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The scroll of tangents of an elliptic quartic curve

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1. The equation \mathcal{C} of the scroll R of tangents of the curve Γ of intersection of two quadrics in general position in [3] was found by Cayley in 1850 ((1), p. 50). Cayley also considered special positions of the quadrics but we shall not be concerned with these. \mathcal{C} was reproduced in Salmon's treatise ((8), p. 191) and Salmon, on this same page, provides an equation in covariant form and so applicable to pairs of quadrics whether in canonical form, or in general position, or not.

The procedure now proposed for finding the equation of R takes advantage from the outset of every quadric through Γ being invariant under an elementary abelian group E of eight projectivities; when, as below, the quadrics are in canonical form E is generated by sign changes of the homogeneous coordinates. The basic fact is that all points not in any face of the common self-polar tetrahedron U of the quadrics are distributed in octads, and these octads can be handled as entities.

It is shown in §3 how the equation now to be found for R can be written so as to disclose the existence of a singly infinite system, of index 2, of quartic surfaces Q_μ having R for their envelope. Q_μ has fixed nodes at the vertices of the four cones through Γ and has also nodes at an octad on Γ that varies with μ ; it meets R in Γ (reckoned twice because it is cuspidal on R) and touches R along an elliptic curve of order 12.

The opportunity is taken, in the second part (§§6–7) of the paper, of placing R among the combinants of the pencil of quadrics which contain Γ , the complete system of combinants being at hand in Todd's paper (9) of 1947. The quartic surface F , whose covariant relation to the quadrics was discovered by Todd (10) and explained ((4), pp. 4, 5) by Dye, is placed in the present context.

In the third part (§§8–15) of the paper R is transformed into a quartic surface, W having a cuspidal line γ and four nodes. The transformation has been used before, but never applied to R ; and although W was discovered long ago by Plücker in his pioneering researches in line geometry it can perhaps more easily be studied in the

present less elaborate setting. It can be generated by more than one quite elementary geometrical process and one such, involving twisted cubics, is used in § 11 to obtain a mapping of W on a plane. The figure of a line and four points naturally suggests, quite apart from their being the main features of W , using the quadric Φ for which the points are vertices of a self-polar tetrahedron and which contains the line; W proves to be its own polar reciprocal with respect to Φ .

I

2. The curve Γ is defined by

$$\Omega_0 = \Omega_1 = 0, \quad (2.1)$$

where

$$\Omega_k \equiv \sum_{i=0}^3 a_i^k x_i^2$$

and no two of the four a_i are equal. Write

$$\begin{aligned} f(\phi) &\equiv (\phi - a_0)(\phi - a_1)(\phi - a_2)(\phi - a_3) \\ &\equiv \phi^4 - e_1\phi^3 + e_2\phi^2 - e_3\phi + e_4 \end{aligned}$$

and

$$s_k \equiv \sum a_i^k / f'(a_i)$$

the sums being, save for one or two exceptions mentioned later, all of four terms. These s_k clearly satisfy the recurrence relation

$$s_k - e_1 s_{k-1} + e_2 s_{k-2} - e_3 s_{k-3} + e_4 s_{k-4} = 0 \quad (k \geq 4) \quad (2.2)$$

with the initial conditions, verified at once by using the partial fractions for $\phi^n/f(\phi)$ with $n \leq 3$,

$$s_0 = s_1 = s_2 = 0, \quad s_3 = 1. \quad (2.3)$$

In particular

$$s_4 = e_1, \quad s_5 = e_1^2 - e_2. \quad (2.4)$$

All solutions x_i^2 of (2.1) are linearly dependent on the two

$$x_i^2 = 1/f'(a_i), \quad x_i^2 = a_i/f'(a_i).$$

For these are solutions, by (2.3), and there cannot be more than two linearly independent solutions. Hence the octads on Γ are parametrized by

$$\xi_i^2 = (\theta + a_i)/f'(a_i). \quad (2.5)$$

Since the tangent to Γ at $x_i = \xi_i$ is $\sum x_i \xi_i = \sum a_i x_i \xi_i = 0$ its points x_i are obtained, if no ξ_i is zero, by varying λ in

