Boundedness results for groups of birational self-maps

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1. The Jordan property

Throughout this talk all fields are of characteristic zero. As a motivation, let me pose the following naive problem:

**Question 1.1.** Let $Γ$ be some huge “transformation group”. What are finite subgroups of $Γ$?

Our first main example will be the general linear group over $k$.

1.1. **Case** $Γ = GL_n(k)$. There are two fairly different situations.

- $k$ is a number field, say $k = \mathbb{Q}$. Then a classical theorem of Minkowski states that there exists a constant $B = B(GL_n(k))$ depending only on $GL_n(k)$ such for every finite subgroup $G \subset GL_n(k)$ one has $|G| \leq B$.

- $k$ is an algebraically closed field of characteristic zero, say $k = \mathbb{C}$. Clearly, in this situation Minkowski’s theorem cannot be true, as there exist arbitrary large abelian subgroups inside $(k^*)^n \subset GL_n(k)$. However, in a sense, this is the worst thing that can happen, as the following holds:

  **Theorem 1.2** (C. Jordan, 1878). *For any finite subgroup $G \subset GL_n(k)$ there exists a normal abelian subgroup $A \triangleleft G$ with $[G : A] \leq J$, where $J$ is some constant depending only on $n$.*

Jordan’s theorem motivates the following definition, first introduced by V. L. Popov:

**Definition 1.3.** A group $Γ$ is called Jordan (we also say that $Γ$ has Jordan property) if there exists a positive integer $m$ such that every finite subgroup $G \subset Γ$ contains a normal abelian subgroup $A \triangleleft G$ of index at most $m$. The minimal such $m$ is called the Jordan constant of $Γ$ and is denoted by $J(Γ)$.

**Example 1.4.** It follows from Theorem 1.2 that every linear algebraic group has Jordan property.

**Non-example 1.5.** Clearly, the infinite symmetric group $S_\infty$ is not Jordan, as it contains a copy of each $A_n$, which are simple for $n \geq 5$.

Let us move to the next example of $Γ$.

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1.2. **Case** $\Gamma = \text{Diff}(M)$. According to D. Fisher, a lot of questions about the analogy between linear groups and diffeomorphism groups of smooth manifolds are due to É. Ghys. He asked, for example, the following

**Question 1.6.** *Let $M$ be a smooth compact manifold. Is $\text{Diff}(M)$ Jordan?*

Although the answer to this question is known to be negative in general, $\text{Diff}(M)$ is indeed Jordan in many interesting cases, e.g. when

- $\dim(M) \leq 3$ (B. Zimmermann);
- $\chi(M) \neq 0$ (Mundet i Riera);
- $M$ is a homology sphere or $\mathbb{T}^n$.

1.3. **Case** $\Gamma = \text{Bir}(X)$. In fact all this recent activity around the Jordan property started with the following result:

**Theorem 1.7** (J.-P. Serre, 2009). *The Cremona group $\text{Bir}(\mathbb{P}^2_k)$ is Jordan.*

**Remark 1.8.** This result perfectly fits into general philosophy which says that Cremona groups enjoy many properties of linear algebraic groups. Moreover, it can be generalized as follows.

Let us say that a group $\Gamma$ has the **Jordan-Schur property**, if each periodic$^2$ subgroup $G \subset \Gamma$ contains a normal abelian subgroup $A \subset G$ of index bounded by some constant $S = S(\Gamma)$ depending only on $\Gamma$. A classical **Jordan-Schur theorem** states that the group $\text{GL}_n(\mathbb{C})$ has the Jordan-Schur property. It is interesting to notice that the same holds for almost all birational automorphism groups of complex projective surfaces:

**Observation 1.9.** *The group $\text{Bir}(\mathbb{P}^2_k)$ has the Jordan-Schur property.*

As was shown by Terence Tao, the Jordan-Schur theorem for $\text{GL}_n(\mathbb{C})$ can be deduced from two facts:

- Jordan’s theorem;
- Schur’s theorem, which claims that every finitely generated periodic subgroup of a general linear group $\text{GL}_n(\mathbb{C})$ is finite.

An analog of Schur’s theorem for groups of birational automorphisms was established by Serge Cantat. Let $X$ be a Kähler compact surface. It was shown by Cantat that $\text{Bir}(X)$ satisfies the Tits alternative and from this he deduces

**Theorem 1.10** (S. Cantat). *Let $X$ be a Kähler compact surface. Then every finitely generated$^3$ periodic subgroup of $\text{Bir}(X)$ is finite.*

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$^2$A group is called periodic if each its element has finite order.

