AN OPERAND FOR A GROUP OF ORDER 1512

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1. The tableau on p. 105 built with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 is one of a possible 1920 and was compiled in the year 1963 at about the time when [5] was published. Whether it is new or not, its construction, described below, with the help of the geometry of Study's quadric $\mathfrak{S}$, in projective space of 7 dimensions over the finite field GF(2), may be. One can observe the visual effects on the tableau of 1512 even permutations of the group H for which it is, as a whole, invariant. H could, in this context, be identified as the intersection of three alternating groups of degree 9 derivable from each other by using the triality on $\mathfrak{S}$. But H and its simple subgroup h of order 504, isomorphic to the linear fractional group LF(2, 2^3), occur also in a less elaborate geometry. LF(2, 2^3) is the group of projectivities on a 9-point line λ and is extended to a group isomorphic to H by the automorphism of period 3 of GF(2^3) which, replacing each mark by its square, leaves three points (those with parameters 0, 1, ∞) of λ unmoved while permuting the others in two cycles of three—a permutation unattainable by projectivities because any projectivity which leaves three points all unmoved can only be the identity.

The most recent appearance of H in the literature may be Dye's recognition [3] of it as a maximal subgroup, of index 960, of the group of the bitangents. Among its other appearances one might mention that as leader [8; p. 452] of a series of groups —all, save itself, simple—of order $q^3(q^3+1)(q-1)$ where $q = 3^{2n+1}$. But, as permutation groups of degree 9, both H and h were identified by Cole [1] and one may grant his claim [1; p. 250] to be the first to have asserted, and proved, that h is simple. His claim, however, to be the original discoverer of this group must be disallowed because it had been found some 30 years earlier by Kirkman. Certainly Kirkman lumps together all permutations of the same cycle type, so distinguishing conjugate classes only in the symmetric group $\mathscr{S}_9$; he does not refine the distinction in any of its subgroups. Yet H and h, along with other groups, are plainly recorded [6; pp. 147 and 146]. Kirkman's essay appears to have escaped observation even in the telescopic sights of Coxeter and Moser.

2. The notation used is that of [5]. $\mathfrak{S}$ is Study's non-singular quadric consisting of 135 points m in projective space [7] over GF(2); at each m there is a prime M tangent to $\mathfrak{S}$. The m are vertices of 960 enneads $\mathscr{E}$ [5; pp. 6, 7], no two vertices of the same $\mathscr{E}$ are ever conjugate for $\mathfrak{S}$. $\mathfrak{S}$ is ruled by two systems of solids ω, ω' each with 135 members, and the existence of the enneads $\mathscr{E}$ implies, by triality, that of enneagrams η, η' each consisting of nine solids of the same system all skew to one another. It is with such η that we shall be concerned, and they will be obtained by elementary arguments which do not invoke triality.

The equation of $\mathfrak{S}$ is $\sum_{i<j} x_i x_j = 0$, with its left-hand side the sum of the 28 products of pairs of eight homogeneous coordinates x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7. The simplex of reference consists of all but one of the vertices of an ennead $\mathscr{E}_0$, the remaining vertex of $\mathscr{E}_0$ being the unit point U.

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The intersection of any two of the nine primes M tangent to $\mathfrak{S}$ at the vertices of an ennead is a [5] C, the polar of the chord c that joins the contacts. The section of $\mathfrak{S}$ by C is ruled by planes, but no solid lies wholly on it. Thus all 270 solids on $\mathfrak{S}$ are accounted for by 30 in each of nine M: that M does contain 30 follows because it meets $\mathfrak{S}$ in a cone projecting a Klein quadric from a point m outside the [5] containing it, and on a Klein quadric there are, over GF(2), 30 planes.