$$x_i \xi_i = (\lambda + a_i)/f'(a_i), \quad (2.6)$$

and x_i traces R when λ and θ both vary. It follows from (2.5) and (2.6) that the equation of R is obtained by eliminating λ , θ , κ between the four equations

$$x_i^2(\theta + a_i) = \kappa(\lambda + a_i)^2/f'(a_i), \quad (2.7)$$

equations which imply, by (2.3) and (2.4),

$$\theta\Omega_0 + \Omega_1 = 0, \quad \theta\Omega_1 + \Omega_2 = \kappa, \quad \theta\Omega_2 + \Omega_3 = \kappa(e_1 + 2\lambda),$$

$$\theta\Omega_3 + \Omega_4 = \kappa(e_1^2 - e_2 + 2\lambda e_1 + \lambda^2).$$

These, in turn, imply

$$\begin{aligned} e_1(\Omega_0 \Omega_2 - \Omega_1^2) - (\Omega_0 \Omega_3 - \Omega_1 \Omega_2) &= -2\lambda(\Omega_0 \Omega_2 - \Omega_1^2), \\ e_2(\Omega_0 \Omega_2 - \Omega_1^2) - e_1(\Omega_0 \Omega_3 - \Omega_1 \Omega_2) + \Omega_0 \Omega_4 - \Omega_1 \Omega_3 &= \lambda^2(\Omega_0 \Omega_2 - \Omega_1^2) \end{aligned}$$

so that, eliminating λ ,

$$\begin{aligned} &\{\Omega_0 \Omega_3 - \Omega_1 \Omega_2 - e_1(\Omega_0 \Omega_2 - \Omega_1^2)\}^2 \\ &= 4(\Omega_0 \Omega_2 - \Omega_1^2) \{\Omega_0 \Omega_4 - \Omega_1 \Omega_3 - e_1(\Omega_0 \Omega_3 - \Omega_1 \Omega_2) + e_2(\Omega_0 \Omega_2 - \Omega_1^2)\} \quad (2\cdot8) \end{aligned}$$

3. This equation shows that either factor on the right represents, when equated to zero, a quartic surface containing Γ (and whose intersection with R therefore includes this cuspidal curve of R twice over) and touching R all along their residual intersection, this curve C of contact lying on the quartic surface obtained by equating the left-hand side of (2·8) to zero. A similar situation is apparent from Salmon's covariant equation, but he was not concerned with this. And (2·8) is only one particular instance of a form of equation that exhibits a whole family of contact quartic surfaces of R ; one is reminded of the standard situation in the study of the contact conics of a plane quartic curve. For, if

$$Q_\mu \equiv \Omega_0 \Omega_4 - \Omega_1 \Omega_3 - (\mu + e_1)(\Omega_0 \Omega_3 - \Omega_1 \Omega_2) + (\mu^2 + e_1 \mu + e_2)(\Omega_0 \Omega_2 - \Omega_1^2), \quad (3\cdot1)$$

(2·8) is equivalent, whenever $\mu \neq \nu$, to

$$\left\{ \Omega_0 \Omega_4 - \Omega_1 \Omega_3 - \left(\frac{\mu + \nu}{2} + e_1 \right) (\Omega_0 \Omega_3 - \Omega_1 \Omega_2) + \left(\mu\nu + e_1 \frac{\mu + \nu}{2} + e_2 \right) (\Omega_0 \Omega_2 - \Omega_1^2) \right\}^2 = Q_\mu Q_\nu,$$

the only change being that (2·8) has been multiplied by $\frac{1}{4}(\mu - \nu)^2$.

One may take the liberty of calling the surface $Q_\mu = 0$ simply Q_μ , and similar liberties may be taken later.

