Smooth varieties with torus actions

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k	algebraically closed field of characteristic zero
$\mathbb{G}_{\mathrm{m}}$	multiplicative group of k
$\mathrm{T} = \mathrm{T}_n$	algebraic torus of dimension n over k
M	character lattice of T
$M_{\mathbb{Q}}$	$M \otimes_{\mathbb{Z}} \mathbb{Q}$
N	1-parameter subgroup lattice of T
$N_{\mathbb{Q}}$	$N \otimes_{\mathbb{Z}} \mathbb{Q}$
σ	Strongly convex polyhedral cone in $N_{\mathbb{Q}}$
$\sigma^{\vee}$	dual cone of σ in $M_{\mathbb{Q}}$
Σ	fan in $N_{\mathbb{Q}}$
$X(\sigma)$	toric variety asociated to σ
$X(\Sigma)$	toric variety asociated to Σ

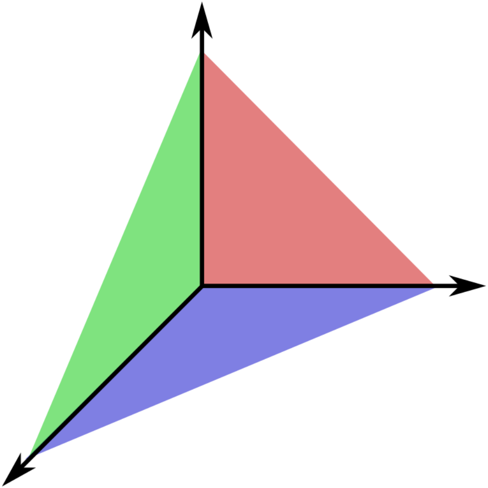
Normal varieties with torus actions

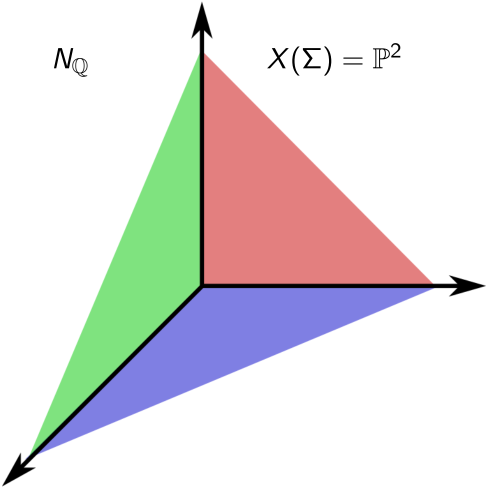
Definition

A $\mathbb{T}$-variety X is a normal variety with a faithful torus action
The complexity of X is the codimension of a generic orbit

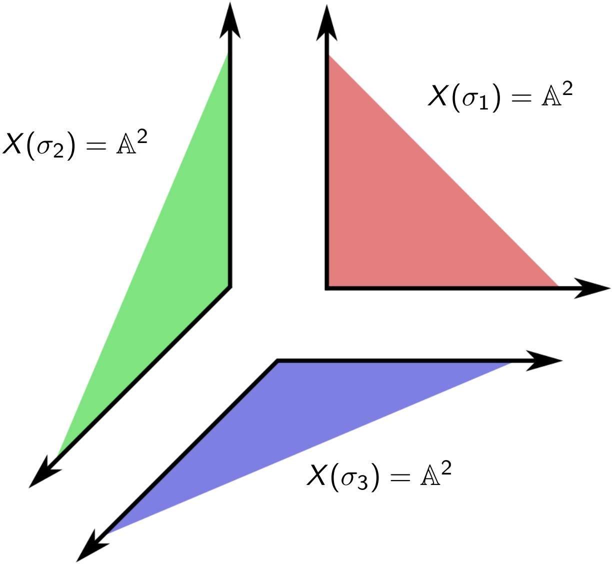
The best known examples are toric varieties (complexity 0)
They have a quite simple combinatorial description (fans)
Many geometrical property can be read from these data

Starting from 2003, Almann, Hausen, Süß, Ilten and many other have developed a similar theory for higher complexity

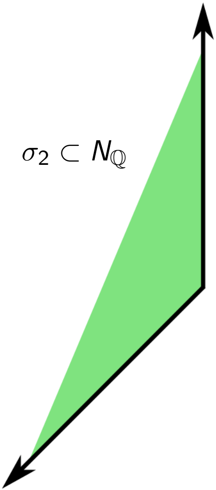


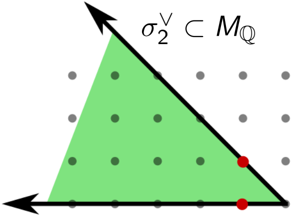


$N_{\mathbb{Q}}$
$X(\Sigma) = \mathbb{P}^2$



$X(\sigma_1) = \mathbb{A}^2$
$X(\sigma_2) = \mathbb{A}^2$
$X(\sigma_3) = \mathbb{A}^2$



