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A footnote on the mystic hexagram

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1. Cremona's projection of 15 lines on a nodal cubic surface $\mathscr{C}$ onto a plane π is a striking example of the resolution of a highly complicated figure into its constituent simplicities, for it explains, almost at a glance, so many properties of Pascal's mystic hexagram. These may appropriately be called ((**12**); (**10**), p. 162) the Veronese properties and have been described on several occasions ((**12**); (**6**); (**10**); (**1**), pp. 219–236; (**2**), pp. 349–355).

The figure in space is free of irrelevancies because, though two lines in a projective plane always meet, two lines in space need not; it is the actual intersections in space of these 15 lines with one another, and with other lines constructed from them, that furnish the points in π relevant to the hexagram. Moreover, ((**10**), p. 166), since these intersections exist in space the centre of projection need not be the node of $\mathscr{C}$; there are figures in π of greater generality than Pascal's yet endowed still with the Veronese properties.

Cremona does not allude to Sylvester's writings on the combinatorial properties of six objects, but his work is really beyond any criticism save that of Richmond. It is proposed, while submitting the following pages by way of a slightly anticipatory centenary tribute, to specialize the hexad, on a conic Γ, from which the Veronese properties take their rise, by using the intersections of Γ with the sides of a self-polar triangle. It at once appears, without leaving the plane, that there are coincidences among the different Pascal lines. Cremona, who seems to regard the plane figure as merely incidental, would surely have explained this by the occurrence of Steiner trihedra having line-axes instead of merely point-vertices; we therefore so explain the phenomenon here. Equations of the 15 tritangent planes of an appropriate nodal cubic surface $\mathscr{C}$ are given explicitly in Table 2 on p. 34; the space figure is of course deliberately chosen so that the specialized plane figure is a projection of it. Sylvester's relevant findings are responsible for Table 1 on p. 34.

The 15 lines (other than those through its node) on $\mathscr{C}$ are also projected, from a point on $\mathscr{C}$ other than its node N, into 15 of the 16 bitangents of a nodal plane quartic. Explicit equations for these bitangents of one such quartic are listed in Table 3 on p. 38; the Veronese properties, or rather what they become when certain among the Pascal lines coincide, can thence be verified.

Finally a description is given of the analogous figure over the Galois field $GF(5)$; it involves Clebsch's diagonal surface. The finite plane figure was encountered 20 years ago ((**8**), p. 117; (**9**), pp. 373–4).

2. Cremona's procedure is as follows. Through the node N of $\mathscr{C}$ pass six lines,

$$a,\ b,\ c,\ a',\ b',\ c', \tag{2·1}$$

lying wholly on $\mathscr{C}$ as well as on the quadric cone of nodal tangents; they meet a plane π not containing N at six points,

$$A,\ B,\ C,\ A',\ B',\ C',$$

of a conic Γ, the section by π of the nodal cone. Our notation distinguishes the six as three pairs of opposites because the geometry is to involve such pairing.

Each of the 15 planes spanned by two of the lines (2·1) meets $\mathscr{C}$ in a third line; these 15 further lines on $\mathscr{C}$ are therefore conveniently labelled by the 15 duads $ab, ac, \ldots, b'c'$; their projections from N are the 15 chords $AB, AC, \ldots, B'C'$ of Γ.

The plane spanned by a and a' meets that spanned by b and b' in a line through N and so meeting $\mathscr{C}$ in only one more point; this is on both aa' and bb': *lines labelled by disjoint duads intersect.* Thus each of the 15 synthemes (to use Sylvester's term for three duads together accounting for all six of a hexad) corresponds to a set of three coplanar lines; three such lines compose the whole section of $\mathscr{C}$ by their plane, which is a *tritangent plane.*

Now partition the lines (2·1) into complementary triads in any of the ten possible ways: say $abc|a'b'c'$. Then each line in either of the triads

$$bc,\ ca,\ ab \quad \text{and} \quad b'c',\ c'a',\ a'b'$$

meets all three in the other; these six lines are on a quadric and compose its complete intersection with $\mathscr{C}$. The remaining nine lines can be arranged in the scheme

$$\left.\begin{matrix} aa' & bc' & b'c \\ bb' & ca' & c'a \\ cc' & ab' & a'b \end{matrix}\right\}. \qquad (2\cdot2)$$

The rows provide three tritangent planes forming a trihedron T with vertex σ; the columns provide a companion trihedron T' with vertex σ'; each of the nine lines in the scheme is common to a face of T and a face of T'. There are ten such *pairs of Steiner trihedra* for $\mathscr{C}$, one for each partition of the lines (2·1) into complementary triads.

The projection from N onto π of the line common to the planes provided by the last two columns of (2·2) contains the intersections of

$$BC' \text{ and } B'C, \quad CA' \text{ and } C'A, \quad AB' \text{ and } A'B;$$

these intersections are indeed projections of *actual intersections in space.* This is Cremona's proof of Pascal's theorem for the ordered hexagon $AB'CA'BC'$. Each pair of columns of (2·2) thus gives a Pascal line, as does each pair of rows. Furthermore: the three lines that are projections of the three lines common to pairs of planes provided by the rows of (2·2) concur at the projection S of σ, while the three that are projections of the three lines common to pairs of planes provided by the columns concur at the projection S' of σ'. This is Cremona's proof of Steiner's result: the 60 Pascal lines concur in threes at 10 pairs of points.