4. Before describing the geometry it may be remarked that, with Σ denoting a sum of six terms throughout this § 4,

$$\left. \begin{aligned} \Omega_0 \Omega_2 - \Omega_1^2 &\equiv \Sigma(a_i - a_j)^2 x_i^2 x_j^2, \\ \Omega_0 \Omega_3 - \Omega_1 \Omega_2 &\equiv \Sigma(a_i - a_j)^2 (a_i + a_j) x_i^2 x_j^2, \\ \Omega_0 \Omega_4 - \Omega_1 \Omega_3 &\equiv \Sigma(a_i - a_j)^2 (a_i^2 + a_i a_j + a_j^2) x_i^2 x_j^2 \end{aligned} \right\} \quad (4\cdot1)$$

so that the coefficient of $x_0^2 x_1^2$ in Q_μ is the product of $(a_0 - a_1)^2$ and

$$a_0^2 + a_0 a_1 + a_1^2 - (\mu + e_1)(a_0 + a_1) + \mu^2 + e_1 \mu + e_2 = \mu^2 + \mu(a_2 + a_3) + a_2 a_3$$

so that, writing the leading term of the six summed,

$$Q_\mu \equiv \Sigma(a_0 - a_1)^2 (\mu + a_2) (\mu + a_3) x_0^2 x_1^2, \quad (4\cdot2)$$

and the equation of R is, for any unequal pair μ and ν ,

$$\begin{aligned} &\{\Sigma(a_0 - a_1)^2 [\mu\nu + \frac{1}{2}(a_2 + a_3)(\mu + \nu) + a_2 a_3] x_0^2 x_1^2\}^2 \\ &= \{\Sigma(a_0 - a_1)^2 (\mu + a_2) (\mu + a_3) x_0^2 x_1^2\} \{\Sigma(a_0 - a_1)^2 (\nu + a_2) (\nu + a_3) x_0^2 x_1^2\}. \end{aligned}$$

In particular, taking $\mu = \infty$ and $\nu = 0$, the equation becomes

$$\{\Sigma(a_0 - a_1)^2 (a_2 + a_3) x_0^2 x_1^2\}^2 = 4\Sigma(a_0 - a_1)^2 x_0^2 x_1^2 \Sigma(a_0 - a_1)^2 a_2 a_3 x_0^2 x_1^2. \quad (4.3)$$

Cayley found \mathcal{C} by taking U as tetrahedron of reference. If the unit point is then chosen so that one quadric through Γ is $\Omega_0 = 0$ then $a' = b' = c' = d' = 1$ in \mathcal{C} and the outcome is equivalent to (4.3).

5. The identities (4.1) show that the surfaces Q_μ have nodes at the vertices of U , and they have other nodes too. It is apparent from (2.2), or from the necessary linear dependence of the rows of the zero determinant

$$\begin{vmatrix} \Omega_0 & 1 & 1 & 1 & 1 \\ \Omega_1 & a_0 & a_1 & a_2 & a_3 \\ \Omega_2 & a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ \Omega_3 & a_0^3 & a_1^3 & a_2^3 & a_3^3 \\ \Omega_4 & a_0^4 & a_1^4 & a_2^4 & a_3^4 \end{vmatrix}$$

that
$$\Omega_4 \equiv e_1 \Omega_3 - e_2 \Omega_2 + e_3 \Omega_1 - e_4 \Omega_0 \quad (5.1)$$

and so, substituting for Ω_4 in (3.1),

$$Q_\mu \equiv (\mu\Omega_0 + \Omega_1)(\mu\Omega_2 + e_1\Omega_2 - \Omega_3) - e_4\Omega_0^2 + e_3\Omega_0\Omega_1 - (\mu^2 + e_1\mu + e_2)\Omega_1^2, \quad (5.2)$$

showing that Q_μ has nodes wherever

$$\Omega_0 = \Omega_1 = (\mu + e_1)\Omega_2 - \Omega_3 = 0,$$

an octad of points on Γ . It is thus what Cayley aptly calls ((3), p. 133) an *octadic surface* and is, incidentally, a refutation of his over-hasty assertion ((3), p. 153) that an octadic surface cannot have more than two additional nodes.

The four intersections of a tangent of Γ with Q_μ consist of the contact with Γ and a contact with this octadic surface. The curve C_μ of contact of Q_μ with R is therefore a directrix, unisecant to the generators, and so elliptic. Since the vertices of U are nodes of Q_μ and quadruple points of R they are multiple points of C_μ .