3. Consider now those ω in M_8, the tangent M

$$x_0+x_1+x_2+x_3+x_4+x_5+x_6+x_7 = 0 \tag{3.1}$$

to $\mathfrak{S}$ at U. Any such ω is determined as the intersection of four linearly independent primes; among these one may or may not include (3.1). It will best serve our purpose to denote a prime by the symbol composed of the suffixes of such coordinates as are present on the left-hand side of its equation; the order of these suffixes does not matter. For instance, the unit prime is 01234567 or any permutation of these digits, while 0357 denotes $x_0+x_3+x_5+x_7 = 0$, and so forth. Symbols may, just as left-hand sides of equations may, be added; any digit that occurs an even number of times in an added set of symbols disappears, any that occurs an odd number of times survives.

One set of four linearly independent primes whose intersection is on $\mathfrak{S}$ consists of

0357, 2571, 4713, 6135.

This is seen, for instance, by remarking that each prime contains the four linearly independent points

(.1 1.1.1.), (1..1 1.1.), (1.1..1 1.), (1.1.1..1),

all of which are on $\mathfrak{S}$ and every pair of them conjugate. Alternatively, one may simply refer to 10.3 in [5]. The sums of pairs of the four symbols are

0123, 4567, 0145, 2367, 0167, 2345;

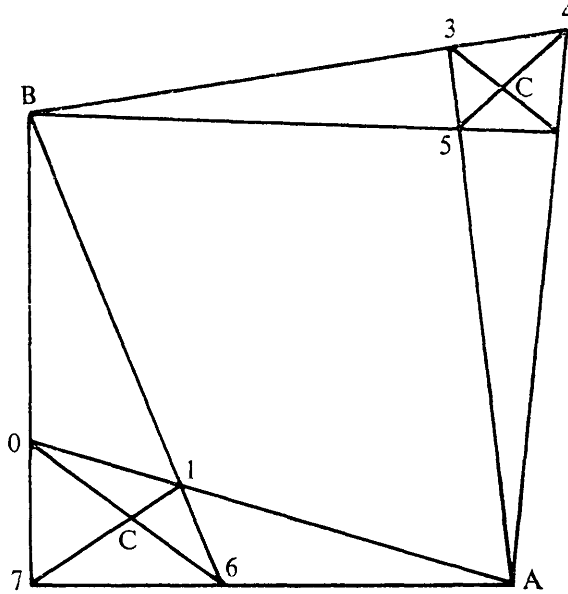
their sums in threes

1246, 0346, 0256, 0247,

while the sum 01234567 of all four completes the set of fifteen primes which contain the solid. The same solid ω is determined by any four linearly independent symbols among those fifteen. If the octad is omitted the other symbols consist of seven complementary pairs of tetrads, and tetrads that are not complementary always share two digits.

4. It is helpful momentarily to regard these tetrads as labelling sets of four coplanar points in [3]. For, over $GF(2)$, there are eight points of [3] not in a given plane π; through each of the seven lines in π pass two planes other than π itself, so that the eight points fall into complementary tetrads in seven ways. Non-complementary tetrads share two digits because the line common to the corresponding planes is not in π and so consists of one point in π and two points outside π.

Call the points outside π 0, 1, 2, 3, 4, 5, 6, 7 and let the planes 0167, 2345 meet π in the line ABC; A, B, C are the collinear diagonal points of both quadrangles and these quadrangles, after the manner of desmic tetrahedra, are in perspective from each point of π not on ABC. We may suppose that



01,	24,	35,	67	pass through A,
07,	16,	25,	34	pass through B,
06,	17,	23,	45	pass through C.

The seven separations into complementary coplanar tetrads are

$$\begin{matrix} 0124 & 0135 & 0167 & 0236 & 0257 & 0347 & 0456 \\ 3567 & 2467 & 2345 & 1457 & 1346 & 1256 & 1237 \end{matrix} \qquad (4.1)$$

No permutation of any tetrad alters the meaning of this scheme. But the perspectivities just mentioned serve to build a less arbitrary scheme, for one can prescribe that every pair of complementary tetrads has the digits so ordered that the joins of points in the same column are four concurrent lines; once the digits of either tetrad are placed this permits four orderings of the other. And the arbitrariness can be further restricted by demanding that the seven centres of perspective, one for each pair of tetrads, are *all* the points of π; none serves for more than one pair of tetrads. The columns of the scheme then account for all twenty-eight pairs of the eight digits.