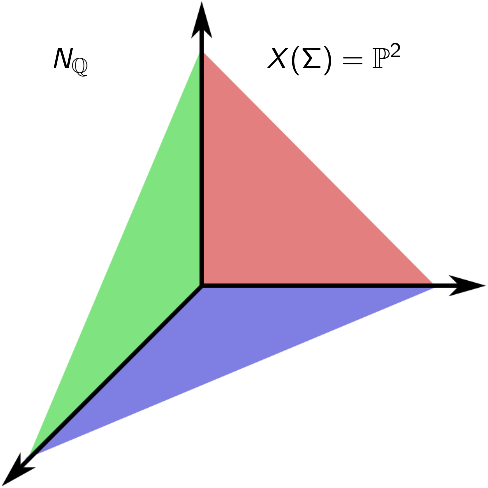


$$k[\sigma_2^{\vee} \cap M] = \bigoplus_{m \in \sigma_2^{\vee}} k \cdot \chi^m$$

$$k[\sigma_2^{\vee} \cap M] = k[x, y]$$

$$x = \chi^{(-1,0)}$$

$$y = \chi^{(-1,1)}$$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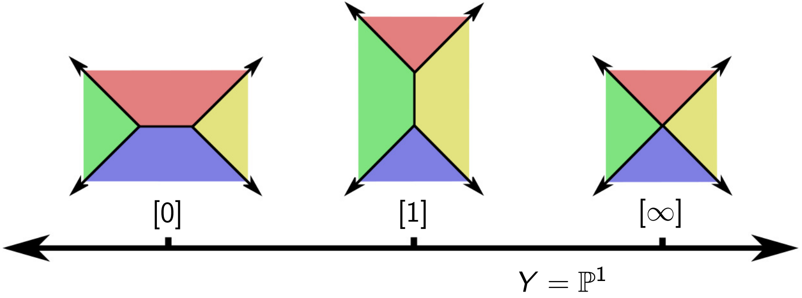
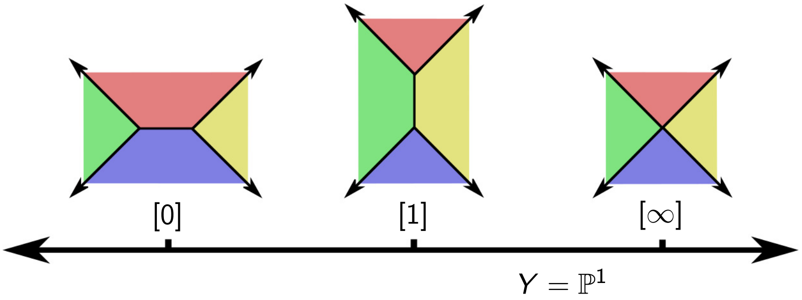
A divisorial fan

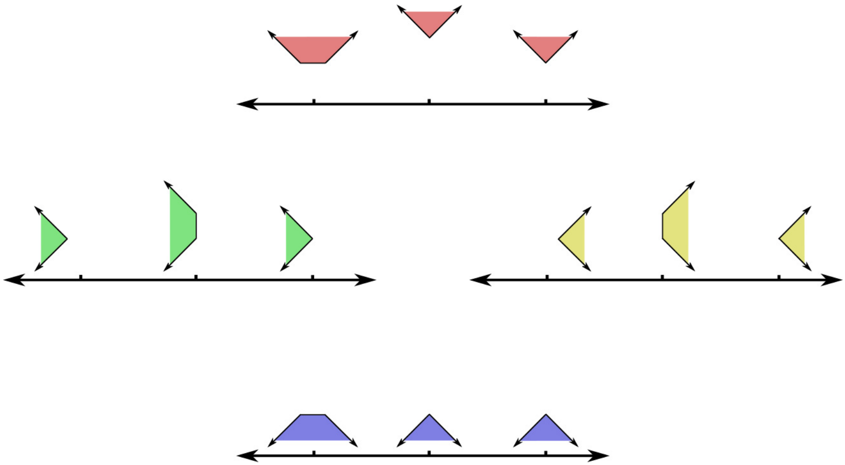


A divisorial fan

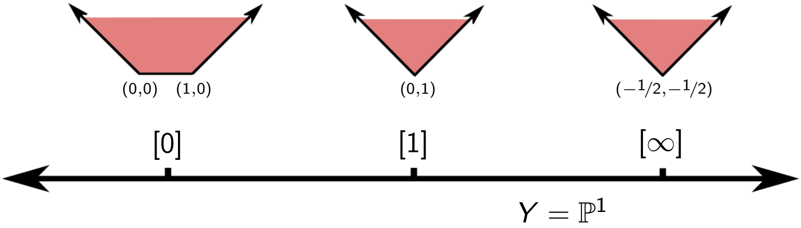


Represents a T_2-variety of complexity 1
It is separated and complete
Corresponds to the the smooth quadric in $\mathbb{P}^4$

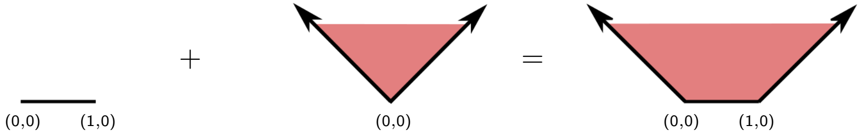
p-divisors on the projective line



A p-divisor on the projective line



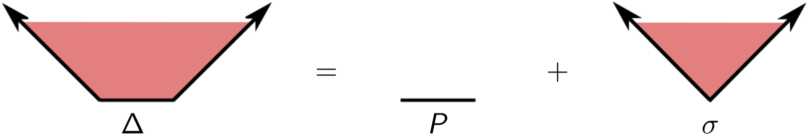
Minkowski sum



Tailed polyhedra

$\Delta \subset N_{\mathbb{Q}}$ polyhedon
The tail cone of Δ is the unique cone σ such that

$$\Delta = P + \sigma, \qquad \text{with } P \text{ a polytope}$$



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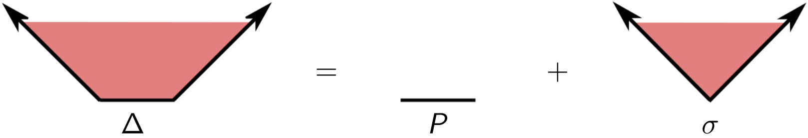
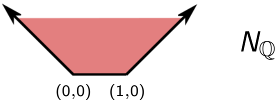


$\mathrm{Pol}(\sigma, N_{\mathbb{Q}})$ is the set of all polyhedra with fixed tail cone σ
$\Delta \in \mathrm{Pol}(\sigma, N_{\mathbb{Q}})$ is called a σ-polyhedra
$\mathrm{Pol}(\sigma, N_{\mathbb{Q}})$ is a semigroup under Minkowski sum
σ is the neutral element in $\mathrm{Pol}(\sigma, N_{\mathbb{Q}})$

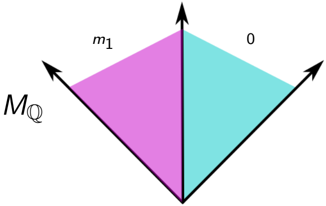
Support function

Let $\Delta \in \mathrm{Pol}(\sigma, N_{\mathbb{Q}})$
The support function of Δ is the map

$$\sigma^{\vee} \to \mathbb{Q}, \quad m \mapsto \min_{v \in \Delta} \langle m, v \rangle$$