3. Were, now, a trihedron of a Steiner pair to have not merely a point-vertex σ but a line-axis p three Pascal lines would coincide on the projection of p. If T has an axis p the section of $\mathscr{C}$ by each plane of T' is not three sides of a triangle but three concurrent lines. A cubic surface does not, in general, have any three of its lines concurrent save at

a node; if it does the concurrence is called an Eckardt point (7), E-point for short, though the possible occurrence of such points was first noted by Cayley (3). So p cuts $\mathscr{C}$ in three E-points. And since concurrent lines on $\mathscr{C}$ are projected into concurrent lines in π the six points on Γ can be partitioned in three ways as a pair of perspective triangles, the three centres of perspective being collinear on the projection of p.

4. This specialization of a hexad on Γ occurs in quadruplicate for its six intersections with the sides of a self-polar triangle. Take such a triangle XYZ and let the pairs of points of Γ on YZ, ZX, XY be, respectively, A, A'; B, B'; C, C'. Since the quadrangle $BCB'C'$ is inscribed in Γ its diagonal-point triangle is self-polar for Γ; hence the intersections $BC.B'C'$ and $B'C.BC'$ are on the polar YZ of the third diagonal point X. Thus there are concurrencies

$$\alpha \text{ of } BC, B'C', AA'; \quad \beta \text{ of } CA, C'A', BB'; \quad \gamma \text{ of } AB, A'B', CC';$$
$$\alpha' \text{ of } BC', B'C, AA'; \quad \beta' \text{ of } CA', C'A, BB'; \quad \gamma' \text{ of } AB', A'B, CC';$$

there are six synthemes in the hexad of points whose three duads are in involution on Γ.

The Pascal line of the ordered hexad $AB'CA'BC'$ contains α', β', γ' which are therefore collinear; this same line occurs for $AA'CC'BB'$ and $AC'CB'BA'$, so that $\alpha'\beta'\gamma'$ accounts for three Pascal lines as, likewise, do $\alpha'\beta\gamma$, $\alpha\beta'\gamma$, $\alpha\beta\gamma'$; this last is the Pascal line for $AA'BB'C'C$, $AB'BCC'A'$, $ACBA'C'B'$.

It is natural to take XYZ as triangle of reference and $\alpha'\beta'\gamma'$ as the unit line

$$x+y+z = 0.$$

Then α, the harmonic conjugate of α' with respect to Y and Z, is $(0, 1, 1)$. But α, α' are also conjugate for Γ, as are Y, Z so that the pair A, A', harmonic both to Y, Z and α, α', is $(0, 1, \pm i)$. Similar statements hold on $y = 0$ and on $z = 0$; the equation of Γ is

$$x^2+y^2+z^2 = 0$$

passing through

$$A(0,1,i), \quad B(i,0,1), \quad C(1,i,0), \quad A'(0,1,-i), \quad B'(-i,0,1), \quad C'(1,-i,0).$$

This plane figure is invariant under the group $\mathscr{G}$ of 24 projectivities that are imposed by the three-rowed monomial matrices, of determinant $+1$, whose non-zero entries are ± 1. Each of the six permutation matrices yields four of these, and the diagonal matrices impose the projectivities of the self-conjugate four-group V. Any line not containing any of X, Y, Z belongs to a quadrilateral having XYZ for diagonal triangle, its four sides undergoing all $4!$ permutations under $\mathscr{G}$. One such quadrilateral has α, α'; β, β'; γ, γ' as its pairs of opposite vertices; its sides $\alpha'\beta\gamma$, $\alpha\beta'\gamma$, $\alpha\beta\gamma'$ are

$$x-y-z = 0, \quad y-z-x = 0, \quad z-x-y = 0.$$

When the coordinates of a point or the equation of a line are given one can apply the operations of $\mathscr{G}$ in the expectation of finding further points or lines; one can permute x, y, z cyclically and/or multiply any two of them by -1. The 15 joins of the six points on Γ are

$$AA'\colon x = 0, \quad BB'\colon y = 0, \quad CC'\colon z = 0$$

and 12 others obtainable from any one of them by using $\mathscr{G}$.

$$\begin{array}{lll} BC'\colon ix+y+z=0 & CA'\colon iy+z+x=0 & AB'\colon iz+x+y=0 \\ B'C\colon ix-y-z=0 & C'A\colon iy-z-x=0 & A'B\colon iz-x-y=0 \\ BC\colon -ix+y-z=0 & CA\colon -iy+z-x=0 & AB\colon -iz+x-y=0 \\ B'C'\colon -ix-y+z=0 & C'A'\colon -iy-z+x=0 & A'B'\colon -iz-x+y=0 \end{array}$$

All these lines must be obtained as projections of 15 lines on a nodal cubic surface; so to this, and cognate matters, we now proceed.

