Edge days - Arithmetic of cubic threefolds

Theorem (Torelli) ε)

Let C_1, C_2 be smooth projective curves over a field k of genus $g \geq 2$.

If $J(C_1) \overset{?}{\cong} J(C_2)$ as principally polarised abelian varieties, then

$$C_1 \overset{?}{\cong} C_2.$$

Analogue false for $g = 0, 1$!

Generalisation? Say: Obvious analogue of $J(C)$ in higher dimensions is the albanese variety. However for many classes of varieties this is trivial. So use intermediate Jacobians!

Cubic threefolds / $\mathbb{C}$

Let $X \subseteq \mathbb{P}^5$ be a smooth cubic threefold $/\mathbb{C}$.

Hodge Diamond:

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 5 & & 5 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}$$

$\leftarrow$ Looks like H^1(ab. var) of $\mathbb{C}$

- We define the <u>intermediate Jacobian</u> of X to be
$$J(X) := H^3(X,\mathbb{R}) / H^3(X,\mathbb{Z})$$
equipped with its natural complex structure.

This is a principally polarised [illegible] abelian fivefold with $H^1(J(X),\mathbb{Z}) \cong H^3(X,\mathbb{Z}(1))$

Say:
- polarisation comes from cup-product.
- Need special Hodge structure ($H(2)$) to get an abelian variety, rather than just a complex torus.

Clemens–Griffiths ('72): Analogue of Torelli theorem
- X is not rational.

Cubic ~~threefolds over any field k~~ (char(k) ≠ 2)

Let X be a smooth cubic threefold $/k$.

Intermediate Jacobian: (Bombieri – Swinnerton-Dyer '6?)

Let $F(X)$ be a Fano variety of lines of X.
This is a smooth surface of general type.

$$J(X) := \mathrm{Alb}(F(X))$$

Clemens–Griffiths: these def^n agree $/\mathbb{C}$

Deligne '72, Achter '14: Intermediate Jacobian in families

(2)

Beauville '82: Torelli theorem / $\bar{k}$.

Q

Let X be a smooth cubic threefold over an algebraically closed field k with $\mathrm{char}(k) = 2$.

Is X non-rational?

Theorem 1 (~~C~~ Javanpeykar – ~~Loong~~ ('15))

~~[illegible]~~

Let X_1, X_2 be smooth cubic threefolds over a field k with $\mathrm{char}(k) \neq 2$.

If $J(X_1) \cong J(X_2)$ as principally polarised abelian varieties

then $X_1 \cong X_2$.

proof Assume $k \subset \mathbb{C}$ for simplicity (in positive characteristic, require ~~e~~ ℓ-adic cohomology instead of Betti cohomology)

Let: $\mathcal{C}$ = moduli stack of cubic threefolds over $\mathbb{Q}$

$= $ Hilbert scheme of cubic threefolds / PGL_5

say every cut linear.

say stack remembers the automorphisms.

$\binom{n+d}{d}$ $n=4$, $d=3$ $\binom{7}{3} = 35$

$\mathcal{A}_5$ = moduli stack of ppav of dim 5 over $\mathbb{Q}$

not schemes due to presence of non-trivial automorphisms

Intermediate Jacobian

$$J : \mathcal{C} \to \mathcal{A}_5$$

Torelli / $\mathbb{C}$: J is injective on (isomorphism classes of) $\mathbb{C}$-points

If $\mathcal{C}, \mathcal{A}_5$ were schemes, would be done! Not true, but fibres <u>are schemes</u>.

<u>Proposition</u>

The geometric fibres of J are schemes
"J is a representable morphism".

<u>proof</u>

What stops a stack from being a scheme (or an algebraic space): the presence of non-trivial automorphisms. So it suffices to show that the fibre has no nontrivial automorphisms.

fibre over A classifies pairs (X, φ) with
- X smooth cubic threefold
- $\varphi : J(X) \cong A$

with automorphisms coming from $\mathrm{Aut}\, X$.
for $\sigma \in \mathrm{Aut}\, X$ to preserve φ, need σ to act trivially on $J(X)$.

