Boundedness results for groups of birational self-maps

Egor Yasinsky*

EDGE Days 2017, Edinburgh

1. The Jordan property

Throughout this talk all fields are of characteristic zero. As a motivation, let me pose the following naive problem:

Question 1.1. *Let Γ be some huge "transformation group". What are finite subgroups of Γ?*

Our first main example will be the general linear group over $\mathbb{k}$.

1.1. **Case $\Gamma = \mathrm{GL}_n(\mathbb{k})$.** There are two fairly different situations.

- $\mathbb{k}$ is a number field, say $\mathbb{k} = \mathbb{Q}$. Then a classical theorem of Minkowski states that there exists a constant $B = B(\mathrm{GL}_n(\mathbb{k}))$ depending only on $\mathrm{GL}_n(\mathbb{k})$ such for every finite subgroup $G \subset \mathrm{GL}_n(\mathbb{k})$ one has
$$|G| \leqslant B.$$
- $\mathbb{k}$ is an algebraically closed field of characteristic zero, say $\mathbb{k} = \mathbb{C}$. Clearly, in this situation Minkowski's theorem cannot be true, as there exist arbitrary large abelian subgroups inside $(\mathbb{k}^*)^n \subset \mathrm{GL}_n(\mathbb{k})$. However, in a sense, this is the worst thing that can happen, as the following holds:

  Theorem 1.2 (C. Jordan, 1878)**.** *For any finite subgroup $G \subset \mathrm{GL}_n(\mathbb{k})$ there exists a normal abelian subgroup $A \subset G$ with $[G : A] \leqslant J$, where J is some constant depending only on n.*

Jordan's theorem motivates the following definition, first introduced by V. L. Popov:

Definition 1.3. A group Γ is called *Jordan* (we also say that Γ has *Jordan property*) if there exists a positive integer m such that every finite subgroup $G \subset \Gamma$ contains a normal abelian subgroup $A \lhd G$ of index at most m. The minimal such m is called the *Jordan constant* of Γ and is denoted by $\mathrm{J}(\Gamma)$.

Example 1.4. It follows from Theorem 1.2 that every linear algebraic group has Jordan property.

Non-example 1.5. Clearly, the infinite symmetric group $\mathfrak{S}_\infty$ is not Jordan, as it contains a copy of each $\mathfrak{A}_n$, which are simple for $n \geqslant 5$.

Let us move to the next example of Γ.

*yasinskyegor@gmail.com. This is a slightly expanded version of author's talk at the workshop *EDGE days 2017*, June 26-30, Edinburgh. In particular these notes are quite rough, be careful!

1.2. **Case** $\Gamma = \mathrm{Diff}(M)$**.** According to D. Fisher, a lot of questions about the analogy between linear groups and diffeomorphism groups of smooth manifolds are due to É. Ghys. He asked, for example, the following

Question 1.6. *Let M be a smooth compact manifold. Is* $\mathrm{Diff}(M)$ *Jordan?*

Although the answer to this question is known to be negative in general, $\mathrm{Diff}(M)$ is indeed Jordan in many interesting cases, e.g. when

- $\dim(M) \leqslant 3$ (B. Zimmermann);
- $\chi(M) \neq 0$ (Mundet i Riera);
- M is a homology sphere or $\mathbb{T}^n$.

1.3. **Case** $\Gamma = \mathrm{Bir}(X)$**.** In fact all this recent activity around the Jordan property started with the following result:

Theorem 1.7 (J.-P. Serre, 2009)**.** *The Cremona group* $\mathrm{Bir}(\mathbb{P}^2_{\Bbbk})$ *is Jordan.*

Remark 1.8. This result perfectly fits into general philosophy which says that Cremona groups enjoy many properties of linear algebraic groups. Moreover, it can be generalized as follows.

Let us say that a group Γ has the *Jordan-Schur property*, if each periodic[2] subgroup $G \subset \Gamma$ contains a normal abelian subgroup $A \subset G$ of index bounded by some constant $S = S(\Gamma)$ depending only on Γ. A classical *Jordan-Schur theorem* states that the group $\mathrm{GL}_n(\mathbb{C})$ has the Jordan-Schur property. It is interesting to notice that the same holds for almost all birational automorphism groups of complex projective surfaces:

Observation 1.9. *The group* $\mathrm{Bir}(\mathbb{P}^2_{\Bbbk})$ *has the Jordan-Schur property.*

As was shown by Terence Tao, the Jordan-Schur theorem for $\mathrm{GL}_n(\mathbb{C})$ can be deduced from two facts:

- Jordan's theorem;
- Schur's theorem, which claims that every finitely generated periodic subgroup of a general linear group $\mathrm{GL}_n(\mathbb{C})$ is finite.

An analog of Schur's theorem for groups of birational automorphisms was established by Serge Cantat. Let X be a Kähler compact surface. It was shown by Cantat that $\mathrm{Bir}(X)$ satisfies the Tits alternative and from this he deduces

Theorem 1.10 (S. Cantat)**.** *Let X be a Kähler compact surface. Then every finitely generated*[3] *periodic subgroup of* $\mathrm{Bir}(X)$ *is finite.*

[2]A group is called periodic if each its element has finite order.