$^3$Note that there do exist periodic groups of birational automorphisms which are not finitely generated. For example, the additive group $\mathbb{Q}/\mathbb{Z}$ embeds into $\text{GL}_2(\mathbb{C})$, hence in $\text{Bir}(\mathbb{P}^2_k)$. Taking direct products with finite non-abelian subgroups of $\text{Bir}(\mathbb{P}^2_k)$ one also gets non-abelian examples.
Now the Jordan-Schur theorem for Bir(Bir$_k^2$) follows from Cantat’s theorem, Tao’s argument and the Jordan property of Bir($\mathbb{P}_k^2$).

Serre’s theorem was substantially generalized by Yu. Prokhorov and C. Shramov:

**Theorem 1.11** (Yu. Prokhorov, C. Shramov, 2014). *For every positive integer $n$, there exists a constant $I = I(n)$ such that for any rationally connected variety $X$ of dimension $n$ defined over an arbitrary field $\mathbb{k}$ of characteristic 0 and for any finite subgroup $G \subset \text{Bir}(X)$ there exists a normal abelian subgroup $A \subset G$ of index at most $I$.*

**Corollary 1.12.** *The group $\text{Cr}_n(\mathbb{k})$ is Jordan for each $n \geq 1$.***

**Remark 1.13.** This theorem was initially proved modulo so-called Borisov-Alexeev-Borisov conjecture, which states that for a given positive integer $n$, Fano varieties of dimension $n$ with terminal singularities are bounded, i.e. are contained in a finite number of algebraic families. This conjecture was settled in dimensions $\leq 3$ a long time ago, but in its full generality was proved only recently in a preprint of Caucher Birkar.

2. **Classification of varieties with Jordan group of birational automorphisms**

The following theorem is due to V. L. Popov ($b \Rightarrow a$) and Yu. G. Zarhin ($a \Rightarrow b$).

**Theorem 2.1** (V. L. Popov, Yu. G. Zarhin, 2010). *Assume that $\mathbb{k} = \bar{\mathbb{k}}$. Let $X$ be an irreducible variety of dimension $\leq 2$. Then the following two properties are equivalent:

(a) the group Bir$(X)$ is Jordan;

(b) the variety $X$ is not birational to $\mathbb{P}^1 \times E$, where $E$ is an elliptic curve.

**Sketch** of proof. We may assume that $X$ is smooth projective and minimal. If $\text{kod}(X) = 2$, then Bir$(X)$ is finite by Matsumura’s theorem. If $X$ is rational, then Bir$(X) = \text{Bir}(\mathbb{P}_k^2)$ is Jordan by Serre’s result. If $X$ is nonrational ruled surface, then $X$ is birational to $\mathbb{P}^1 \times B$, where $g(B) \geq 1$. Moreover, $\text{Bir}(X) \cong \text{PGL}_2(\mathbb{k}(B)) \rtimes \text{Aut}(B)$.

If $g(B) \geq 2$, then Aut$(B)$ is finite, so we are done. The case $g(B) = 1$ is discussed below.

Finally, for all other types of surfaces $K_X$ is nef, so Bir$(X) = \text{Aut}(X)$. The latter group is known to be a locally algebraic group. Using some structure theory for them, one concludes that Aut$(X)$ is Jordan in this case.

2.1. **Zarhin’s counterexample.** Here we have the following

**Theorem 2.2** (Yu. Zarhin, 2010). *Let $A$ be an abelian variety of positive dimension and $X \cong A \times \mathbb{P}^1$. Then the group Bir$(X)$ is non-Jordan.*
First, let me recall what is a Heisenberg group. For any commutative ring $R$ with unity and an ideal $I \subset R$ define

$$\Gamma(R, I) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in I \right\}.$$ 

Then for each $n \geq 1$ the group $\Gamma(\mathbb{Z}, n\mathbb{Z})$ is normal in $\Gamma(\mathbb{Z}, \mathbb{Z})$, so put $\Gamma_n = \Gamma(\mathbb{Z}, \mathbb{Z})/\Gamma(\mathbb{Z}, n\mathbb{Z})$. It is easy to see that a homomorphism $\Gamma_n \to \mathbb{Z}/n \times \mathbb{Z}/n$ sending a class of matrix above to $([x], [y])$ is surjective and has a kernel isomorphic to $\mathbb{Z}/n$. So, $\Gamma_n$ fits into a short exact sequence

$$1 \to \mathbb{Z}/n \to \Gamma_n \to \mathbb{Z}/n \times \mathbb{Z}/n \to 1.$$ 

The group $\Gamma_n$ is called a Heisenberg group. Its crucial property is that for each abelian subgroup $A \subset \Gamma_n$ one has $[\Gamma_n : A] \geq n$.