Convex polyhedron



Concave piecewise linear function

polyhedral divisors

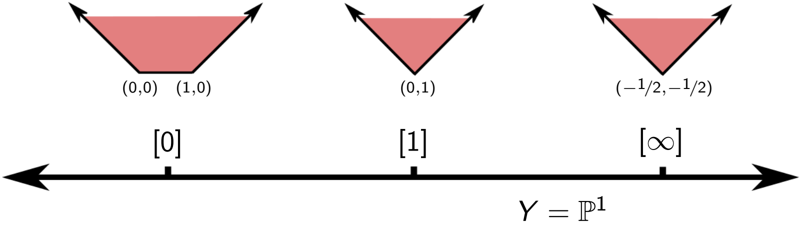
Let Y be a normal semiprojective variety, i.e.
The morphism $Y \to \operatorname{Spec} H^0(Y, \mathcal{O}_Y)$ is projective
Let σ be a fixed strongly convex tail cone

Definition

A polyhedral divisor on Y is a formal sum

$$\mathfrak{D} = \sum_{Z \subset Y} \Delta_Z \cdot Z$$

where Δ_Z are σ-polyhedra and all but finitely many Δ_Z are σ



polyhedral divisors

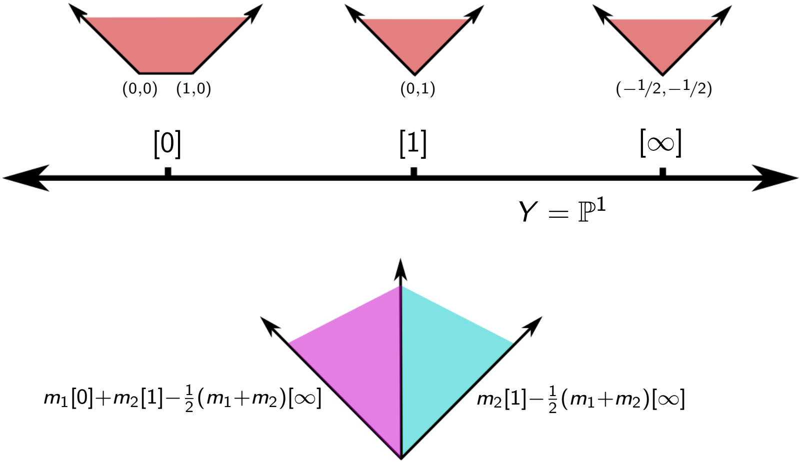
Let $\mathfrak{D} = \sum_{Z \subset Y} \Delta_Z \cdot Z$ be a polyhedral divisor
Let h_Z be the support function of Δ_Z
We can see $\mathfrak{D}$ as a function to Weil $\mathbb{Q}$-divisors

$$\begin{aligned} \mathfrak{D} : \sigma &\longrightarrow \mathrm{WDiv}_{\mathbb{Q}}(Y) \\ m &\longmapsto \mathfrak{D}(m) = \sum_{Z \subset Y} h_Z(m) \cdot Z \end{aligned}$$

$\mathfrak{D}$ is piecewise linear and concave.

$$\mathfrak{D}(m) + \mathfrak{D}(m) \leq \mathfrak{D}(m + m')$$

polyhedral divisors



polyhedral divisors

Let $\mathfrak{D} = \sum_{Z \subset Y} \Delta_Z \cdot Z$ be a p-divisor. We define

$$A(\mathfrak{D}) = \bigoplus_{m \in \sigma^\vee \cap M} H^0(Y, \mathcal{O}(\mathfrak{D}(m))) \cdot \chi^m \quad \text{and} \quad X(\mathfrak{D}) = \operatorname{Spec} A(\mathfrak{D})$$

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Let $f \in H^0(Y, \mathcal{O}(\mathfrak{D}(m)))$ and $g \in H^0(Y, \mathcal{O}(\mathfrak{D}(m')))$
The multiplication map is given by

$$f\chi^m \cdot g\chi^{m'} = fg\chi^{m+m'}$$

It is well defined since

$$\begin{aligned} \operatorname{div}(f) + \mathfrak{D}(m) + \operatorname{div}(g) + \mathfrak{D}(m') &\geq 0 \\ \operatorname{div}(fg) + \mathfrak{D}(m) + \mathfrak{D}(m') &\geq 0 \\ \operatorname{div}(fg) + \mathfrak{D}(m + m') &\geq 0 \end{aligned}$$

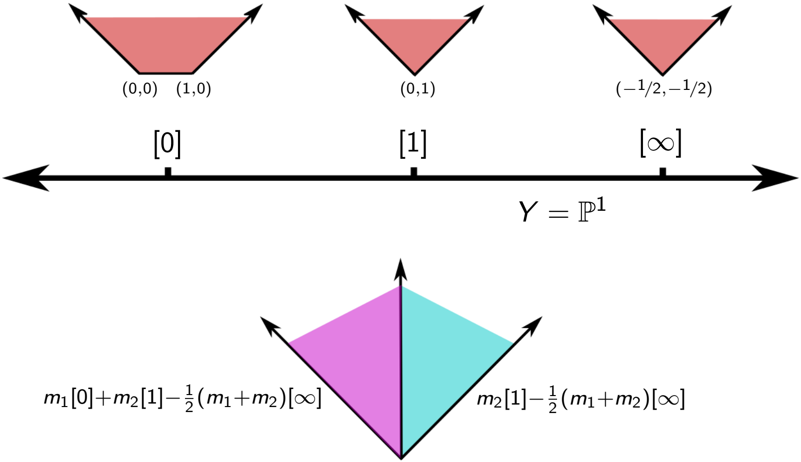
polyhedral divisors

Definition

A polyhedral divisor $\mathfrak{D}$ is called a p-divisor if

- $\mathfrak{D}(m)$ is $\mathbb{Q}$-Cartier and semiample, $\forall m \in \sigma^\vee$
- $\mathfrak{D}(m)$ is big, $\forall m \in \operatorname{rel.int}(\sigma^\vee)$