[3]Note that there do exist periodic groups of birational automorphisms which are not finitely generated. For example, the additive group $\mathbb{Q}/\mathbb{Z}$ embeds into $\mathrm{GL}_2(\mathbb{C})$, hence in $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. Taking direct products with finite non-abelian subgroups of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ one also gets non-abelian examples.

Now the Jordan-Schur theorem for $\mathrm{Bir}(\mathrm{Bir}^2_{\Bbbk})$ follows from Cantat's theorem, Tao's argument and the Jordan property of $\mathrm{Bir}(\mathbb{P}^2_{\Bbbk})$.

Serre's theorem was substantially generalized by Yu. Prokhorov and C. Shramov:

Theorem 1.11 (Yu. Prokhorov, C. Shramov, 2014)**.** *For every positive integer n, there exists a constant $I = I(n)$ such that for any rationally connected variety X of dimension n defined over an arbitrary field $\Bbbk$ of characteristic 0 and for any finite subgroup $G \subset \mathrm{Bir}(X)$ there exists a normal abelian subgroup $A \subset G$ of index at most I.*

Corollary 1.12. *The group $\mathrm{Cr}_n(\Bbbk)$ is Jordan for each $n \geqslant 1$.*

Remark 1.13. This theorem was initially proved modulo so-called Borisov-Alexeev-Borisov conjecture, which states that for a given positive integer n, Fano varieties of dimension n with terminal singularities are bounded, i. e. are contained in a finite number of algebraic families. This conjecture was settled in dimensions $\leqslant 3$ a long time ago, but in its full generality was proved only recently in a preprint of Caucher Birkar.

2. Classification of varieties with Jordan group of birational automorphisms

The following theorem is due to V. L. Popov ($b \Rightarrow a$) and Yu. G. Zarhin ($a \Rightarrow b$).

Theorem 2.1 (V. L. Popov, Yu. G. Zarhin, 2010)**.** *Assume that $\Bbbk = \overline{\Bbbk}$. Let X be an irreducible variety of dimension $\leqslant 2$. Then the following two properties are equivalent:*

(a) the group $\mathrm{Bir}(X)$ is Jordan;
(b) the variety X is not birational to $\mathbb{P}^1 \times E$, where E is an elliptic curve.

Sketch[4] *of proof.* We may assume that X is smooth projective and minimal. If $\mathrm{kod}(X) = 2$, then $\mathrm{Bir}(X)$ is finite by Matsumura's theorem. If X is rational, then $\mathrm{Bir}(X) = \mathrm{Bir}(\mathbb{P}^2_{\Bbbk})$ is Jordan by Serre's result. If X is nonrational ruled surface, then X is birational to $\mathbb{P}^1 \times B$, where $g(B) \geqslant 1$. Moreover,

$$\mathrm{Bir}(X) \cong \mathrm{PGL}_2(\Bbbk(B)) \rtimes \mathrm{Aut}(B).$$

If $g(B) \geqslant 2$, then $\mathrm{Aut}(B)$ is finite, so we are done. The case $g(B) = 1$ is discussed below.

Finally, for all other types of surfaces K_X is nef, so $\mathrm{Bir}(X) = \mathrm{Aut}(X)$. The latter group is known to be a locally algebraic group. Using some structure theory for them, one concludes that $\mathrm{Aut}(X)$ is Jordan in this case. □

2.1. Zarhin's counterexample.

Here we have the following

Theorem 2.2 (Yu. Zarhin, 2010)**.** *Let A be an abelian variety of positive dimension and $X \cong A \times \mathbb{P}^1$. Then the group $\mathrm{Bir}(X)$ is non-Jordan.*

First, let me recall what is a Heisenberg group. For any commutative ring R with unity and an ideal $I \subset R$ define

$$\Gamma(R, I) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \ x, y, z \in I \right\}.$$

Then for each $n \geqslant 1$ the group $\Gamma(\mathbb{Z}, n\mathbb{Z})$ is normal in $\Gamma(\mathbb{Z}, \mathbb{Z})$, so put $\Gamma_n = \Gamma(\mathbb{Z}, \mathbb{Z})/\Gamma(\mathbb{Z}, n\mathbb{Z})$. It is easy to see that a homomorphism $\Gamma_n \to \mathbb{Z}/n \times \mathbb{Z}/n$ sending a class of matrix above to $([x], [y])$ is surjective and has a kernel isomorphic to $\mathbb{Z}/n$. So, Γ_n fits into a short exact sequence

$$1 \to \mathbb{Z}/n \to \Gamma_n \to \mathbb{Z}/n \times \mathbb{Z}/n \to 1.$$

The group Γ_n is called a *Heisenberg group*. Its crucial property is that for each abelian subgroup $A \subset \Gamma_n$ one has $[\Gamma_n : A] \geqslant n$.