5. Let $\mathscr{C}$ be the cubic surface

$$t(x^2+y^2+z^2)+2\lambda xyz = 0, \tag{5·1}$$

λ being any non-zero constant. It has a node N at $(0,0,0,1)$, the nodal cone being $x^2+y^2+z^2 = 0$. The six lines through N on $\mathscr{C}$ are those generators of this cone in the three planes $xyz = 0$, two in each plane; label them a, a' in $x = 0$, b, b' in $y = 0$, c, c' in $z = 0$. They meet $t = 0$ at A, A', B, B', C, C' composing the hexad in which the conic Γ, $x^2+y^2+z^2 = 0$, is cut by the sides of a self-polar triangle.

$\mathscr{C}$, like Γ, is invariant under a group, which we may still call $\mathscr{G}$, of 24 projectivities. These all leave the plane $t = 0$ fixed but impose all 4! permutations on the planes

$$x+y+z = 0, \quad x-y-z = 0, \quad y-z-x = 0, \quad z-x-y = 0$$

through N. $\mathscr{G}$ can now be imposed by 4-rowed monomial matrices one of whose non-zero entries is at the bottom right-hand corner; this entry is 1 or -1 according as its complementary 3-rowed monomial matrix has an even or an odd number of negative entries. The permutations of the pairs a, a', b, b', c, c' correspond to those of the pairs of opposite vertices of a regular octahedron when subjected to its group of 24 rotations. The self-conjugate four-group V of $\mathscr{G}$ consists of identity and those three biaxial harmonic inversions whose axes are the pairs of opposite edges of the tetrahedron of reference.

One tritangent plane of $\mathscr{C}$ is, clearly, $t = 0$. Another is $t = \lambda x$, for the substitution of λx for t in (5·1) leads to

$$\lambda x(y+z+ix)(y+z-ix) = 0;$$

the intersections of $t = \lambda x$ with the planes

$$x = 0, \quad y+z+ix = 0, \quad y+z-ix = 0$$

are lines on $\mathscr{C}$. Indeed these are the planes joining N to AA', BC', $B'C$; they have in common the line $N\alpha'$. If, now, x, y, z and, in consequence, A, B, C as well as A', B', C' are permuted cyclically it appears that

$$t = \lambda x, \quad t = \lambda y, \quad t = \lambda z$$

are the faces of T, the trihedron provided, exactly as in (2·2), by the rows of

$$\left.\begin{array}{lll} aa' & bc' & b'c \\ bb' & ca' & c'a \\ cc' & ab' & a'b \end{array}\right\}. \tag{5·2}$$

The columns of this scheme provide the faces of the companion trihedron T', the first column providing $t = 0$. Since each column is unchanged under the cyclic permutation one expects the second and third columns to provide tritangent planes

$$t+k(x+y+z) = 0;$$

if so, T' has a line-axis $t = x+y+z = 0$ meeting $\mathscr{C}$ in three E-points. Since T and T' are (reducible) members of the pencil of cubic surfaces through the nine lines of intersection of their faces there will be some identity.

$$(t-\lambda x)(t-\lambda y)(t-\lambda z)-t\{t+k(x+y+z)\}\{t+k'(x+y+z)\}$$
$$\equiv -\tfrac{1}{2}\lambda^2\{t(x^2+y^2+z^2)+2\lambda xyz\} \quad (5\cdot3)$$

which does indeed hold when

$$k+k'+\lambda = 0 \quad \text{and} \quad kk' = \tfrac{1}{2}\lambda^2;$$

the values of k and k' are $-\frac{1}{2}\lambda(1\pm i)$ and the faces of T' are

$$t = 0, \quad t-\tfrac{1}{2}\lambda(1+i)(x+y+z) = 0, \quad t-\tfrac{1}{2}\lambda(1-i)(x+y+z) = 0.$$

The projectivites of V applied to T and T' produce three more pairs of Steiner trihedra and three identities analogous to (5·3); all 15 tritangent planes are thereby accounted for. They are

$$t = 0, \quad t\pm\lambda x = 0, \quad t\pm\lambda y = 0, \quad t\pm\lambda z = 0,$$
$$t-\tfrac{1}{2}\lambda(1\pm i)(\pm x\pm y\pm z) = 0$$

with either none or two minus signs in $\pm x\pm y\pm z$. $\mathscr{C}$ has the six points $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ in $t = 0$ all E-points.

It will be noted that the vertex $(1, 1, 1, \lambda)$ of T is projected into $(1, 1, 1, 0)$, the pole of the axis $\alpha'\beta'\gamma'$ of T' with respect to Γ. One is allowed to use $t = 0$ as the plane π since it does not pass through N.

While each of the 15 tritangent planes contains three of the 15 lines it is also true that each of the 15 lines lies in three of the 15 tritangent planes; this is because each duad belongs to three synthemes.

6. Those lines in space of which the Pascal lines are projections are common to pairs of tritangent planes but are not on $\mathscr{C}$. If one of the 15 planes is chosen then, since each of the three lines in it lies in two more, those which cut it in lines whose projections are Pascal lines number $15-1-2.3 = 8$, so that there are, in general, $\frac{1}{2}.15.8 = 60$ Pasca lines ((6), p. 855). But for each Steiner trihedron with a line-axis there is a coincidence of three Pascal lines.