Example of p-divisor



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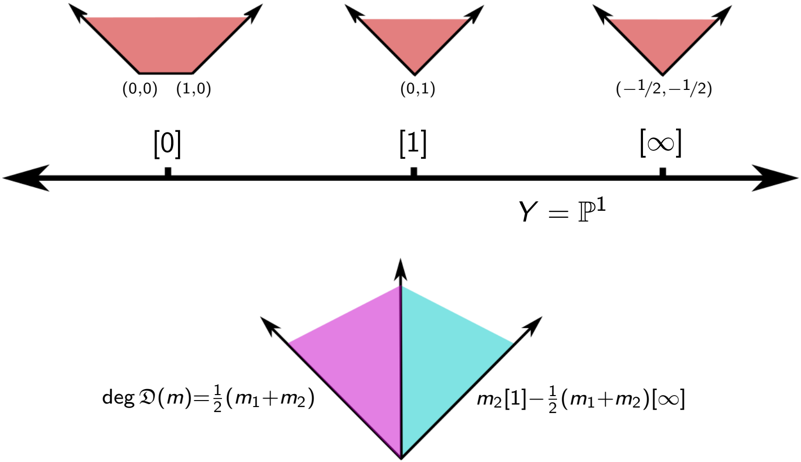

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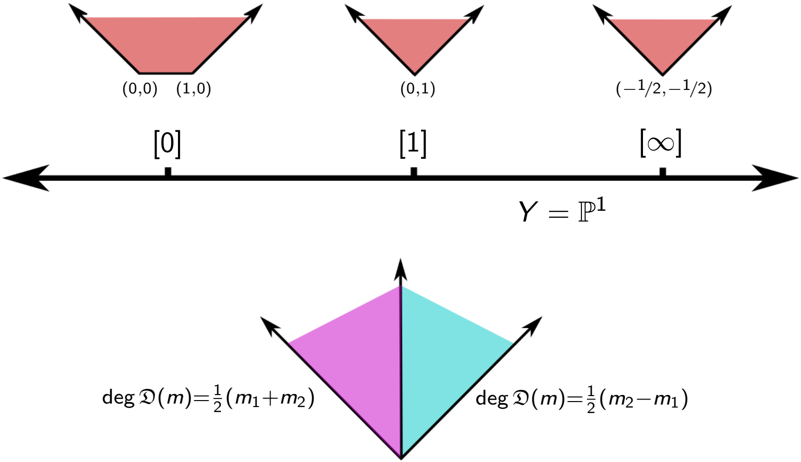

Theorem (Altmann and Hausen)

Let $\mathfrak{D}$ be a p-divisor on a semiprojective normal variety Y. Then $X(\mathfrak{D})$ is a normal affine T-variety of complexity $\dim Y$

Conversely, every normal affine T-variety is equivariantly isomorphic to $X(\mathfrak{D})$ for some p-divisor $\mathfrak{D}$ on some semiprojective normal variety Y

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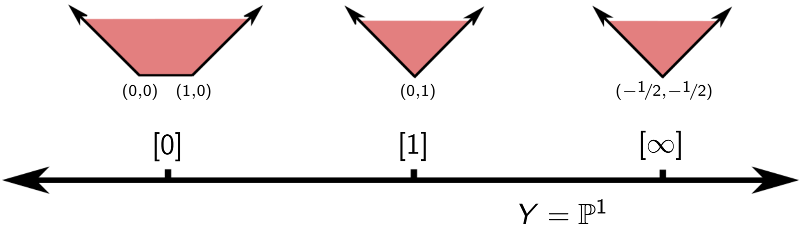
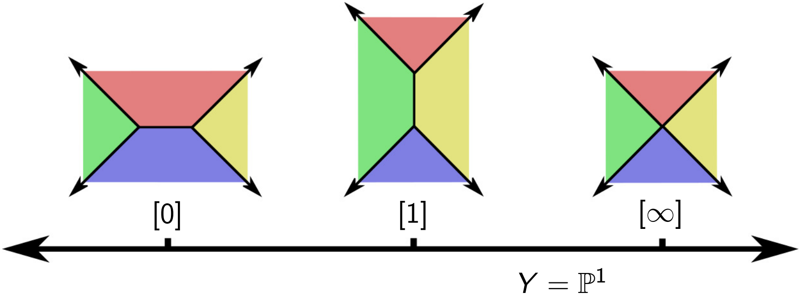


Then $X(\mathfrak{D})$ is isomorphic to $\mathbb{A}^3$ with the complexity 1 T_2-action

$$\mathrm{T}_2 \times \mathbb{A}^3 \longrightarrow \mathbb{A}^3$$
$$(t_1, t_2) \times (x_1, x_2, x_3) \longmapsto (t_1^{-1} t_2 \cdot x, t_1^1 t_2^{-1} \cdot y, t_2 \cdot z)$$

Gluing of affine pieces

The nice gluing process in these pictures only works in complexity 1



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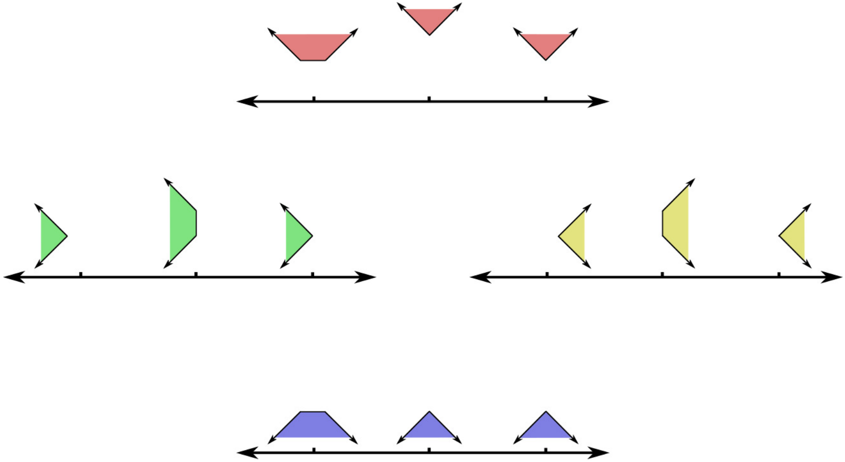

Overview of known results

Combinatorial description	Altmann, Hausen, Süß
Nice pictures for cp 1	Ilten, Süß
Divisors, Cl(X), Pic(X)	Petersen, Süß
Cox rings	Hausen, Huggenberger, Süß
	Altmann, Petersen, Wiśniewski
Singularities	L., Süß (Laface, L., Moraga)
Projective T-varieties	Ilten, Süß
Deformations	Altmann, Ilten, Hochenegger, Vollmert
Generators	Ilten, Kastner
$\mathbb{G}_a$-actions	Langlois, L.