Sketch of proof of Zarhin's theorem. Let $\mathscr{L}$ be a very ample line bundle on A. Its total space T is birational to X. For a given n there is a finite group $A[n] \cong (\mathbb{Z}/n)^{\dim A}$ of points of A such that the corresponding translations preserve $\mathscr{L}$, provided that $\mathscr{L}$ is ample enough. Further, the group $A[n]$ has an extension

$$1 \to \mathbb{Z}/n \to \widetilde{A[n]} \to A[n] \to 1,$$

acting on T, hence acting on X by birational automorphisms. Moreover, $\widetilde{A[n]} \cong \Gamma_n$. Going through this construction for arbitrary large n, we embed an infinite series of Heisenberg group into $\mathrm{Bir}(X)$. The rest is clear. □

Remark 2.3. Let X be a variety over $\Bbbk$. Clearly the Jordanness of $\mathrm{Bir}(X \otimes \overline{\Bbbk})$ implies the Jordanness of $\mathrm{Bir}(X)$. However, the *classification* of varieties with Jordan/non-Jordan Bir depends on the base field. For example, for some fields (e.g. $\Bbbk = \mathbb{R}$) the group of birational self-maps of a surface is *always* Jordan.

Remark 2.4. Complex threefolds with Jordan $\mathrm{Bir}(X)$ were recently classified by Prokhorov and Shramov. Namely, $\mathrm{Bir}(X)$ is Jordan if and only if X is birational either to $E \times \mathbb{P}^2$ (E is an elliptic curve), or to $S \times \mathbb{P}^1$, where S is one of the following: an abelian surface, a bielliptic surface or a surface of Kodaira dimension 1 such that the Jacobian fibration of the pluricanonical fibration $S \to B$ is locally trivial.

Remark 2.5. Let us back to the C^∞-category. The first counterexample to Ghys' question was obtained by B. Csikós, L. Pyber, and E. Szabó and is actually based on Zarhin's construction:

Theorem 2.6. *The group* $\mathrm{Diff}(\mathbb{S}^2 \times \mathbb{T}^2)$ *is not Jordan.*

3. Jordan constants

After establishing that a given group is Jordan, the next natural question is to estimate its Jordan constant. This can be highly non-trivial: the precise values of $\mathrm{J}(\mathrm{GL}_n(\Bbbk))$ for all n and $\Bbbk = \overline{\Bbbk}$ were found only in 2007 by M. J. Collins. As the group $\mathfrak{S}_{n+1}$ has a faithful n-dimensional representation, one has

$$(n+1)! \leqslant \mathrm{J}(\mathrm{GL}_n(\Bbbk)), \quad \text{when } n \geqslant 4.$$

The equality holds for all $n \geqslant 71$. For the remaining ns one has some sporadic values of J. For example $\mathrm{J}(\mathrm{GL}_2(\mathbb{C})) = 60$, $\mathrm{J}(\mathrm{GL}_3(\mathbb{C})) = 360$.

For the plane Cremona group the Jordan constant was computed only recently.

Theorem 3.1 (E. Y., 2016)**.** *One has*

$$\mathrm{J}(\mathrm{Bir}^2_{\mathbb{C}}) = 7200, \qquad \mathrm{J}(\mathrm{Bir}^2_{\mathbb{R}}) = 120, \qquad \mathrm{J}(\mathrm{Bir}^2_{\mathbb{Q}}) = 120.$$

Sketch of proof. Take a finite subgroup $G \subset \mathrm{Bir}(\mathbb{P}^2_{\Bbbk})$. Regularizing its action, we may assume that G acts biregularly and minimally on a smooth rational surface X. Moreover, X is either a del Pezzo surface with $\mathrm{rk}\,\mathrm{Pic}(X)^G = 1$ or a G-equivariant conic bundle $X \to B$ with $\mathrm{rk}\,\mathrm{Pic}(X)^G = 2$. In the first case all possible automorphism groups are basically known (at least when $\Bbbk = \overline{\Bbbk}$), so one easily gets all possible values of J. In the conic bundle case one has a short exact sequence

$$1 \to G_F \to G \to G_B \to 1,$$

where G_B is the image of G in $\mathrm{Aut}(B)$ and G_F acts by automorphisms of the generic fiber. Note that both G_F and G_B are finite subgroups of $\mathrm{PGL}_2(\Bbbk)$. Using some group theoretic arguments and a bit of geometry, one computes J in this case.

Finally, let me notice that the value 7200 is achieved for the group $G = (\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2$ acting on $\mathbb{P}^1 \times \mathbb{P}^1$. The values 120 are achieved for the group $\mathfrak{S}_5$ which is the automorphism group of a del Pezzo surface of degree 5 (both over $\mathbb{R}$ and $\mathbb{Q}$). □

Finally, in dimension 3 one has

Theorem 3.2 (Yu. Prokhorov, C. Shramov)**.** *Suppose that the field* $\Bbbk$ *has characteristic 0. Then every finite subgroup of* $\mathrm{Bir}(\mathbb{P}^3_{\Bbbk})$ *has an abelian (not necessarily normal!) subgroup of index at most* 10368*. Moreover, this bound is sharp if* $\Bbbk$ *is algebraically closed.*

It is known that the previous theorem implies $\mathrm{J}(\mathrm{Bir}(\mathbb{P}^3_{\Bbbk})) \leqslant 10368^2$. However, this bound seems to be extremely far from being sharp.