Edinburgh
June 2017

NO bodies

$(X, D) \leftarrow$ Cartier divisor (big)
smooth proj var $/\mathbb{C}$
$\dim X = n$

$\rightsquigarrow$ convex body $\Delta_v(D)$ extends notion of Newton polytope (for toric vars)

$v : H^0(X, \mathcal{O}_X(D)) \setminus \{0\} \to \mathbb{Z}^n$

- [Okounkov 1996-2003]
- [Kaveh-Khovanskii 2008] [Lazarsfeld-Mustață 2008] (extended systematically)

aim: translating questions for (X, D) to $\Delta_v(D)$

- valuation given by a flag of subvarieties

$Y_\bullet : X = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{pt\}$ s.t. Y_i irred and smooth at pt, $\operatorname{cod} Y_i = i$

- $s \overset{\neq 0}{\in} H^0(D)$ D big

$\nu_{Y_\bullet}(s) = (v_1(s), \dots, v_n(s))$: -) $v_1(s) = \operatorname{ord}_{Y_1}(s)$

-) $v_n(s) = \operatorname{ord}_{Y_1}(s)$

-) $v_2(s) = \operatorname{ord}_{Y_2}(s_1|_{Y_1})$

-) etc.

(after choosing loc eqs for $Y_1 \subset X$, can choose $s_1 \in H^0(X, D - v_1 Y_1)$ s.t. $s_1 \neq 0$ along Y_1) , $\tilde{s}_1 \in H^0(Y, \mathcal{O}_Y(\ldots))$ $v_2 = \operatorname{ord}(\tilde{s}_1)$

Define $\Gamma_{Y_\bullet}(D) = \{ (m, v(s)) \mid s \in H^0(X, mD) \setminus \{0\} \} \subset \mathbb{N} \times \mathbb{Z}^n$ VALUE SEMIRING

Define $\operatorname{Cone}_{Y_\bullet}(D) =$ closure of convex hull $\subset \mathbb{R} \times \mathbb{R}^n$

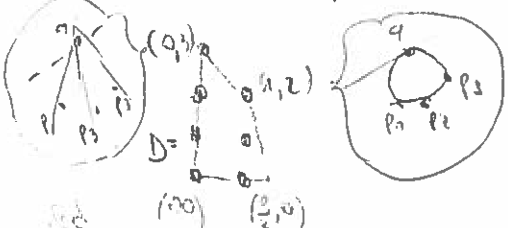
Define $\Delta_{Y_\bullet}(D) = \operatorname{Cone}_{Y_\bullet}(D) \cap \{\{1\} \times \mathbb{R}^d\}$

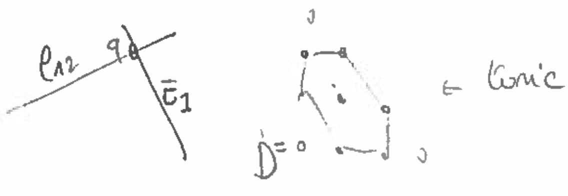
$\mathbb{N}$ $\mathcal{O}_{\mathbb{P}^1}(2)$

Ex : deg 6 del Pezzo $(Y = Bl_3 \mathbb{P}^2, K_X)$ $\Delta_{Y_\bullet}(-K_X)$:

$Y_\bullet :$ $\ell \ni q$ gen'l

$Y_\bullet = (\ell_{12}, \overbrace{L_{12} \cap E_1}^{q})$





- { NO bodies encode properties of (X, D) }   e.g. ($h^0(D) = \# \Delta_{Y_\bullet}(D) \cap \mathbb{Z}$ / ... Newton-Okounkov

e.g. LM: $\exists$ closed convex cone $\Delta_{Y_\bullet}(X) \subseteq \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$ s.t.

fibers over big divisors is $\Delta_{Y_\bullet}(D)$

(global NO body : determined of volume fn.