Overview of known results

SL_2-actions	Arzhantsev, L.
Automorphism groups	Arzhantsev, Hausen, Huggenberger, L.
G-varieties	Altmann, Kiritchenko, Petersen
	Langlois, Terpereau, Perepechko
Vector bundles	Ilten, Süß
Kahler-Einstein metrics	Ilten, Süß
Frobenius Splitting	Achinger, Ilten, Süß
Topology and Chow group	Laface, L., Moraga
Uniform rationality	Petitjean
Okunkov bodies and Well-poisedness	Ilten, Manon, Petersen

Overview of known results

Cox rings on proyectivized TVB	Gonzalez, Hering, Payne, Süß
Flexibility	Michalek, Perepechko, Süß
Smoothness	Liendo, Petitjean

Uniqueness of the combinatorial description

The description $X \simeq X(\mathfrak{D})$ is not unique

- Automorphism of T reflected in a base change in N
- Choice of an equivariant rational map $X \dashrightarrow \mathrm{T}$
- Automorphism of the base

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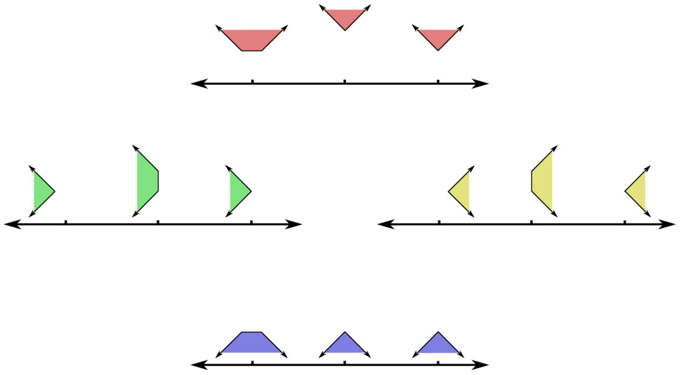

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- Automorphism of T reflected in a base change in N
- Choice of an equivariant rational map $X \dashrightarrow \mathrm{T}$
- Automorphism of the base
- Birational map of the base

$$\varphi : Y \longrightarrow Y' \text{ projective birational,} \quad \mathfrak{D} = \varphi^*(\mathfrak{D}')$$

Then

$$X(\mathfrak{D}') \simeq X(\mathfrak{D}) \text{ equivariantly}$$

This phenomenon does not arrive in complexity 1

Minimal p-divisors

Definition

Let $\mathfrak{D}$ and be a p-divisor on Y.
Then $\mathfrak{D}$ is minimal if $\mathfrak{D} = \varphi^*(\mathfrak{D}')$ with $\varphi : Y \longrightarrow Y'$ projective birational implies φ is an isomorphism

For every T-variety X, there is a canonical way (GIT) to construct a rational quotient Y (called Chow quotient) where there is a minimal p-divisor $\mathfrak{D}$ such that $X \simeq X(\mathfrak{D})$

Smoothness criteria

Report on a joint work with Charlie Petitjean from Dijon

Before, the known results were very special cases

- Y projective (Kambayashi, Russell, 1982)
- $X(\mathfrak{D})$ of complexity 1 (Flenner, Zaidenberg, 2002)

Y projective

Let $X = X(\mathfrak{D})$ with $\mathfrak{D}$ minimal p-divisor on Y

$k[X]^{\mathrm{T}} = H^0(Y, \mathcal{O}(\mathfrak{D}(0)))$

Y projective $\iff$ the algebraic quotient $X/\!/\mathrm{T}$ is a point

Kambayashi and Russell in "On linearizing algebraic torus actions"

X is smooth $\iff$ X is equivariantly isomorphic to the affine space with a linear torus action

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In p-divisors language

X is smooth $\iff$

- $Y = X(\Sigma)$ is toric
- $\mathfrak{D}$ is supported in toric divisors and
- the cone spanned in $(N \oplus N_Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ by $(\sigma, 0)$ and (Δ_ρ, ρ), for all $\rho \in \Sigma(1)$ is smooth

$X(\mathfrak{D})$ of complexity 1

Let $X = X(\mathfrak{D})$ with $\mathfrak{D}$ minimal p-divisor on Y

X is smooth
Y projective $\iff$

- $Y = \mathbb{P}^1$
- $\mathfrak{D}$ is supported in $[0]$ and $[\infty]$
- the cone spanned in $N_{\mathbb{Q}} \oplus \mathbb{Q}$ by $(\sigma, 0)$, $(\Delta_0, 1)$ and $(\Delta_\infty, -1)$ is smooth

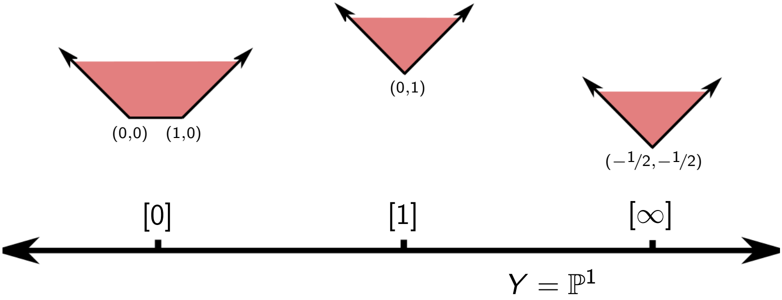
$X(\mathfrak{D})$ of complexity 1

Let $X = X(\mathfrak{D})$ with $\mathfrak{D}$ minimal p-divisor on Y

X is smooth
Y affine $\iff$

- Y is anything
- $\mathfrak{D}$ is supported anywhere
- the cone spanned in $N_{\mathbb{Q}} \oplus \mathbb{Q}$ by $(\sigma, 0)$ and $(\Delta_z, 1)$ is smooth for all $z \in Y$