Any two of 03, 15, 26, 47 meet since they are in a plane containing one of A, B, C; but such a plane does not contain either of the other two joins, so that all four are concurrent. Other sets of four concurrent joins are

05, 13, 27, 46; 04, 12, 56, 37; 02, 14, 36, 57.

So one rebuilds (4.1) thus:

$$\begin{matrix} 0124 & 0135 & 0167 & 0236 & 0257 & 0347 & 0456 \\ 6735 & 2467 & 5342 & 7541 & 4163 & 1526 & 3712 \end{matrix} \qquad (\Omega_8)$$

5. The primes whose symbols are the fourteen tetrads in Ω_8 intersect in a solid, which we now label ω_8, on $\mathfrak{S}$; ω_8 lies in M_8 and contains U. Both $\mathfrak{S}$ and M_8 have equations symmetric in the coordinates, so that any scheme derived from Ω_8 by permuting the digits represents a solid ω or ω', on $\mathfrak{S}$, in M_8, containing U. That there are thirty such solids is corroborated by noting how many permutations leave Ω_8 unchanged, i.e. permute its tetrads among themselves; the group Γ of projectivities in [3] for which π is invariant has [**4**; p. 335] order 1344, and $8!/1344 = 30$. Alternatively

one can show that there are 30 schemes such as Ω_8 determining solids on $\mathfrak{S}$ in M_8 without alluding to Γ. For Ω_8 evolves from any of its seven blocks by linking the three bisections of either tetrad one with each bisection of its complement. Take, say, 0167 and 5342. Then Ω_8 evolves by linking

01.67 with 24.35, 06.17 with 23.45, 07.16 with 25.34

so that another block consists of 0124 and 3567, yet another of 0135 and 2467, and so on. Since there are six ways of linking the bisections of a tetrad with those of its complement, and since a scheme can so evolve from any of its seven blocks, the number of schemes is $\frac{1}{2}.{}^8C_4.6/7 = 30$.

When two of the eight digits are transposed, six tetrads in Ω_8 are unchanged: the solids designated by Ω_8 and the altered scheme lie together in seven primes—the unit prime and those six symbolised by the unchanged tetrads—linearly dependent on three among them. Thus the two solids lie together in a [4] and are in opposite systems on $\mathfrak{S}$. A scheme got from Ω_8 by permuting the digits designates a solid belonging to the same system as ω_8 or to the opposite system according as the permutation is even or odd.

6. The discussion has so far been confined to the solids on $\mathfrak{S}$ which lie in M_8. But consider now solids on $\mathfrak{S}$ which lie in the tangent prime at some vertex of $\mathscr{E}_0$ other than U, say in the tangent prime M_0 at X_0; they all contain X_0. The equation of M_0 being

$$x_1+x_2+x_3+x_4+x_5+x_6+x_7 = 0,$$

its symbol, according to the convention adopted in §3, is 1234567 and, since this is the sum of 347, 567 and 127, M_0 contains the solid ω_0' common to the primes

$$136, \quad 347, \quad 567, \quad 127. \tag{6.1}$$

This solid ω_0' is seen to be on $\mathfrak{S}$ by arguments like those of §3. The symbols of the ten primes, other than M_0 and the four in (6.1), that contain ω_0' are the sums

3456	1357	1467
2367	1234	1256

of pairs of triads (6.1), together with the sums

235 246 145

of three triads that include 136, and the sum 2457 of all four. So, of the fourteen primes other than M_0 through ω_0', seven have tetrads and seven have triads for their symbols. But let the triads all be augmented by a dummy digit 8. Then ω_0' is determined by the scheme

1234	1256	1278	1357	1368	1548	1467	(6.2)
7856	3478	6543	8642	4275	2637	5823	

and the thirty solids on $\mathfrak{S}$ through X_0 appear on permuting the eight digits. In building this scheme it is again helpful to regard the digits as labels of points outside a plane in [3]; but this is a mere accessory and not the same as the [3] used in §4.