Each tritangent plane, as remarked in § 2, can be labelled by a syntheme. Now Sylvester explained (11) that one can, in six different ways, choose five synthemes so that the duads, three in each, account for all 15 duads of the original six objects: such a collection of five 'disjoint' synthemes he called a *synthematic total.* Each syntheme appears in two of the six totals; indeed once the six totals are known the 15 synthemes can be identified as each common to one of the 15 pairs of totals. These, naturally, have been exhibited before ((1), p. 221; (5), p. 280); but to serve our immediate purpose they

Table 1 (after Sylvester)

	1	2	3	4	5	6
1	—	$aa'.bb'.cc'$	$b'c.c'a.a'b$	$bc'.ca.a'b'$	$b'c'.ca'.ab$	$bc.c'a'.ab'$
2	$aa'.bb'.cc'$	—	$bc'.ca'.ab'$	$b'c.c'a'.ab$	$bc.c'a.a'b'$	$b'c'.ca.a'b$
3	$b'c.c'a.a'b$	$bc'.ca'.ab'$	—	$aa'.bc.b'c'$	$bb'.ca.c'a'$	$cc'.ab.a'b'$
4	$bc'.ca.a'b'$	$b'c.c'a'.ab$	$aa'.bc.b'c'$	—	$cc'.ab'.a'b$	$bb'.ca'.c'a$
5	$b'c'.ca'.ab$	$bc.c'a.a'b'$	$bb'.ca.c'a'$	$cc'.ab'.a'b$	—	$aa'.bc'.b'c$
6	$bc.c'a'.ab'$	$b'c'.ca.a'b$	$cc'.ab.a'b'$	$bb'.ca'.c'a$	$aa'.bc'.b'c$	—

Table 2. *Tritangent planes of* $t(x^2+y^2+z^2)+2\lambda xyz=0$

	1	2	3	4	5	6
1	—	t	$t-\frac{1}{2}\lambda(1-i)(x+y+z)$	$t-\frac{1}{2}\lambda(1-i)(x-y-z)$	$t-\frac{1}{2}\lambda(1-i)(y-z-x)$	$t-\frac{1}{2}\lambda(1-i)(z-x-y)$
2	t	—	$t-\frac{1}{2}\lambda(1+i)(x+y+z)$	$t-\frac{1}{2}\lambda(1+i)(x-y-z)$	$t-\frac{1}{2}\lambda(1+i)(y-z-x)$	$t-\frac{1}{2}\lambda(1+i)(z-x-y)$
3	$t-\frac{1}{2}\lambda(1-i)(x+y+z)$	$t-\frac{1}{2}\lambda(1+i)(x+y+z)$	—	$t+\lambda x$	$t+\lambda y$	$t+\lambda z$
4	$t-\frac{1}{2}\lambda(1-i)(x-y-z)$	$t-\frac{1}{2}\lambda(1+i)(x-y-z)$	$t+\lambda x$	—	$t-\lambda z$	$t-\lambda y$
5	$t-\frac{1}{2}\lambda(1-i)(y-z-x)$	$t-\frac{1}{2}\lambda(1+i)(y-z-x)$	$t+\lambda y$	$t-\lambda z$	—	$t-\lambda x$
6	$t-\frac{1}{2}\lambda(1-i)(z-x-y)$	$t-\frac{1}{2}\lambda(1+i)(z-x-y)$	$t+\lambda z$	$t-\lambda y$	$t-\lambda x$	—

must be displayed again here, in Table 1. Table 2 gives, in corresponding positions, linear forms which, when equated to zero, give equations for the tritangent planes. It may be noted, since it has relevance below, that the sum of the five forms in any row or column of Table 2 is $5t$.

Table 1 shows that the synthemes composing the columns of (5·2) are those identified by the pairs *12*, *23*, *31* of totals while those composing the rows are identified by the pairs *56*, *64*, *45*; the ten pairs of Steiner trihedra correspond not only to the ten partitions into complementary triads of a, b, c, a', b', c' but also to the ten analogous partitionings of *1*, *2*, *3*, *4*, *5*, *6*. Just to give a second example: the columns of

$$\begin{matrix} aa' & bb' & cc' \\ bc & c'a & a'b' \\ b'c' & ca' & ab \end{matrix}$$

are identified by *34*, *46*, *63* and the rows by *12*, *25*, *51*.

$\mathscr{G}$ is seen to leave both totals *1* and *2* invariant, although it may permute the five linear forms or multiply any of them by -1; it imposes all 4! permutations on *3*, *4*, *5*, *6*. For example, the cyclic permutation (*3546*) on these latter is induced by $(bcb'c')\,(a)\,(a')$, and this permutation of lines on $\mathscr{G}$ and its nodal cone occurs when x, y, z, t are, respectively, replaced by $x, -z, y, -t$. Since the trihedron whose faces are labelled by the synthemes common to the pairs of totals in *123* has a line-axis the same is, on applying $\mathscr{G}$, true of the trihedra whose faces are labelled by the synthemes common to the pairs of totals in *124*, in *125* or in *126*.