Those four Q_μ for which $\mu = -a_i$ are reducible. When, say, $\mu = -a_3$ three of the six terms of the sum in (4.2) are zero while the others all have the non-zero factor

$$(a_0 - a_3)(a_1 - a_3)(a_2 - a_3);$$

Q_{-a_3} consists of the repeated plane $x_3 = 0$ and the cone

$$\Omega_1 - a_3\Omega_0 \equiv (a_0 - a_3)x_0^2 + (a_1 - a_3)x_1^2 + (a_2 - a_3)x_2^2 = 0.$$

This cone touches R all along those four tangents of Γ that concur at $x_0 = x_1 = x_2 = 0$. The rest of the duodecimic curve of contact consists of the nodal plane quartic of R in $x_3 = 0$, reckoned twice.

C_μ is of order 12 because the complete intersection of R , which has order 8, and Q_μ , which has order 4, consists of C_μ and Γ both counted twice: C_μ because it is a curve of contact, Γ because it is cuspidal on R .

II

6. R is a combinant of the pencil $\lambda\Omega_0 + \Omega_1$ of quadratic forms; the complete list of irreducible combinants, complete in the sense that every combinant is expressible in

terms of those in his list, was obtained by Todd ((9), p. 487). Since, as has been seen above, R has order 8 in the x_i and degree 6 in the coefficients of Ω_1 , and so combined degree 12 in the coefficients of any two forms on which the pencil may be based, it must be a linear combination, in Todd's notation, of his

$$F^2, \quad \Delta(3, 1), \quad H(4, 0); \tag{6.1}$$

say
$$R = pF^2 + q\Delta(3, 1) + rH(4, 0). \tag{6.2}$$

Now whenever the quadrics are referred to their common self-polar tetrahedron no powers x_i^8 occur in R . The forms for $\Delta(3, 1)$ and $H(4, 0)$ appear in the antepenultimate and penultimate lines of Todd's equation (35), while the forms S, S', Q, Q' which he uses appear in his earlier equations (22), (28), (29). His F is in (30), and easy calculations show that the coefficients of x_0^8 in the three forms (6.1) are

$$36a^2b^2(a^4 - b^4)^2, \quad -3a^2b^2(a^4 - b^4)^2, \quad 3a^2b^2(a^4 - b^4)^2.$$

Hence
$$12p - q + r = 0 \tag{6.3}$$

Examine, next, the terms in $x_0^4x_1^4$. That in F^2 has coefficient $-72a^2b^2(a^4 - b^4)^2$; the calculations for the other two forms are somewhat longer but the coefficients prove to be, on using Todd's equations,

$$\begin{aligned} -2a^2b^2(a^8 - 34a^4b^4 + b^8) & \text{ in } \Delta(3, 1), \\ 2a^2b^2(5a^8 + 22a^4b^4 + 5b^8) & \text{ in } H(4, 0). \end{aligned}$$

Now take Salmon's equation ((8), p. 191) for R and adapt it to that canonical form which, as Todd explains, is got by choosing one of its three pairs of mutually apolar quadrics as base for the pencil. What does Salmon's coefficient $(cd')^2(ab')^4$ become in these circumstances? Since, using the apolar pair, Todd replaces Salmon's

$$\begin{pmatrix} a & b & c & d \\ a' & b' & c' & d' \end{pmatrix}$$

by
$$\begin{pmatrix} a & a & b & b \\ b & -b & a & -a \end{pmatrix}$$

the coefficient sought is $(-2ab)^2(-2ab)^4 = 64a^6b^6$ and the identity (6.2) implies, on dropping the common factor $2a^2b^2$ and using the value for $12p$ given by (6.3),

$$\begin{aligned} 32a^4b^4 & \equiv -3(q-r)(a^4 - b^4)^2 - q(a^8 - 34a^4b^4 + b^8) + r(5a^8 + 22a^4b^4 + 5b^8) \\ & \equiv -q(4a^8 - 40a^4b^4 + 4b^8) + r(8a^8 + 16a^4b^4 + 8b^8), \end{aligned}$$

requiring $q = 2r$ and $r = \frac{1}{3}$. Hence

$$R \equiv \frac{1}{36}F^2 + \frac{2}{3}\Delta(3, 1) + \frac{1}{3}H(4, 0).$$

7. This placing of R among the octavic combinants of degree 12 enables one to base them on three linearly independent ones of known geometrical significance. For F was interpreted recently (4) by Dye, and Todd's definition ((9), p. 485) of $H(4, 0)$ shows, on referring to his equations (23) and (24), that it is, among the quadrics through Γ , the

Hessian tetrad of the four cones. $\Delta(3, 1)$ perhaps lies outside the domain of geometrically significant forms; one can perhaps include it among those many forms for which (7), p. 287) 'it commonly happens that the significance of some members of the system is so remote as to render them of little geometrical importance'. But this could surely not apply to F since it is unique: the only combinant of as low a degree as 6 in the coefficients. Yet it had to wait 30 years after Todd discovered it for any mathematician to provide the geometrical interpretation.

