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Seminar
EDGE day
June 2017

Recent developments in bir geom.

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- Context

- MMP

$$X = X_1 \overset{\text{bir contr.}}{\dashrightarrow} X_2 \overset{\text{flips}}{\dashrightarrow} X_3 \dashrightarrow \cdots \dashrightarrow X_n = Y$$

sing.

$Y \to Z$

$\begin{cases} K_Y \text{ ample} \\ \text{or} \\ Y \to Z \quad \text{Fano or CY fib.} \end{cases}$

- Problems

$\boxed{\text{Bir geom}}$

Can. pol.
- Eff bir
- moduli

CY's
- Eff. Iitaka ?
- moduli ?

Fano's
- Eff. bir
- complements
- BAB
- let of |-K| (Ambro conj)
- moduli ?
- stabilities

Sing
- ACC for lct's
- ACC for mld's
- complements (local)
- lct of lin systems

MMP
- Termination
- Abundance

Bir Aut groups
- Jordan property of Bir(X)

Arithmetic
- Manin's conj
- density of rational points

Complements

- $X \to Z$ proj morphism, $z \in Z$.
- An <u>m-comp. for K_X over z</u> is as $K_X + \Delta$ s.t. $\begin{cases} (X,\Delta) \text{ lc } /z \\ m(K_X+\Delta) \sim 0 \ /z \end{cases}$
- <u>Rem</u> $m\Delta \in |-mK_X|$ over z.

Thm (Boundedness of comps.) (B, 2016)

$\forall\, d \in \mathbb{N}$, $\exists\, m \in \mathbb{N}$ s.t. if $\begin{cases} X \xrightarrow{\text{proj}} Z \ni z \\ X \text{ klt} \\ -K_X \ \underset{\text{nef \& big}}{\underline{\text{ample}}} /Z \end{cases}$ then $\exists$ m-comp of K_X over z.

- Rem
  - Conj of Shokurov (90's, originates in 70's).
  - $d=2$ proved by Shokurov
  - $d=3$ partial proved by Prokhorov–Shokurov.

- <u>Global case</u>: $z = \mathrm{pt}$. ($\Leftrightarrow$ X Fano).
  - Thm $\Rightarrow$ $|-mK_X| \neq \emptyset$ and contains element with good sing.
  - Exa: X = toric, $m=1$, Δ = sum inv. divs.
  - Exa:
    - $X = \mathbb{P}^2$, $m=1$, $\Delta =$  or $\Delta =$ .
    - $X = \mathbb{P}^2$, $m=2$, $\Delta = \tfrac{1}{2}$  $+ \tfrac{1}{2}$ smooth cubic.

- Exa: exceptional Fanos. ~~eg some surfaces in weighted proj space~~
  - $\exists$ sing. Del Pezzo surfaces X s.t. (X, G) klt $\forall$ $0 \leq G \sim_{\mathbb{R}} -K_X$.
  - In particular, $\nexists$ 1-comp for K_X.
  - eg, $X \subseteq \mathbb{P}(2,3,5,9)$ of deg 18. [Cheltsov–Park–Shramov]

X smooth 3-fold

- **Local case:** $X \to Z$ identities, $x = z \in X$. klt sing. (3)

  - In general, Cartier index of K_X not bnd.
  - $\exists$ m-comp $K_X + \Delta$ where Cartier index is bnd.

- Ex: X toric, $m=1$, Δ = sum of inv. div.

- Ex: X surface, ~~canonical, $m=1$, $\Delta = 0$~~ ( )

  $x \in X$ has type
  - smooth or A-sing $\Rightarrow$ $\exists$ 1-comp of K_X
  - D-sing $\Rightarrow$ $\exists$ 2-comp
  - E_6-sing $\Rightarrow$ $\exists$ 3-comp
  - E_7-sing $\Rightarrow$ $\exists$ 4-comp
  - E_8-sing $\Rightarrow$ $\exists$ 6-comp.

  (E_6, E_7, E_8: exceptional)

Effective bir

Thm (B, 2016) $\forall\, d \in \mathbb{N}$, $\varepsilon \in \mathbb{R}^{>0}$, $\exists\, m \in \mathbb{N}$ s.t. if X is ε-lc Fano dim $= d$, then $|-mK_X|$ bir.

- Rem:
  - Not true without ε-lc condition: $\exists$ Fano surfaces with $\mathrm{vol}(-K)_{a.c.}$
  - $d = 3$ proved by Jiang,
  - Excl ——— Cascini–McKernan

  } different methods.

BAB

Thm (B. 2016) $\forall\, d \in \mathbb{N}$, $\epsilon \in \mathbb{R}^{>0}$, the set

$$\{ X \mid X \ \epsilon\text{-lc Fano}, \dim = d \}$$ is bdd.

Rem: • known as BAB conj. (90's)
- $d=2$ proved by Alexeev. (& Alexeev–Mori).
- $d=3$, term. by Kawamata,
- smooth, Kollár–Miyaoka–Mori, (rational curves)
- toric, Borisovs. (combinatorics)
- $d=3$, canonical, KMMT.
- special cases HMX.