7. Veronese's original contribution to the geometry of the figure in π was to show that it consists of six Desargues figures each of ten Pascal lines concurrent by threes at ten points K, there being three K on each line. This was duly clarified in Cremona's treatment when he observed ((6), p. 857) that the 15 tritangent planes in space form six pentahedra, each plane being common to one pair of pentahedra. This is, of course, tantamount to Sylvester's distribution of the 15 synthemes among the six totals. The ten edges and ten vertices of any pentahedron are projected into the lines and points of a Desargues figure in π.

In general, no two of these six figures have any common line, so that all 60 Pascal lines are accounted for; but in the special circumstances which concern us this is far from being so. As the Pascal lines are the intersections of $t = 0$ with the planes joining edges of the pentahedra to N their equations are found by subtracting from each other any two entries in the same column of Table 2. It is clear, then, that the Desargues figures *1* and *2* are the same; the ten lines, may, in view of their relations to the other four figures, be separated as the four sides

$$x+y+z = 0, \quad x-y-z = 0, \quad y-z-x = 0, \quad z-x-y = 0$$

of a quadrilateral R and the six joins

$$y+z = 0, \quad z+x = 0, \quad x+y = 0, \quad y-z = 0, \quad z-x = 0, \quad x-y = 0$$

of the poles of the sides of R with respect to Γ.

Each of the other four figures includes among its ten lines one side of R and the three lines through its pole, R yielding one side to each of four figures *3*, *4*, *5*, *6*; this, too, is clear on observing the differences between linear forms in any column of Table 2. The 60 Pascal lines of the mystic hexagram thus coalesce as four, each reckoned three times, six, each reckoned four times, and 24 others.

This confluence of parts of a figure that are in general distinct occurs not only in π but also, though naturally to a lesser degree, in space. For the hexahedron which Cremona discovered and used so effectively ((6), pp. 854, 860) has now two of its faces coincident with $t = 0$; the other four can be shown to be $t+\lambda(\pm x \pm y \pm z) = 0$ with an even number of *minus* signs. And the identity

$$\{t+\lambda(x+y+z)\}^3+\{t+\lambda(x-y-z)\}^3+\{t+\lambda(y-z-x)\}^3+\{t+\lambda(z-x-y)\}^3+\tfrac{1}{16}(-4t)^3$$
$$\equiv 12\lambda^2\{t(x^2+y^2+z^2)+2\lambda xyz\} \qquad (7\cdot1)$$

shows that in Sylvester's canonical sum, of multiples of the cubes of five linear forms summing identically to zero, four of the five numerical multipliers are equal.

8. Since the relevant intersections of Pascal lines in π are projections of actual intersections of lines in space it follows ((10), p. 163) that the Veronese properties are possessed by sets of 15 lines other than the joins of six points on a conic; one need only project from a point other than N ((10), p. 167).

Take, then, a point O on $\mathscr{C}$; it is not, however, to lie on any line of $\mathscr{C}$. Its coordinates will be

$$(\xi, \eta, \zeta, -2\lambda\xi\eta\zeta/\sigma),$$

where, since O is not on the nodal cone,

$$\sigma = \xi^2+\eta^2+\zeta^2 \neq 0.$$

The polar quadric Q of O is

$$\Sigma \equiv \xi(tx+\lambda yz)+\eta(ty+\lambda zx)+\zeta(tz+\lambda xy)-\lambda\xi\eta\zeta(x^2+y^2+z^2)/\sigma = 0$$

and has the same tangent plane

$$\Lambda \equiv 2\lambda\{\eta\zeta(\sigma-2\xi^2)x+\zeta\xi(\sigma-2\eta^2)y+\xi\eta(\sigma-2\zeta^2)z\}+\sigma^2 t = 0$$

at O as does $\mathscr{C}$; Q and $\mathscr{C}$ meet in a sextic $\mathscr{S}$ with nodes at O and N, and the chords of $\mathscr{S}$ through O generate a quartic cone with nodal generator ON. The section of this cone by any plane not containing O is an apparent contour of $\mathscr{C}$ from O.

The quartic surfaces of the pencil

$$\{t(x^2+y^2+z^2)+2\lambda xyz\}\Lambda = k\Sigma^2 \qquad (8\cdot1)$$

all touch each other along the composite curve, consisting of $\mathscr{S}$ and two lines, in which any one of them cuts Q; one surface of the pencil is the cone of chords of $\mathscr{S}$ through O and is identified by containing ON. The plane $t = 0$ cuts the surfaces in a pencil of quartic curves, and one apparent contour of $\mathscr{C}$ from O is that curve of this pencil which contains the intersection $(\xi, \eta, \zeta, 0)$ of ON with $t = 0$. The necessity to contain ON requires $k = \sigma$ in (8·1); the section of the cone by $t = 0$ is the plane quartic

$$4\sigma\{\eta\zeta(\sigma-2\xi^2)x+\zeta\xi(\sigma-2\eta^2)y+\xi\eta(\sigma-2\zeta^2)z\}xyz$$
$$= \{\sigma(\xi yz+\eta zx+\zeta xy)-\xi\eta\zeta(x^2+y^2+z^2)\}^2. \qquad (8\cdot2)$$

This curve q has a node at (ξ, η, ζ), while the four linear factors on the left of (8·2) indicate manifest bitangents.