III

8. Now replace the squares x_i^2 by X_i in the above account of R and its contact octadic surfaces. The $[3]S$ containing R is thereby changed into another $[3]\sigma$, each octad in S being mapped on a point of σ save for three exceptions.

(i) Each vertex A_i of the tetrahedron of reference u in σ maps a single point, the corresponding vertex of U .

(ii) Each point, other than a vertex, on an edge of u maps a pair on the corresponding edge of U , the pair being harmonic to the vertices of U on this edge.

(iii) Each point in a face, but not on an edge, of u maps the vertices of a quadrangle in the corresponding face of U ; the vertices of U in this face are the diagonal points of the quadrangle.

This transformation is applied to Γ , though not to R , by Enriques ((6), p. 298); the quadrics $\Omega_k \equiv \sum a_i^k x_i^2 = 0$ in S become, as he says, planes $\omega_k \equiv \sum a_i^k X_i = 0$ in σ , so that the octads on Γ are mapped by the points of the line

$$\gamma: \omega_0 = \omega_1 = 0.$$

The intersections A'_i of γ with the faces of u opposite to its vertices A_i map the tetrads in which Γ meets the faces of U .

The transform of Q_μ is, from (5.2),

$$q_\mu \equiv (\mu\omega_0 + \omega_1)(\mu\omega_2 + e_1\omega_2 - \omega_3) - e_4\omega_0^2 + e_3\omega_0\omega_1 - (\mu^2 + e_1\mu + e_2)\omega_1^2 = 0,$$

a quadric cone whose vertex

$$\omega_0 = \omega_1 = (\mu + e_1)\omega_2 - \omega_3 = 0$$

is the point on γ mapping the octad of nodes of Q_μ on Γ ; q_μ contains A_i as well as γ . The transform of R is a quartic surface W , the envelope of the cones q_μ ; its equation, by (4.3), is

$$\{\sum(a_0 - a_1)^2(a_2 + a_3)X_0X_1\}^2 = 4\sum(a_0 - a_1)^2X_0X_1\sum(a_0 - a_1)^2a_2a_3X_0X_1, \quad (8.1)$$

each sum here involving six terms. W has nodes at the vertices of u and γ is a double line, which will presently be seen to be cuspidal, on W . Since γ counts twice as a component of the complete intersection of W and q_μ their residual contact has order $\frac{1}{2}(4 \cdot 2 - 2 \cdot 1) = 3$ and so is a twisted cubic through A_0, A_1, A_2, A_3 .

9. There is only one quadric cone, with a given vertex P on γ , having $\gamma, PA_0, PA_1, PA_2, PA_3$ for generators so that one can define W ((5), p. 359) simply as the envelope of those cones that belong to the net of quadrics containing γ and circumscribing u .

Since any two such cones meet, apart from γ , in that twisted cubic which contains A_0, A_1, A_2, A_3 and their two vertices the contacts of the cones with W are those twisted cubics which contain A_0, A_1, A_2, A_3 and touch γ ; so W may be defined as generated by such cubics ((12), p. 1105, fn. 885).

Those contact cones of W whose vertices are the four points A'_i have three coplanar generators and so are plane pairs. If P is at A'_3 the two planes are $A_0 A_1 A_2$ and γA_3 , i.e. $X_3 = 0$ and $a_3 \omega_0 = \omega_1$. This splitting of the cone in σ corresponds to the splitting in S of an octadic surface into the repeated plane $x_3 = 0$ and the cone $a_3 \Omega_0 = \Omega_1$. The planes $X_i = 0$ are tropes of W , touching it along conics; the contact conic of W with $X_3 = 0$ maps the repeated trinodal quartic that is the section of R by $x_3 = 0$. The planes γA_i touch W along the lines $A_i A'_i$; this line of contact maps the four tangents to Γ at its intersections with $x_i = 0$, i.e. the four lines along which $a_i \Omega_0 = \Omega_1$ touches R .