LM: $\boxed{\mathrm{vol}_X(D) = n!\, \mathrm{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(D))}$ Euclidean volume

$\mathrm{vol}_X(D) = \lim_{m \to \infty} \dfrac{H^0(mD)}{m^n/n!}$

does not dep on flag (diff shape, same volume)

(see examples pag 1)

$\mathrm{vol}(-K_X) = -K_X^2 = \deg = 6$, $h^0 = 7$

In some case $\Delta_{Y_\bullet}$ are familiar objects

LM: X smooth proj toric, $Y_\bullet$ torus invariant $\Rightarrow \Delta_{Y_\bullet}(D) = P_{D'}$

where $D' \sim D$ torus inv. (modulo linear transf of $\mathbb{R}^n$)

(Ex 2 del Pezzo)

In general $\Delta_{Y_\bullet}(D)$ are difficult / ugly

→ can be non rational

Nef, Eff circular

ex X ab surf Pic $\rho \geq 3$, $Y_\bullet : X \supset C \ni p$, C ample curve

$\Delta_{Y_\bullet}(D) =$  — slope $-C^2$

$\mu(D)$ = smallest root of $P(t) = (D - tC)^2$

(can be not rational)

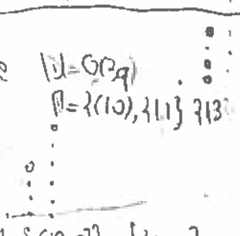
→ even worse: can be round

Lemma: Γ f.g $\Rightarrow \Delta_Y(D)$ not pol

Γ not f.g.

Ex C = ell curve, $|D| = O(2q)$, q

$\Gamma = \{(1,0), (1,1)\}$?13



$|D| = O(3q')$, $q' + q$

$\Gamma' = \{(0,0)\} \cup \{(m,y) : y = 3m-1\}$

ex hilbert ~ (Cutkosky's construction)

$X = \mathbb{P}(\mathcal{E})$ $\mathcal{E} = \mathcal{O}_V(A_1) \oplus \mathcal{O}_V(-B_1) \oplus \mathcal{O}_V(-B_2)$

A ab $\rho \geq 3$

A_1, B_1, B_2 ample

Natural Question: which X admit rat'l polyhedral $\Delta_{Y_\bullet}(D)$?

Motivation: Toric degenerations

THM [Anderson '12] If $\Gamma_{Y_\bullet}(D)$ f.g $\Rightarrow \exists$ toric degeneration of (X, D) to

toric variety (whose normalization is toric var of $\Delta_{Y_\bullet}(D)$)

($\pi : \mathcal{X} \to \mathbb{C}$ flat family, irreducible fibres $X_t \cong X$, X_0 toric

... over $\mathbb{C} \setminus \{0\}$

$j : X \times \mathbb{C}^* \xrightarrow{\sim} \pi^{-1}(\mathbb{C}^*)$)

THM [Harada - Kaveh '15] In above notation,

$X \subseteq \mathbb{P}^N \times \mathbb{C}$ $(X \subseteq \mathbb{P}^N)$

$\pi \downarrow$ Ω Kähler form on $\mathbb{P}^N$ (Fubini - Study form) (multiple of)

$\mathbb{C}$ $\Omega \times \left(\frac{i}{2} dz \wedge d\bar{z}\right)$ on $\mathbb{P}^N \times \mathbb{C}$ (standard)

$\omega_{\mathcal{X}} = \Omega \times (\,)|_{\mathcal{X}}$ and $\omega_t = \omega|_{X_t}$

alg. torus $\mathbb{T} = (\mathbb{C}^*)^n \supseteq$

Ω $T = (S^1)^n$-invariant (compact torus)

1) $\exists$ (continuous) map $X_1 \to X_0$ symplectomorphism (preserving Kähler form) (given by Hamiltonian flow given by $\nabla(\mathrm{Re}(\pi))$ or equiv, specialization map (non-Archimedean geom: Berkovich)

2) $\exists$ integrable system $\{f_1, \dots, f_n\}$ on (X, ω) s.t. comp moment map $(\{f_1, \dots, f_n\} : X \to \mathbb{R}^n)$ has image $\Delta_{Y_\bullet}(D)$ (f_i continuous, df_i lin indep away from critical pts, $\{f_i, f_j\} = 0$ Poisson brackets given by ω)

COR: If (X^D has toric deg), $\exists$ natural map to $\Delta_{Y_\bullet}(D)$

—— Main result this talk ——

$\boxed{\text{THM}}$ [P, Urbinati '17] X smooth Mori Dream space, D Cartier.
$\Rightarrow \exists Y_\bullet$ s.t. $\Gamma_{Y_\bullet}(D)$ f.g. (in partic $\Delta_{Y_\bullet}(D)$, $\Delta_{Y_\bullet}(X)$ rat'l polyhedral) and (X, D) admits toric degeneration.

