}_3$ be the image of $\mathrm{Aut}^\circ(X) \to \mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3$. Then we have the following alternative.

1. *$H = \mathrm{Aut}(\mathbb{P}^2)$ and X is decomposable or equal to $\mathcal{S}_1 \simeq \mathbb{P}(T_{\mathbb{P}^2})$.*
2. *H is conjugated to $\mathrm{Aut}(\mathbb{P}^2, C) \simeq \mathrm{PGL}_2$, where C is a smooth conic in $\mathbb{P}^2$ and $X \simeq \mathcal{S}_b$ for $b \geq 2$.*
3. *H fixes a point in $\mathbb{P}^2$ and there is a $\mathbb{P}^1$-bundle $X' \to \mathbb{F}^1$ and a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\ \psi\ } & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{F}_1 & \xrightarrow[\ \eta\ ]{} & \mathbb{P}^2 \end{array}$$

   where η is the contraction morphism and ψ is a birational morphism such that $\psi^{-1}\mathrm{Aut}^\circ(X)\psi \subset \mathrm{Aut}^\circ(X')$.

The result

Theorem

Let $\pi : X \to S$ be a $\mathbb{P}^1$-bundle over a projective rational surface S. Then, there exists a surface S', being either equal to $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_a$, $a \geq 0$, a $\mathbb{P}^1$-bundle $X' \to S'$, and a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & X' \\ \pi\downarrow & & \downarrow\pi' \\ S & \underset{\eta}{\dashrightarrow} & S' \end{array}$$

such that ψ, η are $\mathrm{Aut}^\circ(X)$-equivariant birational maps (for some biregular actions of $\mathrm{Aut}^\circ(X)$ on X', S, S') and such that the $\mathbb{P}^1$-bundle $X' \to S'$ is of the following type:

(1)	*a decomposable*	$\mathbb{P}^1$-*bundle*	$\mathcal{F}_a^{b,c}$	$\longrightarrow$ $\mathbb{F}_a$	$a, b \geq 0, c \in \mathbb{Z}$;
(2)	*a decomposable*	$\mathbb{P}^1$-*bundle*	$\mathcal{P}_b$	$\longrightarrow$ $\mathbb{P}^2$	$b \geq 0$;
(3)	*an Umemura*	$\mathbb{P}^1$-*bundle*	$\mathcal{U}_a^{b,c}$	$\longrightarrow$ $\mathbb{F}_a$	$a, b \geq 1, c \geq 2$;
(4)	*a Schwarzenberger*	$\mathbb{P}^1$-*bundle*	$\mathcal{S}_b$	$\longrightarrow$ $\mathbb{P}^2$	$b \geq 1$.