A plane $k\omega_0 = \omega_1$ through γ cuts W further in a conic w_k that maps those eight tangents of Γ common to $k\Omega_0 = \Omega_1$ and R . Their contacts with Γ compose an octad and exhaust the whole of their intersections with Γ , so that there is only a single point, the map of this octad, common to w_k and γ ; γ is a tangent to w_k . Just as there is a (1, 1) correspondence between octads on Γ and quadrics through Γ , the tangents to Γ at an octad lying on the corresponding quadric, so is there a (1, 1) correspondence between points on γ and planes through γ , the conics w_k in the planes touching Γ at the corresponding points. w_k is a repeated line if k is any of the four a_i . As the faces of u are tropes of W the lines in which they meet $k\omega_0 = \omega_1$ touch w_k ; w_k is the unique conic touching the sides of the pentagram consisting of γ and the intersections of $k\omega_0 = \omega_1$ with the faces of u , and W may be defined as generated by such conics as k varies ((5), p. 359).

10. The lines l , through a point P on a double line of a surface, having (at least) three-point intersection with the surface at P lie in two planes through the double line; if the surface is a quartic l has (at most) one further intersection. If P is on the double line γ of W there is a plane through γ meeting W in γ (repeated) and a conic touching γ at P ; a line in this plane through P has (at most) one further intersection with W . But any other plane through γ meets W in γ (repeated) and a conic touching γ at a point distinct from P , so that lines through P in this other plane meet W twice apart from P . (Should this plane be one of those four in which the conic is a repeated line the argument stands because there is then a contact with W apart from P .) Thus only a single plane through γ satisfies the condition on lines through P , and γ is a cuspidal line on W .

The conic that completes with γ the section of W by the plane γA_i is, as already remarked, the repeated line $A_i A'_i$; the lines in this plane through A'_i , other than $A_i A'_i$ and γ themselves, have no further intersection with W : A'_i is what Cayley ((2), pp. 339–341) and Zeuthen ((11); especially pp. 479–489) call a close point.

11. Since the plane γA_i is $a_i \omega_0 = \omega_1$ the parameters of the A_i on any twisted cubic h through them which has γ for a chord compose a tetrad projective with (a_0, a_1, a_2, a_3) . Take them, therefore, to be $-a_0, -a_1, -a_2, -a_3$ and h to have the parametric form

$$X_i = \alpha_i(\psi + a_i)^{-1},$$

where the α_i are constants. But then the third intersection of h with $\omega_0 = 0$ has $\psi = \infty$ and so, as the three intersections with $\omega_0 = 0$ are determined by

$$\sum \alpha_i (\psi + a_i)^{-1} = 0,$$

$\sum \alpha_i = 0$. The same condition is imposed by the fact of the third intersection of h with $\omega_1 = 0$ having $\psi = 0$. The parameters of the two intersections of h and γ are therefore the roots of the quadratic

$$\sum \alpha_i \{ (e_1 - a_i) \psi^2 + (e_2 - a_i e_1 + a_i^2) \psi + e_3 - a_i e_2 + a_i^2 e_1 - a_i^3 \} = 0. \quad (11.1)$$

Now four numbers α_i summing to zero are such that, for some p, q, r ,

$$\alpha_i f'(a_i) = p + qa_i + ra_i^2$$

so that, by (2.3) and (2.4),

$$\sum \alpha_i a_i = r, \quad \sum \alpha_i a_i^2 = q + e_1 r, \quad \sum \alpha_i a_i^3 = p + e_1 q + (e_1^2 - e_2) r$$

and (11.1) is, simply,

$$p - q\psi + r\psi^2 = 0.$$

Thus the condition for h to touch γ is $q^2 = 4rp$, or

$$p : q : r = \chi^2 : 2\chi : 1,$$

each value of χ providing one such twisted cubic. Its parametric form is

$$X_i = \frac{(\chi + a_i)^2}{(\psi + a_i) f'(a_i)} \quad (11.2)$$

and its contact with γ has $\psi = \chi$. The four equations (11.2) are a parametric representation of W in terms of ψ and χ just as, analogously, (2.7) is a parametric representation of the octads on R .