MDS $\ni$ toric, del Pezzo surfaces, Fano varieties, log Fano pairs

$\to$ have rat'l pol $\overline{\mathrm{Eff}}$/Mov cones + f.m. nef chambers ($X \dashrightarrow X_i$ SQM)

$\to$ behave well w.r.t. MMP (flips terminate)

<u>Hukeel '2000</u> X MDS $\Leftrightarrow$ $\mathrm{Cox}(X) = \bigoplus_{D \in \mathrm{Pic}(X)} H^0(X, D)$ f.g. (coordinate ring)

$\exists X \hookrightarrow Z$ toric quasi-smooth ($\Leftrightarrow$ simplicial $= \mathbb{Q}$-factorial) s.t.

(1) $\mathrm{Pic}(Z)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Pic}(X)_{\mathbb{Q}}$

(2) $\overline{\mathrm{Eff}}(Z) \xrightarrow{\sim} \mathrm{Eff}(X)$ (induced by (1))

(3) $\forall$ nef chamber of X is union of f. m. chambers of Z

(4) $\forall$ rat'l contraction $\varphi : X \dashrightarrow X'$ is induced by toric map $Z \dashrightarrow Z'$ by restriction.

Proof (sketch) / recipe to construct $\Delta_{Y_\bullet}(D)$

$$\begin{tikzcd}
 & & \bar{Z} \arrow[dl] \arrow[ddr] & \tilde{Z} \arrow[l] \\
 & \bar{X} \subset \bar{T} \arrow[ur, bend left, "j"] \arrow[dl, "h"'] \arrow[dr] \arrow[dll, bend right] & & \\
 X \arrow[rr, dashed, "f_i"', "\text{SQM}"] & & X_i \in \mathcal{Z}_i \circlearrowleft T \cong (\mathbb{C}^*)^N & \\
T \circlearrowright Z \supseteq X \arrow[urr, bend right, "\text{SQM}"'] & & &
\end{tikzcd}$$

X MDS, X_i MDS

1) $\exists\, \bar{X}$ dominates SQM's of X ($h_* \operatorname{Nef}(\bar{X}) = \operatorname{Mov}(X)$) obtained using Tevelev's "tropical compactification" of $X \cap T \subset T$: $\bar{X} \subset \bar{T}$ s.t.

- $\bar{T}$ smooth (non proj) toric var. fan $\Sigma_{\bar{T}}$ supported on trop$(X \cap T)$
- $\bar{X} \setminus X$ boundary is divisorial and s.n.c.

2) $\exists\, \bar{Z}$ (quasi-smooth) proj : $\Sigma_{\bar{Z}}$ complete refinement of $\Sigma_{\bar{T}}$ and $j^* \operatorname{Pic}(\bar{Z}) \simeq \operatorname{Pic}(\bar{X})$ and $\circlearrowleft$

(3) $\exists\, \tilde{Z}$ smooth toric (not isom of Pic) ($\leftarrow$ toric res. of sings = star subdivisions of $\bar{Z}$)

LEMMA: All toric flags (that intersect $\bar{X}$ (i.e those that are in $\bar{T}$)) induce flag on $\bar{X}$ (by restriction + translation to first m elements)

$$\begin{array}{ccccccc}
\bar{Z} = Y_N & \supseteq & Y_{N-1} & \cdots & \supseteq & Y_{N-m} \supseteq \cdots \\
\cup\!\!\mid & & \cup & & & \cup \\
\bar{X} = Y_{N|X} \supseteq & & \cdots & & \supseteq & \{pt\}
\end{array}$$

flag on X

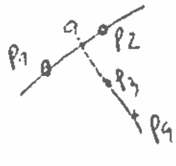
($\leftarrow$ here using properties of trop comp)

$\boxed{\text{THM}}$ NO bodies of D on X is linear projection of NO body of $\mathcal{D}$ on $\bar{Z}$ $\mathbb{R}^N \to \mathbb{R}^m$ toric $P_{\mathcal{D}'}$

$\boxed{\text{COR}}$ $\Delta_{Y_\bullet}(D)$ rat'l pol! (and computable using toric geometry)
Even more $\Gamma_{Y_\bullet}(D)$ is f.g.

Ex : del Pezzo deg 5 : $X = \operatorname{Bl}_4 \mathbb{P}^2$, $D = -K_X$ ($X = \bar{M}_{0,5} = Gr(2,5)$) $h^0 = 6$ vol $= 4$

$Y_\bullet = (\ell_{12}, \ell_{12} \cap \ell_{34})$ — (realisation of pts) degeneration : Bl $\mathbb{P}^2$ in 2 points of co mean pts $[1,1]^2$



$Y_\bullet = (\ell_{1,2}, \ell_{1,2} \cap E_1)$ — degeneration Bl $\mathbb{P}^2$ in 2 pts and 2 co mean pts