So, by using nine digits, one obtains *the same type of scheme* for every solid on $\mathfrak{S}$. Should ω be in M_0, and so contain X_0, the scheme lacks the digit 0; when ω lay in M_8, and so contained U, the digit 8 was absent. For ω in M_i and containing X_i

the digit i will be missing. In any scheme arranged to accord with the prescriptions of §4, the pairs of the eight digits all occur, one pair in each column, once and only once.

7. Take Ω_8 and apply to it repeatedly the cyclic permutation $\mathfrak{P} \equiv (0\,1\,2\,3\,4\,5\,6\,7\,8)$. The resulting schemes are in Table 1; they determine nine ω, one through each of the vertices of $\mathscr{E}_0$. No tetrad occurs more than once in the table; all ${}^9C_4 = 126$ appear, each once and only once. All primes, save one, through any ω are symbolised by the tetrads of its scheme, and there is no prime containing two of the nine ω. So these nine ω are skew and belong to the same system on $\mathfrak{S}$. They compose an enneagram η and, with 15 m in each, account for all 135 m on $\mathfrak{S}$. The scheme Ω_0 that leads the table is (6.2) with 4 and 5 transposed.

TABLE 1

1235	1246	1278	1347	1368	1458	1567	(Ω_0)
7846	3578	6453	8652	5274	2637	4823	
2346	2357	2380	2458	2470	2560	2678	(Ω_1)
8057	4680	7564	0763	6385	3748	5034	
3457	3468	3401	3560	3581	3671	3780	(Ω_2)
0168	5701	8675	1874	7406	4850	6145	
4568	4570	4512	4671	4602	4782	4801	(Ω_3)
1270	6812	0786	2085	8517	5061	7256	
5670	5681	5623	5782	5713	5803	5012	(Ω_4)
2381	7023	1807	3106	0628	6172	8367	
6781	6702	6734	6803	6824	6014	6123	(Ω_5)
3402	8134	2018	4217	1730	7283	0478	
7802	7813	7845	7014	7035	7125	7234	(Ω_6)
4513	0245	3120	5328	2841	8304	1580	
8013	8024	8056	8125	8146	8236	8345	(Ω_7)
5624	1356	4231	6430	3052	0415	2601	
0124	0135	0167	0236	0257	0347	0456	(Ω_8)
6735	2467	5342	7541	4163	1526	3712	

8. Any permutation of the nine digits that leaves η as a whole unchanged must, since the ω stay in the same system, be even. How many of them leave unchanged not only η as a whole but also three of its individual members—say Ω_6, Ω_7, Ω_8? Since the triad 678 is accompanied in Ω_0 by 4, in Ω_4 by 3, in Ω_3 by 0: in Ω_1 by 2, in Ω_2 by 5, in Ω_5 by 1, the permutation (043)(125) is suggested, and the suggestion is reinforced on observing that the duads which accompany

78 in Ω_6 are 02, 45, 31;
86 in Ω_7 are 05, 41, 32;
67 in Ω_8 are 01, 42, 35.

Moreover, the triad 043 is accompanied in

	Ω_1	Ω_5	Ω_2	Ω_6	Ω_7	Ω_8
by	5	2	1	8	6	7

while 125 is accompanied in

	Ω_0	Ω_3	Ω_4	Ω_6	Ω_7	Ω_8
by	3	4	0	7	8	6

Indeed, (043)(125) does leave each of Ω_6, Ω_7, Ω_8 unchanged and permutes the remaining members of η, with their suffixes, in two cycles of three. There is a corresponding statement when any three members of η are selected to be invariant; the three-fold stabilisers are of order three. Since any two of them share only the identity permutation, 168 operations of period 3 are accounted for in H.