12. The equations (11.2), whose rationalized form is led by

$$X_0 = (\chi + a_0)^2 (\psi + a_1) (\psi + a_2) (\psi + a_3) / f'(a_0), \quad (12.1)$$

map W on the affine plane coordinatized by (ψ, χ) . Plane sections of W are mapped by quintic curves which have

- (a) a triple point M 'at infinity' at the concurrence of the parallels to $\psi = 0$,
- (b) a node N at the concurrence of the parallels to $\chi = 0$,
- (c) simple points L_i at $\psi = \chi = -a_i$, touching each other there with common tangents $\psi = -a_i$.

They therefore, as they should, have genus $6 - 3 - 1 = 2$ and grade

$$5^2 - 3^2 - 2^2 - 2 \cdot 4 = 4.$$

They could be changed to quartics by Cremona transformation but this, while lessening the degree of the map, would mar its symmetry.

The line $\psi = \chi$ maps γ .

Lines through N map twisted cubics, those four mapped by the NL_i being composite; that mapped by NL_0 consists of the conic in the trope $A_1 A_2 A_3$ and of the line $A_0 A'_0$.

If one writes $-a_0$ for χ in (12.1) it appears that the conic in the trope $X_0 = 0$ is parametrized as

$$X_1(a_1 - a_2)(a_1 - a_3) = (a_1 - a_0)(\psi + a_2)(\psi + a_3) \quad (12.2)$$

together with the two equations derived by cyclically permuting the suffixes 1, 2, 3. These equations may also be written as

$$X_1 f'(a_1) = (a_1 - a_0)^2 (\psi + a_2)(\psi + a_3) \quad (12.3)$$

with its two companion equations.

13. Expressions for the ω_k in terms of ψ and χ are quickly found on writing $\psi + a_i + \chi - \psi$ for $\chi + a_i$ in (12.1). For then, writing Π for the product

$$(\psi + a_0)(\psi + a_1)(\psi + a_2)(\psi + a_3),$$

$$\begin{aligned} X_i f'(a_i) &= (\psi + a_i) \Pi + 2(\chi - \psi) \Pi \\ &\quad + (\chi - \psi)^2 \{ \psi^3 + (e_1 - a_i) \psi^2 + (e_2 - a_i e_1 + a_i^2) \psi + e_3 - a_i e_2 + a_i^2 e_1 - a_i \} \end{aligned}$$

so that

$$\begin{aligned} \omega_k = \sum a_i^k X_i &= (\psi s_k + s_{k+1}) \Pi + 2(\chi - \psi) \Pi s_k \\ &\quad + (\chi - \psi)^2 \{ (\psi^3 + e_1 \psi^2 + e_2 \psi + e_3) s_k - (\psi^2 + e_1 \psi + e_2) s_{k+1} \\ &\quad + (\psi + e_1) s_{k+2} - s_{k+3} \} \end{aligned}$$

giving immediately

$$\omega_0 = -(\chi - \psi)^2, \quad \omega_1 = \psi(\chi - \psi)^2, \quad \omega_2 = \Pi - \psi^2(\chi - \psi)^2.$$

The multipliers of ψ^5 and ψ^4 in ω_k are zero, whatever k . The plane $k\omega_0 = \omega_1$ meets W in the line $\psi = \chi$, i.e. γ , reckoned twice, and the conic $\psi = -k$.

The close points A'_i are in the planes $X_i = 0$. Any plane through, say, A'_0 has an equation $p\omega_0 + q\omega_1 + rX_0 = 0$ and so cuts W in the section mapped by

$$(q\psi - p)(\chi - \psi)^2 + r(\chi + a_0)^2 (\psi + a_1)(\psi + a_2)(\psi + a_3)/f'(a_0) = 0.$$

This quintic curve has a node at $\psi = \chi = -a_0$, i.e. at L_0 ; its genus is therefore 1, not 2. This agrees with the fact that a plane through a close point gives a section not with an ordinary cusp but with a tacnode ((11), p. 479).

14. At the opening and at the close of § 9 two methods of generating W were described

(a) as an envelope of cones containing pentads of lines through points of γ ,

(b) as a locus of conics inscribed in pentagrams in planes through γ .