Example with Y projective



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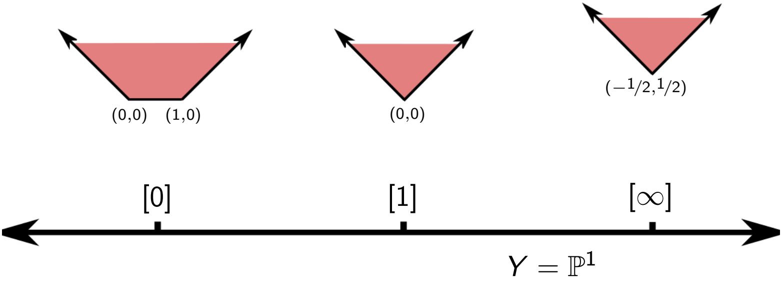

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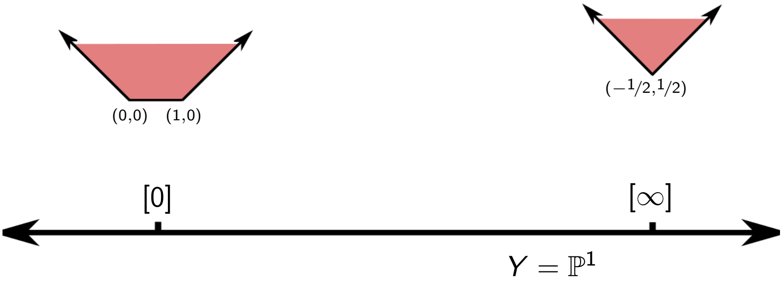

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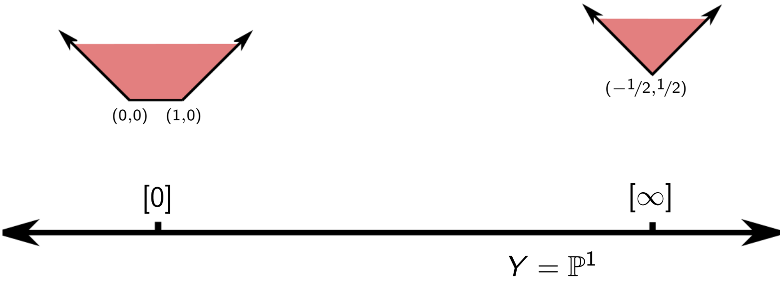
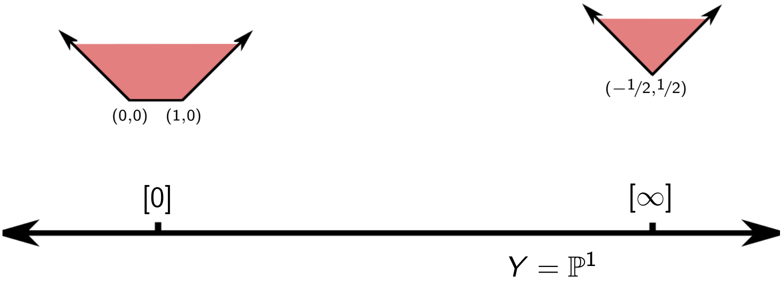


The cone generated by $(\sigma, 0)$, $(\Delta_0, 1)$ and $(\Delta_\infty, -1)$ is spanned by

$$v_1 = (1, 1, 0)$$
$$v_2 = (-1, 1, 0)$$
$$v_3 = (0, 0, 1)$$
$$v_4 = (1, 0, 1)$$
$$v_5 = (-1, 1, -2)$$

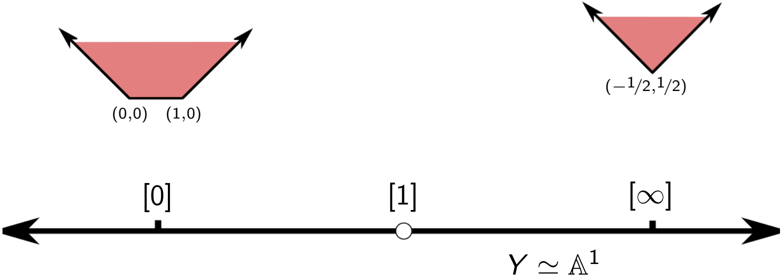
Example with Y projective



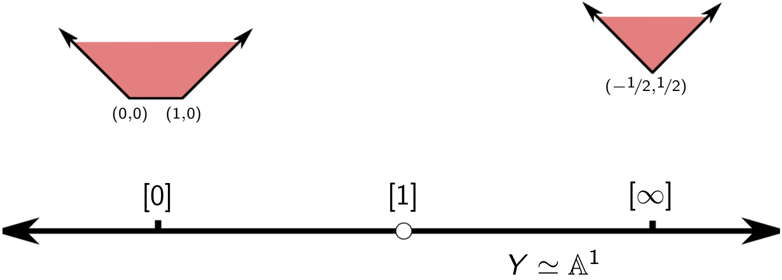
The cone generated by $(\sigma, 0)$, $(\Delta_0, 1)$ and $(\Delta_\infty, -1)$ is spanned by

$$
\begin{aligned}
{\color{red} v_1} &{\color{red} = (1,1,0)} \\
{\color{red} v_2} &{\color{red} = (-1,1,0)} \\
v_3 &= (0,0,1) \\
v_4 &= (1,0,1) \\
v_5 &= (-1,1,-2)
\end{aligned}
\qquad
\det \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & -2 \end{pmatrix} = -1
$$

Example with Y affine



Example with Y affine



The cone generated by $(\sigma, 0)$, $(\Delta_0, 1)$ is spanned by

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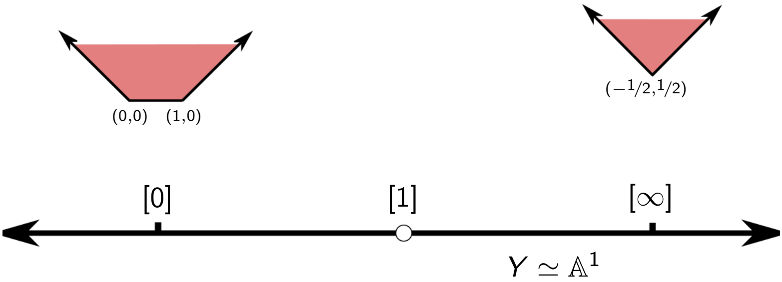
$$v_2 = (-1,1,0)$$
$$v_3 = (0,0,1)$$
$$v_4 = (1,0,1)$$

it is not even a simplicial cone

$X(\mathfrak{D})$ is not smooth

Example with Y affine



The cone generated by $(\sigma, 0)$, $(\Delta_\infty, 1)$ is spanned by

$$v_1 = (1,1,0) \qquad v_2 = (-1,1,0) \qquad v_5 = (-1,1,2)$$

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 2 \end{pmatrix} = -4$$

Towards the general case

The main tool to provide a characterization of smooth affine T-varieties in higher complexity is Luna's Slice Theorem