When any three members, say Ω_i, Ω_j, Ω_k, of η are chosen the residual six congregate automatically into complementary trios, each having its three schemes permuted cyclically by the threefold stabiliser H_{ijk}. The 84 trios thus fall into 28 companion sets of three. Take, to be specific, $i = 0$, $j = 3$, $k = 6$; Table 2 tells which digit accompanies the triad to the left of each row in the scheme heading each column.

TABLE 2

	Ω_0	Ω_3	Ω_6	Ω_1	Ω_4	Ω_7	Ω_2	Ω_5	Ω_8
036	.	.	.	7	1	4	5	8	2
147	3	6	0	.	.	.	8	2	5
258	6	0	3	4	7	1	.	.	.

and implies that H_{036}, H_{147}, H_{258} are subgroups of α, the direct product of any two of them; the two operations of α that do not belong to any threefold stabiliser permute the members of each of the trios cyclically. This accounts for 56 more operations of period 3, two for each of the 28 sets of complementary trios. Nor are there any further [**6**; p. 147] operations of period 3 in H.

9. H_{ijk} is a normal subgroup of G_{ijk}, the subgroup of H that permutes Ω_i, Ω_j, Ω_k among themselves; α subjects them to even permutations. It will be seen in a moment that G_{ijk} is the direct product $\mathscr{C}_3 \times \mathscr{D}_3$ of cyclic and dihedral groups.

Again, to be specific, let $i = 0$, $j = 3$, $k = 6$. In order to identify a permutation, in the stabiliser H_0 of Ω_0, which transposes Ω_3 and Ω_6, note that the column in Ω_0 composed of the pair 36 is in the central block; the involution $R = (18)(27)(36)(45)$ is thereby indicated. It does belong to G_{036}; it transposes Ω_3 with Ω_6 and the other two trios of the companion set with one another. The three involutions

$$(18)(27)(36)(45), \qquad (06)(15)(24)(78), \qquad (03)(12)(48)(57)$$

suggested by the central blocks in Ω_0, Ω_3, Ω_6 all belong to G_{036}, which is generated by R and $S = (036)(285)$. For RS and SR, both of period 6, have the same square and the generating relations for $\mathscr{C}_3 \times \mathscr{D}_3$ are satisfied [**2**; p. 134]. This group includes six operations of period 6 which, with the three involutions, compose the coset of α. As H has 84 subgroups G_{ijk}, it includes 504 operations of period 6 [**6**; p. 147].

10. The elementary Abelian group of order 9 which, while permuting members of each trio cyclically, leaves invariant each companion trio of a set of three is normal in a group of order 54 which, while leaving the whole set invariant, includes every operation that permutes the three companion trios. Those 27 operations that impose odd permutations on the trios

$$\Omega_0,\ \Omega_3,\ \Omega_6; \qquad \Omega_1,\ \Omega_4,\ \Omega_7; \qquad \Omega_2,\ \Omega_5,\ \Omega_8;$$

are the cosets of α in G_{036}, G_{147}, G_{258}; there remain 18 operations that permute the three trios cyclically. Each of these is expressible, in different ways, as a product of operations from any two of G_{036}, G_{147}, G_{258} just as a cyclic permutation of three objects is expressible, in different ways, as a product of two transpositions. All these 18 operations are cyclic, of period 9; $\mathfrak{P}$, used in constructing η, must be among them; indeed

$$\mathfrak{P} = (012345678) = (18)(27)(36)(45).(08)(17)(26)(35),$$

the product of involutions in G_{036} and G_{147}. Such a group of order 54 is duly registered by Kirkman [**6**; p. 145].

The companion trios of the set chosen to illustrate the above discussion had the virtue of visual convenience, proferring the central columnar blocks of Table 1 for scrutiny. But the other 27 companion sets are of equal standing and have the same properties.