**Sketch of proof of Zarhin’s theorem.** Let $\mathcal{L}$ be a very ample line bundle on $A$. Its total space $T$ is birational to $X$. For a given $n$ there is a finite group $A[n] \cong (\mathbb{Z}/n)^{\dim A}$ of points of $A$ such that the corresponding translations preserve $\mathcal{L}$, provided that $\mathcal{L}$ is ample enough. Further, the group $A[n]$ has an extension

$$1 \to \mathbb{Z}/n \to \widehat{A[n]} \to A[n] \to 1,$$ 

acting on $T$, hence acting on $X$ by birational automorphisms. Moreover, $\widehat{A[n]} \cong \Gamma_n$. Going through this construction for arbitrary large $n$, we embed an infinite series of Heisenberg group into $\text{Bir}(X)$. The rest is clear. □

**Remark 2.3.** Let $X$ be a variety over $\mathbb{k}$. Clearly the Jordanness of $\text{Bir}(X \otimes \overline{\mathbb{k}})$ implies the Jordanness of $\text{Bir}(X)$. However, the classification of varieties with Jordan/non-Jordan Bir depends on the base field. For example, for some fields (e.g. $\mathbb{k} = \mathbb{R}$) the group of birational self-maps of a surface is always Jordan.

**Remark 2.4.** Complex threefolds with Jordan $\text{Bir}(X)$ were recently classified by Prokhorov and Shramov. Namely, $\text{Bir}(X)$ is Jordan if and only if $X$ is birational either to $E \times \mathbb{P}^2$ ($E$ is an elliptic curve), or to $S \times \mathbb{P}^1$, where $S$ is one of the following: an abelian surface, a bielliptic surface or a surface of Kodaira dimension 1 such that the Jacobian fibration of the pluricanonical fibration $S \to B$ is locally trivial.

**Remark 2.5.** Let us back to the $C^\infty$-category. The first counterexample to Ghys’ question was obtained by B. Csikós, L. Pyber, and E. Szabó and is actually based on Zarhin’s construction:

**Theorem 2.6.** The group $\text{Diff}(\mathbb{S}^2 \times \mathbb{T}^2)$ is not Jordan.

3. **Jordan constants**

After establishing that a given group is Jordan, the next natural question is to estimate its Jordan constant. This can be highly non-trivial: the precise values of $J(\text{GL}_n(\mathbb{k}))$ for all $n$ and $\mathbb{k} = \overline{\mathbb{k}}$ were found
only in 2007 by M. J. Collins. As the group $S_{n+1}$ has a faithful $n$-dimensional representation, one has

$$(n + 1)! \leq J(GL_n(k)), \text{ when } n \geq 4.$$ 

The equality holds for all $n \geq 71$. For the remaining $n$s one has some sporadic values of $J$. For example $J(GL_2(\mathbb{C})) = 60, J(GL_3(\mathbb{C})) = 360$.

For the plane Cremona group the Jordan constant was computed only recently.

**Theorem 3.1** (E. Y., 2016). One has

$$J(Bir_P^2) = 7200, \quad J(Bir_R^2) = 120, \quad J(Bir_Q^2) = 120.$$ 

**Sketch of proof.** Take a finite subgroup $G \subset Bir(\mathbb{P}^2_k)$. Regularizing its action, we may assume that $G$ acts biregularly and minimally on a smooth rational surface $X$. Moreover, $X$ is either a del Pezzo surface with $\text{rk} \text{Pic}(X)^G = 1$ or a $G$-equivariant conic bundle $X \to B$ with $\text{rk} \text{Pic}(X)^G = 2$. In the first case all possible automorphism groups are basically known (at least when $k = \overline{k}$), so one easily gets all possible values of $J$. In the conic bundle case one has a short exact sequence

$$1 \to G_F \to G \to G_B \to 1,$$

where $G_B$ is the image of $G$ in $\text{Aut}(B)$ and $G_F$ acts by automorphisms of the generic fiber. Note that both $G_F$ and $G_B$ are finite subgroups of $\text{PGL}_2(\overline{k})$. Using some group theoretic arguments and a bit of geometry, one computes $J$ in this case.

Finally, let me notice that the value 7200 is achieved for the group $G = (\mathfrak{S}_5 \times \mathfrak{S}_5) \rtimes \mathbb{Z}/2$ acting on $\mathbb{P}^1 \times \mathbb{P}^1$. The values 120 are achieved for the group $\mathfrak{S}_5$ which is the automorphism group of a del Pezzo surface of degree 5 (both over $\mathbb{R}$ and $\mathbb{Q}$). 

Finally, in dimension 3 one has

**Theorem 3.2** (Yu. Prokhorov, C. Shramov). *Suppose that the field $k$ has characteristic 0. Then every finite subgroup of $Bir(\mathbb{P}^3_k)$ has an abelian (not necessarily normal!) subgroup of index at most 10368. Moreover, this bound is sharp if $k$ is algebraically closed.*

It is known that the previous theorem implies $J(Bir(\mathbb{P}^3_k)) \leq 10368^2$. However, this bound seems to be extremely far from being sharp.