Let $\phi : X \to X'$ be an equivariant morphism of T-varieties

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ \downarrow & & \downarrow \\ X/\!/\mathrm{T} & \xrightarrow{\phi_{/\!/\mathrm{T}}} & X'/\!/\mathrm{T} \end{array}$$

The morphism is called strongly étale if

- ϕ and $\phi_{/\!/\mathrm{T}}$ are étale
- $X \simeq X' \times_{X'/\!/\mathrm{T}} X/\!/\mathrm{T}$ equivariantly

Strongly étale morphism play the role of local isomorphisms

Smooth varieties

Theorem

Smooth varieties X are locally isomorphic to the affine space in the étale topology

For every $x \in X$ there exists:

- A Zariski neighborhood $\mathcal{U} \subset X$ of x
- A Zariski open $\mathcal{V} \subset \mathbb{A}^k$
- A variety Z and étale morphisms $\phi : Z \to \mathcal{U}$ and $\psi : Z \to \mathcal{V}$

$$\mathcal{U} \xleftarrow{\phi} Z \xrightarrow{\psi} \mathcal{V}$$

Smooth $\mathbb{T}$-varieties

Theorem (Ad-hoc Luna's Slice Theorem)

Smooth $\mathbb{T}$-varieties X are equivariantly locally isomorphic in the étale topology to the affine space endowed with a linear $\mathbb{T}$-action. Furthermore, the local isomorphism is realized by strongly étale morphisms

Let $\pi : X \to Y_0 = X/\!/\mathbb{T}$ be a algebraic quotient
For every $y \in Y_0$ there exists:

- A Zariski neighborhood $\mathcal{U}_0 \subset Y_0$ of y_0.
- A linear $\mathbb{T}$-action on $\mathbb{A}^k$
- A $\mathbb{T}$-equivariant Zariski open $\mathcal{V} \subset \mathbb{A}^k$
- A $\mathbb{T}$-variety Z and strongly étale morphisms $\phi : Z \to \mathcal{U}$ and $\psi : Z \to \mathcal{V}$

$$\begin{array}{ccccc}
\pi^{-1}(\mathcal{U}_0) & \xleftarrow{st.\,ét.} & Z & \xrightarrow{st.\,ét.} & \mathcal{V} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{U}_0 & \xleftarrow{ét.} & Z/\!/\mathbb{T} & \xrightarrow{ét.} & \mathcal{V}/\!/G
\end{array}$$

Main result

Theorem

Let $\mathfrak{D}$ be a minimal p-divisor on Y. Then $X(\mathfrak{D})$ is smooth if and only if the combinatorial data $(Y, \mathfrak{D})$ is locally isomorphic in the étale topology to the combinatorial data of the affine space endowed with a linear $\mathbb{T}$-action.

Let $q : Y \to Y_0$. For every $y \in Y_0$ there exists:

- A Zariski neighborhood $\mathcal{U} \subset Y$ of $q^{-1}(y_0)$
- A linear T-action on $\mathbb{A}^k$ given by a minimal p-divisor $\mathfrak{D}'$ en Y'
- A Zariski open $\mathcal{V} \subset Y'$
- A variety Z and étale morphisms $\phi : Z \to \mathcal{U}$ and $\psi : Z \to \mathcal{V}$ such that $\phi^*(\mathfrak{D}) = \psi^*(\mathfrak{D}')$

$$(\mathcal{U}, \mathfrak{D}) \xleftarrow{\ \phi\ } (Z, \mathfrak{D}'') \xrightarrow{\ \psi\ } (\mathcal{V}, \mathfrak{D}')$$

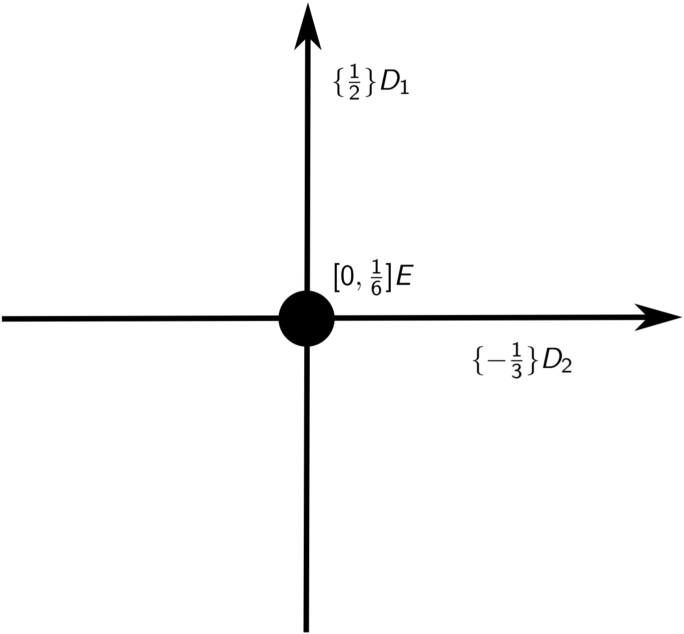
Example: the affine space

Let $N = \mathbb{Z}$
$Y = \mathrm{Bl}_0(\mathbb{A}^2)$
$Y_0 = \mathbb{A}^2$
D_1, D_2 the strict transform of coordinate hyperplanes
E exceptional divisor

$$\mathcal{D} = \left\{\frac{1}{2}\right\} \cdot D_1 + \left\{-\frac{1}{3}\right\} \cdot D_2 + \left[0, \frac{1}{6}\right] \cdot E$$