There are 63 involutions in H, one for each of the 63 pairs of complementary tetrads of digits. Each of the 84 G_{ijk} has been seen to include three involutions; on the other hand, each involution belongs to four G_{ijk}. For instance,

$$(17)(28)(34)(56),$$

corresponding to a complementary pair of tetrads in Ω_0, belongs to G_{017}, G_{028}, G_{034}, G_{056}.

11. One has now accounted for all the operations of H save the 216, shown in Kirkman's partitioning, of period 7. These must compose, with identity, 36 subgroups of order 7; indeed each of the 36 two-fold stabilisers H_{ij} will be found to include one such subgroup.

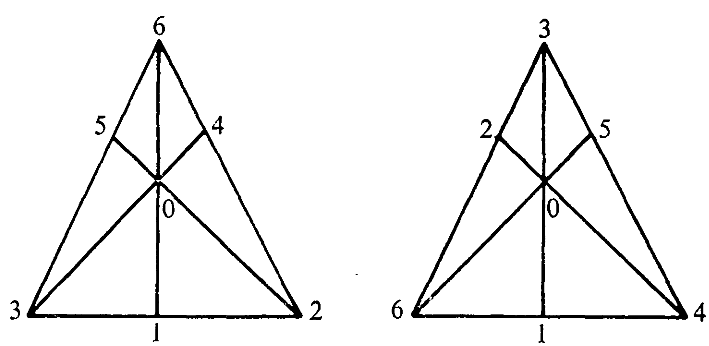
Take $i = 7$, $j = 8$. If Ω_7 and Ω_8 are both stable, the triads

356 246 016 145 025 034 123

that accompany 7 in tetrads of Ω_8 must be permuted among themselves; so must the triads

013 024 056 125 146 236 345

that accompany 8 in tetrads of Ω_7. So the two-fold stabiliser H_{78} belongs to the intersection of two Klein groups, the groups of projectivities of the two seven-point planes depicted thus:



it being understood that 145 is rectilinear in the left-hand, 125 in the right-hand diagram, and that all the visually apparent Euclidean collinearities are valid in the finite planes.

Choose any point in either plane. The other six lie in threes on four lines and one, and only one, of these lines is such that those three of the six points that are not on it are collinear in the other plane. Take, for instance, 0 in the left-hand plane. Those four lines that consist of three points other than 0 are

123, 246, 356, 145

leaving residual triads

456, 135, 124, 236

of which only 236 is a collinear triad in the right-hand plane; the permutations

$$(145)(632) \quad \text{and} \quad (154)(623)$$

belong to both Klein groups, and indeed to the threefold stabiliser H_{078}. So one obtains, common to H_7 and H_8,

$$(145)(263), \quad (024)(356), \quad (016)(345), \quad (025)(164),$$
$$(123)(056), \quad (013)(246), \quad (034)(152)$$

and their inverses: indeed the operations of the seven groups H_{i78}. The product of any two of these permutations belongs to H_{78} and has period 7;

$$\text{e.g.} \quad (024)(356).(145)(263) = (0234651), = S$$

say; then if $T = (123)(056)$, $ST = TS^2$ and the defining relations [**2**; p. 134] for a group of order 21 are satisfied. As H includes 36 two-fold stabilisers they account for 216 operations of period 7.

A stabiliser H_i is of order 168; it has, like Klein's group of this order, 8 subgroups of order 21, namely the twofold stabilisers H_{ij}; but H_i is not a Klein group. H_i includes only 7 involutions, whereas a Klein group includes 21. Nor does a Klein group possess any operations of period 6, whereas the operations RS and SR mentioned in §9 belong one to H_3 and the other to H_6. Moreover, Klein's group is a simple group, whereas H_i has a normal elementary Abelian subgroup composed of its 7 involutions and the identity. If publication dates are to decide priority, Kirkman has been, if only just, anticipated in encountering this group of order 168, since it appears, along with " Klein's group ", on p. 292 of Mathieu's memoir [7].