The crux of the matter is that the projections from O of the 15 lines such as ab are bitangents of q. It is true that there is a sixteenth bitangent – the line $\Lambda = t = 0$, intersection of $t = 0$ with the tangent plane of $\mathscr{C}$ at O and one of the four bitangents indicated by (8·2); but this line is specially related to q. There are six tangents to a nodal quartic δ from its node, and their six contacts are on a conic γ; the chord joining the other two intersections of γ and δ meets δ again in its two further intersections with its nodal tangents. This is so for any nodal quartic; but for q it happens that this latter line is a bitangent, the nodal tangents meeting q again on γ.

If the triangle of reference has its vertices at the node and at the further intersections of δ with its nodal tangents δ has an equation

$$X^2YZ + XU_3 + YZU_2 = 0,$$

U_j being homogeneous of degree j in Y and Z. The identity

$$X^2YZ + XU_3 + YZU_2 + YZ(X^2 - U_2) \equiv X(2XYZ + U_3), \tag{8·3}$$

the cubic on the right being the first polar of the node, shows that the six contacts of δ with the tangents from its node are on the conic $X^2 = U_2$. The special case of $X = 0$ being a bitangent occurs when U_2 is a multiple of YZ.

The equations of the 15 bitangents of q occur on putting zero for t in the equations of the planes joining the lines $ab, \ldots, b'c'$ to O: a routine matter for which the initial details are available in Tables 1 and 2. Since aa', bb', cc' are the lines $x = t = 0, y = t = 0, z = t = 0$ they provide the bitangents $x = 0$, $y = 0$, $z = 0$. When the equation of any of the twelve remaining bitangents has been found others can be derived from it by simultaneous cyclic permutations of ξ, η, ζ and of x, y, z. A single example must suffice to show the procedure. Suppose we require the bitangent that is the projection of ac'. Table 1 shows that the synthemes involving ac' are those common to the pairs 13, 25, 46 of totals; Table 2 then shows that ac' satisfies the equations

$$t = \tfrac{1}{2}(1-i)\,\lambda(x+y+z) = \tfrac{1}{2}(1+i)\,\lambda(y-z-x) = \lambda y.$$

The plane joining this line to O is

$$(t-\lambda y)\,\{\eta + i(\zeta+\xi)\} + \{y + i(z+x)\}\,(\lambda\eta + 2\lambda\xi\eta\zeta/\sigma) = 0,$$

and meets $t = 0$ in the bitangent

$$y\{i(\zeta+\xi) - 2\xi\eta\zeta/\sigma\} = i(z+x)\,(\eta + 2\xi\eta\zeta/\sigma). \tag{8·4}$$

The two actual contacts of a bitangent can, once its equation is known, be found by solving a quadratic. These 15 bitangents are certainly not the joins of six points, but they have the Veronese properties, or at least their specialized form when there are the four triple coincidences among the Pascal lines.

9. If, in particular, $\xi = \eta = \zeta$ one can carry through the whole process without the slightest difficulty. O is now $(1, 1, 1, -\frac{2}{3}\lambda)$ and is the only intersection, other than N, of $\mathscr{C}$ with the line $x = y = z$; it does not lie on the nodal cone, nor is it a zero of any of the entries in Table 2; it is therefore eligible as a centre of projection. The resulting plane quartic is, by (8·2)

$$12(x+y+z)\,xyz = \{3(yz+zx+xy) - (x^2+y^2+z^2)\}^2 \tag{9·1}$$

Table 3. *Fifteen bitangents of the nodal quartic* (9·1)

$b'c$	$4x = (3-i)(y+z)$	bc'	$4x = (3+i)(y+z)$
$c'a$	$4y = (3-i)(z+x)$	ca'	$4y = (3+i)(z+x)$
$a'b$	$4z = (3-i)(x+y)$	ab'	$4z = (3+i)(x+y)$
bc	$2ix = z-y$	$b'c'$	$2ix = y-z$
ca	$2iy = x-z$	$c'a'$	$2iy = z-x$
ab	$2iz = y-x$	$a'b'$	$2iz = x-y$
aa' $x=0$	bb' $y=0$	cc' $z=0$	

having a node at (1, 1, 1). There is complete symmetry in x, y, z. The nodal tangents are those lines through the node which, with the node itself, are invariant under the projectivity which permutes x, y, z cyclically; namely, if ω is a complex cube root of 1,

$$Y \equiv x+\omega y+\omega^2 z = 0, \quad Z \equiv x+\omega^2 y+\omega z = 0.$$