$X(\mathfrak{D}) = \mathbb{A}^3$ with the T_1-action

$$\begin{aligned} \mathrm{T}_1 \times \mathbb{A}^3 &\longrightarrow \mathbb{A}^3 \\ t \times (x, y, z) &\longmapsto (t^2 \cdot x, t^3 \cdot y, t^{-6} \cdot z) \end{aligned}$$


$\{\frac{1}{2}\}D_1$
$[0, \frac{1}{6}]E$
$\{-\frac{1}{3}\}D_2$

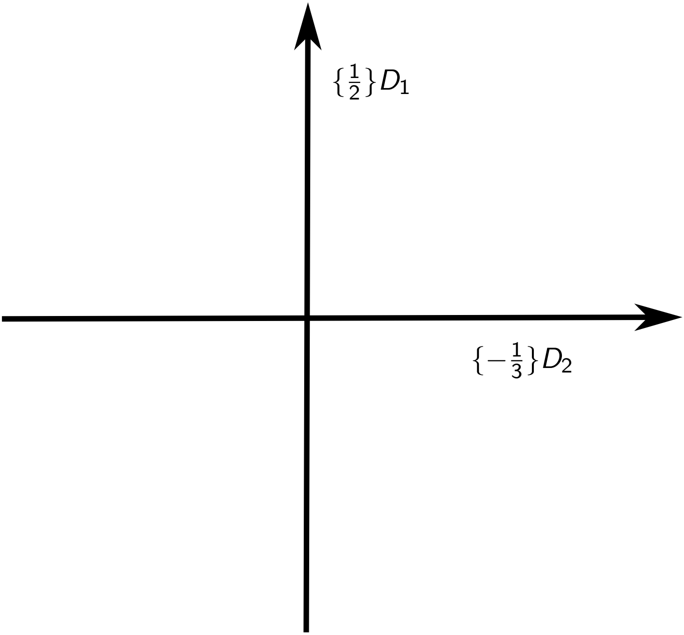
Example: an open set in the affine space

Let $N = \mathbb{Z}$
$Y = \mathbb{A}^2$
D_1, D_2 the coordinate hyperplanes

$$\mathcal{D} = \left\{\frac{1}{2}\right\} \cdot D_1 + \left\{-\frac{1}{3}\right\} \cdot D_2$$

$X(\mathfrak{D}) = \mathbb{A}^2 \times k^*$ with the T_1-action

$$\mathrm{T}_1 \times X(\mathfrak{D}) \longrightarrow X(\mathfrak{D})$$
$$t \times (x, y, z) \longmapsto (t^2 \cdot x, t^3 \cdot y, t^6 \cdot z)$$


$\{\frac{1}{2}\} D_1$
$\{-\frac{1}{3}\} D_2$

Example: non-rational support

Let $N = \mathbb{Z}$
$Y = \mathrm{Bl}_0(\mathbb{A}^2)$
$Y_0 = \mathbb{A}^2(u, v)$
D_1 the strict transform of the affine elliptic curve

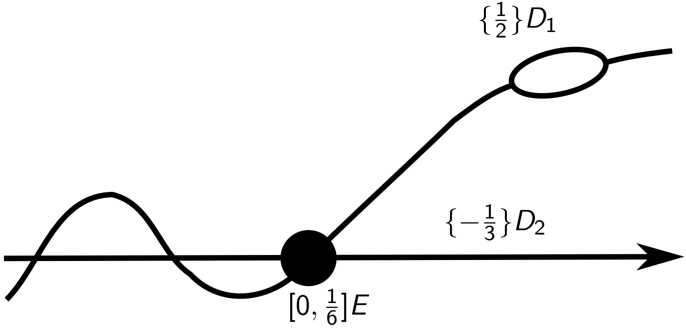
$$\left\{h(u,v) = u^2 - v(v-\alpha)(v-\beta)\right\}$$

D_2 the strict transform of $\{u = 0\}$
E exceptional divisor

$$\mathcal{D} = \left\{\frac{1}{2}\right\} \cdot D_1 + \left\{-\frac{1}{3}\right\} \cdot D_2 + \left[0, \frac{1}{6}\right] \cdot E$$

$$X(\mathfrak{D}) = \left\{\tfrac{1}{z} h(x^3 z, yz) = t^2\right\} \subset \mathbb{A}^4 \qquad \text{weight } (2, 6, -6, 3)$$

and $X(\mathfrak{D})$ is smooth (jacobian criterion)


$\{\frac{1}{2}\}D_1$
$\{-\frac{1}{3}\}D_2$
$[0, \frac{1}{6}]E$

Example: bad crossing

Let $N = \mathbb{Z}$
$Y = \mathrm{Bl}_0(\mathbb{A}^2)$
$Y_0 = \mathbb{A}^2(u, v)$
D_1 the strict transform of the affine rational curve

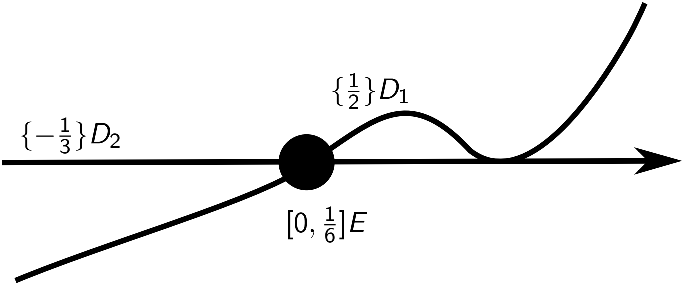
$$\left\{h(u,v) = u - v(v-1)^2\right\} \simeq \mathbb{A}^1$$

D_2 the strict transform of $\{u = 0\}$
E exceptional divisor

$$\mathcal{D} = \left\{\frac{1}{2}\right\} \cdot D_1 + \left\{-\frac{1}{3}\right\} \cdot D_2 + \left[0, \frac{1}{6}\right] \cdot E$$

$$X(\mathfrak{D}) = \left\{x^3 + y(yz-1)^2 = t^2\right\} \subset \mathbb{A}^4 \qquad (2, 6, -6, 3)$$

and $X(\mathfrak{D})$ is not smooth. The point $(0, 1, 1, 0)$ is singular


$\{\frac{1}{2}\}D_1$
$\{-\frac{1}{3}\}D_2$
$[0, \frac{1}{6}]E$

¡Gracias!