The duality of (a) and (b) is a reminder of what is only too obvious without any involvement of W at all: there is a quadric Φ invariantly related to the figure in σ , namely the quadric for which u is self-polar and which contains γ . Its equation is

$$\alpha_0 X_0^2 + \alpha_1 X_1^2 + \alpha_2 X_2^2 + \alpha_3 X_3^2 = 0,$$

where α_i are such that the two distinct points

$$X_i = 1/f'(a_i) \quad \text{and} \quad X_i = a_i/f'(a_i)$$

on γ are both on Φ as well as being conjugate. The three linear conditions so imposed

give, on solving them determinantly, unique values for $\alpha_0: \alpha_1: \alpha_2: \alpha_3$. Or perhaps it is enough to say that, by inspection,

$$\Sigma f'(a_i) X_i^2 = 0$$

satisfies all three necessary conditions.

The polar plane with respect to Φ of the point in $X_0 = 0$ identified by (12·3) is

$$(a_1 - a_0)^2 (\psi + a_2) (\psi + a_3) X_1 + (a_2 - a_0)^2 (\psi + a_3) (\psi + a_1) X_2 \\ + (a_3 - a_0)^2 (\psi + a_1) (\psi + a_2) X_3 = 0$$

and its envelope, as ψ varies, is

$$\{(a_1 - a_0)^2 (a_2 + a_3) X_1 + (a_2 - a_0)^2 (a_3 + a_1) X_2 + (a_3 - a_0)^2 (a_1 + a_2) X_3\}^2 \\ = 4\{(a_1 - a_0)^2 X_1 + (a_2 - a_0)^2 X_2 + (a_3 - a_0)^2 X_3\} \\ \times \{(a_1 - a_0)^2 a_2 a_3 X_1 + (a_2 - a_0)^2 a_3 a_1 X_2 + (a_3 - a_0)^2 a_1 a_2 X_3\},$$

which (8·1) shows to be the nodal cone of W at X_0 . This is just one of the many aspects of W being its own polar reciprocal with respect to Φ . The poles of the points of that conic, in a plane π through γ , which is inscribed in the pentagram formed by γ and the intersections of π with the faces of u are the tangent planes of a cone whose vertex P , the pole of π , is on γ and which contains γ and the joins of P to the vertices of u . Just as the points of the conic are on W , so are the tangent planes of the cone tangent planes of W . Also: just as W is generated by those twisted cubics which circumscribe u and touch γ , so is W enveloped by those cubic developables that are inscribed in u and have γ for an axis.

15. One can use Φ , or indeed any non-singular quadric Ψ for which u is self-polar, as a springboard to return from σ to S . By the reverse of the transformation of § 8 such a quadric $\Sigma \alpha_i X_i^2 = 0$ becomes a quartic surface $K: \Sigma \alpha_i x_i^4 = 0$, each point of Ψ for which no X_i is zero furnishing an octad on K with no x_i zero.

A generator of Ψ , having equations $\Sigma p_i X_i = \Sigma q_i X_i = 0$, becomes a curve $\Sigma p_i x_i^2 = \Sigma q_i x_i^2 = 0$ on K , in general an elliptic quartic; there are two families of such curves on K , curves in opposite families having an octad in common whereas two curves in the same family do not meet at all. But Ψ meets every edge of u in two points and both generators through such a point meet, since u is self-polar, the opposite edge. If a generator meets $X_0 = X_1 = 0$ and $X_2 = X_3 = 0$ it has equations

$$p_0 X_0 + p_1 X_1 = q_2 X_2 + q_3 X_3 = 0$$

and the corresponding 'elliptic quartic' on K is $p_0 x_0^2 + p_1 x_1^2 = q_2 x_2^2 + q_3 x_3^2 = 0$, a skew quadrilateral whose diagonals are the opposite edges $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$ of U . Each family of elliptic quartics thus includes six such quadrilaterals, each pair of opposite edges of U being diagonals of two of them; K contains 48 lines ((9), p. 484). The harmonic relations of the three pairs of skew quadrilaterals in either family to one another ((4), p. 467) are also immediate consequences of those of the pairs of generators of Ψ meeting opposite edges of u .

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