12. It was explained in §15 of [**5**] how the 135 m consist, in relation to any of the 960 $\mathscr{E}$, of the nine vertices of $\mathscr{E}$ and 126 other m; these latter each supplement one of the 126 tetrads of vertices of $\mathscr{E}$ in that the solid λ spanned by the tetrad meets $\mathfrak{S}$ further in the supplementary m, and in no other point. No two of the five m in λ are conjugate. The polar solid λ' of λ also meets $\mathfrak{S}$ in five m, no two conjugate; when λ is spanned by four vertices of $\mathscr{E}$, the m in λ' are those supplementing the tetrads among the residual pentad of vertices of $\mathscr{E}$. It was also seen that the third point on the join of those m which supplement disjoint tetrads in $\mathscr{E}$ is that point of $\mathscr{E}$ not in either tetrad.

The triality on $\mathfrak{S}$ imposes a (1, 1) correspondence between points m and solids ω; if two m are conjugate their correspondents have a common line, or are incident; to an ennead $\mathfrak{S}$, nine m no two of them conjugate, corresponds an enneagram η of mutually skew ω. The 135 ω consist, in relation to any of the 960 η, of the nine ω in η and 126 others; these latter each supplement one of the 126 sets of four ω in η in the sense of being skew to all these four and incident with all the other five—transversal to these five one might say. Those ω which so supplement disjoint sets of four in η

are incident, and the third ω through their common line is the ninth member of η. These facts, consequences by triality of properties of m, can be established directly by using the tableau of Table 1.

A solid transversal to each of five ω in η has, by three points on each of five skew lines, all its 15 points accounted for; it is thus *ipso facto* skew to the other four ω in η. That such a solid exists is seen by constructing the scheme for it.

Suppose that one requires an ω transversal to ω_4, ω_5, ω_6, ω_7, ω_8. It is in Ω_6 that the tetrad 0123 occurs and one links, in one of the six possible ways, the bisections of 0123 with those of 7845. The linking in Ω_6 was

01.23 with 47.58,   03.21 with 48.57,   02.31 with 45.78;

since an ω distinct from ω_6 is required, one shifts the three bisections of 7845 cyclically. But 0148 is in Ω_3, 0157 in Ω_2, 2348 in Ω_0, 2357 in Ω_1, and the scheme that would evolve from any linking of 01.23 with 48.57 would give an ω incident with ω_0, ω_1, ω_2, ω_3. So one links

01.23 with 45.78,   02.31 with 48.57,   03.21 with 47.58

and arrives at

0123	0145	0178	0248	0257	0347	0358	(Ω_{0123})
7845	8732	3245	5713	4813	1285	2147	

The primes here symbolised contain, respectively (see Table 1)

ω_6	ω_7	ω_4	ω_7	ω_8	ω_8	ω_4
ω_6	ω_5	ω_8	ω_4	ω_5	ω_7	ω_5

so that, with ω_6 lying in 01234578, each of ω_4, ω_5, ω_6, ω_7, ω_8 lies with this new ω in three primes of which two are independent. So the ω thus obtained is indeed transversal to ω_4, ω_5, ω_6, ω_7, ω_8 and (therefore) skew to ω_0, ω_1, ω_2, ω_3.

13. The fifth power of the cyclic permutation $\mathfrak{P}$ applied to Ω_{0123} produces a scheme Ω_{5678} that identifies an ω transversal to ω_0, ω_1, ω_2, ω_3, ω_4 and skew to ω_5, ω_6, ω_7, ω_8; this scheme is

5678	5601	5634	5704	5713	5803	5814	(Ω_{5678})
3401	4387	8701	1368	0468	6741	7603	

This ω, supplementing ω_5, ω_6, ω_7, ω_8, and the previous ω, supplementing ω_0, ω_1, ω_2, ω_3 ought to have in common a line in ω_4. So, indeed, they do. The three tetrads

0178   0358   1357

are common to Ω_{0123}, Ω_{5678} and Ω_4; the three solids are all in the [5] common to these three linearly dependent primes and, being all their own polars with respect to $\mathfrak{S}$, contain the polar line of this [5].

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