Suffices to show that $\mathrm{Aut}\, X \to \mathrm{Aut}\, J(X)$ is injective.

$$\downarrow$$

$\mathrm{Aut}\, H^1(J(X), \mathbb{C})$

15 3…

(5) Lefschetz trace formula

char $\neq 0$ ↓ smooth

$\mathrm{Tr}(\sigma \mid H^*(X,\mathbb{C})) = \chi(X^\sigma)$

[If $\sigma^{\neq 1}$ acts trivially on H^*: $1+1-10+1+1 = -6$]

diagonalise action of σ on $V = K^5$

$V = \oplus V_\lambda$ (eigenvalues λ of σ)

Macaulay's theorem: $X^\sigma = \bigsqcup_\lambda (X \cap \mathbb{P}(V^\sigma))$, with X_λ = ϕ, $\mathbb{P}^d$, cubic curve or cubic surface

$\chi = 0, d+1, 0, 9$

$-6 = \sum_\lambda \chi_\lambda(X_\lambda) \geq 0$ *.

<u>Remarks</u>

1. Similar result holds for intersections of 3 quadrics of odd dimension (Torelli / $\mathbb{C}$ Debarre '89).

2. Can't always descend Torelli theorems to arbitrary ground fields!

Int of 2 quadrics of odd dim.

Torelli / $\mathbb{C}$ Donagi '80, but no Torelli over $\mathbb{Q}$!

Let $X: \; a_0 x_0^2 + \dots + a_{n+2} x_{n+2}^2 = 0$
$\quad\;\; b_0 x_0^2 + \dots + b_{n+2} x_{n+2}^2 = 0 \quad \subseteq \mathbb{P}^{n+2}_{\mathbb{Q}}$

n odd.

$J(X) = J(C)$, where ← alternate Fano var of $\left(\frac{n-1}{2}\right)$-planes is an abelian var / $\mathbb{Q}$

$C: \; y^2 = (-1)^{\left(\frac{n-1}{2}\right)} (a_0 x + b_0) \dots (a_{n+2} x + b_{n+2})$

Let $t \in \mathbb{Q}^* \setminus \mathbb{Q}^{*2}$ and let X_t be given by the same equations, but a_0, b_0, a_1, b_1 multiplied by t.

Then $X \not\cong X_t$ but $J(X) \cong J(X_t)$.

Applications

Let X be a smooth projective variety / $\mathbb{Q}$. Say that X has **good reduction** at a prime p if there exists a smooth proper scheme

$$\mathcal{X} \to \operatorname{Spec} \mathbb{Z}_p$$

whose generic fibre is isomorphic to X.

Theorem (Faltings, '83 Shafarevich conjecture)

Let $g \geq 1$ and S a finite set of primes. There are only finitely many g-dim abelian varieties over $\mathbb{Q}$ with good reduction outside of S.

Theorem 2 (Javanpeykar, L. '15)

Let S be a finite set of primes. There are only finitely many smooth cubic threefolds over $\mathbb{Q}$ with good reduction outside of S.

proof Let $\ell \in S$.

$$H^3(X, \mathbb{Q}_\ell(1)) \cong H^1(J(X), \mathbb{Q}_\ell)$$

④

smooth basechange

X good reduction at p ⟹ unramified Galois action on $H^3(X, \mathbb{Q}_\ell)$ at p

⟹ " " " $H^1(J(X), \mathbb{Q}_\ell)$

Néron-Ogg-Shafarevich ⟹ $J(X)$ has good reduction at p.

Now use Faltings's theorem & Theorem 1

□

we have versions of the Shaf. conj for other complete intersections of Hodge level 1, e.g. intersections of two quadrics + quartic threefolds.

But proofs are different as, recall that the Torelli theorem fails here! Also no global Torelli for quartic threefolds, need "quasi-finite Torelli", follows from "infinitesimal Torelli" + Griffiths transversality.