Exa: • $E \subseteq W_n \longrightarrow X_n$ = cone over rational curve of deg n

$$\begin{array}{c} W_n \\ \downarrow {\scriptstyle \mathbb{P}^1\text{-bundle}} \\ \mathbb{P}^1 \end{array}$$

- $E^2 = -n$
- X_n sing Fano (if $n \geq 2$)
- $K_{W_n} + \frac{n-2}{n} E = {}^*K_{X_n}$, $\frac{2}{n}$-lc.
- $\{ X_n \mid_{n \in \mathbb{N}} \}$ not bdd.

• Cremona groups:

Cor (B, 2016)(Prokhorov–Shramov)

X rationally-connected $\Rightarrow$ $Bir(X)$ satisfies Jordan property.

Sings of linear systems

- (X,B) pair, A $\mathbb{R}$-div.
- Define

$$\operatorname{lct}(X,B,|A|_{\mathbb{R}}) = \inf\{\operatorname{lct}(X,B,L) \mid 0 \le L \sim_{\mathbb{R}} A\}$$

(*) **Thm** (B, 2016)

$\forall d, r \in \mathbb{N}$, $\epsilon \in \mathbb{R}^{>0}$, $\exists$ $t \in \mathbb{R}^{>0}$ s.t. if:

- (X,B) proj ϵ-lc,
- A v. ample with $A^d \le r$,
- $A - B$ ample,
- $0 \le G$ with $|A-G|_{\mathbb{R}} \ne \emptyset$,

then $\operatorname{lct}(X,B,G) \ge \operatorname{lct}(X,B,A) \ge t$.

"$\deg_A B$" & "$\deg_A G$" bnd.

Thm (B, 2016)

$\forall d \in \mathbb{N}$, $\epsilon \in \mathbb{R}^{>0}$, $\exists$ $t \in \mathbb{R}^{>0}$ s.t. if (X,B) ϵ-lc, dim $= d$, $A := -(K_X+B)$ nef & big, then

$$\operatorname{lct}(X,B,|A|_{\mathbb{R}}) \ge t.$$

- Rem: Known as Ambro conj but doesn't follow easily from previous thm.

Thm (B, 2016)

Assumption as in $\cdot$. If $\operatorname{lct} \le 1$, then $\exists$ $0 \le L \sim_{\mathbb{R}} A$ s.t.

$$\operatorname{lct}(X,B,|A|_{\mathbb{R}}) = \operatorname{lct}(X,B,L).$$

- Rem: Answers question of Tian (90's).

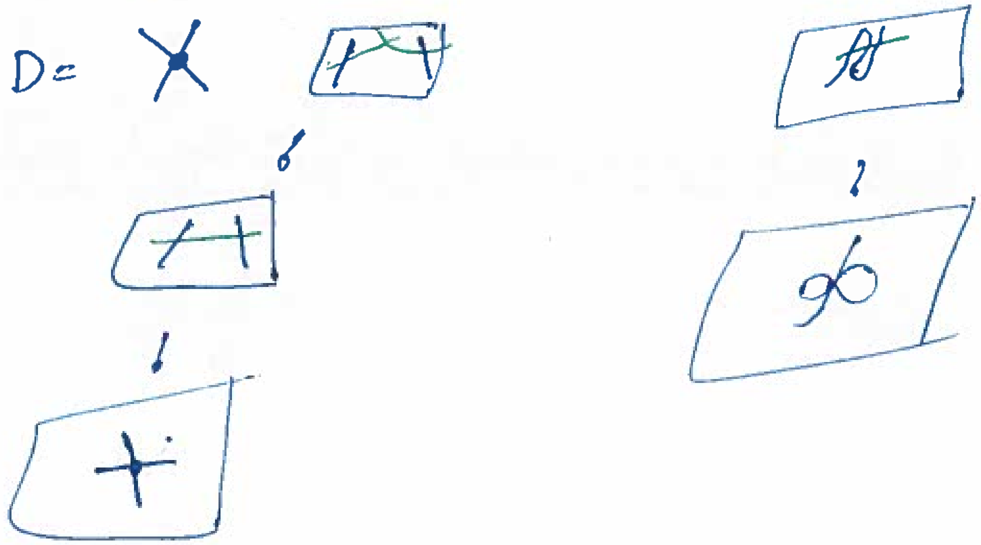
• Rems on Thm (*)

- $X = \mathbb{R}^d$.
- (X, B) ε-lc, coeffs $B \in \Gamma$,
- $G \geq 0$ with coeffs $G \in \Gamma$.
- $Y \xrightarrow[\phi]{\text{bir}} X$ s.t. $K_Y + \underbrace{D}_{\geq 0} = \phi^*(K_X + B)$.

$\Rightarrow$ • thm implies coeffs of $\phi^* G$ bdd from above.

• E.g., toroidal case:

• $D =$



$\Rightarrow$ # blowups bdd.