If $X \equiv x+y+z$ (9·1) is equivalent to

$$8X^2YZ+4X(Y^3+Z^3) = 25Y^2Z^2;$$

the identity corresponding to (8·3) is

$$8X^2YZ+4X(Y^3+Z^3)-25Y^2Z^2+YZ(8X^2+25YZ) \equiv 4X(4XYZ+Y^3+Z^3).$$

The 15 relevant bitangents are shown in Table 3. The symmetry in x, y, z allows equations of further bitangents to be derived from any one, and should a complex coefficient appear in the equation of any bitangent the equation of another is obtained by conjugation. For example: (8·4) serves to show that $c'a$ is $4y = (3-i)(z+x)$; cyclic permutation of x, y, z produces $a'b$ and $b'c$, and then conjugation produces bc', ca', ab'. And, again, the two contacts of any one bitangent can be found explicitly by solving a quadratic equation. Just to give a single instance: the contacts of $b'c'$ are its intersections with the pair of lines

$$(1+2i)y^2-\tfrac{2}{3}yz+(1-2i)z^2 = 0.$$

The Pascal lines, in $t=0$, grouped as six Desargues figures, that these bitangents give rise to do not involve as many coincidences as do those arising from the joins of intersections of Γ with the sides of a self-polar triangle; this dilution is due to the choice of O instead of N for the centre of projection. For O is not invariant under the whole octahedral group $\mathscr{G}$ but only under the dihedral group of permutations of x, y, z; harmonic inversion in a pair of opposite edges of the tetrahedron of reference moves O to another position on $\mathscr{C}$. The six ten-line figures are now distinct, though there is certainly overlapping; indeed those four lines that are each a coincidence of three are *in* the plane $t=0$ apart from any projection and so appear again with the same property.

One sets out, as in § 7, from Table 2. But now it is not mere subtraction of entries from each other that is involved: one requires planes joining lines of intersection of tritangent planes not to N but to O; once the equation of such a plane is known its intersection with $t = 0$ is the Pascal line in question. There is no call to give details; but figure *1* is found to consist of the lines

$$x+y+z = 0, \quad x-y-z = 0, \quad y-z-x = 0, \quad z-x-y = 0,$$

$$y = z, \quad z = x, \quad x = y,$$

$$6x = (5+2i)(y+z), \quad 6y = (5+2i)(z+x), \quad 6z = (5+2i)(x+y),$$

while figure *2* has these last three lines replaced by their complex conjugates.

The first four lines, as has been anticipated, belong not only to *1* and *2* but also, respectively, to *3, 4, 5, 6*. The next three lines belong, all of them, to *1, 2, 3* and one each to one of *4, 5, 6*. Apart from these seven lines, four of them coincidences of three and the other three of four Pascal lines, the remaining Pascal lines are 'simple'; there are three in *1* and *2*, six in *3*, eight in each of *4, 5, 6*.

10. Whereas $\mathscr{C}$ has E-points at the six vertices of a plane quadrilateral Clebsch's diagonal surface $\mathscr{D}$ has this property in quintuplicate having E-points at the ten vertices of a pentahedron $\wp$ ((4), p. 333). When the faces of $\wp$ are chosen as reference planes for supernumerary coordinates X, Y, Z, T, U, and multipliers are adjusted so that the one linear identity between the coordinates is

$$X+Y+Z+T+U \equiv 0, \tag{10·1}$$

the equation of $\mathscr{D}$ is

$$X^3+Y^3+Z^3+T^3+U^3 = 0. \tag{10·2}$$

$\mathscr{D}$ contains the three diagonals of the quadrilateral of lines in which any face of $\wp$ is cut by the other four. Since

$$(X+Y+Z)^3 \equiv 3(Y+Z)(Z+X)(X+Y)+X^3+Y^3+Z^3$$

and

$$(T+U)^3 \equiv 3TU(T+U)+T^3+U^3$$

it follows, in virtue of (10·1), that one of ten forms for the equation of $\mathscr{D}$ is

$$(Y+Z)(Z+X)(X+Y)+TU(T+U) = 0, \tag{10·3}$$

exhibiting a pair of Steiner trihedra, one with point-vertex $(0, 0, 0, 1, -1)$, the other with line-axis $T = U = 0$. This point and line are a vertex and opposite edge of $\wp$.

Any expectation that $\mathscr{D}$ will afford a plane figure endowed in quintuplicate with the specialization of the figure investigated above cannot be fulfilled in Cremona's manner unless $\mathscr{D}$ has a node. Since the conditions for this are

$$X^2 = Y^2 = Z^2 = T^2 = U^2 \tag{10·4}$$

they cannot, subject to (10·1), be satisfied anywhere, let alone on $\mathscr{D}$, unless the field from which the coordinates are chosen has characteristic $p = 3$ or $p = 5$. If $p = 3$ (10·2) is not available and one must use (10·3). A solution of (10·4) is then $(-1, 1, 1, 1, 1)$, but none of the five such points is on $\mathscr{D}$. But if $p = 5$ the unit point satisfies all of (10·1),

Table 4. *Tritangent planes of a (nodal!) diagonal surface over GF(5)*

	1	*2*	*3*	*4*	*5*	*6*
1	—	t	$t+x+y+z$	$t+x-y-z$	$t+y-z-x$	$t+z-x-y$
2	t	—	$t-2(x+y+z)$	$t-2(x-y-z)$	$t-2(y-z-x)$	$t-2(z-x-y)$
3	$t+x+y+z$	$t-2(x+y+z)$	—	$t+x$	$t+y$	$t+z$
4	$t+x-y-z$	$t-2(x-y-z)$	$t+x$	—	$t-z$	$t-y$
5	$t+y-z-x$	$t-2(y-z-x)$	$t+y$	$t-z$	—	$t-x$
6	$t+z-x-y$	$t-2(z-x-y)$	$t+z$	$t-y$	$t-x$	—

$\mathscr{D}: t(x^2+y^2+z^2)+2xyz = 0.$

(10·2), (10·4) so that one may say that, over the Galois Field $\mathscr{F} \equiv GF(5)$, the diagonal surface has a node. Indeed, if

$$\begin{aligned} X &\equiv x-y-z+t, \\ Y &\equiv -x+y-z+t, \\ Z &\equiv -x-y+z+t, \\ T &\equiv \qquad\qquad\quad t, \\ U &\equiv x+y+z+t, \end{aligned}$$

then, over $\mathscr{F}$ which consists of $-2, -1, 0, 1, 2$ with the usual additive and multiplicative properties to modulus 5, not only is (10·1) satisfied but (and one may compare (7·1) with $\lambda = 1$)

$$X^3+Y^3+Z^3+T^3+U^3 \equiv 2\{t(x^2+y^2+z^2)+2xyz\}.$$

11. The tritangent planes of $\mathscr{D}$, the nodal diagonal surface over $\mathscr{F}$, can be found as were those of $\mathscr{C}$ in § 5. The identity analogous to (5·3) is, over $\mathscr{F}$,

$$(t-x)(t-y)(t-z)-t(t+x+y+z)\{t-2(x+y+z)\} \equiv 2\{t(x^2+y^2+z^2)+2xyz\}$$

and the tritangent planes are displayed in Table 4, where, it will be observed, the sum of the five entries in any row or column is identically zero over $\mathscr{F}$.

Each separation of *1, 2, 3, 4, 5, 6* into complementary triads yields a pair of Steiner trihedra for $\mathscr{D}$. For example: the separation *135 246* gives a trihedron of planes

$$t+x+y+z = 0, \quad t-x+y-z = 0, \quad t+y = 0$$

common to the pairs *13, 15, 35* of the totals in Table 4; this trihedron has a line-axis $t+y = x+z = 0$. Its companion trihedron consists of the planes

$$t = 2(x-y-z), \quad t = 2(z-x-y), \quad t = y$$

common to the pairs *24, 26, 46* of totals; this trihedron has the point vertex (1, 0, 1, 0). There is, over $\mathscr{F}$, a corresponding identity

$$\begin{aligned}(t+x+y+z)(t-x+y-z)(t+y)-\{t-2(x-y-z)\}\{t-2(z-x-y)\}(t-y) \\ \equiv -2\{t(x^2+y^2+z^2)+2xyz\},\end{aligned}$$

and a corresponding identity will occur for each such separation of the six totals. The operations of $\mathscr{G}$ are still available and may be used to deduce other identities from any given one.

12. Now that coordinates are no longer complex numbers but are restricted to $\mathscr{F}$ there are further coincidences in the geometry resulting from projection from N; it is still true that the Desargues figures *1* and *2* are the same, but now *all six figures are the same.* For a mere glance at Table 4 shows that the differences between the ten pairs of entries in any column are, whichever column is used, multiples over $\mathscr{F}$ of the same ten linear forms. That this should happen is only to be expected, for each figure consists of ten Pascal lines of a hexad points of Γ. But, over $\mathscr{F}$, this hexad is the whole of Γ, and only ten Pascal lines are available ((8), p. 117). The plane $t = 0$, like any other plane over $\mathscr{F}$, contains 31 lines; of these 15 are chords and 6 'tangents' of Γ; only ten, those lines skew to Γ, remain to serve as Pascal lines.

13. The projection from a point of $\mathscr{D}$ other than its node is not available. Its intersections with the faces of $\wp$ are completely accounted for by 15 lines and it appears that, save for the unit point, there is no point with all coordinates non-zero that satisfies, over $\mathscr{F}$, both (10·1) and (10·2). Since there are only four non-zero marks in $\mathscr{F}$, and since their sum is zero, the sum of five non-zero marks of which one only, say α, is repeated is not zero but α; (10·1) is not satisfied. It is, admittedly, possible for a zero sum to occur with non-zero marks of which three are equal, say $1, 1, 1, -1, -2$; but the sum of their five cubes is not zero; (10·2) is not satisfied. And if four of five marks are equal the fifth must, to ensure a zero sum, also be equal to them and the corresponding point of $\mathscr{D}$ is its node.

All 5! permutations of

$$-2, -1, 0, 1, 2$$

provide coordinate vectors of points on $\mathscr{D}$, but as the coordinates are homogeneous only $\frac{1}{4}.5! = 30$ points occur, six in each face of π; these are the intersections of π with the six lines on $\mathscr{D}$ through its node: over $\mathscr{F}$, identically in μ,

$$(\mu-2)^3+(\mu-1)^3+\mu^3+(\mu+1)^3+(\mu+2)^3 \equiv 0.$$

The remaining points on $\mathscr{D}$ are its node, the ten E-points, vertices of $\wp$, and the diagonal points of the quadrilaterals in the faces of $\wp$.

The complete intersection of a face of $\wp$ with $\mathscr{D}$ consists of the diagonals of the quadrilateral; each diagonal consists of six points, namely the two opposite vertices of the quadrilateral that it joins, its intersections with the other two diagonals, and two more points on lines of $\mathscr{D}$ through its node.

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